Explicit Solution to Green and Ampt Infiltration Equation

By Sergio E. Serrano[1]

Abstract: An explicit solution of the Green and Ampt infiltration equation is presented by constructing a decomposition series. Simple expressions for the cumulative infiltration depth and the infiltration rate are proposed. These expressions are valid for deep homogeneous soils under ponding conditions resulting from intense rainfall events. The solution was compared with the exact implicit solution, and with the Lambert W solution of the Green and Ampt equation. It was found that with three terms in the series, the decomposition solution has a maximum error of about 0.15%. The inclusion of additional terms in the series reduces the error. With four terms, the decomposition series yields an error of about 0.02%. The accuracy of a three-term decomposition solution seems adequate for most practical calculations. Furthermore, its simplicity of implementation makes it suitable for fast hand calculations.

INTRODUCTION

The Green and Ampt equation (Green and Ampt 1911) continues to be a widely used model of time evolution of the cumulative infiltration depth and the infiltration rate in deep homogeneous soils under ponding conditions that develop during intense rainfall events. It is relatively easy to implement; it is physically based, since it arises from a finite-difference application of Darcy's law; and it lends itself to the following applications: transient rainfall conditions (Chu 1978), time-varying depth of ponding (Freyberg et al. 1980), and soils in which the hydraulic conductivity changes with depth (Beven 1984). It has been used as a basic approach to comprehensive models (Schmid 1990).

Among the difficulties in the implementation of the Green and Ampt model are the accurate estimation of the time to ponding conditions (Mein and Larson 1973), and the fact that the Green and Ampt equation gives the cumulative infiltration implicitly in terms of time. To use the equation, the hydrologist must use iteration to calculate the time, t, for a selected value of the cumulative infiltration, F. Once the cumulative infiltration versus time is constructed in this manner, the infiltration rate versus time curve may be inferred. Barry et al. (1993) derived an explicit solution that makes use of the Lambert W function, which is a relatively unknown special function that needs to be evaluated over its series expansion, or based on some approximations.

In the next section, another explicit solution based on the application of the method of decomposition (Adomian 1994) is presented. Decomposition generates a series, much like the Fourier series, that converges fast to the exact solution. Decomposition has been applied to the analytical solution of nonlinear infiltration and transport equations (Serrano 1996, 1998; Serrano and Adomian 1996) without discretization or linearization. In the verification section, a comparison with the exact implicit solution and with the Lambert W solution is presented for both the cumulative infiltration and the infiltration rate.

DECOMPOSITION OF GREEN AND AMPT EQUATION

Consider the infiltration of water into a deep homogeneous soil under an intense constant rainfall rate, p (mm/h), such that a ponding layer of water of thickness H (mm) has developed on the ground surface at time t_p (h) after rainfall began. At any time t (h) after the start of rainfall, the wetting front is located a vertical distance L (mm) from the ground surface. The pressure head at the wetting front, ψ_f (mm), is negative if we take the ambient atmospheric pressure as a reference. Neglecting air entrapment, the portion of the soil between the ground surface and the wetting front is saturated and has a volumetric water content of $\theta = n$, where n is the soil porosity. For the soil deeper than the wetting front at time t, the water content is $\theta = \theta_i$, where θ_i is the initial water content—that is, the water content prior to the storm. Applying Darcy's law, and approximating the hydraulic gradient as a finite difference between the ground surface and the wetting front, the infiltration rate at the ground surface, f (mm/h), is given by

$$f(t) = -K\left(\frac{\psi_f - L - H}{L}\right) \tag{1}$$

where K = saturated hydraulic conductivity (mm/h). Neglecting the ponding head, H, and introducing the cumulative infiltration depth in millimeters as $F(t) = \int_0^t f(\tau)\,d\tau = (n - \theta_i)L$, (1) becomes

$$f = K\left(\frac{a}{F} + 1\right); \quad a = |\psi_f|(n - \theta_i) \tag{2}$$

where $|.|$ denotes the absolute value. Expressing $f(t) = dF(t)/dt$, (2) becomes the ordinary differential equation

$$\frac{dF}{dt} - \frac{aK}{F} - K = 0; \quad t \geq t_p; \quad F(t_p) = F_p \tag{3}$$

where F_p = cumulative infiltration at the time of ponding (mm).

Eq. (3) describes the cumulative infiltration depth at any time $t \geq t_p$. Multiplying by dt; separating variables; integrating t between t_p and t, and F between F_p and F; and rearranging, we obtain the traditional Green and Ampt solution

$$t = \frac{F - F(t_p)}{K} + \frac{a}{K}\ln\left[\frac{F(t_p) + a}{F(t) + a}\right] + t_p; \quad t \geq t_p \tag{4}$$

We call (4) the "exact" solution of the approximate infiltration model [(3)], although F is not given explicitly in terms of t. To obtain the cumulative infiltration function versus time, the hydrologist must select trial values of F and substitute them in (4) to obtain the corresponding time of occurrence t.

Using a special form of the soil-water physical relationships (i.e., the water content versus pressure-head relationship, and the hydraulic conductivity versus pressure-head relationship), Barry et al. (1993) obtained a particular solution of the Richards equation and showed that, under the above assumptions, the cumulative infiltration at the ground surface reduces to the Green and Ampt equation [(4)]. Thus, (3) may be expressed as

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$$F - a\ln(F + a) - B(t) = 0; \quad B(t) = K(t - t_p) + F_p - a\ln(F_p + a) \tag{5}$$

By making the substitution $y = -(F + a)$, (5) reduces to

$$we^w = x; \quad x = -\frac{e^{-(B(t)+a)/a}}{a}; \quad w = \frac{y}{a} \tag{6}$$

The solution to (6) is simply $w = W(-1, x)$, where $W(\ ,\ )$ denotes the Lambert W function. Expanding

$$F(t) = -a - aW\left[-1, -\frac{e^{-(B(t)+a)/a}}{a}\right]; \quad t \geq t_p \tag{7}$$

The Lambert W function is a relatively unknown special function that solves (6), which has an infinite number of solutions for each nonzero value of x (Corless et al. 1996). As a result, Lambert W has an infinite number of branches, especially in the complex plane. Exactly one of these branches is analytic at zero. This branch is referred to as the principal branch of Lambert W, and is denoted by $W(0, x)$. The other branches all have a branch point at zero, and these branches are denoted by $W(k, x)$, where k is any nonzero integer. The real branch that corresponds to the solution of the infiltration equation [(3)] occurs when $k = -1$, as expressed in (7). Barry et al. (1993) showed that the branch when $k = 0$ corresponds to capillary rise at the water table. The Lambert W function is related to the tree generating function, $T(x)$, common in the analysis of algorithms discipline. Whether or not (7) is considered a ''closed-form solution'' is subject to interpretation. The Lambert W function has the series expansion (Corless et al. 1996)

$$W(x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} i^{i-2}}{(i-1)!} x^i = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \frac{54}{5}x^6 + \cdots \tag{8}$$

which converges for $|x| < 1/e$. A useful approximation is given in (9a) and (9b)

$$W(x) \approx 0.665[1 + 0.0195\ln(x + 1)]\ln(x + 1) + 0.04; \quad 0 \leq x \leq 500 \tag{9a}$$

$$W(x) \approx \ln(x - 4) - \left[1 - \frac{1}{\ln(x)}\right]\ln[\ln(x)]; \quad x > 500 \tag{9b}$$

The approximation corresponds to the real branch ($-1/e < x$, $W > -1$); that is, $k = 0$. Corless et al. (1996, 1997) provide an excellent review of approximations, examples, applications, and references to this function that appears in many applications of engineering and science. Symbolic algebra software, such as Maple (Waterloo Maple Inc., Waterloo, Ontario, Canada), now provide approximations to the Lambert W function as intrinsic functions. As of this writing, most calculators do not offer it as a standard function.

The infiltration rate is obtained by differentiating (7) with respect to t as

$$f(t) = \frac{KW\left[-1, -\dfrac{e^{-(B(t)+a)/a}}{a}\right]}{1 + W\left[-1, -\dfrac{e^{-(B(t)+a)/a}}{a}\right]}; \quad t \geq t_p \tag{10}$$

Now attempted is an explicit solution of (3) using the method of decomposition (Adomian 1994), which allows the solution of nonlinear equations. A solution to F is sought in terms of more commonly known functions, such as the natural logarithm. From the many decomposition options available, one could start from the differential equation [(3)] itself and obtain a series solution. However, it seems simpler to start from the Green and Ampt solution [(4)]. Rewriting (4)

$$F = K(t - t_p) + F_p + a\ln\left(\frac{F + a}{F_p + a}\right) \tag{11}$$

which may be written as

$$F = K(t - t_p) + F_p + NF; \quad NF = a\ln\left(\frac{F + a}{F_p + a}\right) \tag{12}$$

Expanding the nonlinear operator NF

$$F = K(t - t_p) + F_p + \sum_{i=0}^{\infty} A_i \tag{13}$$

and the Adomian polynomials are defined as

$$A_0 = NF_0; \quad A_1 = F_1\frac{dNF_0}{dF_0} \tag{14a,b}$$

$$A_2 = F_2\frac{dNF_0}{dF_0} + \frac{F_1^2}{2!}\frac{d^2NF_0}{dF_0^2} \tag{14c}$$

$$A_3 = F_3\frac{dNF_0}{dF_0} + F_1F_2\frac{d^2NF_0}{dF_0^2} + \frac{F_1^3}{3!}\frac{d^3NF_0}{dF_0^3} \tag{14d}$$

The polynomials A_n are generated for each nonlinearity so that A_0 depends only on F_0, A_1 depends only on F_0 and F_1, A_2 depends only on F_0, F_1, F_2, and so on. All of the F_n components are analytic and calculable. The term $\Sigma_{n=0}^{\infty} F_n$ constitutes a generalized Taylor series about the function F_0. Sometimes the magnitude of the soil parameters is such that the series converges rapidly and the n-term partial sum $\Phi_n = \Sigma_{j=0}^{n-1} F_j$, the ''approximant,'' serves as an accurate enough and practical solution. For further discussion on the convergence problem of decomposition series, the reader is referred to Cherruault et al. (1992), Cherruault (1989), and Abbaoui and Cherruault (1994). It is also important to mention the rigorous mathematical framework for the convergence of decomposition series developed by Gabet (1992, 1993, 1994). He connected the method of decomposition to well-known formulations where classical theorems (e.g., fixed point theorem, substituted series, etc.) could be used. For a discussion on the convergence of decomposition series with special reference to infiltration equations, see Serrano (1998). The proposed decomposition expansion differs from the Taylor series expansion proposed by Schmid (1990). The concept behind decomposition constructs each term in the series based on the (nonlinear) operator of the equation and the analytical form of the previous terms in the series.

From (13) and (14)

$$F_0 = K(t - t_p) + F_p \tag{15a}$$

$$F_1 = A_0 = a\ln\left[\frac{K(t - t_p) + F_p + a}{F_p + a}\right] \tag{15b}$$

$$F_2 = A_1 = a\ln\left[\frac{K(t - t_p) + F_p + a}{F_p + a}\right]\left[\frac{a}{K(t - t_p) + F_p + a}\right] \tag{15c}$$

$$F_3 = A_2 = a\ln\left[\frac{K(t - t_p) + F_p + a}{F_p + a}\right]\left\{\frac{a^2}{[K(t - t_p) + F_p + a]^2}\right\} - \frac{a}{2}\ln^2\left[\frac{K(t - t_p) + F_p + a}{F_p + a}\right]\left\{\frac{a^2}{[K(t - t_p) + F_p + a]^2}\right\} \tag{15d}$$

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$$F_4 = A_3 = a \ln\left[\frac{K(t - t_p) + F_p + a}{F_p + a}\right]\left\{\frac{a^3}{[K(t - t_p) + F_p + a]^3}\right\}$$
$$- \frac{3a}{2}\ln^2\left[\frac{K(t - t_p) + F_p + a}{F_p + a}\right]\left\{\frac{a^3}{[K(t - t_p) + F_p + a]^3}\right\}$$
$$+ \frac{a}{3}\ln^3\left[\frac{K(t - t_p) + F_p + a}{F_p + a}\right]\left\{\frac{a^3}{[K(t - t_p) + F_p + a]^3}\right\}$$
$$\vdots \qquad (15e)$$

More terms may be easily derived. Factorizing, one obtains a logarithmic series whose coefficients are time series with a closed-form representation

$$F(t) = F_0(t) + a \ln\left[\frac{F_0(t) + a}{F_p + a}\right]\left[\frac{F_0(t) + a}{F_0(t)}\right]$$
$$- \frac{a}{2}\ln^2\left[\frac{F_0(t) + a}{F_p + a}\right]\left[\frac{F_0(t) + a}{F_0^3(t)}\right] + \frac{a}{3}\ln^3\left[\frac{F_0(t) + a}{F_p + a}\right]$$
$$\cdot\left\{\frac{[F_0(t) + a]^2}{F_0^5(t)}\right\} - \ldots; \quad t \geq t_p \qquad (16)$$

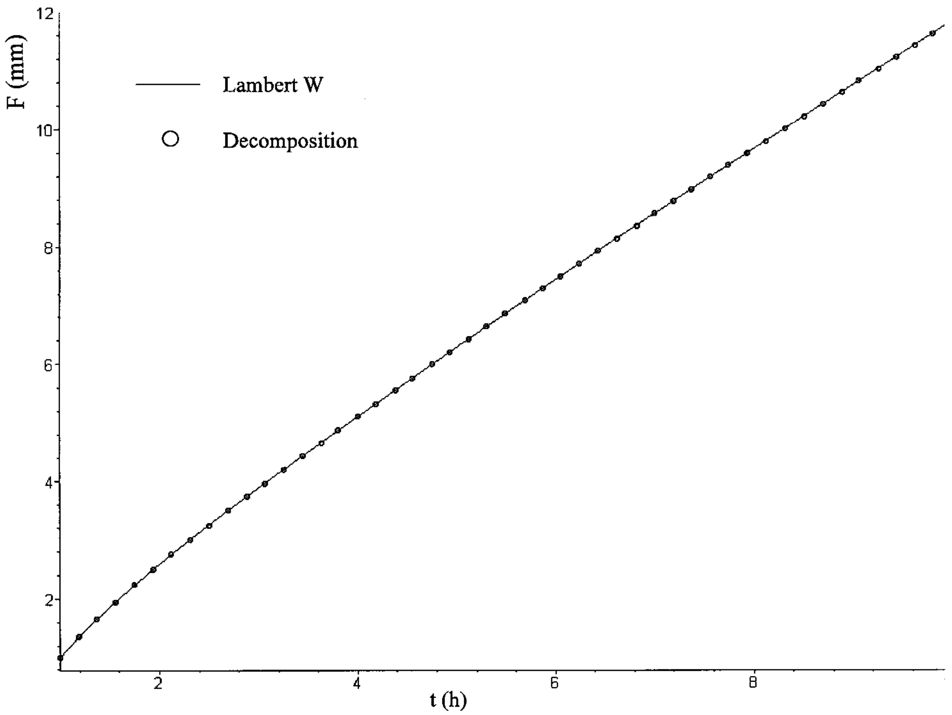

FIG. 1. Comparison between Lambert W and Decomposition Solutions

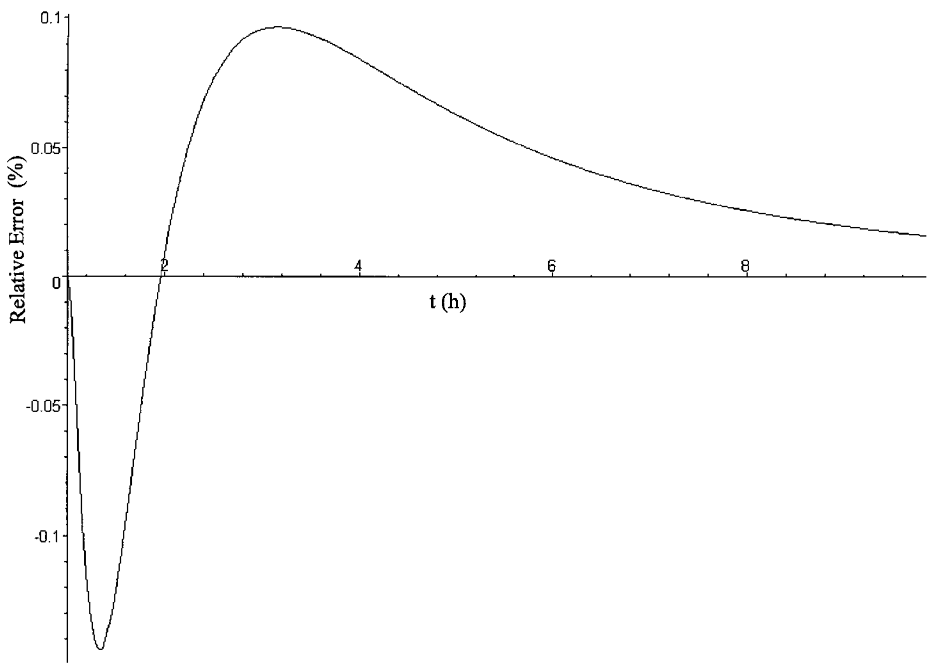

FIG. 2. Relative Error in Percentage

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The infiltration rate is obtained after differentiating (16) with respect to t

$$f(t) = K + \frac{aK}{F_0}\left\{1 + \ln\left(\frac{F_0 + a}{F_p + a}\right)\left[1 - \left(\frac{F_0 + a}{F_0}\right) - \frac{1}{F_0^2}\right]\right\} - \ldots; \quad t \geq t_p \tag{17}$$

VERIFICATION

To gain an idea on the accuracy of (16), assume $K = 1$ mm/h, $F_p = a = 1$ mm, and $t_p = 1$ h. Fig. 1 illustrates a comparison between the Lambert W solution [(7)] and the decomposition solution with three terms on the right side of (16). The agreement between the two solutions is excellent, and can be improved by calculating more terms in the decomposition series. Fig. 2 shows the relative error in percentage between the two solutions for the cumulative infiltration. The maximum relative error is about 0.15%. The maximum relative error is reduced to about 0.02% if one includes the fourth term on the right side of (16). If the hydrologist uses the first two terms in (16) only, the error increases to about 0.9%, which might be acceptable for preliminary hand-calculator estimations.

Now conducted is a more detailed comparison, using the exact solution [(4)] as applied by Dingman (1994, p. 243),

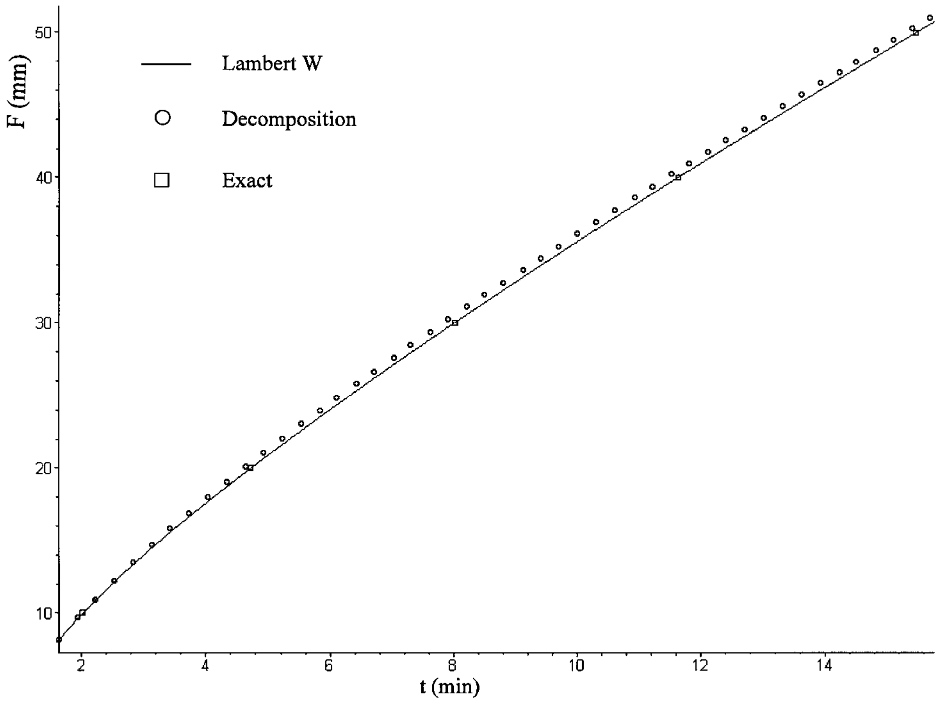

FIG. 3. Comparison among Lambert W, Decomposition, and Exact Solutions

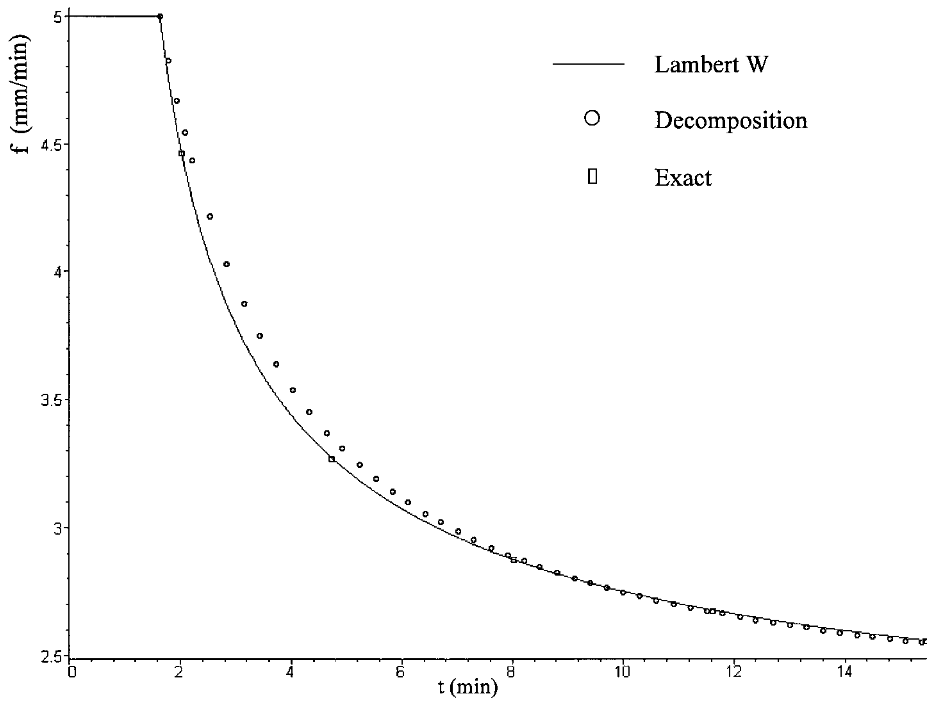

FIG. 4. Infiltration Rates according to Different Solutions

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which consisted of the estimation of the cumulative infiltration and the infiltration rate under constant precipitation intensity. The following are the parameter values after some dimensional transformation (Dingman 1994): p = 5 mm/min, K = 2.082 mm/min, n = 0.419, θ_i = 0.35, and $\psi_f = -166$ mm. The time to ponding was estimated as (Dingman 1994)

$$t_p = \frac{K|\psi_f|(n - \theta_i)}{p(p - K)} = \frac{Ka}{p(p - K)} = 1.633 \text{ min} \tag{18}$$

Accordingly, the cumulative infiltration at the time of ponding is $F_p = pt_p$ = 8.166 mm. The parameter $a = |\psi_f|(n - \theta_i)$ = 11.454 mm.

Fig. 3 shows a comparison among the exact solution [(4)], the Lambert W solution [(7)], and the decomposition solution with three terms on the right side of (16). Again the agreement among the three solutions is excellent. The decomposition solution slightly overestimates the values of cumulative infiltration. This discrepancy could be reduced by including more terms of the series in the calculations. Fig. 4 shows the corresponding evolution of the infiltration rate according to (2) and (4) for the exact solution, (10) for the Lambert W solution, and (17) for the decomposition solution, respectively. Note that $f = p$, when $t < t_p$. As expected, the accuracy of the decomposition solution decreases when differentiating F. For early times after ponding, and for prolonged times after wetting, the decomposition solution seems adequate. By far the most important advantage of the decomposition solution is its simplicity of implementation.

SUMMARY AND CONCLUSIONS

An explicit solution of the Green and Ampt infiltration equation was derived by constructing and truncating a decomposition series. Simple series expressions for the cumulative infiltration depth and the infiltration rate were proposed. The solution was compared with the exact implicit solution, and with the Lambert W solution. It was found that with three terms in the series, the decomposition solution has a maximum error of about 0.15%. The inclusion of additional terms in the series reduces the error. With four terms, the decomposition series yields an error of about 0.02%. The accuracy of a three-term decomposition solution seems adequate for most practical calculations. Furthermore, its simplicity of implementation makes it suitable for fast hand-calculator estimations. The accuracy of the decomposition solution decreases for the infiltration rate, being adequate at the times immediately after ponding, and at prolonged infiltration.

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