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Analytical solutions of the nonlinear groundwater flow equation in unconfined aquifers and the effect of heterogeneity

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Abstract. Using the method of decomposition, new analytical solutions of the nonlinear Boussinesq flow equation and of the exact two-dimensional groundwater flow equation subject to a nonlinear free-surface boundary condition are presented and tested with respect to the linearized Boussinesq equation. It is found that for mild regional gradients and for the range of recharge values usually encountered in the field, the extensively used linearized equation with Dupuit assumptions is a reasonable approximation to the exact solution for the hydraulic heads and the regional flow velocities. Discrepancies occur in the presence of high regional hydraulic gradients, unusually high recharge rates, or regions of low hydraulic conductivity. With this result a new nonperturbation solution of the two-dimensional plan view groundwater flow equation in a heterogeneous (heterogeneity represented as a spatial random field) aquifer is presented, and expressions for the statistical properties of the longitudinal and transverse velocities are derived. The results suggest that scale dependency, or spatial variability, in the flow velocity arises naturally as a result of recharge from rainfall and aquifer heterogeneity and may help explain the scale dependency of aquifer dispersion parameters.

1. Introduction

One of the difficult problems in hydrology is the proper mathematical representation of regional groundwater flow in unconfined aquifers. Variables of interest in groundwater management are the distribution of hydraulic heads and groundwater flow velocities, which in turn are used as the building blocks for capture zone analysis and contaminant dispersion models. The exact differential equation governing groundwater flow in unconfined aquifers is the Laplace's equation. This equation is usually subject to a set of boundary conditions; the most difficult to handle is the free-surface boundary condition, a nonlinear partial differential equation itself which creates the need for its solution (knowledge of its position) prior to the solution of the domain equation. However, part of the domain equation solution is finding the position of the free surface.

Common approaches to solve this problem include trial and error procedures to approximate the location of the free surface in combination with numerical solutions of the differential equations [i.e., *Hunt*, 1983; *Bear and Verruijt*, 1987; *Huyakorn and Pinder*, 1983; *Ligget and Liu*, 1983; *Pinder and Gray*, 1977] and linearization of the free-surface boundary condition and the analytical solution of the differential equation [*Van de Giesen et al.*, 1994; *Polubarinova-Kochina*, 1962]. Two-dimensional problems can sometimes be solved with the use of the inverse velocity hodograph [*Aravin and Numerov*, 1965; *Harr*, 1962; *Polubarinova-Kochina*, 1962], the application of stochastic methods in combination with numerical methods [*Serrano and Unny*, 1986]; or the adoption of Dupuit assumptions of essentially horizontal flow that eliminate the vertical coordinate and the free-surface boundary condition. In the latter the resulting domain equation, the Boussinesq flow equation, is itself a nonlinear equation, which is subsequently linearized or discretized to facilitate its solution [*Knight*, 1981; *Dagan*, 1966; *Kirkham*, 1966; *Charnyi*, 1951; *Muskat*, 1937]. Although it is known that these solutions offer acceptable accuracy for mildly sloping aquifers, the accuracy of the linearized solution with respect to the nonlinear solution and with respect to the exact solution has not been tested, in part because of the scarcity of systematic analytical methods to solve nonlinear equations.

In this article a comparison between the three solutions (i.e., linearized Boussinesq, nonlinear Boussinesq, and exact Laplace's subject to the nonlinear free-surface boundary condition) is attempted for the hydraulic heads and the groundwater flow velocities (section 2). New contributions in the decomposition method [*Adomian*, 1994, 1991, 1986] allow a systematic treatment and solution of nonlinear, deterministic or stochastic, ordinary, partial, or integro-differential equations. The decomposition method does not require linearization or small perturbation assumptions, which are usually adopted to make the mathematics tractable and not necessarily to reflect field conditions. It generates a series solution, much like Fourier series, which usually converge rapidly. Finally, the solution is analytic since no domain discretization, and its associated massive computation, is required. Applications of decomposition methods are beginning to appear in the hydrologic literature [i.e., *Serrano and Unny*, 1987; *Serrano*, 1993] with promising possibilities for the solution of increasingly complex problems.

In section 3 a new nonperturbation solution of the two-dimensional plan groundwater flow equation in a heterogeneous (heterogeneity represented as a spatial random field) aquifer is presented. This solution is not attached to the limitations of existing small-perturbation solutions (variances are not required to be "small"); the actual transmissivity field,

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rather than its logarithm, is considered; and the underlying groundwater flow problem with its regional hydrology (i.e., boundary conditions, recharge from rainfall) is specifically included. Expressions for the statistical properties of the longitudinal and transverse velocities are derived. It is hoped that the resulting statistical properties of the velocity field will aid in the building of scale-dependent groundwater transport models.

2. Analysis of the Groundwater Flow Equation in Unconfined Aquifers

The exact differential equation governing groundwater flow in a steady homogeneous unconfined aquifer is given by

$$\nabla^2\phi = 0, \quad \phi \in \Gamma \tag{1}$$

where $\phi(x, y, z)$ represents the hydraulic head (meters) within a three-dimensional domain, $\Gamma \subset \Re^3$, and (x, y, z) represent the spatial coordinates, with z the vertical dimension. Equation (1) is subject to

$$\begin{gathered} Q\phi = J, \qquad \phi \in \partial\Gamma \\ (\nabla\phi)^2 - \frac{\partial \phi}{\partial z} = \frac{I}{K}\left(\frac{\partial \phi}{\partial z} - 1\right) \qquad \text{on} \quad z = h \\ \phi = h \qquad \text{on} \quad z = h \end{gathered} \tag{2}$$

where Q is a boundary operator, J is a known function, I is the aquifer recharge from rainfall (meters per month), K is the aquifer hydraulic conductivity (meters per month), and $h(x, y)$ is the elevation of the free surface (meters). The first condition in (2) represents the presence of Dirichlet, Neumann (or a combination of the two) boundary conditions, the second condition satisfies continuity at the free surface, and the third condition requires that the hydraulic head at the free surface must equal its elevation head.

For nearly horizontal flow in mildly sloping aquifers the hydraulic head along the vertical is similar to that at the free surface, that is, $\phi \approx h$. This is the common Dupuit assumption which obviates the vertical coordinate, neglects the vertical component of flow, eliminates the need of free-surface boundary conditions in (2), and results in the Boussinesq flow equation

$$\begin{gathered} \nabla_2(Kh\nabla_2 h) = -I \qquad h \in \Gamma_1 \subset \Re^2 \\ Q_1 h = J_1 \qquad h \in \partial\Gamma_1 \end{gathered} \tag{3}$$

where ∇_2 is a horizontal gradient operator, Γ_1 is the two-dimensional areal domain corresponding to Γ, Q_1 is a boundary operator, and J_1 is a known function.

Equation (3) is nonlinear; its solution would require a domain discretization and numerical approximation or a linearization and analytical solution. A linearization scheme conceives the term $T = Kh$ as a vertically averaged hydraulic conductivity, or transmissivity, and reduces (3) to

$$\nabla_2(T\nabla_2 h) = -I \tag{4}$$

An analytical solution would further require the assumption of T as constant.

Few studies have actually compared solutions to (4) with exact solutions to (3) and (1) or numerical solutions to (3) with exact solutions to (3) or (1). This leaves several physical questions unanswered, such as the range of applicability of linearized solutions, their accuracy, the effect of various sets of boundary conditions, and the effect of significant recharge rates on the vertical component of the flow velocity, among others. The present article moves in such a direction by attempting some exact solutions to (1) and (3).

Consider initially a horizontal aquifer of length l_x, bounded by two constant-head boundary conditions, H_1 and H_2, respectively. Locating the origin, $x = 0$, at the left boundary and the datum at the bottom of the aquifer and assuming constant transmissivity, (4) reduces to

$$\begin{gathered} \frac{\partial^2 h}{\partial x^2} = -\frac{I}{T} \qquad 0 \le x \le l_x \\ h(0) = H_1, \quad h(l_x) = H_2 \end{gathered} \tag{5}$$

whose solution is simply

$$h(x) = -(I/2T)x^2 + Ax + B \tag{6}$$

and the constants $A = a + (Il_x/2T)$, $a = (H_2 - H_1)/l_x$, $B = H_1$. If the transmissivity is not constant, instead of (5), the governing equation is, from (4),

$$\partial/\partial x[Kh(\partial h/\partial x)] = -I \tag{7}$$

subject to the same boundary conditions as (5). Let us write this equation as

$$\frac{\partial^2 h}{\partial x^2} = -\frac{I}{Kh} - \frac{1}{h}\left(\frac{\partial h}{\partial x}\right)^2, \tag{8}$$

define the operator $L_x = (\partial^2/\partial x^2)$, and apply its inverse, L_x^{-1}, defined as a double spatial integration, on (8) to obtain

$$\begin{gathered} h = -L_x^{-1}\frac{I}{Kh} - L_x^{-1}\frac{1}{h}\left(\frac{\partial h}{\partial x}\right)^2, \\ h = ax + B - L_x^{-1}Nh, \\ Nh = \frac{1}{h}\left\{\frac{I}{K} + \left(\frac{\partial h}{\partial x}\right)^2\right\}, \end{gathered} \tag{9}$$

As usual in the decomposition method [*Adomian*, 1994, 1991, 1986] for nonlinear equations we define the series solution for (9) as

$$h = \sum_{n=0}^{\infty} h_n = h_0 - L_x^{-1}\sum_{n=0}^{\infty} A_n, \tag{10}$$

where the first term satisfies the homogeneous version of (8), that is,

$$h_0 = ax + B \tag{11}$$

Subsequent terms are defined as

$$\begin{gathered} h_1 = -L_x^{-1}A_0, \\ h_2 = -L_x^{-1}A_1, \\ \vdots \\ h_{n+1} = -L_x^{-1}A_n \end{gathered} \tag{12}$$

and the series expansion A_n for the nonlinear term N in (9) is defined as

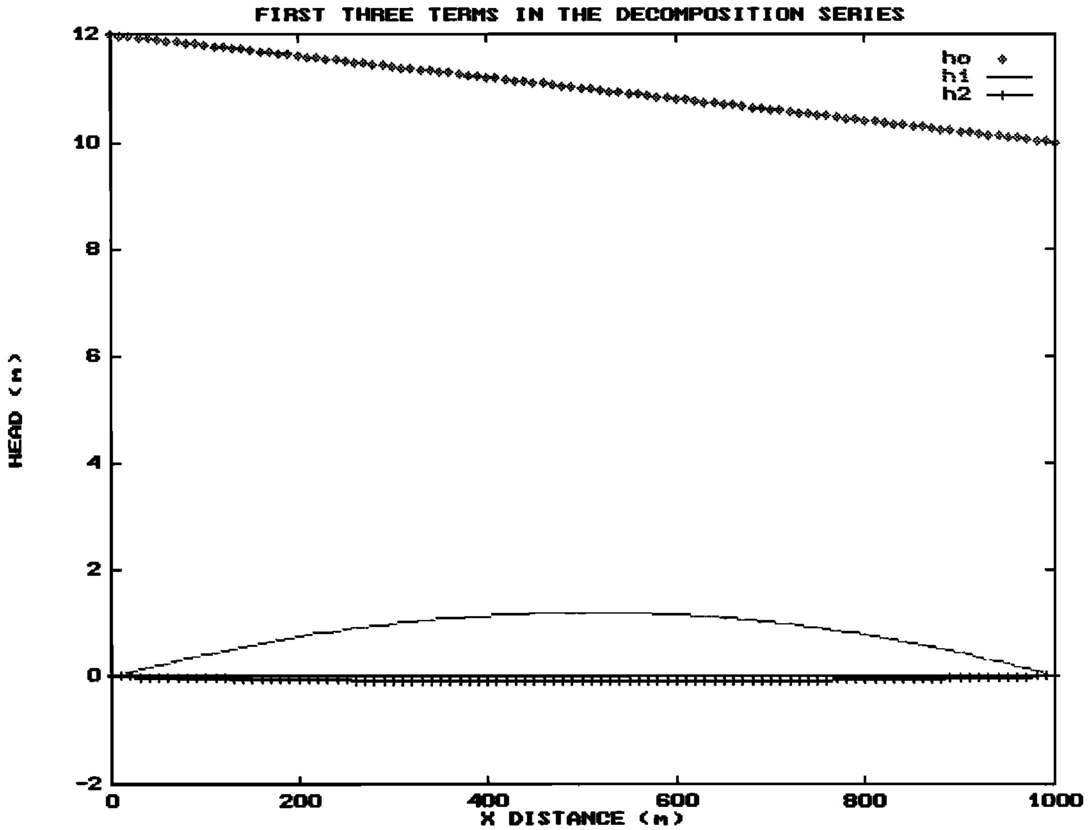

Figure 1. First three terms in the decomposition series.

$$
\begin{aligned}
A_0 &= N(h_0),\\
A_1 &= h_1 \frac{dN(h_0)}{dh_0},\\
A_2 &= h_2 \frac{dN(h_0)}{dh_0} + \frac{h_1^2}{2!}\frac{d^2N(h_0)}{dh_0^2},\\
A_3 &= h_3 \frac{dN(h_0)}{dh_0} + h_1h_2 \frac{d^2N(h_0)}{dh_0^2} + \frac{h_1^3}{3!}\frac{d^3N(h_0)}{dh_0^3}\\
&\vdots
\end{aligned}
\tag{13}
$$

The polynomials A_n are generated for each nonlinearity so that A_0 depends only on h_0, A_1 depends only on h_0 and h_1, A_2 depends only on h_0, h_1, h_2, etc. All of the h_n components are calculable. It is now established that the series $\Sigma_{n=0}^{\infty} A_n$ for Nh is equal to a generalized Taylor series for $f(h_0)$, that $\Sigma_{n=0}^{\infty} h_n$ is a generalized Taylor series about the function h_0, and that the series terms approach zero as $1/(mn)!$, if m is of the order of the highest linear differential operator. Since the series converges and does so very rapidly, the n term partial sum $\Phi_n = \Sigma_{i=0}^{n-1} h_i$ usually serves as an accurate enough and practical solution. Thus from (10) to (13),

$$
\begin{aligned}
h_1 &= -L_x^{-1}\frac{1}{h_0}\left\{\frac{I}{K} + \left(\frac{dh_0}{dx}\right)^2\right\}\\
h_1 &= \hat{h}_1 + Ex + F,\\
\hat{h}_1(x) &= -\frac{W}{a^2}\{h_0(x)Ln[h_0(x)] - h_0(x)\},\quad W = \frac{I}{K} + a^2,\\
F &= -\hat{h}_1(0), \qquad E = \frac{\hat{h}_1(0) - \hat{h}_1(l_x)}{l_x}
\end{aligned}
\tag{14}
$$

Similarly,

$$
\begin{aligned}
h_2 &= L_x^{-1} h_1 \frac{d}{dh_0}\left\{\frac{1}{h_0}\left[\frac{I}{K} + \left(\frac{dh_0}{dx}\right)^2\right]\right\},\\
h_2 &= \hat{h}_2(x) + Gx + M,\\
\hat{h}_2 &= -\frac{W^2h_0(x)Ln^2[h_0(x)]}{2a^4} - \left(\frac{2W}{a} + E\right)\frac{\hat{h}_1(x)}{a}\\
&\quad + \left(\frac{EB}{a} - F\right)\frac{WLn[h_0(x)]}{a^2},\\
M &= -\hat{h}_2(0), \qquad G = \frac{\hat{h}_2(0) - \hat{h}_2(l_x)}{l_x}
\end{aligned}
\tag{15}
$$

An objective way to observe the uniform convergence of the decomposition series and at the same time decide the level of truncation is to calculate the individual terms. For instance, if we set $H_1 = 12.0$ m, $H_2 = 10.0$ m, $l_x = 1000.0$ m, $I = 0.01$ m/month, and $K = 100.0$ m/month and use (11), (14), and (15) to calculate the first three terms in the series (Figure 1), we observe that the series converges very fast and that terms beyond h_2 are negligible. Thus we define the approximant as

$$\Phi_3 \approx h = h_0 + h_1 + h_2 \tag{16}$$

It is interesting to compare the solution to the linearized equation (6) with the solution to the nonlinear Boussinesq flow equation (16) for various parameter values. From extensive simulations the following observations may be noted:

1. For low values of recharge, $I < 0.01$ m/month, and high values of hydraulic conductivity, $K > 100.0$ m/month, the linearized solution is close to the nonlinear solution.

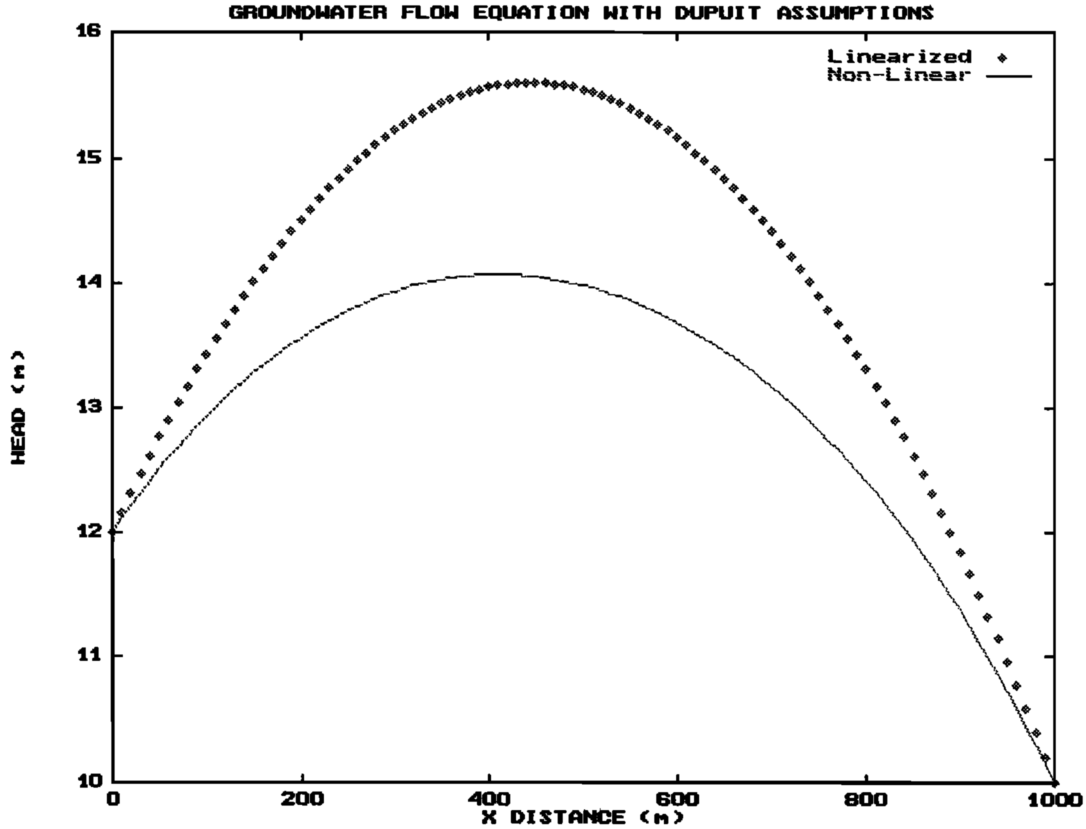

Figure 2. Linearized versus nonlinear groundwater flow equation with Dupuit assumptions.

2. When the above condition is not satisfied, that is, for I large and/or K small, the linearized solution tends to overestimate the magnitude of the hydraulic head. Figure 2 illustrates such a case.

3. For case 2 above the convergence of the decomposition series is slower, and more terms may be needed.

Groundwater flow velocities may be calculated from $u = -K(\partial h)/(\partial x)$. Thus from (6) the flow velocity, or specific discharge, according to the linearized solution is

$$u(x) \approx (2Ix/al_x) - KA \tag{17}$$

From (16) it is easy to observe that the gradient of h_2 is negligible. Thus the flow velocity according to the nonlinear solution is

$$u(x) \approx -K\{a - (W/a)Ln[h_0(x)] + E\} \tag{18}$$

Comparison between (17) and (18) for different parameter values yielded similar conclusions as for the head simulations above; that is, for large recharge rates or small hydraulic conductivity values the linearized equation overestimates the flow velocities. Figure 3 shows such a situation. For mild recharge values in highly transmissive aquifers the linearized equation appears to be a reasonable approximation.

Consider now the exact boundary value problem ((1) and (2)) for the above aquifer. Let us formulate the hypothesis that the free surface is a known function, $h(x)$, given by (6). After deriving a solution of (1) and (2), subject to (6), the accuracy of this hypothesis may be tested by comparing $h(x)$ with $\phi(x, z)$ on $z = h(x)$. This would provide an objective way to assess the accuracy of Dupuit equation solutions with respect to exact solutions. From (1) and (2),

$$\begin{aligned}
&\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad 0 \le x \le l_x,\ 0 \le z \le h(x),\\
&\phi(0, z) = H_1,\ \phi(l_x, z) = H_2,\ \frac{\partial\phi}{\partial z}(x, 0) = 0,\\
&\phi(x, h) = z,\\
&\frac{\partial\phi}{\partial z}(x, h) = \frac{I}{K+I} + \frac{K}{K+I}\left(\frac{\partial\phi}{\partial x}(x, h)\right)^2\\
&\qquad + \frac{K}{K+I}\left(\frac{\partial\phi}{\partial z}(x, h)\right)^2
\end{aligned} \tag{19}$$

where the last two lines in the system (19) represent the free-surface boundary condition. The nonlinear boundary condition is a boundary-value problem in itself that may be written as

$$\begin{aligned}
&\frac{\partial\phi}{\partial z} = g + m\left(\frac{\partial\phi}{\partial x}\right)^2 + m\left(\frac{\partial\phi}{\partial z}\right)^2 \qquad 0 \le x \le l_x,\ z = h(x),\\
&\phi(0, z) = H_1,\ \phi(l_x, z) = H_2,\ \phi(x, 0) = 0
\end{aligned} \tag{20}$$

where $g = I/(K + I)$, $m = K/(K + I)$. The decomposition solution for (20) could be written as $\phi = \phi_0 + \phi_1, \cdots$, where

$$\begin{aligned}
&\phi_0(x, z) = c_0k_0(x) + gz = h(x),\\
&\phi_0 = (1 - g)h + gz,
\end{aligned} \tag{21}$$

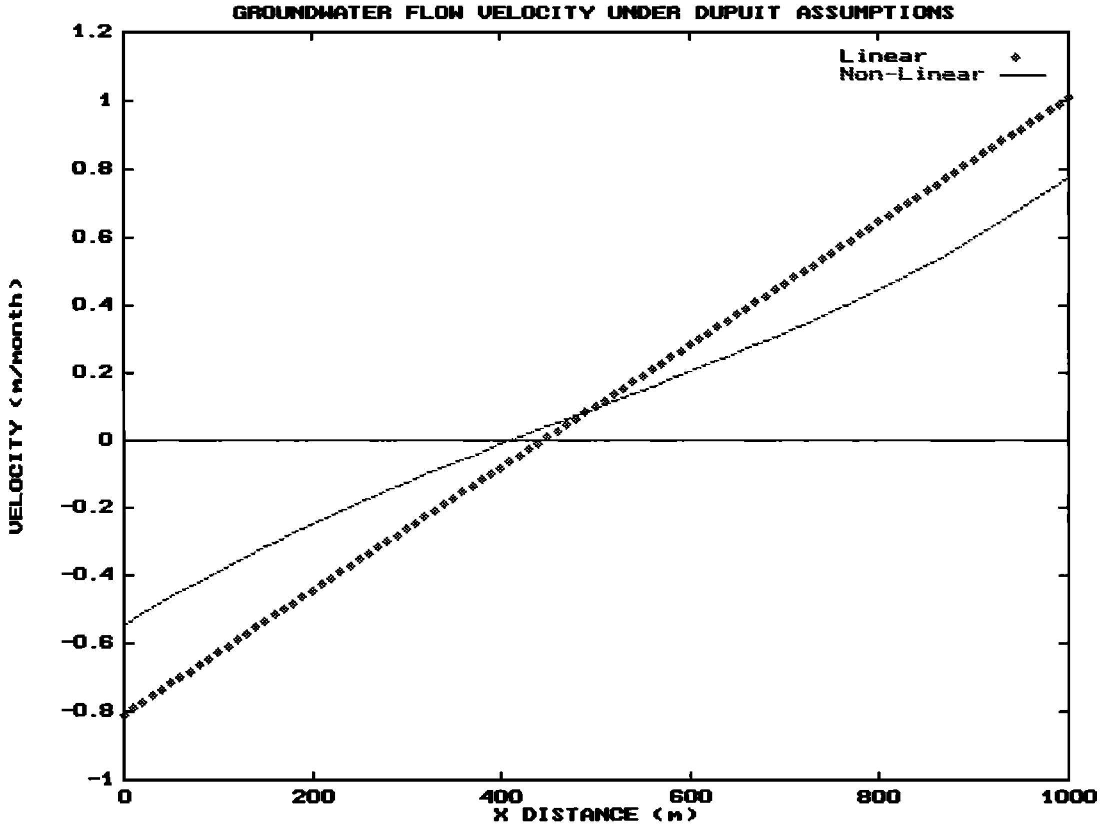

Figure 3. Linear versus nonlinear groundwater flow velocity under Dupuit assumptions.

$$\phi_1 = mL_z^{-1}\left(\frac{\partial \phi_0}{\partial x}\right)^2 + mL_z^{-1}\left(\frac{\partial \phi_0}{\partial z}\right)^2, \quad L_z = \frac{\partial}{\partial z},$$
$$\phi_1 = (m(1-g)^2h'^2 + mg^2)z \tag{22}$$

It is easy to see that terms ϕ_i, $i \geq 2$ are negligible. Thus an approximation of (20) is

$$\phi(x, z) \approx \phi_0 + \phi_1 = (1-g)h + w(x)z \quad \text{on} \quad z = h(x),$$
$$w(x) = [g + mg^2 + m(1-g)^2h'^2]z \tag{23}$$

This is the modified free-surface boundary condition for system (19). We now turn our attention to the solution of (19), subject to (23) in the domain $\Gamma = [0, l_x] \times [0, h]$. Operating on $L_x = \partial^2/\partial x^2$, and $L_z = \partial^2/\partial z^2$, the domain equation (19) generates the following equations, respectively [*Adomian*, 1994]:

$$\phi = c_1k_1(z) + c_2k_2(z)x - L_x^{-1}L_z\phi, \tag{24}$$

$$\phi = c_3k_3(x) + c_4k_4(x)z - L_z^{-1}L_x\phi \tag{25}$$

From (24), and considering the x boundary conditions, the series solution would be

$$\phi_0 = h(x), \ \phi_i = 0 \qquad i \geq 1 \tag{26}$$

From (25)

$$\phi_0 = c_3k_3(x) + c_4k_4(x)z \tag{27}$$

Using the bottom boundary condition in (19) and the top in (23),

$$\phi_0 = [(1-g) + w(x)]h \tag{28}$$

Similarly, from (25), and using (28),

$$\phi_1 = -L_z^{-1}(\partial^2\phi_0/\partial x^2)$$
$$\phi_1 = -[1 + mg^2 + m(1-g)^2(2h''h + 5h'^2)]h''(z^2/2) \tag{29}$$

and the solution of (25) is approximated as $\phi \approx \phi_0 + \phi_1$, where ϕ_0 is given by (28) and ϕ_1 is given by (29). Thus from (26), (28), and (29) the solution to (19) is approximated as

$$\phi(x, z) = (1 + Y)(h/2) - (Vh''z^2/4),$$
$$Y = 1 + mg^2 + m(1-g)^2h'^2,$$
$$V = 1 + mg^2 + m(1-g)^2(2h''h + 5h'^2) \tag{30}$$

Numerical tests indicated that the following approximations to (30) hold:

$$V \approx 1 + (2h''h + 5h'^2), \ Y \approx 1 + h'^2,$$
$$\phi(x, z) \approx (2 + h'^2)(h/2) - (1 + 2h''h + 5h'^2)h''(z^2/4) \tag{31}$$

An assessment of the accuracy of the linearized Dupuit equation as a means to approximate the free-surface profile and the groundwater velocity is accomplished by comparing (6) with (31), evaluated at $z = h$. Figure 4 illustrates an example of the free-surface profiles obtained from the linearized Boussinesq equation and the exact two-dimensional equation. The results confirm that the linearized Boussinesq equation is a reasonable approximation when the regional gradient is small, i.e., $a = (H_2 - H_1)/L \sim 0.001$. The interesting result,

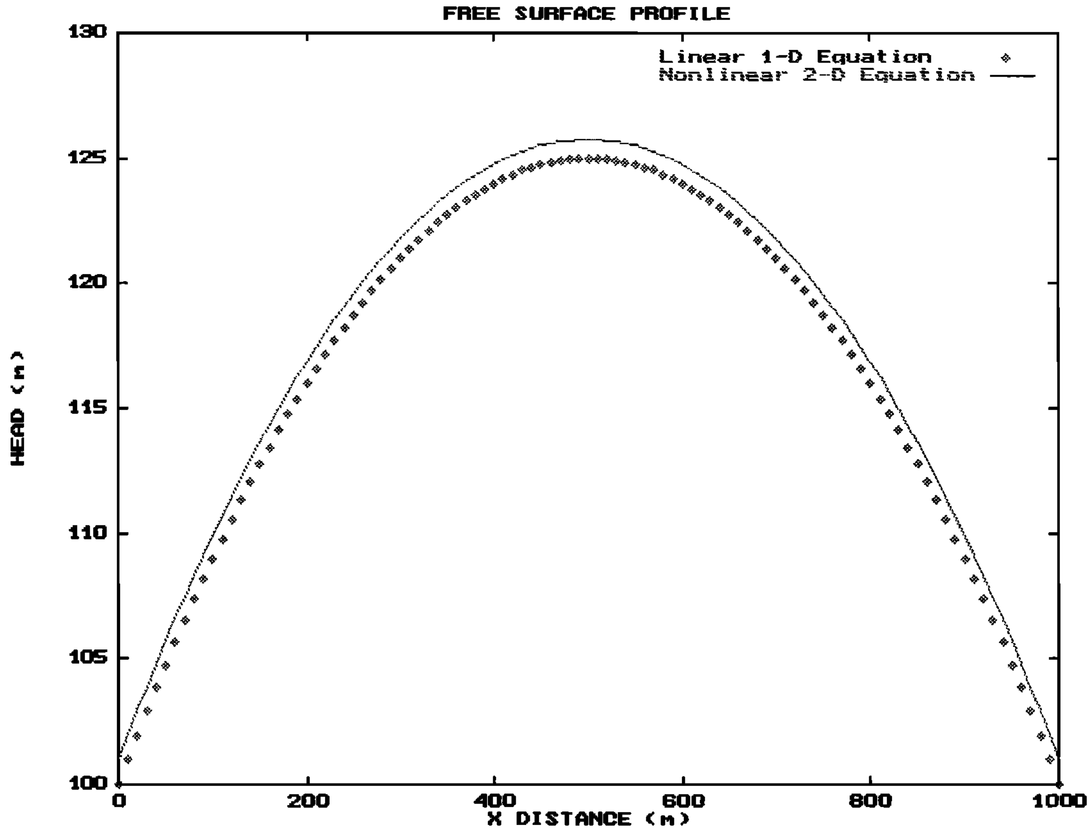

Figure 4. Free-surface profile from the exact two-dimensional and the linearized Boussinesq equations.

however, suggests that the approximation is also valid when the values of K or I are such that strong local gradients are created, provided that a is small.

From (30), horizontal and vertical flow velocities are given as, respectively,

$$u_x = -K\left((1+Y)\frac{h'}{2} + m(1-g)^2h''h'h - 3m(1-g)^2h''^2h'z^2\right),$$

$$u_z = K[1 + mg^2 + m(1-g)^2(2h''h + 5h'^2)]h''\frac{z}{2} \tag{32}$$

From (17), (18), and (32) a comparison of the three versions of the horizontal velocity can be done. Figure 5 shows an example of the horizontal velocity as calculated from the linearized, the nonlinear Boussinesq, and the exact equations, respectively. The best agreement between the three versions occurs in cases of low recharge or high hydraulic conductivity. Figure 5 represents such a case. It can be seen that the disagreement is not substantial. Finally, Figure 6 shows the distribution of the vertical velocity. In most cases the magnitude of the vertical velocity is a fraction of the horizontal. The above observations do not hold in cases of shallow rivers acting as boundaries of deep homogeneous aquifers (the Toth's aquifer). In such aquifers, however, aquifer layering and natural heterogeneity would tend to enhance velocities along the horizontal or the principal direction of permeability.

In conclusion, the linearized Boussinesq equation constitutes a good approximation to the head and groundwater velocity in unconfined aquifers with mild regional hydraulic gradients, for the ranges of recharge commonly encountered in nature and especially in aquifers exhibiting high values of hydraulic conductivity. These constitute the most important aquifers for water resources management and the most sensitive from the groundwater pollution point of view. Finally, aquifer recharge generates a spatially (scale) dependent groundwater flow velocity, and therefore dispersion coefficients functionally defined in terms of velocity should naturally reflect this scale dependency [*Serrano*, 1992].

3. Application to Regional Heterogeneous Aquifers

We now extend the results from the previous section to two-dimensional, plan view, heterogeneous aquifers. Our objective is to derive expressions for the longitudinal and transverse velocity, which constitute the bases to functionally define dispersion parameters. We choose a two-dimensional plan groundwater system for the analysis since, from the previous results, the Dupuit assumptions appear to be reasonable and a detailed representation of the vertical dimension does not seem important at the field scale. However, we remark that the following could be easily extended to general three-dimensional groundwater flow. Thus the two-dimensional version of (4) is

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right) = -I \qquad 0 \le x \le l_x,\ 0 \le y \le l_y,$$

$$h(0, y) = H_1,\ h(l_x, y) = H_2,\ h(x, y) = f(x), \tag{33}$$

where y represents distance in the horizontal direction (meters), l_y is the y dimension of the aquifer, and the rest of the terms are as before. The restriction $h(x, y) = f(x)$ implies an aquifer with a regional gradient in the x direction and

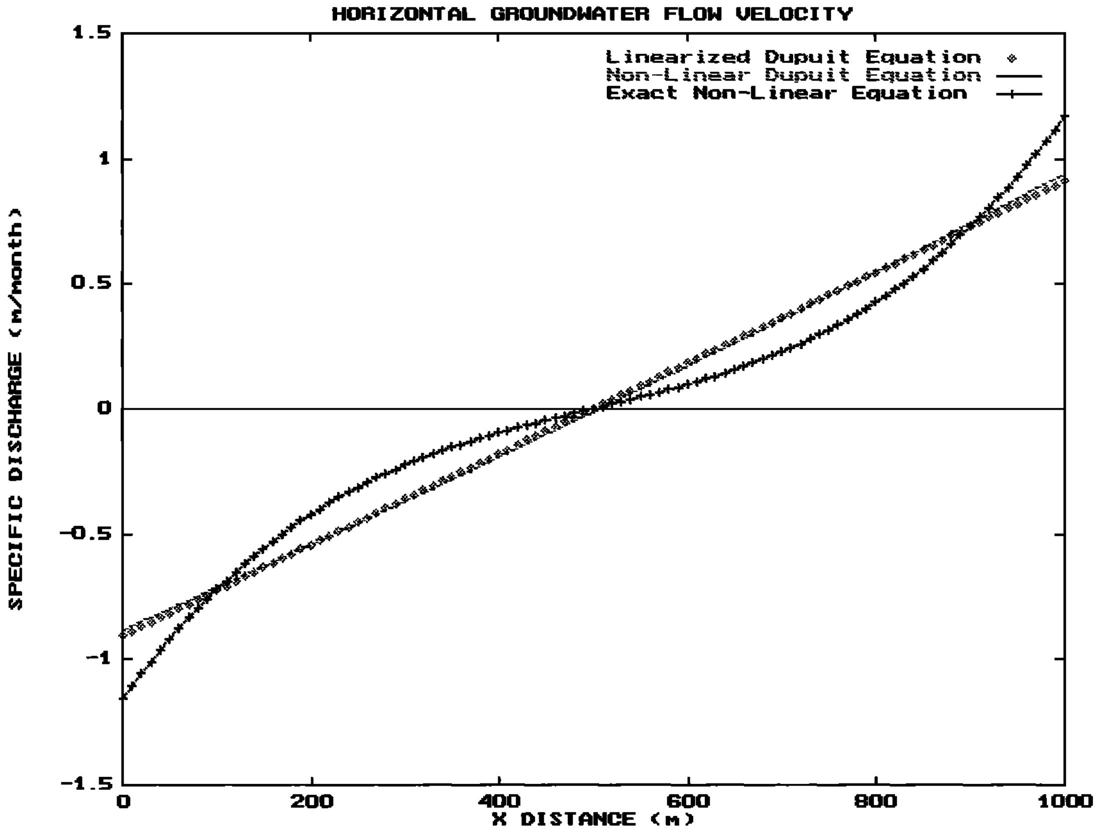

Figure 5. Horizontal groundwater velocity from the exact, the nonlinear, and the linearized Boussinesq equations.

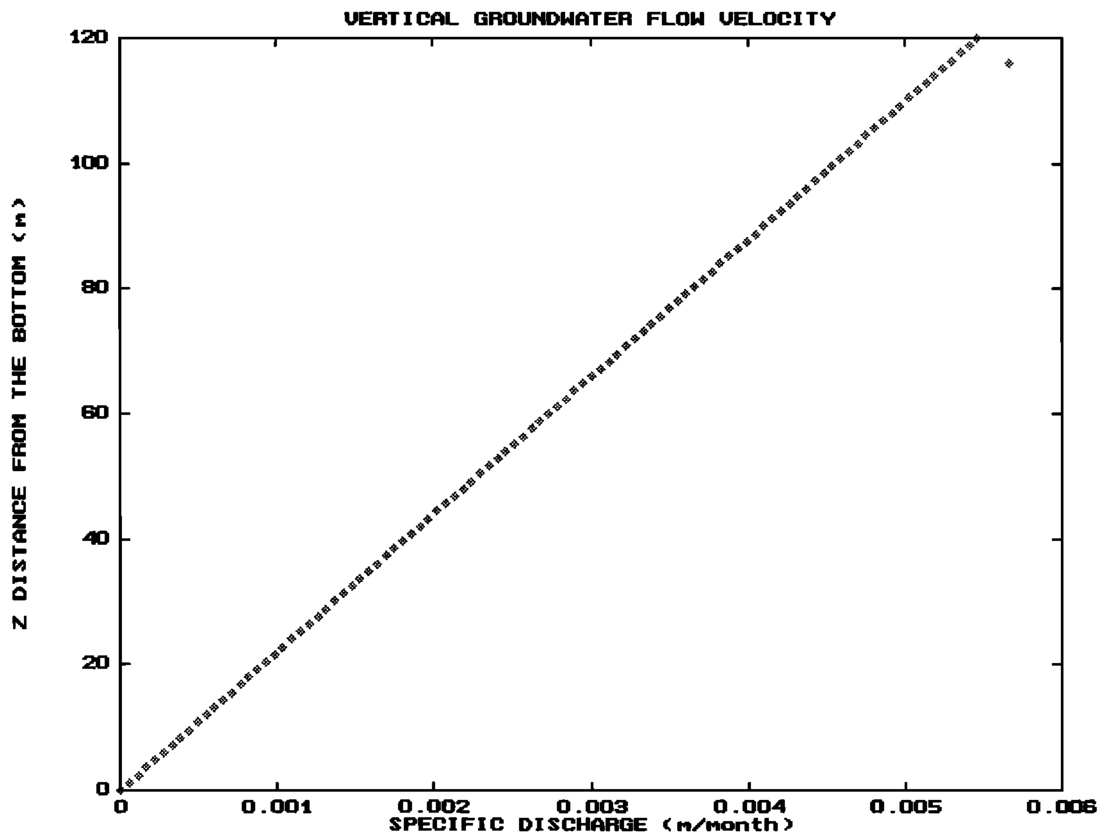

Figure 6. Vertical groundwater velocity according to the exact equation.

negligible in the y direction commonly studied in contaminant hydrogeology.

Aquifer heterogeneity will be characterized as a random field with known second-order statistical properties. This approach has received considerable attention in the last two decades. However, in this study the statistical properties of the "raw" transmissivity, rather than its logarithm, will be considered, and no particular probability law will be assumed. The decomposition solution that follows does not need the artificial logarithmic reduction of the transmissivity variances required by the traditional small-perturbation solutions. From the practical point of view the modeler rarely has sufficient information to characterize a particular probability law, and the second-order statistics are the only information available. Finally, the following procedure does not require stationarity in the random quantities. However, information necessary to characterize nonstationarity is rarely available in practical applications. Thus the transmissivity field is modeled as

$$T(x, y, \omega) = \bar{T} + T'(x, y, \omega),$$
$$E\{T'(x,y)\} = 0, E\{T'(x_1,y_1)T'(x_2,y_2)\} = R_T(d) = \sigma_T^2 e^{-\rho d^2} \tag{34}$$

where $\bar{T}$ is the mean transmissivity (square meters per month), T' is a stationary random field representing the spatial variability in the transmissivity (square meters per month), $E\{\ \}$ is the expectation operator, R_T is the two-point correlation function, σ_T^2 is the variance parameter $(\text{m}^2/\text{month})^2$, ρ is the correlation decay parameter (per square meter), and $d^2 = d_x^2 + d_y^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ is square of the distance (square meters) between the points with coordinates (x_1, y_1) and (x_2, y_2).

Since the regional hydraulic gradients in (33) are oriented along x, the mean groundwater velocity in the y direction should approach zero, leaving room for only random velocity fluctuations. This implies that the condition $E\{\partial h/\partial y\} \sim 0$ should hold.

Now substituting (34) into (33) the following two decomposition equations will result

$$h = f(x) - L_x^{-1}\frac{\partial^2 h}{\partial y^2} - \frac{1}{\bar{T}}L_x^{-1}\frac{\partial T'}{\partial x}\frac{\partial h}{\partial x} - \frac{1}{\bar{T}}L_x^{-1}\frac{\partial T'}{\partial y}\frac{\partial h}{\partial y}, L_x = \frac{\partial^2}{\partial x^2}, \tag{35}$$

$$h = f(x) - L_y^{-1}\frac{\partial^2 h}{\partial x^2} - \frac{1}{\bar{T}}L_y^{-1}\frac{\partial T'}{\partial x}\frac{\partial h}{\partial x} - \frac{1}{\bar{T}}L_y^{-1}\frac{\partial T'}{\partial y}\frac{\partial h}{\partial y}, L_y = \frac{\partial^2}{\partial y^2} \tag{36}$$

From (35) the first few terms in the decomposition series are

$$h_0 = f(x) = -\frac{I}{2\bar{T}}x^2 + Ax + B,$$
$$h_1 = -\frac{1}{\bar{T}}L_x^{-1}\frac{\partial T'}{\partial x}\frac{\partial h_0}{\partial x}, \tag{37}$$
$$\vdots$$
$$h_n = -\frac{1}{\bar{T}}L_x^{-1}\frac{\partial T'}{\partial x}\frac{\partial h_{n-1}}{\partial x},$$

where A is given after (6) with $\bar{T}$ substituted for T. Given second-order statistics for T, it is possible to calculate the statistics of the first two terms of the h series. More terms could be estimated if one assumes a particular probability law. Given the fast convergence of the decomposition series in dissipative systems, the first two terms usually produce a reasonable approximation, except in cases of very large variances in the transmissivity (see *Serrano and Unny* [1987] for a comparison with exact solutions). Thus taking the first to terms in (37), differentiating with respect to x to obtain the hydraulic gradient, and multiplying by $(\bar{T} + T')/h_0$, one obtains an expression for the longitudinal velocity:

$$u_x = -\left(\frac{\bar{T} + T'}{h_0}\right)\frac{\partial h_0}{\partial x} + \frac{1}{h_0}\left(1 + \frac{T'}{\bar{T}}\right)\int \frac{\partial T'}{\partial x}\frac{\partial h_0}{\partial x}dx \tag{38}$$

On taking expectations to (38) and using the rules of expectation and integration operators, the mean longitudinal velocity is given by

$$\bar{u}_x = \frac{1}{h_0}\left\{Ix - \bar{T}A - \frac{\sigma_T^2}{\bar{T}}\left(\left(A - \frac{Ix}{\bar{T}}\right)(e^{-x^2} - 1) + \frac{\rho Ix}{\bar{T}}e^{-\rho x^2} - \frac{I}{2\bar{T}}\sqrt{\frac{\pi}{\rho}}\operatorname{erf}(\sqrt{\rho x^2})\right)\right\}, \tag{39}$$

where erf ( ) denotes the "error function." Extensive numerical experimentation indicated that several terms in the above equation are negligible, and (39) reduces to

$$\bar{u}_x = [(C_v^2 - 1)/h_0](\bar{T}A - Ix) = (C_v^2 - 1)\bar{T}(h_0'/h_0), \tag{40}$$

where $C_v = \sigma_T/\bar{T}$ is the coefficient of variability of the transmissivity. Now from (38) the random component of the longitudinal velocity is $u_x' = u_x - \bar{u}_x$ or

$$u_x' = \frac{1}{h_0}\left(T'\frac{\partial h_0}{\partial x} + \int \frac{\partial T'}{\partial x}\frac{\partial h_0}{\partial x}dx\right) \tag{41}$$

The process u_x' has the following mean and correlation functions, respectively,

$$E\{u_x'\} = 0,$$
$$R_{u_x} = \frac{\sigma_T^2 e^{-\rho d^2}}{h_0(x_1)h_0(x_2)}\{h_0'(x_1)h_0'(x_2) + h_0'(x_2)[2A - h_0'(x_2)] + h_0'^2\} \tag{42}$$

If $x_1 = x_2 = x$, we obtain the variance of the transmissivity as

$$R_{u_x}(0, 0) = \sigma_{u_x}^2 = (\sigma_T^2 h_0'/h_0^2)(h_0' - 2A) \tag{43}$$

From (38) to (42) it is clear that the longitudinal velocity and its statistical properties are functions of distance, aquifer heterogeneity, and recharge rate. Therefore an aquifer dispersion coefficient functionally defined in terms of the longitudinal velocity should naturally reflect a (scale) dependency on distance. A similar result was obtained by *Serrano* [1992] for one-dimensional homogeneous aquifers.

The above results were obtained from the first decomposition equation (35). Since this equation is in itself a solution, we conclude that the transverse velocity component does not appear to influence the statistical properties of the longitudinal velocity, if we define the longitudinal direction as that parallel to the regional hydraulic gradient.

Information on the properties of the transverse velocity is obtained from the second decomposition equation (36). Following identical steps as for (35), we arrive at an expression for the transverse velocity:

$$u_y = -\left(\frac{(2A - h_0'(x))\dfrac{\partial T'}{\partial x}}{h_0(x)}\right)\left(1 + \frac{T'}{\bar{T}}\right) \tag{44}$$

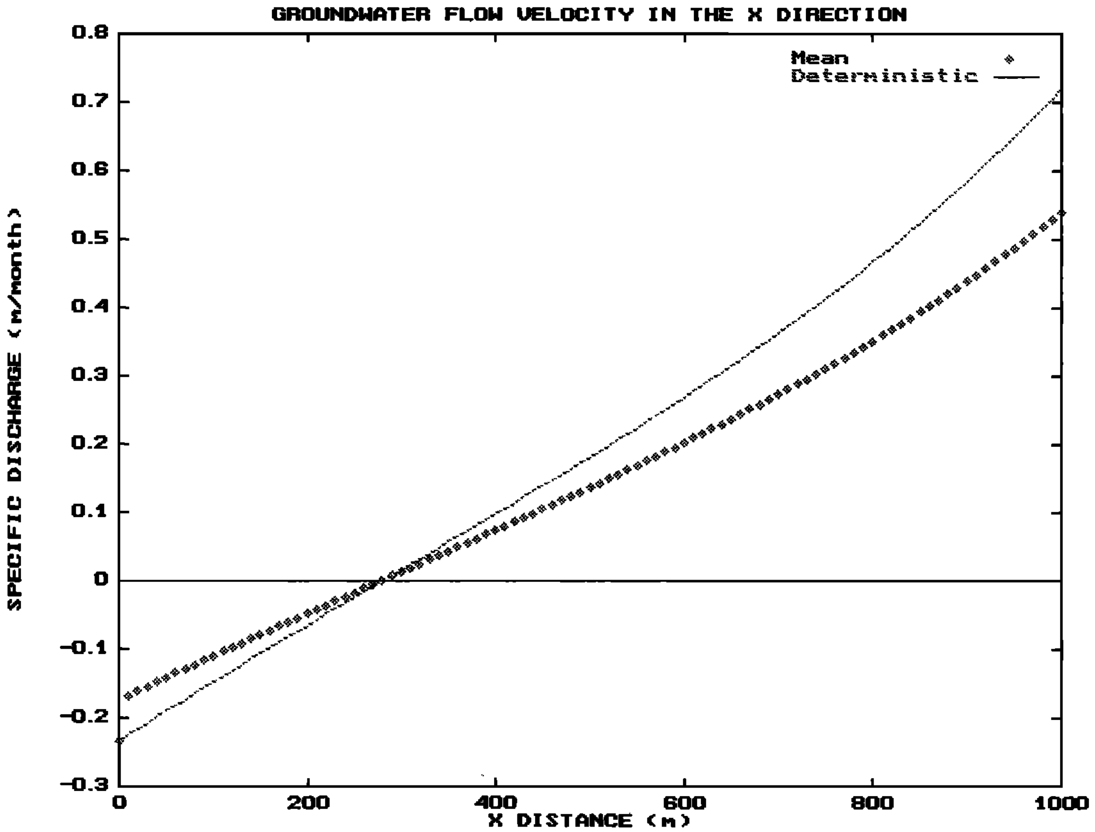

Figure 7. Mean longitudinal velocity versus deterministic velocity.

The mean, correlation function, and variance are given by, respectively,

$$E\{u_y\} = \bar{u}_y = 0,$$

$$R_{u_y}(d_x, d_y) = \frac{2\rho\sigma_T^2 e^{-\rho d^2}(2\rho d_x^2 - 1)h_0'(x_1)h_0'(x_2)}{h_0(x_1)h_0(x_2)}, \tag{45}$$

$$R_{u_y}(0, 0) = \sigma_{u_y}^2 = 2\rho\sigma_T^2\left(\frac{h_0'(x)}{h_0(x)}\right)^2$$

In contrast to the longitudinal velocity, (44)–(45) indicate that the random transverse velocity does not depend on y, except for the correlation function which depends on the distance. The variance is a constant function of y. There is a marked dependency on x, aquifer heterogeneity, and recharge.

Some numerical simulations illustrate the general features of regional groundwater flow in heterogeneous aquifers. Figure 7 is a comparison between the mean longitudinal velocity, (39) or (40), and the deterministic velocity (6) when $C_v = 0.5$, and $\bar{T} = 100.0$ m²/month. The effect of aquifer heterogeneity, as characterized by the variance in the transmissivity, is that of reducing the magnitude of the mean groundwater velocity; that is, the velocity in an equivalent homogeneous aquifer with the same recharge and boundary conditions is higher. For large variances this velocity reduction could be significant. A similar result was found by *Serrano* [1993] for two-dimensional infinite aquifers. Finally, Figure 8 shows the mean groundwater velocity, (40), across the aquifer plus and minus one standard deviation, (43), when $\bar{T} = 10.0$ m²/month and $C_v = 0.3$. As expected the minimum variability (variance) in the velocity occurs near groundwater divides where the gradients are zero, and the maximum variability occurs in zones where the gradients are large. In this example the maximum gradients, and thus velocity variances, occur at the discharge zones near the rivers.

4. Summary and Conclusions

Analytical solutions of the nonlinear Boussinesq flow equation and the exact groundwater flow equation subject to nonlinear free-surface boundary condition were derived and tested with respect to the linearized Boussinesq equation commonly used in groundwater models. The results indicated that for mild regional gradients, the linearized equation is a reasonable approximation to the nonlinear equations even if unusually high recharge rates or unusually low hydraulic conductivity values induce high local hydraulic gradients. The linearized equation deviates from the nonlinear equations in cases of large regional hydraulic gradients and in deep aquifers with shallow boundary conditions (the Toth's aquifer). However, deep homogeneous aquifers rarely occur in nature.

The effect of aquifer heterogeneity was subsequently studied by solving the two-dimensional, plan view, flow equation subject to spatially variable transmissivity as represented by a random field. Expressions for the longitudinal and transverse flow velocities and their statistical properties were derived. It was found that the longitudinal groundwater flow velocity is functionally dependent on distance (scale), aquifer recharge, the boundary conditions, and aquifer heterogeneity and therefore dispersion coefficients defined in terms of velocity should naturally reflect this scale dependency. Velocity variances are directly related to the magnitude of the hydraulic gradient as

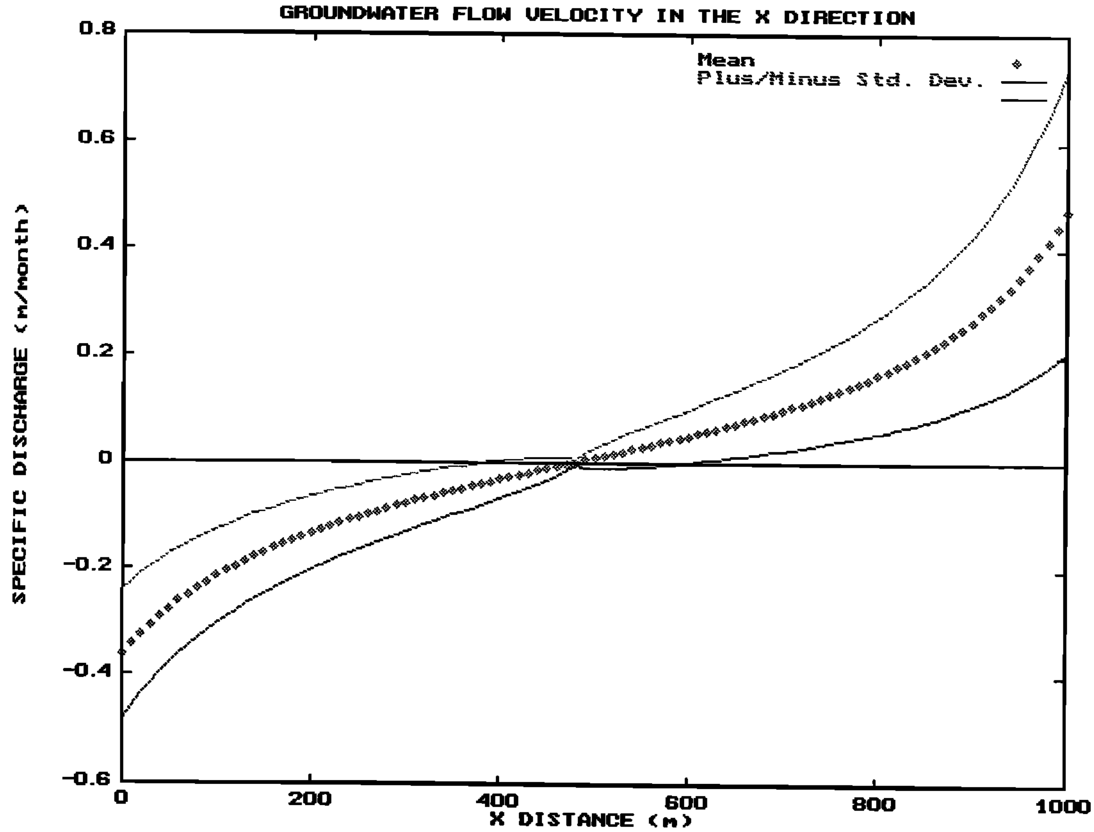

Figure 8. Mean plus and minus one standard deviation of the longitudinal velocity.

well as the above hydrologic and aquifer parameters. It is hoped that the expressions on the statistics of the velocity would help in the development of dispersion models.

The methodology used, the method of decomposition, presents several advantages over the small-perturbation methods. The series solution approximate to the exact nonlinear solution, parameter "smallness," logarithmic assumptions, linearization, or domain discretizations are not required, and the inclusion of the regional aquifer hydrology is possible.

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