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Modeling transient stream/aquifer interaction with the non-linear Boussinesq equation and its analytical solution

Sergio E. Serrano[a,*], S.R. Workman[b]

[a]Department of Civil Engineering, University of Kentucky, Lexington, KY 40506, USA
[b]Department of Biosystems and Agricultural Engineering, University of Kentucky, Lexington, KY 40546, USA

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Abstract

A new analytical solution of the non-linear transient groundwater flow equation subject to time variable river boundaries was used to simulate stream/aquifer interactions in an alluvial valley aquifer. The differential equations were solved using the method of decomposition. The mathematical model required relatively few parameters to simulate groundwater elevations: hydraulic conductivity, specific yield, and recharge. The model was physically based and could simulate the process of a flood wave propagation into an unconfined aquifer. The model was tested using observed water table elevations at three locations across a 2 km wide alluvial valley aquifer. The average daily deviation between observed and simulated water table elevations was approximately 0.09 m. The transient redistribution of water in the aquifer was simulated well with the model. The non-linear form of the Boussinesq equation was shown to better simulate cases when the transmissivity of the aquifer could not be assumed to be a constant.

Keywords: Stream/aquifer interaction; Fluctuating rivers; Unconfined aquifers; Mathematical models; Non-linear equation; Analytical solution

1. Introduction

An important problem in the study of alluvial aquifers is the quantification of stream/aquifer hydraulics. Groundwater levels in alluvial valley aquifers fluctuate with changes in stage levels in the associated surface streams. If the river stage is increased or decreased over a short time period, a flow reversal occurs in the aquifer as a result of the change in gradient between the stream and the aquifer. The distance of influence of the flood wave is dependent on the transmissivity and the porosity of the aquifer, the change in stream stage, and the length of time that conditions for a flow reversal exist. The propagation of a flood wave into the aquifer raises other important questions when the water quality of the river is different from that of the aquifer. The spread of contaminants from the river into the aquifer, or from the aquifer into the river, is a problem intimately related to the hydraulics of the stream–aquifer system.

The hydraulics of the stream–aquifer system could be studied via the solution of the Laplace equation subject to a non-linear free-surface boundary condition, and time-dependent river boundary conditions. Most practical applications of this alternative use simplifying assumptions, such as linearization of the free-surface boundary conditions followed by an analytical or numerical solution of the resulting equations (Kirkham, 1966; Polubarinova-Kochina, 1962; Van de Giesen et al., 1994).

* Corresponding author. Fax: +1 606 257 4404.

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When Dupuit assumptions of zero resistance to vertical flow are valid (Strack, 1984), the non-linear Boussinesq equation appears to be a viable alternative to the use of Laplace's equation. With the Boussinesq equation, the vertical coordinate does not exist, and the free-surface boundary condition is not needed. The result is a simplified model where the effect of time-dependent river boundary conditions can easily be incorporated into the analysis. However, the Boussinesq equation is a non-linear partial differential equation. Until recently, most practical implementations of the Boussinesq equation have used some form of linearization prior to an analytical or numerical solution. Workman et al. (1997) derived a particular solution of the linearized Boussinesq equation using the concepts of analytic semigroups. Verification with observed well hydrographs in an alluvial aquifer showed that the linearized solution of the Boussinesq equation reproduced observed well hydrographs caused by fluctuating river boundaries. This is in agreement with similar results elsewhere that confirm the value of the linearized Boussinesq equation in regions of mild hydraulic gradients. Yet the question remains as to the effectiveness of the linearized Boussinesq equation to simulate cases of large changes in the aquifer transmissivity resulting from high fluctuations in river stage. In such cases, the transmissivity may be a strong function of the hydraulic head and the solution of the linearized equation may not be an accurate model.

The desired model is one based on a solution to the non-linear Boussinesq equation. Until recently, analytical solutions of non-linear partial differential equations were rare, due to the lack of systematic solution methods. With the refinement of decomposition methods (Adomian, 1994), solutions to many non-linear problems in science and engineering are now possible. The method of decomposition generates a series, much like the Fourier series, where the solution of an equation may be approximated to the true non-linear solution. Furthermore, in most dissipative systems the convergence rate is so high that only a few terms in the series are needed to obtain an accurate solution consistent with the resolution of field measurement devices.

Serrano (1995) derived new analytical solutions of the Laplace equation subject to a steady non-linear free-surface boundary condition and of the steady non-linear Boussinesq equation. In the present paper, we extend the analysis to the transient non-linear equation subject to time-varying river boundary conditions. A new analytical solution of the non-linear Boussinesq equation is derived (Section 2). The solution is applied to the simulation of a flood wave propagation into an unconfined aquifer (Section 3). The verification tests consist of comparisons between observed water table elevations at three locations across a 2 km wide alluvial aquifer and simulated levels at corresponding locations using the non-linear Boussinesq equation and the linearized Boussinesq equation. Results and discussion are presented in Section 4.

2. Solution of the transient groundwater flow equation with Dupuit assumptions

Consider a nearly horizontal homogeneous unconfined aquifer of length l_x, bounded by two time-dependent boundary conditions, $H_1(t)$ and $H_2(t)$, respectively. Locating the origin, $x = 0$, at the left boundary, and the datum at the bottom of the aquifer, the governing groundwater flow equation with Dupuit assumptions is:

$$\frac{\partial h}{\partial t} - \frac{1}{S}\frac{\partial}{\partial x}\left(Kh\frac{\partial h}{\partial x}\right) = \frac{I}{S}, \qquad 0 \le x \le l_x, 0 < t \tag{1}$$

$$h(0,t) = H_1(t), \quad h(l_x,t) = H_2(t), \quad h(x,0) = H_0(x)$$

where $h(x,t)$ is the hydraulic head (m); K is the aquifer hydraulic conductivity (m/day); I is the mean daily recharge from rainfall (m/day); S is the aquifer specific yield; $H_1(t)$ and $H_2(t)$ are the time fluctuating heads at the left and right boundaries, respectively; x is the spatial coordinate (m); t is the time coordinate (day); and $H_0(x)$ is the initial head across the aquifer (m).

Among the several decomposition schemes possible, we begin with the t-decomposition solution. Let us write Eq. (1) as:

$$\frac{\partial h}{\partial t} = \frac{I}{S} + \frac{Kh}{S}\frac{\partial^2 h}{\partial x^2} + \frac{K}{S}\left(\frac{\partial h}{\partial x}\right)^2 \tag{2}$$

Defining the operator $L_t = \partial/\partial t$ and pre-multiplying Eq. (2) by the inverse operator L_t^{-1} (i.e. the indefinite t-integration), one obtains:

$$h = H_0 + \frac{It}{S} + L_t^{-1}N(h) \tag{3}$$

where the non-linear operator is given by:

$$N(h)=\frac{K}{S}\left\{h\frac{\partial^2 h}{\partial x^2}+\left(\frac{\partial h}{\partial x}\right)^2\right\} \tag{4}$$

As usual in the decomposition method (Adomian, 1986; Adomian, 1991; Adomian, 1994), for non-linear equations, we define the series solution for Eq. (2) as:

$$h=\sum_{n=0}^{\infty} h_n \tag{5}$$

where the first term satisfies the first term in the right side of Eq. (2) and the initial condition, that is:

$$h_0=H_0(x)+\frac{It}{S} \tag{6}$$

Subsequent terms in the series are given as:

$$h_1=L_t^{-1}A_0 \tag{7}$$

$$h_2=L_t^{-1}A_1$$

$$\vdots$$

$$h_{n+1}=L_t^{-1}A_n$$

and the series expansion, A_n for the non-linear term, N, in Eq. (3) is defined as:

$$A_0=N(h_0) \tag{8}$$

$$A_1=h_1\frac{\mathrm{d}N(h_0)}{\mathrm{d}h_0}$$

$$A_2=h_2\frac{\mathrm{d}N(h_0)}{\mathrm{d}h_0}+\frac{h_1^2}{2!}\frac{\mathrm{d}^2N(h_0)}{\mathrm{d}h_0^2}$$

$$A_3=h_3\frac{\mathrm{d}N(h_0)}{\mathrm{d}h_0}+h_1h_2\frac{\mathrm{d}^2N(h_0)}{\mathrm{d}h_0^2}+\frac{h_1^3}{3!}\frac{\mathrm{d}^3N(h_0)}{\mathrm{d}h_0^3}$$

$$\vdots$$

The polynomials A_n are generated for each non-linearity so that A_0 depends only on h_0, A_1 depends only on h_0 and h_1, A_2 depends only on h_0, h_1, h_2, etc. All of the h_n components are calculable. It is now established that the series $\Sigma_{n=0}^{\infty} A_n$ for $N(h)$ is equal to a generalized Taylor series for $N(h_0)$, that $\Sigma_{n=0}^{\infty} h_n$ is a generalized Taylor series about the function h_0, and that the series terms approach zero as $1/(mn)!$, if m is the order of the highest linear differential operator. Since the series converges and does so very rapidly, the n-term partial sum $\Phi_n=\Sigma_{i=0}^{n-1} h_i$ usually serves as an accurate enough and practical solution. Thus, from Eqs. (7) and (8), the second term in the series is:

$$h_1=\frac{K}{S}L_t^{-1}\left\{h_0\frac{\partial^2 h_0}{\partial x^2}+\left(\frac{\partial h_0}{\partial x}\right)^2\right\} \tag{9}$$

$$h_1=\frac{K}{S}L_t^{-1}\left\{\left(H_0(x)+\frac{It}{S}\right)\frac{\partial^2 H_0(x)}{\partial x^2}+\left(\frac{\partial H_0(x)}{\partial x}\right)^2\right\} \tag{10}$$

Higher terms in the series are similarly derived. The third term would require information on the third-order spatial derivative of the initial condition. Clearly, $H_0(x)$ must be sufficiently smooth for the calculation of its first- and second-order spatial derivative. In practical applications, a smooth surface should be fitted through the heads measured at individual wells. The spatial derivatives are then calculated analytically or numerically. In most applications, the first two or three terms in the decomposition series constitute an accurate solution (for comparisons between decomposition and exact solutions see Serrano, 1992; Serrano and Adomian, 1996; Serrano and Unny, 1987). Thus the two-term approximant, $\phi[h]_2$, of the solution to Eq. (2) is:

$$\phi[h]_2\approx h=h_0+h_1 \tag{11}$$

where h_0 is given by Eq. (6) and h_1 is given by Eq. (10). An interesting feature is that this solution makes use of the initial condition.

Normally, the rate of convergence of decomposition series is so high that two- or three-term approximants usually represent a good approximation to the exact solution. Several works in the past have been devoted to the assessment of the above conclusion. Serrano (1995) showed that two terms in the decomposition solution are sufficient to obtain an accurate solution of the steady state Boussinesq equation. Exceptions to this rule are in situations where the hydraulic conductivity was less than 100 m/month, or when the recharge rate was abnormally high (i.e. $I>0.01$ m/month). Other works showing the uniform convergence of groundwater decomposition series and the accuracy of two- and three-term approximants as compared with the exact solution include Serrano and Adomian (1996) and Serrano (1992). It is also important to consider the rigorous mathematical framework for the convergence of decomposition series developed by Gabet (1992); Gabet (1993); Gabet (1994); Abbaoui and Cherruault (1994); Cherruault (1989); and Cherruault et al. (1992).

Other decomposition solutions to the non-linear Boussinesq solutions are possible (see Appendix A). The selection of the most appropriate solution for a particular model depends on computational considerations or the availability of field data.

3. Application of the model

As an application of the above development, the t-partial solution [Eqs. (6), (10) and (11)] was chosen over the x-partial solution (see Appendix A) due to its simplicity of implementation. The mathematical model was evaluated for use in two lateral flow problems. In the first case, the model was compared to solutions of the linearized Boussinesq equation and to observed data from a stream/aquifer interaction study. In the second case, the model was applied to a hypothetical problem of drainage of a shallow perched aquifer by a drainage canal.

An extensive data set of daily groundwater levels and river elevations was collected at the Ohio Management Systems Evaluation Area (OMSEA) (Jagucki et al., 1995; Ward et al., 1994; Workman et al., 1991) over the period of August 1991 to December 1995. A portion of the data, October 1991 to September 1992, was discussed in a report by Jagucki et al. (1995) and used to test an analytical solution to the linearized Boussinesq equation (Workman et al., 1997). The OMSEA data showed rapid movement in the unconfined aquifer coincident with changes in stage of the Scioto River.

The hydrogeology of the OMSEA site and the areas adjacent to the site have been studied extensively (Jagucki et al., 1995; Norris, 1983a; Norris, 1983b; Norris and Fidler, 1969; Nortz et al., 1994). These studies found the lateral hydraulic conductivity of the aquifer to range from 122 to 152 m/day with a mean value of 142 m/day. The specific yield of the site was estimated to be 0.18 to 0.22 with a mean of 0.2. At the OMSEA site, the unconfined aquifer was 18–20 m thick and extended across the width of the valley (approximately 2 km; see Fig. 1). Note that the vertical scale of Fig. 1 was greatly exaggerated to depict the cross-section of the aquifer. The actual ver-

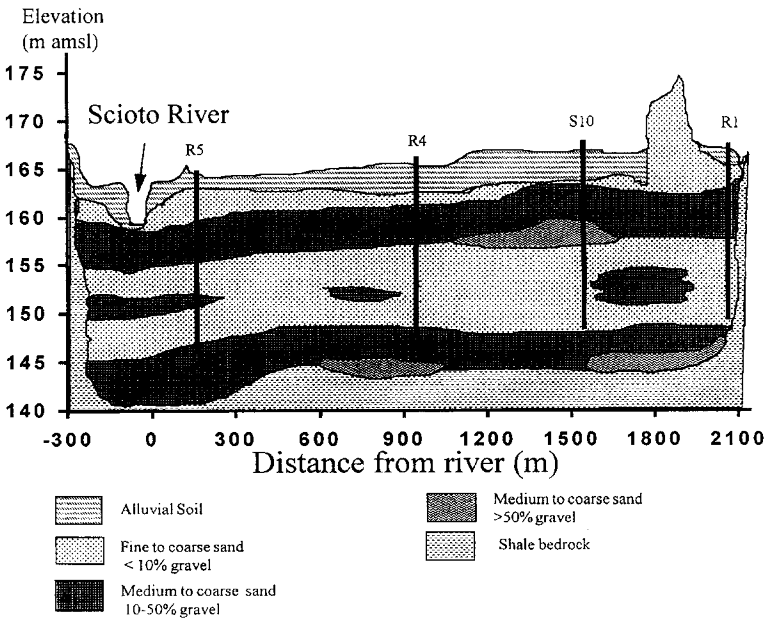

Fig. 1. Cross-section of the Scioto River alluvial valley aquifer. The average hydraulic conductivity of the aquifer materials was 142 m/day.

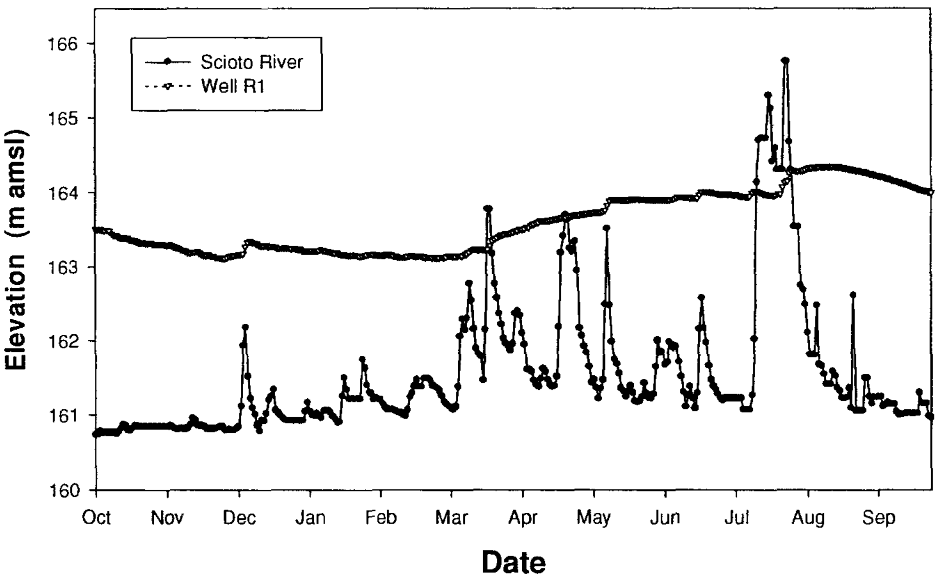

Fig. 2. Daily elevation of the Scioto River located at the right boundary of the OMSEA and well R1 located at the left boundary of the OMSEA from October 1991 to September 1992.

tical dimension would be approximately twice the thickness of the x-axis line.

The left boundary of the problem was taken to be the daily recorded elevation of the Scioto River. The transient nature of flow in the river can be seen in Fig. 2. The Scioto River drains much of central Ohio and forms the western boundary of the OMSEA site. A stream gage at Higby, Ohio (approximately 21 km upstream from the OMSEA) has monitored flow in the 13,290 km^2 watershed for 60 years (Nortz et al., 1994). Over the 60-year period, the mean flow rate was 130 m^3/s with a minimum of 6.9 m^3/s and a maximum flow of 5012 m^3/s. The change in river stage between maximum and minimum flows was 7.4 m. The National Weather Service (NWS) operates a wire-weight gage at Piketon, Ohio. Scioto River elevations adjacent to the OMSEA site were used to develop a gradient correction factor of 1.52 m between the NWS gage site and the OMSEA site (Jagucki et al., 1995). The total change in stage recorded during the period of October 1991 to September 1992 was approximately 4.9 m (Fig. 2).

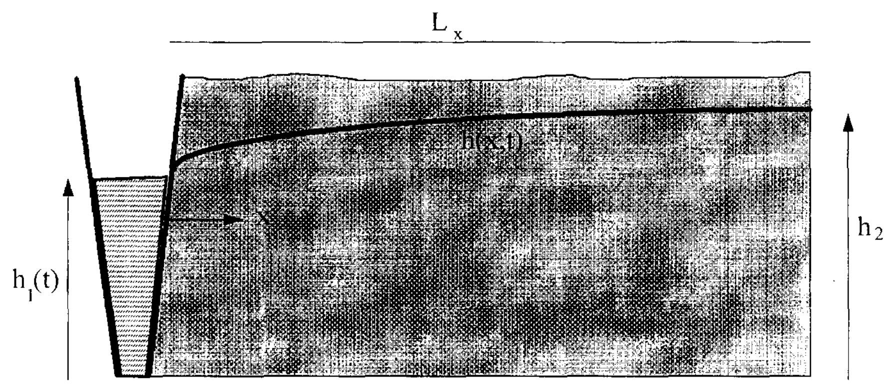

Fig. 3. Sketch of a drainage canal that penetrates the saturated thickness of a perched aquifer.

A well (R1) located at the right boundary of the flow domain shows a gradual rise in elevation over the water year (Fig. 2). Big Beaver Creek, located near well R1, drains an area composed predominantly of the poorly permeable bedrock uplands from which runoff was rapid and bank storage was minimal (Jagucki et al., 1995). The conditions produced rapid stage fluctuations in the creek during storm events but only intermittent flow during the dry summer months.

Eleven wells were constructed over a 260 ha area surrounding the OMSEA site to monitor fluctuations

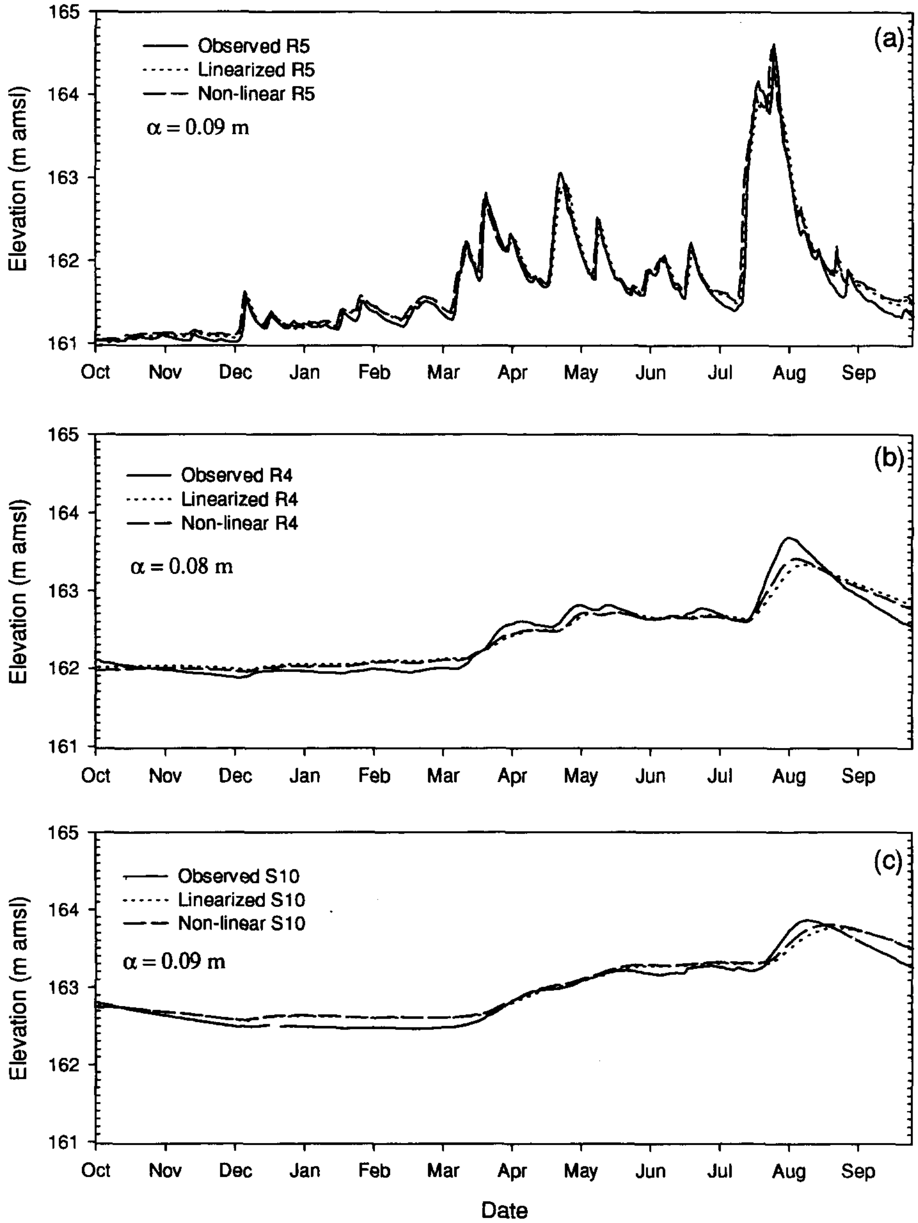

Fig. 4. Observed and simulated water table elevations at locations of 215 (a), 975 (b), and 1525 m (c) from the Scioto River.

in the alluvial valley aquifer. The wells were constructed with 152 mm diameter PVC casing. A 6.1 m long, 2.54 mm slotted, PVC screen was positioned to bracket the highest and lowest expected water table elevations in each well (Jagucki et al., 1995). All the water table wells were instrumented with shaft encoders and electronic dataloggers that recorded hourly water levels. The three wells (R5, R4, and S10) shown in Fig. 1 lie on a flow path from the eastern edge of the OMSEA site to the Scioto River (Jagucki et al., 1995). These wells are located 215, 975, and 1525 m from the Scioto River, respectively.

The aquifer underlying the OMSEA site exhibited many characteristics that made it appropriate for application to the Boussinesq equation; the regional groundwater flow in the Scioto River alluvial valley aquifer was predominantly in the horizontal direction; the aquifer mean thickness was small compared to its horizontal dimension; the water table exhibited mild slopes; the seasonal fluctuation in the water level of the Scioto River was important and appeared to directly affect the groundwater levels in areas near the river (Fig. 2); and except for the top-soil sediments, the aquifer was composed of various types of sand and gravel with high transmissivity values.

For purposes of testing the model, measured values from the previous characterization of the aquifer were used for each of the parameters in the mathematical model. The hydraulic conductivity was 142 m/day, the base of the aquifer was located 143 m above mean sea level (amsl), the average thickness of the aquifer was 18.3 m, and the specific yield was 0.2. The propagation of a flood wave through the Scioto River creates an important variability in the river stage. The effect of temporal fluctuations in the river boundary on aquifer heads seems more important than that of recharge. For this reason, recharge was neglected in the simulations. Comparison of simulated heads with measured well hydrographs confirmed the hypothesis that, in this particular case, temporal variability in the boundary explains to a large extent aquifer head variability. In aquifer with steady boundary conditions, the role of recharge is more important.

An estimate of the ability of the model to simulate the observed water levels was determined by computing an average absolute deviation (α):

$$\alpha = \frac{\sum_{i=1}^{n} |P_i - O_i|}{n} \tag{12}$$

where P is the predicted water elevation, O is the observed water elevation, and n is the number of days simulated.

A primary assumption for using a linearized form of the Boussinesq equation is a constant transmissivity value. At the OMSEA site, the aquifer thickness was large compared to the change in river elevation and a solution to the linearized equation worked well (Workman et al., 1997). There are important problems in the study of lateral saturated flow of water where the saturated thickness is not large compared to the change in boundary conditions. One such problem is the prediction of water table heights adjacent to canals used to drain shallow perched aquifers similar to the sketch in Fig. 3. A simulation was conducted to show the difference between water table elevations predicted with the linearized Boussinesq equation and elevations predicted with the non-linear Boussinesq equation. A soil with a saturated thickness of 2 m, hydraulic conductivity of 2 m/h, and porosity of 0.2 was simulated to have a constant right boundary elevation of 2 m. The left boundary was simulated to fall from a height of 2 m to the base of the canal (0 m) over a four day period. The left boundary was held constant at the base of the canal for 48 days before allowing it to rise to the 2 m starting elevation. The right boundary was assumed to be 400 m from the canal.

4. Results and discussion

The performance of the non-linear model in predicting the stream/aquifer interaction at the OMSEA was very similar to the results obtained with the linearized model. The simulated results from both models are shown along with the observed water levels in three wells located 215, 975, and 1525 m from the Scioto River for the water year from October 1991 to September 1992 in Fig. 4(a–c). The average absolute deviation between simulated and observed values ranged from 0.08 to 0.09 m for the non-linear model.

The summer of 1991 was one of the driest on record in south-central Ohio. The Scioto River was under base-flow conditions with very little deviation in the hydrograph at the beginning of the study (Fig. 2). Rainfall events in the Scioto River watershed upstream of the OMSEA site from March to August 1992 caused fluctuations in the hydrograph of the river at the OMSEA site that were simulated well with both models. The water level in the Scioto River never exceeded the carrying capacity of the channel and overbank flow did not occur.

A large flow event occurring at the end of July 1992 was modeled well for locations near the river with both models. The non-linear model was able to better predict the rise in the aquifer during the flow event at both the R4 and S10 well locations [Fig. 4(b,c)]. The saturated thickness of the aquifer was assumed to be 18.3 m in the linearized model, however, during this particular flow event the saturated thickness became approximately 20 m thick. The non-linear model was able to simulate the change in transmissivity caused by the increase in saturated thickness of the aquifer. Although the non-linear model predicted a faster response of the aquifer to the flow event than the linearized model, neither model predicted the response to be as large or as quick as the observed response.

The non-linear model does not have the limitation of a constant transmissivity. The limitation was only evident for a few days during the simulation of stream/aquifer response at the OMSEA. For conditions when there are large changes in transmissivity, the non-linear model should be expected to better simulate the dynamic response of water table elevations. Spatial variability in the hydraulic conductivity or large variability in heads, especially near the boundaries, accentuate the importance of non-linearity in the differential equation. Fig. 5 shows the simulated water table response 100 m from a drainage canal. For the first few days after drainage was initiated, the simulated water table response was similar between the two models because the assumed constant transmissivity and the transmissivity computed by the non-linear model were nearly equal. Once the profile began to drain, the actual transmissivity became less than the assumed constant value, causing the linearized model to overpredict the drop in water table elevation. After 48 days of drainage, the two solutions approached steady state values. The steady state solution for the linearized solution was quite different from the steady state solution of the non-linear solution because the former was a function of h whereas the latter was a function of h^2. The steady state solution for the problem in Fig. 5 was 0.5 m for the linearized model and 1.0 m for the non-linear model. Upon filling of the canal, the linearized model predicted a faster response of the water table because of the higher assumed transmissivity value.

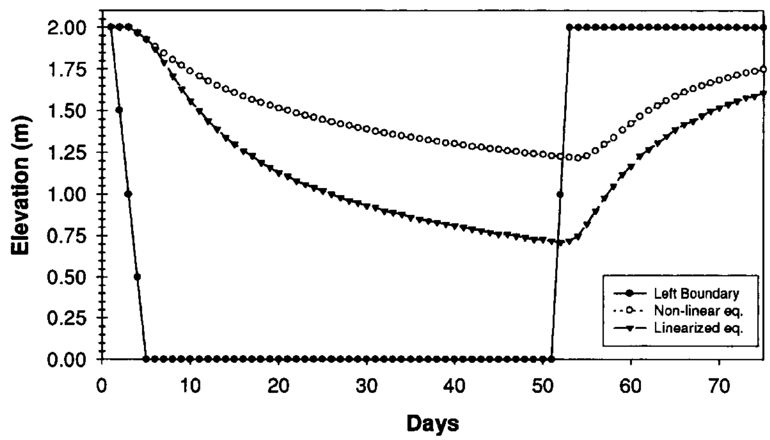

Fig. 5. Simulated response of the water table located a distance of 100 m from a drainage canal that is allowed to drain and then fill over a 48 day period.

5. Summary and conclusions

A mathematical model has been developed to simulate stream/aquifer interactions in an alluvial valley aquifer subject to the assumptions of the non-linear form of the Boussinesq equation. The model implemented a new analytical solution of the transient non-linear Boussinesq equation subject to time-dependent boundary conditions. The differential equations were solved using the method of decomposition. The mathematical model required relatively few parameters to simulate groundwater elevations: hydraulic conductivity, specific yield, and recharge. The model was physically based and could simulate the process of a flood wave propagation into an unconfined aquifer.

The model was tested using observed water table elevations at three locations across a 2 km wide alluvial valley aquifer. The average daily deviation between observed and simulated water table elevations was approximately 0.09 m. The transient redistribution of water in the aquifer was simulated well with the model.

The non-linear form of the Boussinesq equation was shown to better simulate cases when the transmissivity of the aquifer could not be assumed to be a constant. For cases when the change in aquifer thickness is large, the steady state solutions for the linearized and non-linear forms of the Boussinesq equation are not equal.

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Appendix A Other decomposition solutions to the non-linear Boussinesq equation

Other decomposition solutions to the non-linear Boussinesq equation are possible. The x-partial solutions result from writing Eq. (1) as:

$$\frac{\partial^2 h}{\partial x^2} = -\frac{I}{Kh} - \frac{1}{h}\left(\frac{\partial h}{\partial x}\right)^2 + \frac{S}{Kh}\frac{\partial h}{\partial t} \quad \text{(A1)}$$

Defining the operator $L_x = \partial^2/\partial x^2$ and pre-multiplying Eq. (A1) by the inverse operator L_x^{-1} (i.e. the indefinite double x-integration), one obtains:

$$h = c_1 k_1(t) + c_2 k_2(t)x - L_x^{-1}N(h) \quad \text{(A2)}$$

where the constants c_1 and c_2 and the functions $k_1(t)$ and $k_2(t)$ must satisfy the boundary conditions, and the non-linear operator is given by:

$$N(h) = \frac{1}{h}\left\{\frac{I}{K} + \left(\frac{\partial h}{\partial x}\right)^2 - \frac{S}{K}\frac{\partial h}{\partial t}\right\} \quad \text{(A3)}$$

As before, we define the series solution of Eq. (A1) as:

$$h = \sum_{n=0}^{\infty} h_n = h_0 - L_x^{-1} \sum_{n=0}^{\infty} A_n \quad \text{(A4)}$$

where the first term satisfies the homogeneous version of Eq. (A1), that is:

$$h_0 = c_1 k_1(t) + c_2 k_2(t)x \quad \text{(A5)}$$

Using the boundary conditions in Eq. (1) this becomes:

$$h_0 = H_1(t) + a(t)x, \quad a(t) = \frac{H_2(t) - H_1(t)}{l_x} \quad \text{(A6)}$$

Subsequent terms in the series of Eq. (A4) are defined as:

$$h_1 = -L_x^{-1}A_0 \quad \text{(A7)}$$

$$h_2 = -L_x^{-1}A_1$$

$$\vdots$$

$$h_{n+1} = -L_x^{-1}A_n$$

From Eqs. (A6), (A7) and (8):

$$h_1 = -L_x^{-1}\frac{1}{h_0}\left\{\frac{I}{K} + \left(\frac{\partial h_0}{\partial x}\right)^2 - \frac{S}{K}\frac{\partial h_0}{\partial t}\right\} \quad \text{(A8)}$$

Using Eq. (A6) this becomes:

$$h_1 = -L_x^{-1}\left[\frac{1}{H_1(t)+a(t)x}\left\{\frac{I}{K}+a^2(t)-\frac{Sx}{Kl_x}\times\left(\frac{\partial H_2(t)}{\partial t}-\frac{\partial H_1(t)}{\partial t}\right)-\frac{S}{K}\frac{\partial H_1(t)}{\partial t}\right\}\right] \quad (A9)$$

Solving, the second term in the series solution is given by:

$$h_1 = F(t)\hat{h}_1(x,t)+A(t)x^2+B(t)x+C(t) \quad (A10)$$

where

$$F(t) = 1+\frac{S}{a^2K}\left\{\frac{I}{S}-\left(1+\frac{H_1(t)}{l_xa}\right)\frac{\partial H_1(t)}{\partial t}+\frac{H_1(t)}{l_xa}\frac{\partial H_2(t)}{\partial t}\right\} \quad (A11)$$

$$\hat{h}_1(x,t) = h_0(x,t)\ln[h_0(x,t)]-h_0(x,t) \quad (A12)$$

$$B = -\frac{F(t)\hat{h}_1(l_x,t)+A(t)l_x^2+C(t)}{l_x} \quad (A13)$$

$$C(t) = -F(t)\hat{h}_1(0,t) \quad (A14)$$

The third term in the series of Eq. ((A4)) may be obtained in a similar manner. However, this term contains second derivatives with respect to time of the boundary conditions, $\partial H_1(t)/\partial t$ and $\partial H_2(t)/\partial t$. From the applied standpoint, the hydrologist usually has discrete stage values of the river heads at given intervals of time, Δt. While the calculation of the first time derivative of the boundary conditions in Eq. (A11) is plausible, an estimation of second time derivatives may prove unstable, especially when the river hydrographs are noisy. Thus the two-term approximant of the solution to Eq. (A1) is given by Eq. (11), where h_0 is given by Eq. (A6), h_1 is given by Eq. (A10).

An alternative procedure to write Eq. (1) in terms of discharge potentials (Strack, 1989) yields:

$$\frac{\partial^2\Phi}{\partial x^2} = -I+\frac{S}{Kh}\frac{\partial\Phi}{\partial t} \quad (A15)$$

where the discharge potential is defined as:

$$\Phi = \frac{1}{2}Kh^2 \quad (A16)$$

The advantage of Eq. (A15) with Eq. (A16) is that when Eq. (A15) is linearized by replacing h on the right side of Eq. (A15) by some average head h_0, the steady state solution remains non-linear. Eq. (A15) may be solved by decomposition. Under some circumstances the differences between the linear and non-linear solutions will be significant and the non-linear solution becomes important.

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