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Analytical decomposition of the nonlinear unsaturated flow equation

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Abstract. An analytical procedure for the solution of the nonlinear unsaturated flow equation is proposed. The horizontal and vertical flow equations subject to constant boundary conditions and continuous wetting are approximated using analytical decomposition series. The results are verified with the *Philip* [1955] and the *Parlange* [1971] classical solutions and with experimental data. A convergence theorem with proof and discussion on the numerical accuracy of decomposition series are included. Applications to the modeling of infiltration subject to hysteresis in the soil-water functional relationships, soil heterogeneity, and time variable point rainfall are also discussed. Other potential applications in water management models under more realistic conditions are possible. The methodology offers advantages over the common small-perturbation solutions: the possibility to study nonstationarity in the random quantities; statistical separability from the physics of the problem (i.e., independence between the system parameters and the input quantities), rather than forced through closure approximations; and the construction of an analytical series solution that converges uniformly to the true nonlinear solution. Finally, because of stability and computational economy with respect to linearized numerical solutions, an analytical solution appears promising.

1. Introduction

The phenomenon of infiltration in natural unsaturated soils is an important element in the hydrologic cycle. Time and space variability of infiltration variables such as water content, given certain soil properties; soil-water physical relationships; boundary conditions; and initial conditions are necessary in several applications. These applications include storm water management, soil irrigation, and drainage; aquifer recharge estimations; groundwater flow analysis; contaminant transport predictions in the unsaturated and saturated zones; and soil-water redistribution and evaporation.

Unfortunately, the differential equations governing infiltration have soil-water parameters that are strong nonlinear functions of the dependent variables, even under ideal laboratory conditions of continuous wetting. When partial wetting is followed by partial redistribution of moisture, such as in the case of rainfall periods followed by dry periods in natural watersheds, the soil-water physical relationships exhibit hysteresis. Furthermore, a realistic model should include the inherent soil heterogeneity and a mathematical description of the time variability of water content in the root zone as characterized by the erratic time distribution of precipitation in the area. The resulting general infiltration equation is a highly nonlinear transient partial differential equation subject to random parameters and erratic in time boundary conditions.

In the past, soil physicists have generated several special solutions of the horizontal and vertical infiltration equations under controlled laboratory conditions [*Parlange*, 1971; *Philip*, 1972, 1955; *Philip and Knight*, 1974]. Recently, exact solutions for constant flux infiltration using Lie-Backlund transformations were reported [*Broadbridge and White*, 1988; *Sander et al.*, 1988]. Attempts to apply infiltration models to watershed conditions have adopted a domain discretization and a numerical approximation with special treatment [i.e., *Freeze*, 1971] or a linearization and an analytical solution. It has been known, however, that solutions to the linearized equation are poor predictors of the soil-water content. Although stochastic analyses have been helpful in characterizing soil-parameter variability of infiltration equations, the customary treatments of the stochastic infiltration equation lead to lack of closure, small perturbation, and concomitant assumptions that we wish to avoid [*Dagan*, 1983; *Yeh et al.*, 1985]. *Serrano* [1990a, b] presented a stochastic analysis of horizontal and vertical infiltration using the method of decomposition.

In this article a new set of analytical solutions of the horizontal and vertical infiltration equations under various conditions is presented using recent refinements of the method of decomposition [*Adomian*, 1994, 1991, 1986]. With decomposition, systematic solutions of a broad range of nonlinear differential equations are now possible without resorting to linearization or small perturbation assumptions. It generates a series solution, much like Fourier series, which usually converge rapidly to the exact solution. Other decompositions solutions are already available in the area of saturated groundwater flow [*Serrano*, 1995a] and in the area of contaminant dispersion in porous media [*Serrano*, 1996, 1995b; *Serrano and Adomian*, 1996].

In section 2 solutions to the classical horizontal and vertical infiltration equations are presented. In section 3 these solutions are verified with two independent laboratory experiments. In section 4 a solution of the infiltration equation under general conditions that include hysteresis in the soil-water functional relationships, random time variability in point rainfall, and soil heterogeneity is discussed.

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2. Analysis of Groundwater Flow in Unsaturated Soils

The differential equation governing groundwater flow in unsaturated homogeneous soils is given by

$$\frac{\partial \theta}{\partial t} - \nabla \cdot (D(\theta) \cdot \nabla \theta) + \frac{\partial K(\theta)}{\partial \theta} \frac{\partial \theta}{\partial z} = 0 \qquad \theta \in \Gamma \times [0, T]$$
$$Q\theta = J \qquad \theta \in \partial \Gamma \tag{1}$$

where $\theta(x, y, z, t)$ represents the volumetric soil-water content within a three-dimensional domain $\Gamma \subset \Re^3$; (x, y, z) represent the spatial coordinates (meters), with z the vertical dimension; t is the time coordinate (hours); $0 < T < \infty$; $D(\theta)$ is the soil-water diffusivity (m^2/h); $K(\theta)$ is the unsaturated hydraulic conductivity of the soil (m/h); Q is a boundary operator; and J is a known function.

Let us first consider the classical one-dimensional horizontal infiltration equation for continuous wetting in a semi-infinite media. From (1)

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x}\left(D(\theta) \frac{\partial \theta}{\partial x}\right) = 0 \qquad \theta < \theta_s,\ 0 \le x < \infty,\ 0 < t$$
$$\theta(0, t) = \theta_l,\ \theta(\infty, t) = \theta_r,\ \theta(x, 0) = \theta_r \tag{2}$$

where the left and right boundary conditions, θ_l and θ_r, respectively, are constants for now. Depending on the soil type, D is usually a function that increases nonlinearly with θ, except for very dry soils where the opposite is true. As $\theta \to \theta_s$, with θ_s the water content at saturation, the unsaturated flow equation (1) is no longer valid. It is physically reasonable to assume that D approaches a maximum constant value near saturation. Therefore if $D \to D_s$ as $\theta = \theta_0 \to \theta_s$, (2) reduces to a linear heat flow equation in a semi-infinite domain

$$\frac{\partial \theta_0}{\partial t} - D_s \frac{\partial^2 \theta_0}{\partial x^2} = 0, \qquad \theta_0 \approx \theta_s,\ 0 \le x < \infty,\ 0 < t$$
$$\theta_0(0, t) = \theta_l,\ \theta_0(\infty, t) = \theta_r,\ \theta_0(x, 0) = \theta_r \tag{3}$$

with the well-known classical solution [i.e., *Zauderer*, 1983]

$$\theta_0(x, t) = (\theta_l - \theta_r) \operatorname{erfc}\left(\frac{x}{\sqrt{4D_s t}}\right) + \theta_r \tag{4}$$

where erfc( ) denotes the "error function complement."

Indeed, the analytical and experimental verification in section 3 indicates that the linearized unsaturated flow equation with a limiting maximum value for the soil-water diffusivity is a plausible model for the regions near saturation. This suggests the unsaturated water content could be expressed as its near saturation value, θ_0, as modeled by (4), minus the nonlinear component. The latter approaches zero near saturation and becomes more important in the dryer parts of the soil.

With the above observations, we express the soil-water diffusivity as $D(\theta) = D_s - D'(\theta)$, with $D'(\theta)$ becoming zero at saturation. Substituting into (2),

$$\frac{\partial \theta}{\partial t} = D_s \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial D'}{\partial x} \frac{\partial \theta}{\partial x} - D' \frac{\partial^2 \theta}{\partial x^2} \tag{5}$$

Now define the operator $L_t = \partial/\partial t$ and apply its inverse, L_t^{-1}, defined as the integral from zero to t, on (5) to obtain

$$\theta = \theta_0 - L_t^{-1} N\theta$$
$$N\theta = \frac{\partial D'}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right)^2 + D' \frac{\partial^2 \theta}{\partial x^2} \tag{6}$$

As usual in the decomposition method [*Adomian*, 1994, 1991, 1986] for nonlinear equations, we define the series solution of (6) as

$$\theta = \sum_{n=0}^{\infty} \theta_n = \theta_0 - L_t^{-1} \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the first term is given by (4). Subsequent terms are defined as

$$\theta_1 = -L_t^{-1} A_0,$$
$$\theta_2 = -L_t^{-1} A_1,$$
$$\vdots$$
$$\theta_{n+1} = -L_t^{-1} A_n \tag{8}$$

and the series expansion, A_n, for the nonlinear term, N, in (6) is defined as

$$A_0 = N(\theta_0),$$
$$A_1 = \theta_1 \frac{dN(\theta_0)}{d\theta_0},$$
$$A_2 = \theta_2 \frac{dN(\theta_0)}{d\theta_0} + \frac{\theta_1^2}{2!} \frac{d^2N(\theta_0)}{d\theta_0^2}, \tag{9}$$
$$A_3 = \theta_3 \frac{dN(\theta_0)}{d\theta_0} + \theta_1\theta_2 \frac{d^2N(\theta_0)}{d\theta_0^2} + \frac{\theta_1^3}{3!} \frac{d^3N(\theta_0)}{d\theta_0^3}$$
$$\vdots$$

The polynomials A_n are generated for each nonlinearity so that A_0 depends only on θ_0; A_1 depends only on θ_0 and θ_1; A_2 depends only on θ_0, θ_1, and θ_2; etc. All of the θ_n components are calculable. It is now established that the series $\sum_{n=0}^{\infty} A_n$ for $N\theta$ is equal to a generalized Taylor series for $f(\theta_0)$, that $\sum_{n=0}^{\infty} \theta_n$ is a generalized Taylor series about the function θ_0, and that the series terms approach zero as $1/(mn)!$, if m is the order of the highest linear differential operator. Since the series converges and does so very rapidly, the n-term partial sum $\Phi_n = \sum_{i=0}^{n-1} \theta_i$ usually serves as an accurate enough and practical solution [*Adomian*, 1991]. See Appendix A for additional discussion on the convergence question.

Before the series can be calculated, an analytical relationship for $D(\theta)$ is needed. From the several expressions proposed in the literature, we propose a modification of the *Gardner and Mayhugh* [1958] equation:

$$D(\theta) = c_1 \exp(\lambda \theta^\alpha) - 1, \tag{10}$$

where λ and α are empirical constants fitted from experimental data, and $c_1 = 1$ m^2/h. This equation appears to describe several types of soils, except in the very dry range.

For example, the second term in the series solution (6) is, from (7)–(10),

$$\theta_1 = -L_t^{-1} \frac{\partial D'(\theta_0)}{\partial \theta} \left(\frac{\partial \theta_0}{\partial x}\right)^2 - L_t^{-1} D'(\theta_0) \frac{\partial^2 \theta_0}{\partial x^2}$$

$$\theta_1 = c_1 \alpha \lambda \int_0^t \theta_0^{\alpha-1} e^{\lambda \theta_0^\alpha} \frac{(\theta_l - \theta_r)^2}{\pi D_s t'} e^{-[x^2/(2D_s t')]} \, dt'$$

$$- \int_0^t (D_s - c_1 e^{\lambda \theta_0^\alpha} + 1) \frac{(\theta_l - \theta_r) x}{\sqrt{4\pi D_s^3 t'^3}} e^{-[x^2/(4D_s t')]} \, dt' \qquad (11)$$

$$\theta_1 \approx c_1 \alpha \lambda \theta_0^{\alpha-1} e^{\lambda \theta_0^\alpha} \frac{(\theta_l - \theta_r)^2}{\pi D_s} E_t\left(\frac{x^2}{2D_s t}\right)$$

$$- (D_s - c_1 e^{\lambda \theta_0^\alpha} + 1) \frac{(\theta_l - \theta_r)}{D_s} \operatorname{erfc}\left(\frac{x}{\sqrt{4D_s t}}\right),$$

where $E_t(a) = \int_a^\infty (e^{-\xi}/\xi) \, d\xi$ denotes de exponential integral and θ_0 has been taken out of the integral. This approximation has been adopted under the consideration that θ_0 represents a limiting, steady water content. This approximation is not necessary if one uses numerical integration. Repeated numerical experimentation revealed that the approximation is reasonable. From (6)–(9) a third term in the series could be derived. However, for a highly nonlinear equation such as (2), this results in integrals difficult to evaluate analytically. This limitation could be overcome by solving them numerically [*Serrano*, 1992; S. E. Serrano anmd S. R. Workman, Analytical solution of the nonlinear transient groundwater flow equation subject to time variable river boundaries, submitted to *Journal of Hydrology*, 1997]. See Appendix A for a discussion on the accuracy of two-term decomposition series.

If the water content is approximated as only two terms in the decomposition series, then the solution is given by

$$\begin{aligned} \theta &= \theta_0 + \theta_1 \qquad \theta < \theta_s \\ \theta &= \theta_0 \qquad \theta \to \theta_s \end{aligned} \qquad (12)$$

where θ_0 is given by (4) and θ_1 is given by (11). Since the unsaturated flow equation is not valid as $\theta \to \theta_s$, (2) and (11) are no longer valid when the water content approaches saturation. Thus sequential calculations using (12) should start at a high abscissa, where the water content is low, and proceed backwards until the water content approaches saturation. At that point θ_0 governs.

Let us now consider the one-dimensional vertical infiltration equation. From (1)

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z}\left(D(\theta) \frac{\partial \theta}{\partial z}\right) + \frac{dK}{d\theta} \frac{\partial \theta}{\partial z} = 0, \qquad (13)$$

$$\theta < \theta_s, \; 0 \le z < \infty, \; 0 < t$$

$$\theta(0, t) = \theta_t, \; \theta(\infty, t) = \theta_b, \; \theta(z, 0) = \theta_b,$$

where θ_t and θ_b are the constant water content values at the top and the bottom of the soil.

Following a procedure similar to that for the horizontal infiltration equation and adopting a hydraulic conductivity versus water content relation of the form

$$K(\theta) = c_2 e^{\beta \theta^\gamma} - 1, \qquad (14)$$

where $c_2 = 1$ m/h, the decomposition solution of (13) is given by (7), with

$$\theta_0(z, t) = (\theta_t - \theta_b) \operatorname{erfc}\left(\frac{z}{\sqrt{4D_s t}}\right) + \theta_b, \qquad (15)$$

$$\theta_1 = -L_t^{-1} \frac{\partial D'(\theta_0)}{\partial \theta} \left(\frac{\partial \theta_0}{\partial z}\right)^2 - L_t^{-1} D'(\theta_0) \frac{\partial^2 \theta_0}{\partial z^2}$$

$$- L_t^{-1} \frac{dK}{d\theta}\Big|_{\theta=\theta_0} \frac{\partial \theta_0}{\partial z}$$

$$\theta_1 \approx c_1 \alpha \lambda \theta_0^{\alpha-1} e^{\lambda \theta_0^\alpha} \frac{(\theta_t - \theta_b)^2}{\pi D_s} E_t\left(\frac{z^2}{2D_s t}\right) - (D_s - c_1 e^{\lambda \theta_0^\alpha} + 1)$$

$$\cdot \frac{(\theta_t - \theta_b)}{D_s} \operatorname{erfc}\left(\frac{z}{\sqrt{4D_s t}}\right) - c_2 \beta \gamma \theta_0^{\gamma-1} e^{\beta \theta_0^\gamma} (\theta_t - \theta_b)$$

$$\cdot \left[\frac{z}{D_s} \operatorname{erfc}\left(\frac{z}{\sqrt{4D_s t}}\right) - \sqrt{\frac{4t}{\pi D_s}} e^{-\frac{z^2}{4D_s t}}\right] \qquad (16)$$

More terms in the series can be calculated from (8) and (9). Note that the first two terms on the right side of (16) are mathematically identical to the horizontal flow equation (11). This illustrates the fact that the soil-water content during vertical infiltration is equal to the pure suction plus the gravitational effect. If only two terms are used, then θ is given by (12), where θ_0 is given by (15) and θ_1 is given by (16).

3. Experimental and Numerical Verification

We first attempt an experimental verification of the solution to the horizontal infiltration equation. For that purpose we used horizontal infiltration experiments conducted on a Guelph clay loam composed of 17% sand, 48% silt, and 35% clay [*Serrano*, 1990b].

For each experiment sectioned lucite columns were packed uniformly with soil having an initial water content, $\theta_r = 0.086$. A glass bead base, approximately 2 mm thick, with two ports in the entry plate, one for the water and one to allow the escape of air, were helpful in establishing a constant left boundary condition at $x = 0$. After about 30 s the initially small positive inflow head was lowered to approximately -1 mbar. The laboratory temperature for all the experiments varied between 21°C and 23°C. The cumulative volume of infiltrated water, which was measured with a horizontal burette, as well as the distance from the origin to the wetting front was recorded as a function [time$^{1/2}$]. A relationship between the above two variables indicated the preservation of similarity with regard to water flow, in agreement with previous research. At the end of each infiltration experiment the time was recorded, the column was sliced into sections, and the soil in each section, of known volume, was used to calculate the bulk density and the volumetric water content. A summary of the water content profiles for each experiment is presented by *Serrano* [1990b, Figure 1].

Sample values of the soil-water diffusivity were obtained by numerically integrating the water content profiles according to the classical procedure [*Philip*, 1955]. Figure 1 shows the observed values of $D(\theta)$ and the ones predicted by (10) with $\lambda = 500.$ and $\alpha = 11$. The value of D_s was estimated as the diffusivity at $\theta_s = \theta_l = 0.458$. Figure 2 illustrates a comparison between observed values of θ with respect to x 1 hour after the experiment began, and the corresponding simulated values at the same time (equation (12)). Only two terms in the series (7) were used in the simulations. Figure 2 also illustrates the re-

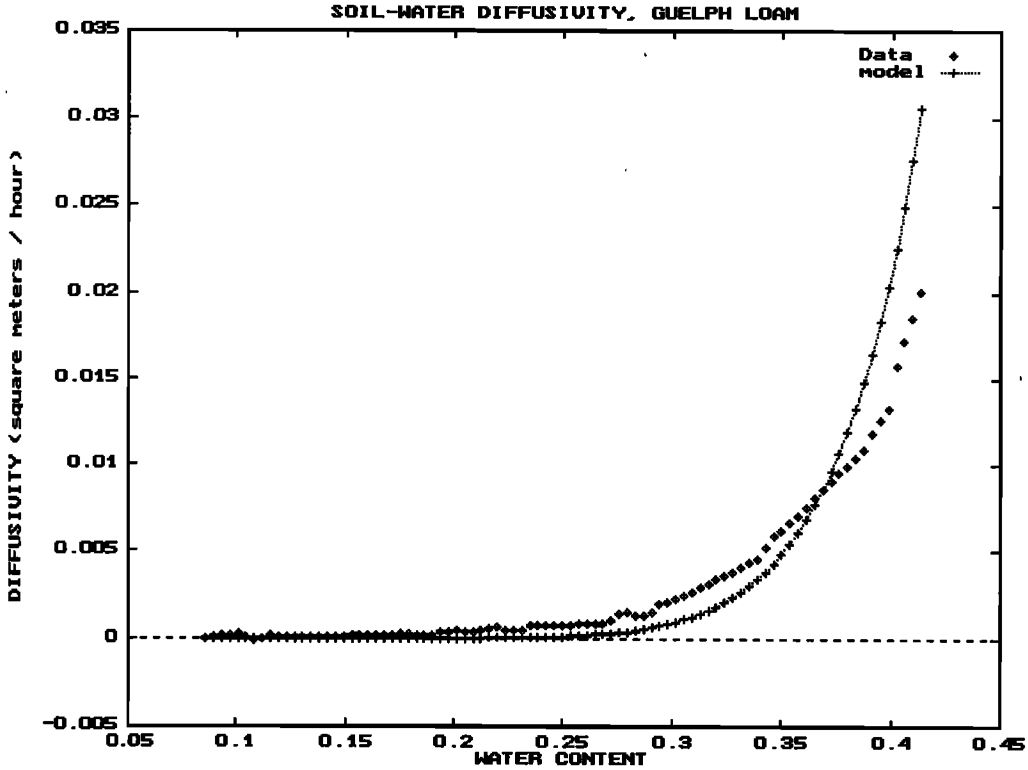

Figure 1. Soil-water diffusivity, Guelph loam.

sults of applying the *Philip* [1955] solution and the *Parlange* [1971] solution to the same problem.

Figure 2, and other comparisons between experiments and simulations at different times, suggests that the decomposition solution of the horizontal infiltration equation agrees reasonably well with experimental data and with other classical solutions, especially in the regions near saturation and in the location of the wetting front. The observed wetting front is

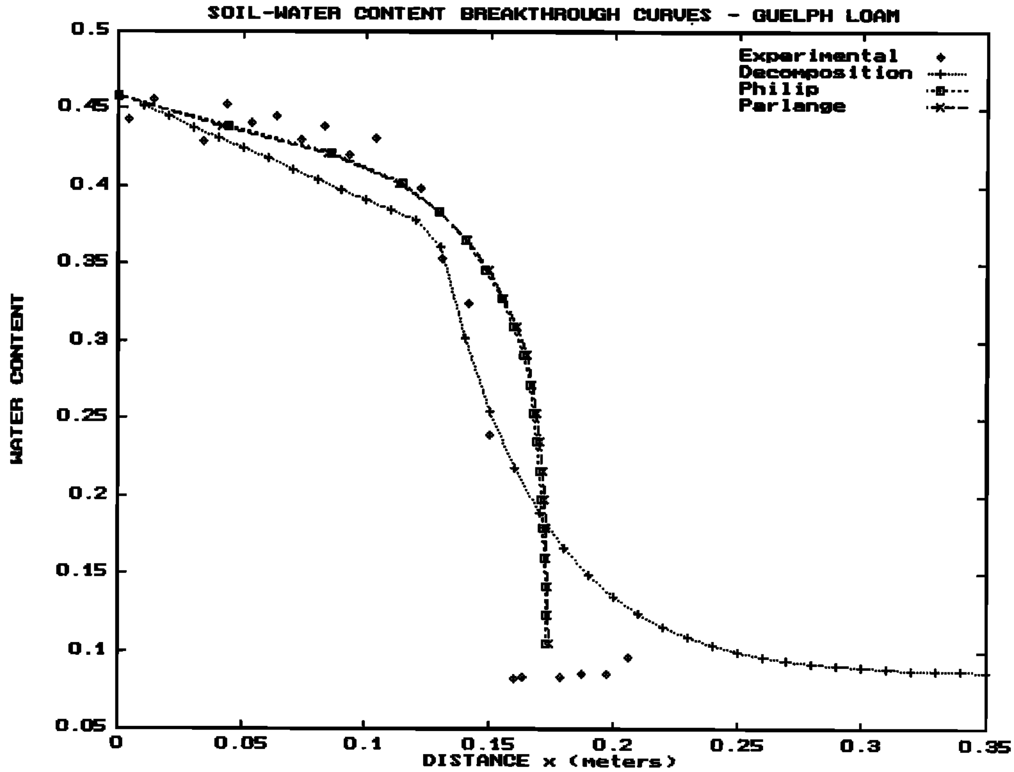

Figure 2. Water content breakthrough curves, Guelph loam.

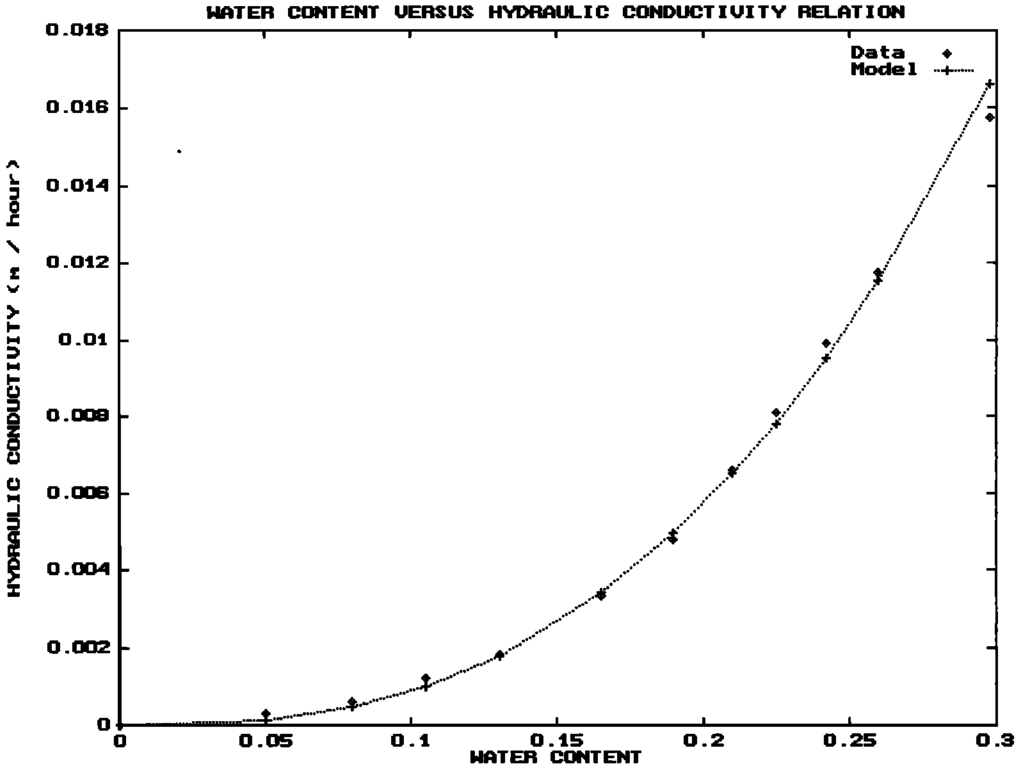

Figure 3. Water content versus hydraulic conductivity relation.

steeper than that predicted by the model. This feature might be corrected with the inclusion of a third term in the decomposition series. The *Philip* [1955] and *Parlange* [1971] solutions are very close to one another and they reproduce better the water content at near saturation regions and the steep position of the wetting front. The decomposition solution agrees better with the experimental data on the location of the wetting front than the *Philip* [1955] and *Parlange* [1971] solutions.

For the verification of the solution to the vertical infiltration equation, we use the well-documented vertical infiltration experiments conducted by *Liakopoulos* [1965]. In one those experiments the saturated water content, $\theta_s = \theta_t = 0.2975$ was maintained at the top of a column of very fine dry sand while the water was allowed to infiltrate for 1 hour. Water content versus depth profiles at 5, 10, 15, 30, 45, and 60 min were estimated from a network of tensiometers inserted in the column [*Liakopoulos*, 1965, Figure 6].

For the estimation of $D(\theta)$ we used the main wetting curve of *Liakopoulos* [1965, Figures 1 and 3, respectively]; measured water content versus pressure head relationship, ψ; and the water content versus hydraulic conductivity relationship. Figure 3 of the present work shows the water content versus hydraulic conductivity relationship measured by Liakopoulos and predicted by (14) with $\beta = 0.43$ and $\gamma = 2.69$. Figure 4 shows the soil-water diffusivity of Liakopoulos data, as calculated from an approximation to $D(\theta) = K(\theta)(\partial\psi/\partial\theta)$, and the one predicted by (10) with $\lambda = 4.5 \times 10^5$ and $\alpha = 12$.

Figure 5 shows the simulated water content versus depth profiles as simulated by (7), (12), (15), and (16). Again we have used two terms in the decomposition series only, which are relatively easy to calculate. The accuracy will improve as more terms are calculated. However, in this case the analytical calculation of a third term is difficult. The results indicate that the decomposition solution predicts reasonably well the evolution of the water content versus depth profile for the earlier times [*Liakopoulos*, 1965, Figure 6]. For large times the decomposition solution underestimates the position of the wetting front and the need for a third term in the series increases. As with the horizontal infiltration the observed profiles appear to exhibit sharper water fronts than those simulated. This observation may be apparent since the observed profiles were taken after an interpolation of measured values from tensiometers located every 10 cm in the column. The inclusion of additional terms may reduce such discrepancy.

4. Applications to Infiltration Analyses in Hysteretic Soils Subject to Natural Rainfall

The methodology presented above is sufficiently general to permit the solution of the nonlinear infiltration equation under the normal cycles of drainage and redistribution caused by the erratic time variability of natural rainfall.

The study of infiltration in natural watersheds significantly increases the complexity of the governing differential equations. First, the modeler needs to determine the functional form of the soil-water content at the root zone as a result of the random variability of point rainfall. Given the erratic nature of natural rainfall, this requires a stochastic analysis of point rainfall in the area of interest and its relationship with the time variability of the water content at the root zone. Second, the modeler needs the functional form of the soil-water diffusivity that results from the cyclic (random) rainfall-dry periods. This requires an investigation of the effect of hysteresis in the in-

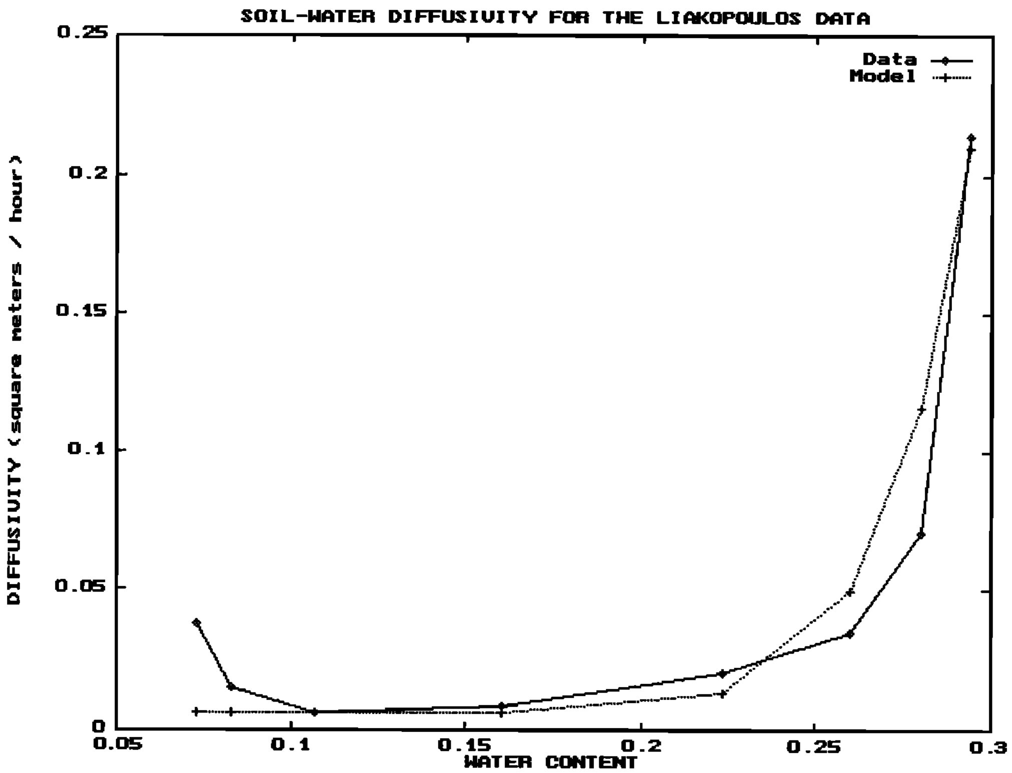

Figure 4. Soil-water diffusivity for the *Liakopoulos* [1965] data.

dividual scanning pathways of the soil-water physical relationships on the functional form of the diffusivity. Given the fact that every point in the soil is pursuing a different pathway, the result would be a large family of diffusivity functions, which are best analyzed stochastically. Finally the modeler may attempt the solution of the infiltration equation subject to a random top boundary condition (the water content at the root zone) and a random diffusivity.

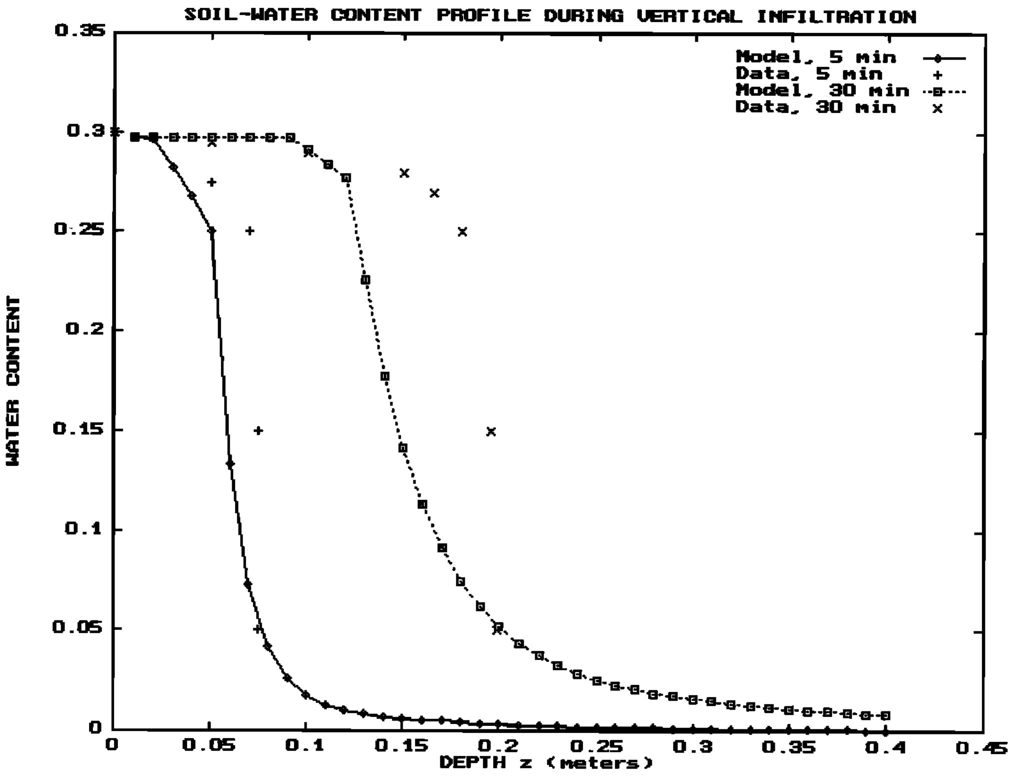

Figure 5. Soil-water content profile during vertical infiltration.

One of the few studies attempting such a solution was presented by *Serrano* [1990a]. That study showed that the effect of time random variability in point rainfall produces a water content in the upper soil layer which can be modeled as a shot noise process and that the hysteretic loops resulting from the natural wetting and drying cycles generate a random soil-water diffusivity process. A particular form of the diffusivity correlation structure was proposed and a solution of the infiltration equation was presented.

In this section we briefly extend the concepts of section 2 and the results presented by *Serrano* [1990a] to the general case when the root zone water content, the diffusivity, and the hydraulic conductivity are random functions of known properties. This case would represent the infiltration phenomenon in heterogeneous soils of natural watersheds.

Consider (13) in a finite soil:

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z}\left(D(z, t, \omega)\frac{\partial \theta}{\partial z}\right) + u(z, t, \omega)\frac{\partial \theta}{\partial z} = 0 \quad (17)$$

$$0 < z < Z, \ 0 < t,$$

$$\theta(0, t) = \theta_t(t, \omega)$$

$$\theta(Z, t) = \theta_b$$

$$\theta(z, 0) = f(z)$$

where $D(z, t, \omega)$, $u(z, t, \omega) = [dK(z, t, \omega)]/d\theta$, and $\theta_t(t, \omega)$ are stochastic processes; ω is the probability variable; and Z is the unsaturated zone thickness (m).

We will take advantage of the finite soil domain and attempt a z-partial decomposition solution [*Adomian*, 1994]. Equation (17) may be written as

$$\frac{\partial^2 \theta}{\partial z^2} = -D^{-1}\frac{\partial \theta}{\partial t} + D^{-1}\frac{\partial D}{\partial z}\frac{\partial \theta}{\partial z} - uD^{-1}\frac{\partial \theta}{\partial z} \quad (18)$$

Now define the operator $L_z = \partial^2/(\partial z^2)$, and L_z^{-1} as the double integral with respect to z. Equation (18) becomes

$$\theta = \theta_0 - L_z^{-1}D^{-1}\frac{\partial \theta}{\partial t} + L_z^{-1}D^{-1}\frac{\partial D}{\partial z}\frac{\partial \theta}{\partial z} - L_z^{-1}uD^{-1}\frac{\partial \theta}{\partial z}, \quad (19)$$

where

$$\theta_0 = C_0(t) + C_1(t)z$$

$$C_0(t) = \theta_t \quad (20)$$

$$C_1(t) = \frac{\theta_b - \theta_t}{Z}$$

The decomposition series will be $\theta = \theta_0 + \theta_1 + \theta_2 + \cdots$, where

$$\theta_1 = -L_z^{-1}D^{-1}\frac{\partial \theta_0}{\partial t} + L_z^{-1}D^{-1}\frac{\partial D}{\partial z}\frac{\partial \theta_0}{\partial z} - L_z^{-1}uD^{-1}\frac{\partial \theta_0}{\partial z}$$

$$\theta_2 = -L_z^{-1}D^{-1}\frac{\partial \theta_1}{\partial t} + L_z^{-1}D^{-1}\frac{\partial D}{\partial z}\frac{\partial \theta_1}{\partial z} - L_z^{-1}uD^{-1}\frac{\partial \theta_1}{\partial z} \quad (21)$$

$$\vdots$$

$$\theta_{n+1} = -L_z^{-1}D^{-1}\frac{\partial \theta_n}{\partial t} + L_z^{-1}D^{-1}\frac{\partial D}{\partial z}\frac{\partial \theta_n}{\partial z} - L_z^{-1}uD^{-1}\frac{\partial \theta_n}{\partial z}$$

We can then write the approximant

$$\phi_m[\theta] = \sum_{n=0}^{m-1} \theta_n \quad (22)$$

which converges to θ. Of interest are the statistical properties of the water content. For stationary or nonstationary cases the mean water content can be found by writing out the series and taking term-by-term expectations without closure approximations or perturbation. Thus if $\langle \ \rangle$ denotes the expectation operator,

$$\langle\theta\rangle = \langle\theta_0\rangle + \langle\theta_1\rangle + \langle\theta_2\rangle + \cdots + \langle\theta_n\rangle + \cdots,$$

$$\langle\theta_0\rangle = \langle\theta_t\rangle + \frac{\theta_b - \langle\theta_t\rangle}{Z}z,$$

$$\langle\theta_1\rangle = -L_z^{-1}\langle D^{-1}\rangle\frac{\partial\langle\theta_0\rangle}{\partial t} + L_z^{-1}\left\langle D^{-1}\frac{\partial D}{\partial z}\right\rangle\frac{\partial\langle\theta_0\rangle}{\partial z} - L_z^{-1}\langle u\rangle \cdot \langle D^{-1}\rangle\frac{\partial\langle\theta_0\rangle}{\partial z} \quad (23)$$

$$\vdots$$

$$\langle\theta_{n+1}\rangle = -L_z^{-1}\langle D^{-1}\rangle\frac{\partial\langle\theta_n\rangle}{\partial t} + L_z^{-1}\left\langle D^{-1}\frac{\partial D}{\partial z}\right\rangle\frac{\partial\langle\theta_n\rangle}{\partial z} - L_z^{-1}\langle u\rangle \cdot \langle D^{-1}\rangle\frac{\partial\langle\theta_n\rangle}{\partial z}$$

D, u, and θ_t could be general stochastic processes. Second-order statistics can be found easily. Because of the fast convergence of the series it is usually accurate to obtain $\phi_3[t]$ and $\phi_3[t']$ and average to derive the correlation function (the errors are discussed by *Adomian* [1986]). The results are easily generalized to three dimensions and, by assuming Gaussian behavior, to higher moments.

It is clear that the infiltration phenomenon may be analyzed under general conditions in several dimensions and, after suitable sensitivity analyses, produce new simplified infiltration models. For example, one may conclude that the most important uncertainty factor in regional infiltration models is the random variability of the water content in the root zone. A similar conclusion was reached by *Serrano* [1990a] following the observation that variance of the diffusivity substantially decreased with depth, that hysteresis is less important with increasing depth, and that the controlling uncertainty factor was the random top boundary condition. Thus if D and u are deterministic and constant and $f(z) = \theta_b = 0$, then (19) reduces to

$$\theta = \theta_0 - L_z^{-1}D^{-1}\frac{\partial \theta}{\partial t} - L_z^{-1}uD^{-1}\frac{\partial \theta}{\partial z} \quad (24)$$

The first three terms in the decomposition series are

$$\theta_0 = \theta_t\left(1 - \frac{z}{Z}\right),$$

$$\langle\theta_0\rangle = \langle\theta_t\rangle\left(1 - \frac{z}{Z}\right),$$

$$\theta_1 = D^{-1}\frac{\partial \theta_t}{\partial t}\left(\frac{z^3}{Z3!} - \frac{z^2}{2!}\right) + uD^{-1}\frac{\theta_t}{Z}\frac{z^2}{2!} + C_2 + C_3 z$$

$$\langle\theta_1\rangle = D^{-1}\frac{\partial\langle\theta_t\rangle}{\partial t}\left(\frac{z^3}{Z3!} - \frac{z^2}{2!}\right) + uD^{-1}\frac{\langle\theta_t\rangle}{Z}\frac{z^2}{2!} + C_2 + C_3 z$$

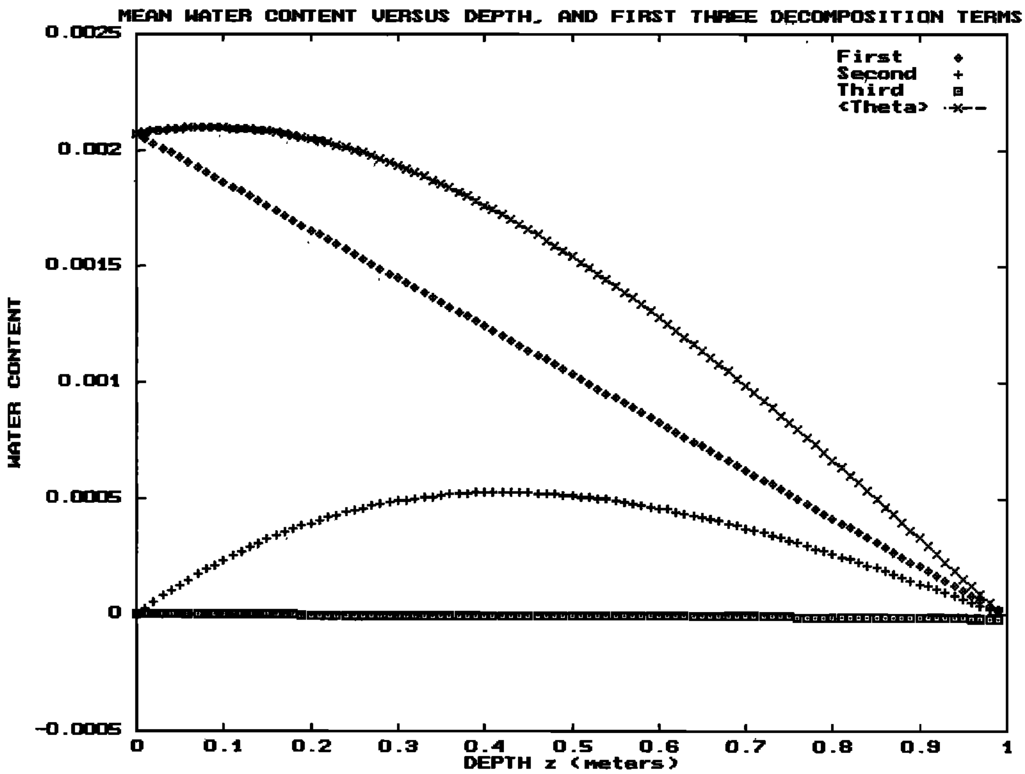

Figure 6. Mean water content versus depth, and first three decomposition terms.

$$\langle\theta_2\rangle = -D^{-2}\frac{\partial^2\langle\theta_t\rangle}{\partial t^2}\left(\frac{z^5}{Z5!}-\frac{z^4}{4!}\right) - uD^{-2}\frac{\partial\langle\theta_t\rangle}{\partial t}\left(\frac{z^4}{Z4!}-\frac{z^3}{3!}\right)$$
$$- u^2D^{-2}\langle\theta_t\rangle\frac{z^3}{Z3!} + C_4 + C_5z \tag{25}$$

$$\vdots$$

where the constants of integration C_2, C_3, C_4, and C_5 must satisfy the boundary conditions. If we now represent the root zone boundary condition as a shot noise process with a mean given by *Serrano* [1990a, equation (9)],

$$\langle\theta_t\rangle = \frac{\rho}{\kappa\zeta}(1-e^{-\kappa t}), \tag{26}$$

where ρ is the Poisson process parameter of the number of storms in the interval $[0, t]$ (day^{-1}); κ is the unit impulse response parameter for the root zone, which is significantly greater for sands than for clays (day^{-1}); and ζ is the exponential distribution parameter for the infiltrated water magnitude per storm (dimensionless if the water content is a ratio). See work by *Serrano* [1990a] for a description on the physical interpretation of these parameters. Thus, from (25) the mean water content is approximated as

$$\langle\phi_3[\theta]\rangle = \langle\theta_0\rangle + \langle\theta_1\rangle + \langle\theta_2\rangle,$$

$$\langle\theta_0\rangle = \frac{\rho}{\kappa\zeta}(1-e^{-\kappa t})\left(1-\frac{z}{Z}\right)$$

$$\langle\theta_1\rangle = \frac{D^{-1}\rho e^{-\kappa t}}{\zeta}\left(\frac{z^3}{Z3!}-\frac{z^2}{2!}\right) + \frac{uD^{-1}\rho}{\kappa\zeta Z}$$
$$\cdot(1-e^{-\kappa t})\frac{z^2}{2!} + C_2 + C_3z$$

$$C_2 = 0,\ C_3 = \frac{D^{-1}\rho e^{-\kappa t}Z}{3} - \frac{uD^{-1}\rho}{2\kappa\zeta}(1-e^{-\kappa t}) \tag{27}$$

$$\langle\theta_2\rangle = \frac{\rho u D^{-2}}{\kappa\zeta}(1-e^{-\kappa t})\frac{z^3}{Z3!} + C_4 + C_5z$$

$$C_4 = 0,$$

$$C_5 = \frac{\rho\kappa D^{-2}e^{-\kappa t}Z^3}{30\zeta} - \frac{\rho u D^{-2}e^{-\kappa t}Z^2}{4\zeta} + \frac{\rho u^2D^{-2}(1-e^{-\kappa t})Z}{6\kappa\zeta}$$

The following parameter values were used in simulations: $\rho = 0.01$ day^{-1}, $\zeta = 0.6$, $\kappa = 0.1$ day^{-1}, $D = 2.0$ m^2/d, $u = 0.0216$ m/d, $\theta_b = 0$, and $Z = 1.0$ m. Figure 6 illustrates the mean water content with depth at $t = 3$ hours as simulated with (27). The mean water content at a given time is a smooth curve in between the mean root zone value and that at the bottom of the soil. Figure 6 also shows the first three terms in the decomposition series. They illustrate the typical features of decomposition: fast convergence and a good approximation with the first few terms. In this case the third term is negligible at the scale of observation of water content and at the requested time. See Appendix A for more discussion on the convergence question. Models of this fashion could be employed to estimate groundwater recharge rates that consider the time variability of point rainfall. Given the value of a sample water content at the root zone, the model could also be used to forecast the water content with depth at a future date. Obviously, the variance of the forecast increases with time [*Serrano*, 1990a].

5. Summary and Conclusions

A general analytical procedure has been proposed for the solution of the nonlinear infiltration equations. The horizontal infiltration equation and the vertical infiltration equation subject to constant boundary conditions and continuous wetting were approximated using analytical decomposition series and verified experimentally and numerically. Applications to the modeling of infiltration subject to hysteresis in the soil-water functional relationships and time variable point rainfall were also discussed. Other potential applications in water management models under more realistic conditions are possible.

It appears that at near-saturation values of the water content, a linearized differential equation with a limiting high diffusivity value is a reasonable model. In any other region the nonlinearity effect is strong.

The methodology offers advantages over the common small-perturbation solutions: The possibility to study nonstationarity in the random quantities; statistical separability from the physics of the problem (i.e., independence between the system parameters and the input quantities), rather than forced through closure approximations; and the construction of an analytical series solution that converges uniformly to the true nonlinear solution. Finally, because of stability and computational economy with respect to linearized numerical solutions, an analytical solution appears promising.

Appendix: Accuracy and Convergence of Decomposition Series

In the preceding sections we studied decomposition solutions of infiltration equations, which were compared with scattered (limited) data and other analytical solutions. Since the unsaturated flow equation is highly nonlinear, only two terms in the series were calculated with relative simplicity. The question arises as to the accuracy of two-term or three-term decomposition solution. In this section we establish the basis of convergence of decomposition series and show its accuracy with respect to well-known exact solutions.

The following theorem concentrates in a one-dimensional domain in an infinite soil for simplicity. It is easy to show that similar theorems may be constructed for the general three-dimensional domain. Consider (5) at near-saturation conditions when $\partial D'/\partial x$ is maximum and $D' \to 0$:

$$\frac{\partial \theta}{\partial t} - D_s \frac{\partial^2 \theta}{\partial x^2} = u' \frac{\partial \theta}{\partial x} \qquad -\infty < x < \infty,\ 0 < t,\ t \in [0, T]$$

$$\theta(\pm\infty, t) = 0,\ \theta(x, 0) = f(x) \tag{A1}$$

where $u' = \partial D'/\partial x$ and $f(x)$ is an initial water content distribution. Let us write the solution of (A1) as

$$\theta(x, t) = J_t f - \int_0^t J_{t-t'} u' \sum_{n=1}^{\infty} \phi_n(x, t')\, dt' \tag{A2}$$

where the decomposition series $\phi_n = \partial \theta_{n-1}/\partial x$ and J_t is the semigroup operator given by [*Serrano*, 1988]

$$J_t f = \frac{1}{\sqrt{4\pi D_s t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x - u't - x')^2}{4D_s t}\right] f(x')\, dx' \tag{A3}$$

According to decomposition theory, (A2) is subject to

$$\phi_1(x, t') = \frac{\partial}{\partial x}[J_t f] \tag{A4}$$

$$\phi_{n+1}(x, t') = -\int_0^{t'} J_{t'-\tau} u'(x) \phi_n(x, \tau)\, d\tau$$

A1. Theorem

If $|u'(x)| \leq M$ and $\| \quad \|_2$ is the norm of the space

$$L^2(0, T; V) = \{g:[0, T] \to V: \int_0^T \|g\|_V^2 < \infty\} \tag{A5}$$

where $V = H^1$ is the first-order Sobolev Space of L^2-valued functions [*Griffel*, 1981], then a sufficient condition for the almost sure convergence of the series $\Sigma_0^\infty\ \phi_n(x, t')$ in the space $L^2(0, T; V)$ is that $tM/2 < 1$.

A2. Proof

It follows from the integral form of the Minkowski inequality that

$$\begin{aligned}
\|\phi_{n+1}(x, t')\|_2 &= \left\| -\int_0^{t'} J_{t'-\tau} u'(x) \phi_n(x, \tau)\, d\tau \right\|_2 \\
&\leq M \left\| \int_0^{t'} J_{t'-\tau} \phi_n(x, \tau)\, d\tau \right\|_2 \\
&\leq M \|\phi_n(x, t')\|_2 \left\| \int_0^{t'} J_{t'-\tau}\, d\tau \right\|_2 \\
&\leq M \|\phi_n(x, t)\|_2 \sup_{x \in V, t'\tau \in [0,T]} \int_0^{t'} |J_{t'-\tau}\, d\tau| \qquad \text{(A6)} \\
&\leq M \|\phi_n(x, t)\|_2 \int_0^t J_{t-\tau}\, d\tau \\
&\leq M \|\phi_n(x, t)\|_2 \frac{1}{2} \int_0^t d\tau \\
&\leq M \|\phi_n(x, t)\|_2 \frac{t}{2}
\end{aligned}$$

Since $L^2(0, T; V)$ is a complete space, the convergence of $\Sigma_0^\infty\ \phi_n(x, t')$ follows from the convergence of $\Sigma_0^\infty\ \|\phi_n(x, t)\|_2$. If $Mt/2 < 1$, the convergence of the decomposition series can be used to construct approximate solutions to (A1), which was to be demonstrated.

The above result indicates that it is always possible to manipulate the simulation time step (i.e., a modeling decision) in order to obtain convergent decomposition series. A rigorous framework for the convergence of decomposition series has also been developed by *Gabet* [1994, 1993, 1992] by connecting the method to well-known formulations where classical theorems (fixed point theorem, substituted series, etc.) could be used. Other rigorous work on the convergence was published by *Abbaoui and Cherruault* [1994], *Cherruault* [1989], and *Cherruault et al.* [1992].

Whether or not a decomposition series may be written in closed form, the important feature to the practical hydrologic modeler is to have a sufficiently accurate method. Consider

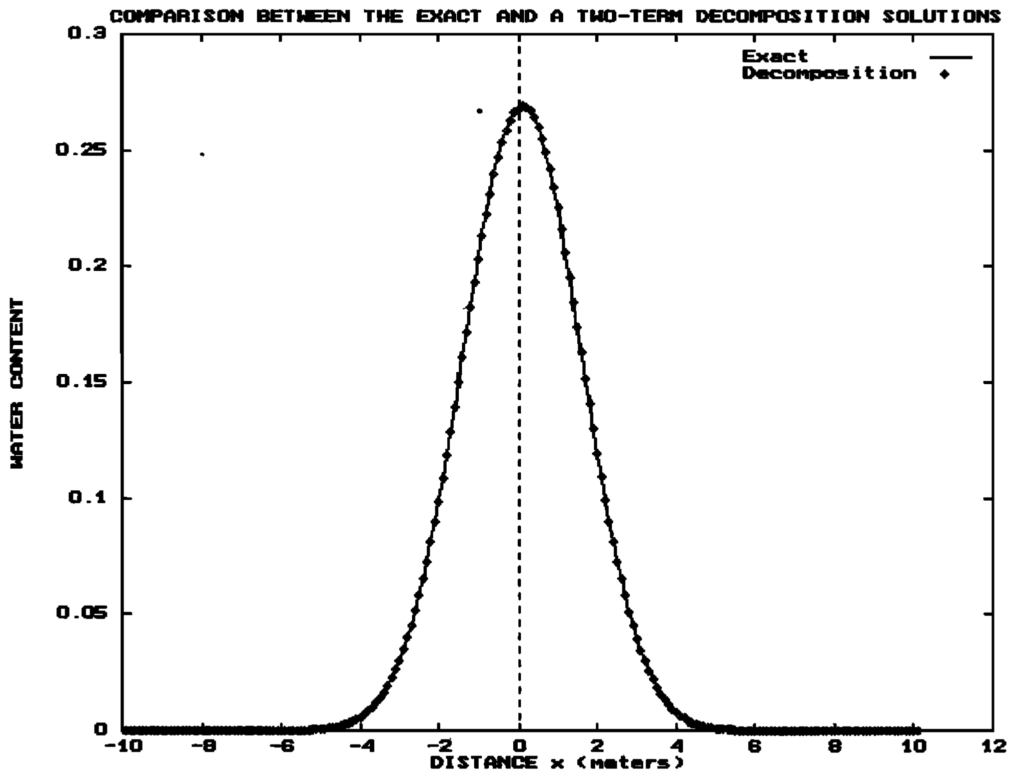

Figure 7. Comparison between the exact and two-term decomposition solutions.

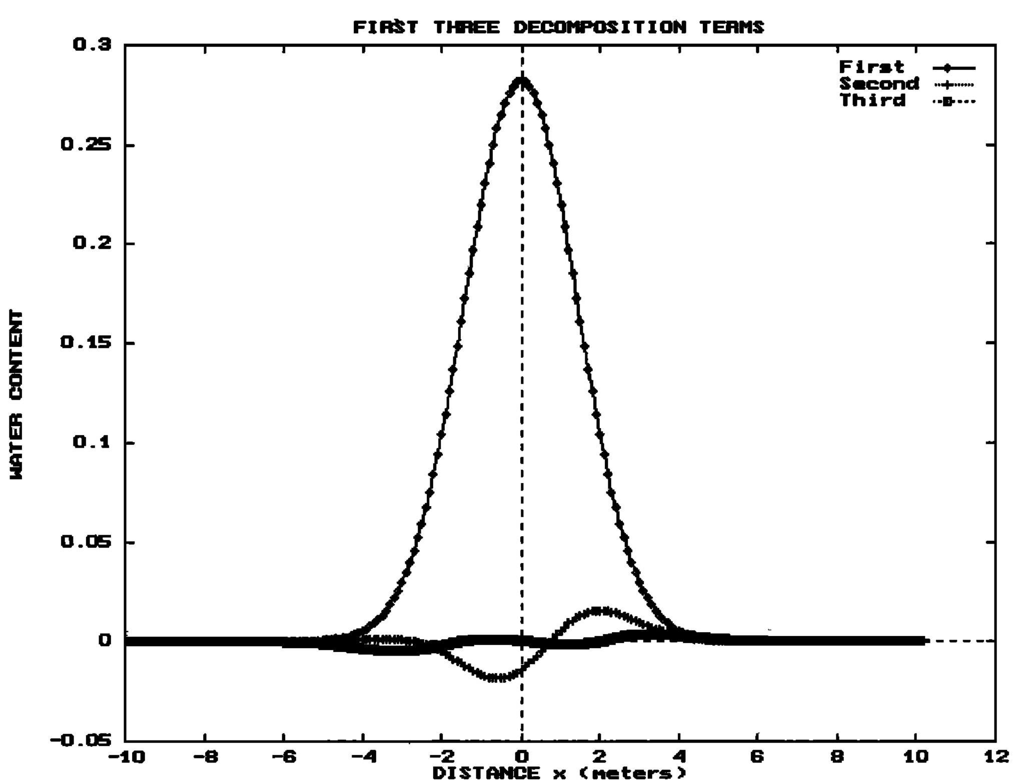

Figure 8. First three decomposition terms.

(A1) for the case when $u' = u =$ constant. The decomposition series are

$$\begin{aligned}\theta_0 &= f(x),\\ \theta_1 &= D_s t f''(x) - u t f'(x),\\ &\vdots\\ \theta_n &= D_s L_t^{-1} \frac{\partial^2 \theta_{n-1}}{\partial x^2} - u L_t^{-1} \frac{\partial \theta_{n-1}}{\partial x}.\end{aligned} \tag{A7}$$

As an example, let us assume an initial condition of the form

$$f(x) = \theta_0 = \frac{e^{-x^2/4}}{\sqrt{4\pi}} \tag{A8}$$

From the above equations the first terms in the series are

$$\begin{aligned}\theta_1 &= \frac{t f(x)}{2} \left(D_s \left(\frac{x^2}{2} - 1 \right) + u x \right),\\ \theta_2 &= \frac{t^2 f(x)}{2} \left(D_s^2 \left(\frac{3}{4} - \frac{3}{4} x^2 - \frac{3}{8} x^3 + \frac{x^5}{8} \right) \right.\\ &\quad \left. - 2 u D_s \left(\frac{3}{4} x - \frac{x^3}{4} \right) + u^2 \left(\frac{x^2}{4} - \frac{1}{2} \right) \right)\\ &\vdots\end{aligned} \tag{A9}$$

The exact solution of the equation is, in this case,

$$\begin{aligned}\theta &= \frac{1}{\sqrt{4\pi D_s t}} \int_{-\infty}^{\infty} e^{-[(x-ut-x')^2]/(4D_s t)} \frac{1}{\sqrt{4\pi}} e^{-x'^2/4}\, dx'\\ &= \frac{1}{\sqrt{4\pi(1 + D_s t)}} e^{-[(x-ut)^2]/[4(1+D_s t)]}\end{aligned} \tag{A10}$$

Let us assume that $D_s = 0.05$ m²/h and $u = 0.05$ m/h. Figure 7 illustrates a comparison between the exact solution at $t = 2$ hours and the decomposition solution as approximated by $\phi_2[\theta]$ with the first two terms in the series. In this case the accuracy of the decomposition with two terms is remarkable. Figure 8 is a graph of the individual terms, θ_0, θ_1, and θ_2 as a function of distance for the same simulation time, which demonstrates numerically the fast and uniform convergence of the decomposition series. For other comparisons between exact and decomposition solutions see work by *Serrano* [1992], *Serrano and Adomian* [1996], and *Serrano and Unny* [1987].

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