The Theis Solution in Heterogeneous Aquifers

by Sergio E. Serrano[a]

Abstract

A new solution of the unsteady confined radial flow equation subject to aquifer heterogeneity, as characterized by the statistics of the transmissivity, is presented. Simple applications to the direct and inverse problems useful to field hydrogeologists are included. The results use the classical Theis solution as an initial condition and the method of decomposition, which does not require many of the restrictions of the traditional small perturbation methods. Given the mean and variance of the transmissivity, the direct problem aims at the prediction of the mean and the variance of the drawdown. Given the mean and the variance of the drawdown, the indirect problem aims at the estimation of the mean and the variance of the transmissivity.

Introduction

One of the most widely used tools in ground-water hydrology is the classical Theis solution (Theis, 1935) to the equation of unsteady confined radial ground-water flow. Given the values of aquifer parameters and the pumping rate, the Theis solution may be used to forecast the drawdown as a function of time and radial distance from the well. Most commonly, the Theis solution is used to estimate aquifer parameters given drawdown measurements from a pumping test.

Nevertheless, the Theis solution is strictly applicable to homogeneous aquifers with uniform storage coefficient and uniform transmissivity. Given the small relative effect of variability in the storage coefficient, the assumption of uniform storage coefficient may be reasonable. However, it is known that spatial variations in transmissivity may be large. Spatial variations in the transmissivity could significantly alter the magnitude and distribution of the drawdown. The consideration of aquifer heterogeneity faces the usual problems of spatial characterization of the transmissivity.

One way to characterize aquifer heterogeneity in an aquifer with no apparent geographic trend in the transmissivity is to express the spatial variability of the transmissivity in a statistical fashion. By taking samples at random and determining the transmissivity at discrete locations, one could calculate statistics such as the mean and the variance. These statistics represent not only the transmissivity spatial variability, but also the numerical uncertainty resulting from the impossibility to sample the entire aquifer. Regardless of the philosophical discussions of stochastic analyses, in the context of this article we define "random" as erratically varying in space. Having the aquifer heterogeneity characterized in a statistical form implies that the hydraulic head should be expressed in a corresponding statistical form as well. In particular, having the mean transmissivity and its variance, we would like to predict the mean drawdown and its variance (the direct problem). If the solution to the flow equation is analytical in form, one could also attempt the solution to the inverse problem. That is, having the mean drawdown and its variance at an observation well and a fixed time, estimate the mean transmissivity and its variance. The latter exercise would constitute a practical approach to the problem of characterization of aquifer heterogeneity. Obviously, the price of increasing our knowledge of aquifer heterogeneity, and of any statistical experiment, is a repetitive sampling of the drawdown under identical conditions.

The objective of the present article is to derive an analytical solution of the flow equation subject to a variable transmissivity represented in statistical form. In ground water, the traditional approach to solving stochastic partial differential equations has been the application of small perturbation methods. Under this approach the logarithm of the transmissivity, logT, is assumed to follow a Gaussian probability distribution, the spatial variability of logT is assumed to be "small," and members of the differential equation are eliminated to obtain a closed form solution (Bakr et al., 1978; Dagan, 1979; Gutjahr and Gelhar, 1981). An interesting recent development in this category is by Cheng and Ouazar (1995), who presented an extension to the Theis solution. While the logarithmic transformation and the adoption of normality is convenient to accommodate the small perturbation solution, these assumptions may not reflect general conditions encountered in the field. It has been found that hydraulic conductivity does not, in general, follow a lognormal distribution (Unlu et al., 1989). From the practical point of view, the hydrogeologist seldom has sufficient field information to fit a particular probability distribution.

The present article attempts to derive a solution to the flow equation that significantly departs from the traditional approach

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to solve stochastic equations. We adopt the method of decomposition (Adomian, 1994) because it allows the derivation of accurate analytical solutions without the restrictions imposed by the traditional methods. Our approach is based on a modification of the decomposition method (Serrano, 1987; Serrano, 1995b) previously applied to regional unconfined flow and ground-water pollution problems. The decomposition method has also been used to obtain analytical solution of the nonlinear ground-water flow equation in unconfined aquifers (Serrano, 1995a). In the following sections, we briefly describe the solution to the flow equation. Technical jargon and mathematical details are kept to a minimum, and emphasis is placed on the practical field applications to the direct and inverse problems.

Solution of the Confined Flow Equation in Heterogeneous Aquifers

Ground-water flow towards a fully penetrating well in a confined aquifer is governed by equation (1) (Bear, 1979).

$$\frac{\partial s}{\partial t} - \frac{1}{rS}\frac{\partial}{\partial r}\left(rT\frac{\partial s}{\partial r}\right) = 0, \quad r_w < r < \infty, 0 < t$$
$$\frac{\partial s}{\partial r}(r \to r_w, t) = -\frac{Q}{2\pi rT}, \quad s(\infty, t) = 0, \quad s(r, 0) = 0, \tag{1}$$

where s(r, t) is the drawdown (m); S is the aquifer storage coefficient; T is the aquifer transmissivity (m^2/hr); r is distance from the center of the pumping well (m); t is the time after pumping began (hr); r_w is the radius of the well (m); and Q is the pumping flow rate (m^3/hr).

For constant T, S, Q, and aquifer thickness b, Theis obtained the classical solution

$$s(r, t) = \frac{Q}{4\pi T} W\left(\frac{r^2}{4\pi \frac{T}{S} t}\right), \tag{2}$$

where

$$W(u) = \int_u^\infty \frac{e^{-\xi}}{\xi} d\xi$$

is the well function. If the parameters, S or T, are not constant, then equation (2) is not generally applicable. The range of variability of S is small (of the order of 10^{-4}), and small changes in S may not have a significant effect on the magnitude of s. However, the range of spatial variability of T could be large (of the order of 10.0 m^2/hr) and this may have an important effect on the magnitude of s. Thus, while the assumption of constant S may be reasonable, the assumption of constant T might not be. Figure 1 illustrates the sensitivity of s in equation (2) to variability in T and S. The relative effect of an increase of 50% in the magnitude of T is more pronounced than that produced by an increase of 50% in the magnitude of S.

We would like to obtain a new analytical solution to (1) that considers aquifer heterogeneity, as characterized by the spatial variability in the transmissivity. While a numerical solution could be obtained for a specific discrete set of measured transmissivity, it appears desirable to derive an analytical solution that includes aquifer heterogeneity in a general form. Such solution would allow the modeler to predict drawdown given the aquifer parameters (the direct problem), or estimate the parameters given the drawdown (the indirect problem).

From the practical point of view, the hydrogeologist usually has a collection of sample point transmissivity values from which the general transmissivity field must be derived. There are two important sources of variability in the transmissivity estimation: uncertainty associated with measurement devices, such as pumps, transducers, transcription errors, etc., and variability in the aquifer heterogeneity derived from the lack of resolution, and the economic limitations of not being able to cover an entire aquifer with a detailed measurement network. Measurement uncertainty may be reduced by the careful calibration and periodic maintenance of instruments. Heterogeneity variability may be defined by increasing the number of samples, that is by increasing the number of wells across the field where pumping tests designed to estimate point values of the transmissivity are performed (i.e., increasing resolution). Measurement uncertainty may be numerically described by a random variable of constant statistics of the field in question, provided that the same instrumentation, personnel, and techniques are employed. Heterogeneity variability may be numerically described as a random field whose statistics are functions of the unknown spatial distribution of the transmissivity. If a change in aquifer thickness is significant, its trend should be added to the random variability in the transmissivity.

It appears that the best general representation of the transmissivity spatial distribution, resulting from both sources of variability, is in statistical form. In other words, the hydrologist should assume that the spatial variability in the transmissivity is characterized by an unknown random field whose statistics are reflected in the mean and the variance of the individual sample points.

Consider a transmissivity random field T(r), with a mean $E\{T(r)\} = \bar{T}$, where E{} denotes the expectation operator, and a spatial correlation between two points r_1 and r_2 $E\{T(r_1)T(r_2)\} = \bar{T}^2 + \sigma_T^2 f(r_1, r_2)$, where σ_T^2 is the variance of the transmissivity, and the function f, with $f(r_1, r_2) = 1$ for $r_1 = r_2$, is generally unknown. Many studies (i.e., Bakr et al., 1978; Dagan, 1979) in the past have developed perturbation solutions to ground-water equations based on a logarithmic transformation of the transmissivity and the assumption of a Gaussian distribution for log

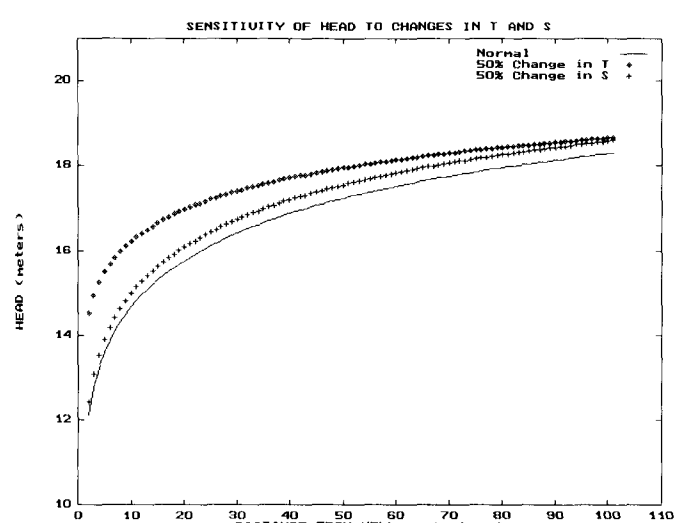

Fig. 1. Sensitivity of head to changes in T and S.

transmissivity. It appears that those assumptions represent ideal situations adopted to make the small perturbation method of solution possible, rather than general conditions found in the field. From the practical point of view, the hydrologist only has the mean and the variance of the transmissivity, and the information is usually insufficient to estimate a particular probability distribution. Therefore, we will refrain from assuming a particular probability distribution or a specific correlation function. It is important to remark that we do not neglect the existence of f, but rather do not assume a particular form of f to obtain a solution of the equations. As will be seen in the following analysis, an assumption on the form of f is not necessary to obtain expressions for the drawdown mean and variance, unless the hydrogeologist is interested in the form of the correlation structure of s. The latter is usually a research, as opposed to a practical, project in most circumstances. As long as the underlying transmissivity process is stationary, and we wish to obtain the mean and variance of the drawdown only, the decomposition series used in the present work produce cross correlation terms reducible to zero.

The mean and variance of the transmissivity may be estimated from a collection of point pumping tests from a collection of wells across the aquifer and applying the usual statistical formulae. Clearly the greater the number of samples, the higher the resolution of the T estimate, and the more accurate the statistics. With a random transmissivity characterized by its mean and variance, equation (1) is a random partial differential equation, and the drawdown is a random function. Our objective is to obtain a solution from which the mean and the variance of the drawdown may be calculated. With a spatially variable transmissivity, and adopting the classical Theis solution as an initial condition for an early time t_0, equation (1) may be written as

$$\frac{\partial s}{\partial t} = \frac{T}{S}\frac{\partial^2 s}{\partial r^2} + \frac{1}{S}\left(\frac{\partial T}{\partial r} + \frac{T}{r}\right)\frac{\partial s}{\partial r},$$

$$s(r, t_0) = s_0 = \frac{Q}{4\pi\bar{T}}\, W\left(\frac{r^2}{4\pi\frac{\bar{T}}{S}t_0}\right) \qquad (3)$$

Using the decomposition method (Adomian, 1994), we define the operator $L_t = (\partial/\partial t)$ and multiply both sides of (3) by its inverse L_t^{-1} (defined as the integral from t_0 to t) to obtain

$$s(r, t) = s_0 + L_t^{-1}\frac{T}{S}\frac{\partial^2 s}{\partial r^2} + L_t^{-1}\frac{1}{S}\left(\frac{\partial T}{\partial r} + \frac{T}{r}\right)\frac{\partial s}{\partial r} \qquad (4)$$

The decomposition solution could be written as the series

$$s = \sum_{n=0}^{\infty} s_n,$$

with s_0 given by the initial condition,

$$s_1 = L_t^{-1}\frac{T}{S}\frac{\partial^2 s_0}{\partial r^2} + L_t^{-1}\frac{1}{S}\left(\frac{\partial T}{\partial r} + \frac{T}{r}\right)\frac{\partial s_0}{\partial r},$$

$$s_2 = L_t^{-1}\frac{T}{S}\frac{\partial^2 s_1}{\partial r^2} + L_t^{-1}\frac{1}{S}\left(\frac{\partial T}{\partial r} + \frac{T}{r}\right)\frac{\partial s_1}{\partial r}, \qquad (5)$$

$$s_{i+1} = L_t^{-1}\frac{T}{S}\frac{\partial^2 s_i}{\partial r^2} + L_t^{-1}\frac{1}{S}\left(\frac{\partial T}{\partial r} + \frac{T}{r}\right)\frac{\partial s_i}{\partial r}.$$

Thus, each term in the series may be derived from the previous one. For example, the second term in the series is, from equations (4) and (5),

$$s_1 = \frac{(t - t_0)\,Q}{2\pi\bar{T}}\, e^{-[r^2/4(\bar{T}/S)t_0]}\left(\frac{T}{2\bar{T}t_0} - \frac{1}{rS}\frac{\partial T}{\partial r}\right) \qquad (6)$$

Higher order terms are easily obtained, and each term is analytical. Since the series usually converges very rapidly, only the first few terms are needed to approximate the dependent variable to a desired accuracy. Thus, we define the approximant

$$\phi_m[s] = \sum_{n=0}^{m-1} s_n.$$

Now, since we only know the mean and variance of T, we may only derive expressions for the mean and variance of s. Thus defining $s \approx \phi_3[s] = s_0 + s_1 + s_2$, and taking expectations, the mean drawdown, or the ensemble mean, is given by $E\{s\} \approx E\{s_0\} + E\{s_1\} + E\{s_2\}$. From (4), (5), and (6) and using the rules of expectation and differentiation (Papoulis, 1965),

$$E\{s(r, t)\} = \frac{Q}{4\pi\bar{T}}\left\{W(u) + \left(\frac{t - t_0}{t_0}\right)e^{-u}\left[1 + \left(\frac{t - t_0}{2t_0}\right)(1 + C_v^2)(u - 1)\right]\right\}, \quad \frac{Q}{2\pi\bar{T}}\left(\frac{t - t_0}{t_0}\right) < 1 \qquad (7)$$

where $u = [r^2/4(\bar{T}/S)t_0]$; and $C_v = \sigma_T/\bar{T}$ is the coefficient of variability of the transmissivity.

In order to approximate the drawdown variance, σ_s^2, we may only include two terms in the decomposition series because the third one would be a function of the third moment of the transmissivity. Therefore, the drawdown variance estimate outlined below is less accurate than its mean estimate, equation (7). If we had information on higher order moments of the transmissivity we could include more decomposition terms. In fact, if we made the assumption of normality in the probability distribution, we could in principle obtain an exact solution (i.e., infinite terms). As explained before this is an assumption we do not wish to adopt, in general.

Using a two-term approximant in the decomposition series and the definition of variance as $\sigma_s^2 = E\{s^2\} - E^2\{s\}$, we arrive at an estimate for the variance of the drawdown:

$$\sigma_s^2(r) = \left[\frac{QC_v e^{-u}}{4\pi\bar{T}}\left(\frac{t - t_0}{t_0}\right)\right]^2, \quad \frac{Q}{2\pi\bar{T}}\left(\frac{t - t_0}{t_0}\right) < 1 \qquad (8)$$

This equation implies that the drawdown variance is a function of the transmissivity mean and variance, the distance from the well, the storage coefficient, and the relative time after a known initial condition. Equations (7) and (8) could be applied to either the direct or the inverse problems as explained below. It is important to understand the meaning of some of the parameters. The initial time t_0 represents the time at which the deterministic initial condition, the Theis solution, is given as characterized by a measured drawdown. Equation (7) would predict the mean drawdown at a subsequent time t, with $t > t_0$. Successive calculation of the mean and variance of the drawdown for increasing t_0, and correspondingly higher t, may provide an understanding of the statistics of s as the cone of depression covers a greater portion of the aquifer. Some care should be exercised with the selection of the interval $(t - t_0)$. If this interval is too large the convergence rate of the decomposition series decreases, and

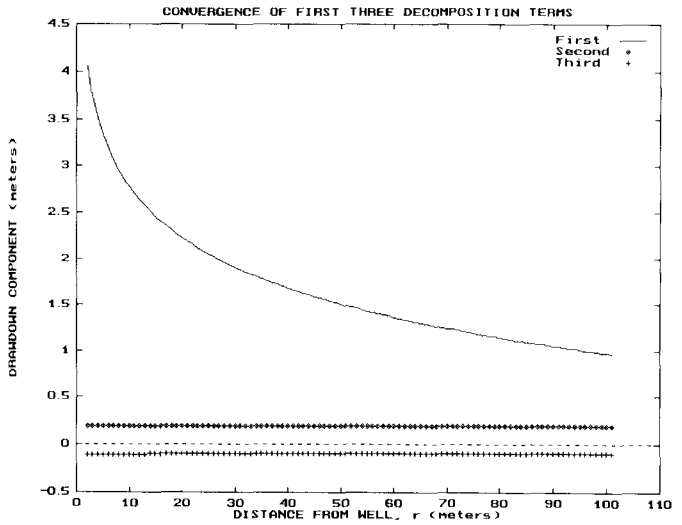

Fig. 2. Convergence of first three decomposition terms.

eventually the series diverges for very large intervals. On the other hand, as the interval $(t - t_0) \rightarrow 0$, or as $C_v \rightarrow 0$, the solution approaches to the deterministic initial condition. To select the appropriate simulation interval the convergence criterion given by the inequalities given in equations (7) and (8) must be satisfied. See Serrano (1992) for a discussion on the convergence of decomposition series in ground-water equations. Figure 2 shows the first three terms in the decomposition series of equation (7), using the parameter values described in the next section. Clearly the fast and uniform convergence of the series may be appreciated. Using the three terms of the series assures a resolution of the order of centimeters in drawdown calculations.

Applications to Drawdown Prediction

The direct problem could be stated as: Given the mean transmissivity and its variance, the radius, the pumping rate, the storage coefficient, the initial time, and the desired time, predict the mean drawdown and its variance.

Example: Several point samples taken at random from a confined aquifer of thickness $b = 10.0$ m are taken and analyzed for hydraulic conductivity. The mean transmissivity is $\bar{T} = 20.0$ m^2/hr; the coefficient of variability is $C_v = \sigma_T/\bar{T} = 1.0$ (a highly heterogeneous aquifer with a transmissivity standard deviation $\sigma_T = 20.0$ m^2/hr); $Q = 100.0$ m^3/hr; $S = 0.01$; $t_0 = 24.0$ hr; and $t = 36.0$ hr. For a fixed value of r, substituting the above values in equations (7) and (8) and using a hand calculator, we may find $E\{s(r)\}$ and $\sigma_s(r)$. With a short computer program we may calculate the drawdown statistics as a function of distance from the well. Figure 3 illustrates the mean of the hydraulic head (for a head before pumping of $H = 20.0$ m), the mean head plus and minus one standard deviation. The head standard deviation is a measure of the uncertainty, and the variability, associated with individual "sample" head predictions.

Applications to the Estimation of Aquifer Transmissivity

The inverse problem may be stated as: Given the well pumping rate, the aquifer storage coefficient, the initial time, the desired final time, the mean drawdown and its standard deviation at some known distance from the well, estimate the mean aquifer transmissivity and its standard deviation.

Example: Several observation wells are drilled in circle around a well at a distance $r = 50.0$ m; $Q = 100.0$ m^3/hr; $S = 0.01$; the initial time is arbitrarily chosen as $t_0 = 24.0$ hr; at $t = 36.0$ hr after pumping begins, drawdown measurements are simultaneously taken at all the observation wells. At that time the pumping ends and the heads are allowed to recover fully. Then a new pumping test is initiated and at $t = 36.0$ hr after pumping begins, drawdown measurements are taken again in all the observation wells. Once again the pump is stopped and the levels are allowed to recover. The pumping test is repeated several times "under identical conditions" until it is decided that a statistically representative sample of the drawdown is collected. If the cost of repeating the pumping test is competitive with that of drilling, then the hydrogeologist may increase the number of observation wells to collect information on the drawdown statistics at the desired time t. In such case, only one pumping test is required and the drawdown mean and variance are calculated from as many wells as possible using regular statistical formulae. Assume that from the collected sample, a mean drawdown $E\{s(r = 50.0\} = 1.6023$ m, with a standard deviation of $\sigma_s(r = 50.0) = 0.1964$ m is obtained. In this example, we used the drawdown statistics calculated in the direct problem (previous section) for the same distance.

From equation (8), an expression for the transmissivity coefficient of variability, C_v is obtained. Next substitute for C_v in equation (7). The modified equation (7) has only $\bar{T}$ as an unknown. After a simple trial and error procedure, the estimated mean transmissivity is $\bar{T} = 19.9983$ m^2/hr, which represents less than 0.01% of relative error with respect to the true value of 20.0 m^2/hr given in the direct problem. Finally substituting the estimated transmissivity in equation (8) we find a transmissivity standard deviation of $\sigma_T = 19.4856$ m^2/hr, which represents a relative error of 2.57% with respect to the true value of 20.0 m/hr given in the direct problem. As stated before, the error in the variance estimate is greater than that of the mean because of our limitation in knowledge of higher order statistics.

It is interesting to know that transmissivity estimation errors will increase as the observation distance decreases. This is consistent with the Theis solution whose validity is limited as distance approaches the well boundary condition (zone of turbulent flow). Estimation errors decrease as the final observation time increases. Finally, it is important to know that the choice of the initial time t_0 appears to have little effect on the estimated

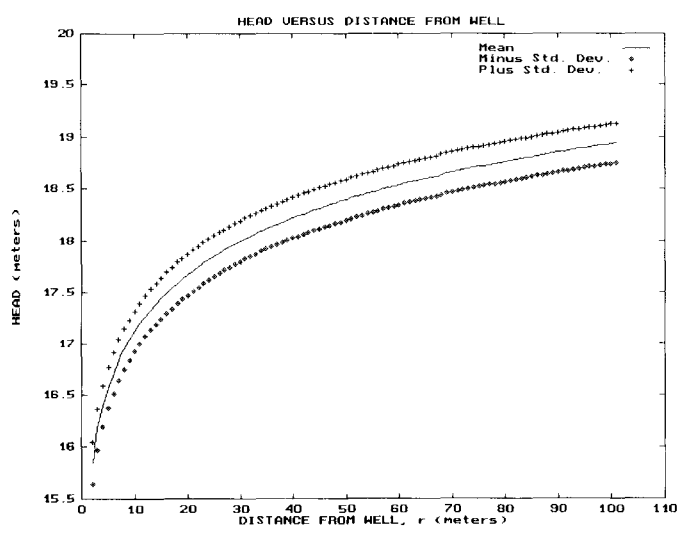

Fig. 3. Head versus distance from well.

parameters, as long as the convergence criterion is not violated. For instance, in the above example, if the initial time is changed to $t_0 = 28.0$ hr, then $\overline{T} = 19.9983\ m^2/hr$ and $\sigma_T = 19.5517\ m^2/hr$. By simultaneously increasing t_0 and the drawdown mesurement time t, it is possible to obtain a quantitative picture of the spatial variability in the statistics of the transmissivity as the cone of depression occupies larger portions of the aquifer.

The present results could be compared to an exact solution where higher order moments are known, and to Monte Carlo numerical analysis of the same problem (Butler, 1994). Figure 2 suggests that the present solution is rapidly convergent and accurate. A comparison of field-measured drawdown means and variances with those given by equations (7) and (8) is desirable at this point.

Conclusions

An analytical solution of the unsteady radial flow equation in a heterogeneous confined aquifer was derived. Practical applications to the direct and inverse problems were discussed. The following conclusions were drawn from the exercise:

1. It is useful to characterize the spatial variability in the transmissivity due to the combined aquifer heterogeneity and measurement uncertainty in a statistical form. This implies the repetitive sampling of point transmissivity at random locations in the aquifer in order to estimate its statistics.

2. In practical field applications, only the mean and the variance of the transmissivity are given, particularly when a limited number of point sample values are available. Information available is usually insufficient to fit a particular probability distribution.

3. Given the mean and the variance of the transmissivity, the direct problem aims at the prediction of the mean and the variance of the drawdown.

4. Given the mean and the variance of the drawdown, as calculated from repetitive measurement of point head under identical conditions, the inverse problem aims at the estimation of the mean and the variance of the transmissivity. These estimates represent the statistics of aquifer heterogeneity. By increasing the pumping time and applying the above procedure, it is possible to obtain the transmissivity statistics of larger portions of the aquifer.

5. A statistical analysis of variability in S could be done by following a similar procedure to that of the transmissivity.

6. The deomposition method offers several advantages over small perturbations methods, which require restrictive assumptions to make the solutions to ground-water equations possible. Assumptions such as "small" parameter variability, logarithmic transformation of the parameters, and Gaussian probability distribution in the log transformed parameters represent ideal conditions generally not found in nature.

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