FIXED POINT THEORY AND INVARIANT SETS: INTRODUCTION AND NEW EXAMPLES

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Contents

1. Abstract

This document surveys fixed point theory and presents original results.

2. Introduction

Definition: A **fixed point** of a function $f : D \to D$ is a point that gets mapped to itself. That is to say, x is a fixed point if $f(x) = x$.

When do we get a fixed point? The following theorems show us how certain properties of the function f or domain D (for example, continuous functions on norm-compact and convex domains) guarantee the existence of a fixed point for any transformation of that domain possessing those properties.

Brouwer's Theorem: Every continuous function from a closed, bounded, and convex (c.b.c.) set $K \subseteq \mathbb{R}^n$ (equipped with the canonical topology induced by the Euclidean distance) into K has a fixed point.

We will prove Brouwer's Theorem in one dimension.

Proof. If $n = 1$ and K is closed, bounded, convex, and nonempty then $K = [a, b]$ for some real numbers $a < b$. Consider the Intermediate Value Theorem: For f a continuous function over $[a, b]$ and for m between $f(a)$ and $f(b)$, then $\exists c \in [a, b]$ such that $f(c) = m$. Now fix a continuous function $f : [a, b] \to [a, b]$. In order to have a fixed point, we want $f(x) = x$. Now, consider $g(x) = f(x) - x$. Note that g is also a continuous function. Here, we want $g(x) = 0$.

A trivial solution to $g(x) = 0$ could occur at one of our end points, i.e. $f(a) = a$ and $f(b) = b$. If this is not the case, $f(a) > a$ or $f(b) < b$. Then, $g(a) > 0$ and $g(b) < 0$. Apply I.V.T. to g. Hence, for some $c \in [a, b]$, it follows that $g(c) = 0$ which implies $f(c) = c$. Therefore, there exists a fixed point.

When generalizing to higher dimensions, the proof of Brouwer's Theorem becomes vastly more complicated. One such proof was written by Takejiro Seki [10]. A generalization of the theorem itself is due to Schauder in the context of Banach spaces is as follows [7].

Definition: A Banach space (B-space) $(X, \|\cdot\|)$ is a vector space X and a norm $\|\cdot\|$ on X so that X is complete in the topology induced by $\|\cdot\|$.

Schauder's Theorem: Every continuous function from a compact, convex domain D (in a B-space) back into D has a fixed point.

Before continuing with results like these, let us see some counter-examples indicating that the geometric domain restrictions are indeed necessary.

Consider $\rho : S^1 \to S^1$ given by $\rho(\cos(\theta), \sin(\theta)) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2))$. This function rotates the circle S^1 90 °. In this case, the lack of convexity is what allows us to have a fixed point free map that is so simple.

Consider $s : \mathbb{R} \to \mathbb{R}$ given by $s(x) = x + 1$. This is not a bounded domain. Here, we suppose s has a fixed point. This then implies that $x = s(x) = x+1 \iff 0 = 1$. We achieve a contradiction. The lack of boundedness allows for a simple fixed point free map.

Consider $h : (0, 1) \to (0, 1)$ given by $h(x) = x/2$. This is not a closed domain in the canonical topology on $\mathbb{R}$. Suppose a fixed point exists. Then, the only solution $x = h(x) = x/2$ would be $x = 0$. Thus, the lack of a closed domain allows for another fixed point free map.

Even though ρ, s, and h are continuous, we have now seen that closedness, boundedness, convexity, and compactness themselves do not inherently imply that a fixed point exists, but combinations of conditions can guarantee the existences of a fixed point (or, absence of a condition can allow for a very simple fixed point free map).

Now, consider the following map from Brailey Sims [8]:

In what follows, c_0 is the set of sequences which converge to 0. We equip this vector space with the sup-norm $\|\cdot\|_\infty$ to make it a B-space.

Let $C = B^+_{c_0} := \{(x_n) \in c_0 : 0 \le x_n \le 1, \forall n\}$, where all sequences, x_n, in c_0 approach 0 as $n \to \infty$. We define the following affine map:

$$S_1 : C \to C : (x_1, x_2, x_3, ...) \mapsto (1, x_1, x_2, ...)$$

Now, to prove that this map is fixed point free. By way of contradiction, assume that there exists a fixed point. Then, $S_1(x_n) = (1, 1, 1, ...)$. However, then $\lim_{n\to\infty}(x_n) = 1$ which proves that (x_n) is not in c_0. Thus, S_1 must be fixed point free.

Another example, due to Brailey Sims, is as follows [8]. Here $\{e_1, e_2, e_3, ...\}$ is the standard basis for c_0 with $e_1 = (1, 0, 0, ...)$, $e_2 = (0, 1, 0, 0, ...)$, etc. This is an example of a fixed point free contractive map, which is a central focus of this document.

Definition. Let (X, d) be a metric space. A function f from X to itself is a **nonexpansive** if $\forall x, y \in X$, it follows that $d(f(x), f(y)) \le d(x, y)$.

Definition. Let (X,d) be a metric space. A function f from X to itself is a **contractive** map if $\forall x, y \in X$, it follows that $d(f(x), f(y)) < d(x,y)$.

Note that all contractive functions are nonexpansive. Note also that every nonexpansive function is continuous. In fact, such maps are Lipschitz continuous with a Lipschitz constant of 1.

Example: Let λ_n be a decreasing sequence of real numbers that converge to 1. Define $C := \{\sum_{n=1}^{\infty} t_n \lambda_n e_n : (t_n) \in c_0, 0 \le t_n \le 1, \forall n\}$ and S to be an affine map on C such that $S(\sum_{n=1}^{\infty} t_n \lambda_n e_n) := \lambda_1 e_1 + \sum_{n=1}^{\infty} t_n \lambda_{n+1} e_{n+1}$. We will show that S is fixed point free for the specific case of $\lambda_n = \frac{n+1}{n}$ [8].

Now, consider $\sum_{n=1}^{\infty} t_n \lambda_n e_n = (t_1\lambda_1, t_2\lambda_2, t_3\lambda_3, ...) = (2t_1, \dfrac{3}{2}t_2, \dfrac{4}{3}t_3, ...)$. Then, $S(\sum_{n=1}^{\infty} t_n \lambda_n e_n) = (2, \dfrac{3}{2}t_1, \dfrac{4}{3}t_2, ...)$. This is essentially a modified shift. Thus, S is a mapping of C into C that is nonexpansive and contractive, given that (λ_n) is strictly decreasing.

Do we have a fixed point? Recalling the definition, we know that we achieve a fixed point when $\sum_{n=1}^{\infty} t_n \lambda_n e_n = S(\sum_{n=1}^{\infty} t_n \lambda_n e_n)$, or when $(2t_1, \frac{3}{2}t_2, \frac{4}{3}t_3, ...) = (2, \frac{3}{2}t_1, \frac{4}{3}t_2, ...)$. This only occurs when $t_n = 1 \forall n$. Thus, the only possible fixed point here is $(2, \frac{3}{2}, \frac{4}{3}, ...) = (\lambda_1, \lambda_2, \lambda_3, ...)$ which is not in c_0 since $(\lambda_n) \nrightarrow 0$ as $n \to \infty$.

We next prove that S is contractive.

With $\lambda_n = \frac{n+1}{n}$, we want to show that $\exists 0 \le k < 1$ such that $kd(t_n, t_m) \ge d(S(t_n, t_m))$.

$$\left| S(\sum_{n=1}^{\infty} t_n \lambda_n e_n) - S(\sum_{m=1}^{\infty} t_m \lambda_m e_m) \right| = \left| (2, \tfrac{3}{2}t_1, \tfrac{4}{3}t_2, ...) - (2, \tfrac{3}{2}t_1, \tfrac{4}{3}t_2, ...) \right| = 0$$

We know that $kd(t_n, t_m)$ cannot be negative. Therefore, S is contractive.

Banach's Contraction Mapping Theorem, also known as the Banach Fixed Point Theorem, guarantees the existence and uniqueness of a fixed point for certain mappings within complete metric spaces. These mappings are referred to as contraction maps or strict contractions.

Definition. The function $f : D \to D$ is a **contraction map** if the following condition is satisfied:

$$\exists 0 \le k < 1 \text{ with } k \in \mathbf{R} \text{ such that } \forall x, y \in X \text{ it follows that } d(f(x), f(y)) \le kd(x,y)$$

Banach's Contraction Mapping Theorem: Every contraction mapping on a complete metric space has a unique fixed point.

Proof: Define a contraction map $f : X \to X$ with (X,d) being our metric space. We want to show that there exists at least one fixed point of f. We define a sequence $\{x_n\}$ with $x_0 \in X$ given and $x_j = f(x_{j-1})$. We notice that

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \le kd(x_0, x_1)$$
$$d(x_2, x_3) \le kd(x_1, x_2) \le k^2 d(x_0, x_1)$$

and so on...

$$d(x_n, x_{n+1}) \le kd(x_{n-1}, x_n) \le k^n d(x_0, x_1)$$

Assume $n < m$. We know two facts:

(1) $d(x_n, x_{n+1}) \le k^n d(x_0, x_1)$

(2) $d(x_n, x_{n+2}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$
$\le k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) = (k^n + k^{n+1}) d(x_0, x_1)$

Consider the following,

$$d(x_n, x_m) \le (k^n + k^{n+1} + \ldots + k^{m-1}) d(x_0, x_1) = k^n (1 + k + k^2 + \ldots + k^{(m-1)-n}) d(x_0, x_1)$$
$$< k^n (1 + k + k^2 + \ldots) d(x_0, x_1) = k^n \frac{1}{1-k} d(x_0, x_1)$$

We can rewrite it in this fashion because this is a geometric series.
Because $k < 1$, $k^n \frac{1}{1-k} d(x_0, x_1)$ converges to 0. Therefore, $\{x_n\}$ Cauchy in a complete metric space. Hence, $\{x_n\}$ converges to a single point, x. Note that f^n refers to the nth iterate of f in what follows. Now, to show that x is a fixed point of f:

We have by definition of our sequence $x_n = f^n(x_0)$. Then by the continuity of f,

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n(x_0) = \lim_{n\to\infty} f(f^{n-1}(x_0)) = f(\lim_{n\to\infty} f^{n-1}(x_0)) = f(x)$$

Now, to show that this fixed point is unique: By way of contradiction, suppose there exist two fixed points of X, i.e. $f(x_1) = x_1$ and $f(x_2) = x_2$. Then, $d(x_1, x_2) = d(f(x_1), f(x_2)) \le k d(x_1, x_2)$ However, we know $0 \le k < 1$ so $d(x_1, x_2)$ must equal 0, which implies that x_1 and x_2 are the same point.
Thus, f contains only one unique fixed point.

Kirk's Theorem: Let K be a nonempty, bounded, closed, and convex subset of a reflexive Banach space X, and suppose that K has a normal structure. If ϕ is a nonexpanisve mapping of K into itself, then ϕ has a fixed point.

More information (including definitions) regarding reflexive spaces and normal structure can be seen in Appendix 8.

2.1. **Alspach's Map.** Fixed point free maps are also of particular interest to mathematicians. In 1981, Dale Alspach [2] provided an example of a fixed point free map of a weakly compact and convex set, $C_{1/2}$. Note that the elements here are really equivalence classes of functions, with additional details on this equivalence in Appendix A. Below, $L^1[0,1] = L^1([0,1], \Sigma, \mu)$ where μ is Lebesgue measure and Σ is the sigma-algebra of Lebesgue-measurable subsets of $[0,1]$.

$$C_{1/2} = \{f \in L^1[0,1] : 0 \le f(t) \le 1 \text{ for all } t \text{ and } \int_0^1 f = \tfrac{1}{2}\}$$

Alspach's Map $T : C_{1/2} \to C_{1/2}$ is given by

$$Tf(t) = \begin{cases} 2f(2t) \wedge 1 & t \in [0, 1/2) \\ (2f(2t-1) - 1) \vee 0 & t \in [1/2, 1) \end{cases}$$

Here, for all $\alpha, \beta \in \mathbb{R}, \alpha \wedge \beta := \min\{\alpha, \beta\}$ and $\alpha \vee \beta := \max\{\alpha, \beta\}$. Alspach's map is a fixed point free isometry. Every isometry is a nonexpansive map, defined below and also known as Lipschitz-1.

Proof that T is fixed point free: This proof has been modified from the writing of Brailey Sims [8].
We let T be defined as it is above. That is to say, that $\forall f \in C$, it follows that $Tf(t) = 1$ iff either $0 \le t \le \frac{1}{2}$ and $\frac{1}{2} \le f(2t) \le 1$ **or** $\frac{1}{2} < t \le 1$ and $f(2t-1) = 1$. We can also note that $\frac{1}{2} \le t \le 1$ and $Tf(t) = 1$ implies that $Tf(t - \frac{1}{2}) = 1$.

Now, we assume that T is not fixed point free. Let f be a fixed point of T. Then,

$$\begin{aligned}
A := &\{t : f(t) = 1\} \\
= &\{t : Tf(t) = 1\} \\
= &\{t : 0 \le t \le \frac{1}{2} \text{ and } \frac{1}{2} \le f(2t) \le 1\} \cup \{t : \frac{1}{2} < t \le 1 \text{ and } f(2t-1) = 1\} \\
= &\{\frac{1}{2}t : \frac{1}{2} \le f(t) \le 1\} \cup \{\frac{1}{2} + \frac{1}{2}t : f(t) = 1\} \\
= &\frac{1}{2}\{t : \frac{1}{2} \le f(t) < 1\} \cup \frac{1}{2}A \cup (\frac{1}{2} + \frac{1}{2}A)
\end{aligned}$$

We can see that the three sets in the above union are mutually disjoint. We can also note that each of the last two sets have a measure that is half of that of A. Therefore, it follows that:

$$B_1 := \{t : \frac{1}{2} \le f(t) < 1\}$$

is a null set. However,

$$B_1 := \{t : \frac{1}{2} \le Tf(t) < 1\} \supset \{\frac{t}{2} : \frac{1}{4} \le f(t) < \frac{1}{2}\}$$

and thus, $B_2 := \{t : \frac{1}{4} \le f(t) < \frac{1}{2}\}$ is also null. We continue this pattern and find that

$$B_n := \{t : \frac{1}{2^n} \le f(t) < \frac{1}{2^{n-1}}\}$$

is also null for $n \in \mathbb{Z}$. Therefore, we can conclude that

$$\{t : 0 < f(t) < 1\} = \bigcup_{n=1}^{\infty} B_n$$

is null and

$$f \equiv \chi_A \qquad\qquad \text{where } \mu(A) = \int_0^1 \chi_A = \frac{1}{2}$$

From the definition of T we have

$$T(\chi_A) = (\chi_{\frac{1}{2}A} + \chi_{(\frac{1}{2}+\frac{1}{2}A)})$$

And thus, up to sets of measure zero,

$$A = \frac{1}{2}A \cup (\frac{1}{2} + \frac{1}{2}A)$$

We continue to iterate T and find that

$$\begin{aligned}
A &= \frac{1}{4}A \cup (\frac{1}{4} + \frac{1}{4}A) \cup (\frac{1}{2} + \frac{1}{4}A) \cup (\frac{3}{4} + \frac{1}{4}A) \\
A &= \frac{1}{8}A \cup (\frac{1}{8} + \frac{1}{8}A) \cup (\frac{1}{4} + \frac{1}{8}A) \cup \ldots
\end{aligned}$$

and so on and so forth.

Thus, the intersection of A with any given dyadic interval has measure half of that of the interval, which is impossible for a set that is not of full measure. Note that the domain of $C_{1/2}$ of T is not a minimal invariant set. This follows from $diam(C) = 1$ as $1 \geq diam(C) \geq \|\chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}\|_1 = 1$ while $\forall f \in C_{1/2}$, we have $-\frac{1}{2} \leq f - \frac{1}{2} \leq \frac{1}{2}$ which implies $\|f - \frac{1}{2}\chi_{[0,1]}\|_1 = \int_0^1 |f - \frac{1}{2}| \leq \frac{1}{2}$.

Therefore, $C_{1/2}$ is not diametral (defined in Appendix B) and hence, not a minimal invariant set. The concept of this proof is simple: We know that iterates of T assume values at either 0 or 1. If T is to have a fixed point, it must be a function that also assumes only values 0 or 1. However, we have just shown that functions that are either constantly 0 or constantly 1 do not lie in $C_{1/2}$ and therefore, T cannot contain a fixed point.

3. Fixed Point Free Contractive Maps

It was unknown until 2014 if there were strictly contractive fixed point free maps on sets like the domain of T. Then such an example was created as defined below. This map acts on $C_{1/2}$ and is built from T:

$$R : C_{1/2} \to C_{1/2} : f \mapsto \sum_{n=0}^{\infty} \frac{T^n f}{2^{n+1}} = \left(\frac{I}{2} + \frac{T}{4} + \frac{T^2}{8} + ...\right)(f).$$

It is known that this map is fixed point free and contractive [3]. Here, I refers to the identity map, which we will also denote as T^0.

Question : If we change the ratio r in the fixed point free map R from $\frac{1}{2}$ to some other number in (0, 1), does the resulting map remain fixed point free and contractive?

It is important to recall that T is Alspach's map and that Alspach's map is an isometry. That is to say, $\|Tf - Tg\|_1 = \|f - g\|_1$ for all $f, g \in C_{1/2}$. For the remainder of this paper, we will use T to refer to Alspach's map and $\|\cdot\| = \|\cdot\|_1$ where domains are subsets of L^1.

We will use the following lemma from [3].

Lemma 1. For every $f, g \in C_{1/2}$ with $\|f - g\| > 0$, there is some $N \in \mathbb{N}$ such that

$$\left\|\frac{I + T^N}{2} f - \frac{I + T^N}{2} g\right\| < \|f - g\|.$$

3.1. Special Cases.

First, we consider a special case. We start with the case $r = 1/3$ where $F_{\frac{1}{3}} f = \frac{1}{2} If + \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} T^n f$.

We want the sum of all coefficients to be equal to 1. Since $\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} = \frac{1}{2}$, we chose $\frac{1}{2}$ to be our coefficient on the I term.

Now, consider $\|F_{\frac{1}{3}} f - F_{\frac{1}{3}} g\|$.

$$\|F_{\frac{1}{3}} f - F_{\frac{1}{3}} g\| = \|\frac{1}{2} I(f - g) + \frac{1}{3} I(f - g) + \frac{1}{9} T(f - g) + \frac{1}{27} T^2(f - g) + \frac{1}{81} T^3(f - g) + ...\| = \star$$

We can rearrange our terms as follows, distributing the I-term coefficient throughout the terms that follow it:

$$\star = \left\|\frac{2}{3}(f - g) + \frac{2}{9}\frac{I + T}{2}(f - g) + \frac{2}{27}\frac{I + T^2}{2}(f - g) + ... + \frac{2}{3^{N+1}}\frac{I + T^N}{2}(f - g) + \frac{2}{3^{N+2}}\frac{I + T^{N+1}}{2}(f - g) + ...\right\|$$

We then simplify using the fact that Alspach's map is an isometry and apply the Triangle Inequality.

$$\star \leq \tfrac{2}{3}\|f - g\| + \tfrac{2}{9}\|f - g\| + \tfrac{2}{27}\|f - g\| + \tfrac{2}{81}\|f - g\| + ... + \tfrac{2}{3^{N+1}}\|\tfrac{I+T^N}{2}\| + \tfrac{2}{3^{N+2}}\|f - g\| + ...$$

We then apply Lemma 1 on the N-th term, getting $\star < (1 - \frac{2}{3^{N+1}})\|f-g\| + \frac{2}{3^{N+1}}\|f-g\| = \|f-g\|$. Hence, $\|F_{\frac{1}{3}}f - F_{\frac{1}{3}}g\| < \|f-g\|$.

Therefore, this map remains contractive for $r = \frac{1}{3}$.

Now, we will show that the map is fixed point free using the technique that worked in [3]:

$$F_{1/3}(Tf) = \frac{1}{2}(Tf) + \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} T^n(Tf) = \frac{T}{2}f + \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} T^{n+1} f = (\frac{T}{2} + \frac{T}{3} + \frac{T^2}{9} + ...)f$$
$$= (\frac{5}{6}T + \frac{T^2}{9} + \frac{T^3}{27} + ...)f$$

Thus, $\frac{F_{1/3}(Tf)}{3} = (\frac{5}{18}T + \frac{T^2}{27} + \frac{T^3}{81} + ...)f$.

Besides our first term, $\frac{F_{1/3}(Tf)}{3}$ looks remarkably similar to $F_{1/3}f$. We can write our $F_{1/3}f$ in terms of $F_{1/3}Tf$ by subtracting our excess amount of T and adding our $\frac{5}{6}I$:

$$F_{1/3}f = \frac{5}{6}If - \frac{1}{6}Tf + \frac{F_{1/3}Tf}{3}$$

Now, consider $\left\|F_{1/3}f - F_{1/3}g\right\| = \|\frac{5}{6}(f-g) - \frac{1}{6}(Tf - Tg) + \frac{1}{3}(FTf - FTg)\| = \heartsuit$

We apply the reverse triangle inequality:

$$\heartsuit \geq \frac{5}{6}\|f-g\| - \frac{1}{6}\|Tf - Tg\| - \frac{1}{3}\|F_{1/3}Tf - F_{1/3}Tg\| \geq \frac{5}{6}\|f-g\| - \frac{1}{6}\|f-g\| - \frac{1}{3}\|f-g\| = \frac{1}{6}\|f-g\|$$

Since $\|f-g\| > 0$, $\|F_{1/3}f - F_{1/3}g\| > 0$ which implies that $F_{1/3}$ is 1-1.

If $F_{1/3}f_0 = f_0$, then $f_0 = \frac{5}{6}f_0 + \frac{1}{9}Tf_0 + \frac{1}{27}T^2f_0 + ...$

$$f_0 = \frac{5}{6}f_0 + \frac{1}{9}Tf_0 + \frac{1}{27}T^2f_0 + ...$$
$$\frac{1}{6}f_0 = \frac{1}{9}Tf_0 + \frac{1}{27}T^2f_0 + ...$$
$$f_0 = \frac{2}{3}Tf_0 + \frac{2}{9}T^2f_0 + ... = F(Tf_0)$$

Since $F_{1/3}$ is 1-1, we have just shown that $f_0 = Tf_0$. However, it is known that T is fixed point free. Thus, we achieve a contradiction. $F_{1/3}$ is fixed point free.

Now, let us consider another attempt using $r = \frac{3}{4}$:

$$F_{3/4} = \sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}} T^n f.$$

The coefficients in the sum is 1. Thus, we do not need an additional I term before our sum. We can rewrite this as follows, breaking the sum up into a portion where the coefficients add to $\frac{1}{4}$ and the remaining terms. By doing this, we ensure that the $\frac{1}{4}I$ can evenly be distributed throughout the T values of that portion.

$$\frac{2}{3}(\frac{3}{16}T + \frac{9}{64}T^2 + ...) + (\frac{1}{4}I + \frac{1}{16}T + \frac{3}{64}T^2 + ...) = (\frac{1}{8}T + \frac{3}{32}T^2 + ...) + (\frac{1}{8}(\frac{I+T}{2}) + \frac{3}{32}(\frac{I+T^2}{2}) + ...)$$

Then,

$$\begin{aligned}
\|F_{3/4}f - F_{3/4}g\| &= \Big\|\Big(\frac{1}{8}T(f-g) + \frac{3}{32}T^2(f-g) + \frac{9}{128}T^3(f-g)...\Big) \\
&+\Big(\frac{1}{8}\frac{I+T}{2}(f-g) + \frac{3}{32}\frac{I+T^2}{2}(f-g) + ... + \frac{3^{N-1}}{2(4^N)}\frac{I+T^N}{2}(f-g) + ...\Big)\Big\| \\
&\leq \Big(\frac{1}{8}\Big\|T(f-g)\Big\| + \frac{3}{32}\Big\|T^2(f-g)\Big\| + \frac{9}{128}\Big\|T^3(f-g)\Big\| + ...\Big) \\
&+\Big(\frac{1}{8}\Big\|\frac{I+T}{2}(f-g)\Big\| + \frac{3}{32}\Big\|\frac{I+T^2}{2}(f-g)\Big\| + \frac{9}{128}\Big\|\frac{I+T^3}{2}(f-g)\Big\| + ... \\
&+\frac{3^{N-1}}{2(4^N)}\Big\|\frac{I+T^N}{2}(f-g)\Big\| + ...\Big)
\end{aligned}$$

Because T is an isometry, we can simplify the first expression to achieve the following: $\|F_{3/4}f - F_{3/4}g\| \leq \frac{1}{2}\|f-g\| + (\frac{1}{8}\|\frac{I+T}{2}(f-g)\| + \frac{3}{32}\|\frac{I+T^2}{2}(f-g)\| + \frac{9}{128}\|\frac{I+T^3}{2}(f-g)\| + +\cdots + \frac{3^{N-1}}{2(4^N)}\|\frac{I+T^N}{2}(f-g)\| + ...)$.

Again, we apply our lemma to the N-th term.

$$\|F_{3/4}f -_{3/4} g\| < \tfrac{1}{2}\|f-g\| + (\tfrac{1}{2} - \tfrac{3^{N-1}}{2(4^N)})\|f-g\| + \tfrac{3^{N-1}}{2(4^N)}\|f-g\|$$

Hence, $\|F_{3/4}f - F_{3/4}g\| < \|f-g\|$. Therefore, this map remains contractive for $r = \frac{3}{4}$.

What would remain to resolve is whether this map is fixed point free.

We know that $F_{3/4}f = \sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}} T^n f$.
$F_{3/4}(Tf) = \sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}} T^{n+1} f = (\frac{1}{4}T + \frac{3}{16}T^2 + \frac{9}{64}T^3 + ...)f$. We can rewrite $F_{3/4}f$ in terms of $F_{3/4}(Tf)$:

$$F_{3/4}f = \tfrac{1}{4}I + \tfrac{3}{4}F_{3/4}(Tf)$$

Now, consider $\|F_{3/4}f - F_{3/4}g\| = \|\frac{1}{4}(f-g) + \frac{3}{4}(F_{3/4}Tf - F_{3/4}Tg)\| = \clubsuit$

The reverse triangle inequality (and the non-expansiveness of $F_{3/4}T$) no longer helps if we try to apply it in the same way:

$$\clubsuit \geq \tfrac{1}{4}\|f-g\| - \tfrac{3}{4}\|F_{3/4}Tf - F_{3/4}Tg\| > \tfrac{1}{4}\|f-g\| - \tfrac{3}{4}\|f-g\| = -\tfrac{1}{2}\|f-g\|.$$

In the previous case, having positive k in the expression $\|F_{1/3}f - F_{1/3}g\| \geq k\|f-g\| > 0$ was crucial. This calculation breaking down in this case greatly informs our general result in the following subsection.

If we were to continue with the assumption that F is 1-1 and assume that $F_{3/4}f_0 = f_0$, then

$$f_0 = \frac{f_0}{4} + \frac{3}{16}Tf_0 + \frac{9}{64}T^2 f_0 + ... = \sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}} T^n f_0.$$

We then rearrange this equation as shown below: First, by subtracting $\frac{f_0}{4}$ from both sides and then by dividing each side by $\frac{3}{4}$.

$$f_0 = \frac{f_0}{4} + \frac{3}{16}Tf_0 + \frac{9}{64}T^2 f_0 + \ldots$$
$$\frac{3}{4}f_0 = \frac{3}{16}Tf_0 + \frac{9}{64}T^2 f_0 + \ldots$$
$$f_0 = \frac{1}{4}Tf_0 + \frac{3}{16}T^2 f_0 + \ldots = F_{3/4}(Tf_0)$$

If $F_{3/4}$ were to be 1-1, we could see how $Tf_0 = f_0$ implies that Alspach's map has a fixed point. Thus, since Alspach's map is fixed point free, this is a contradiction. Therefore, $F_{3/4}$ is fixed point free if $F_{3/4}$ is 1-1. However, because we cannot determine injectivity, we cannot conclude that $F_{3/4}$ is fixed point free.

3.2. **Generalizing What Works.** We would like to generalize the working construction from $F_{1/3}$ to an unknown $r \in (0,1)$. Specifically, for $r \in (0,1)$ we would like to define a selfmap on $C_{1/2}$ given by $W_r f = cf + b\sum_{n=1}^{\infty} r^n T^n f$ for positive real numbers c and b to be determined. Noting that $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ we need $c + b(\frac{r}{1-r}) = 1$ in order for W_r to map f back into $C_{1/2}$.

Then, $W_r(f) = cTf + b(r^2 Tf + r^3 T^2 f + \ldots)$ while $W_r(f) = cf + b(rTf + r^2 T^2 f + \ldots)$.

This implies that $rW_r(f) - W_r(f) = rcTf - rbTf - cf$. Let f_0 be such that $W_r(f_0) = f_0$. Then, $rW_r(Tf_0) = r(c-b)Tf_0 + (1-c)f_0 = \clubsuit$.

This last expression lead to proving $F_{1/3}$ is fixed point free exactly because the coefficient on Tf_0 was 0. To that end, let $c = b$. Then $b + b(\frac{r}{1-r}) = b(1 + \frac{r}{1-r}) = b(\frac{1-r+r}{1-r}) = 1$ which implies $b = 1-r$.

Then, $\clubsuit = r(b-b)Tf_0 + (1-b)f_0 = r(0)Tf_0 + (1-(1-r))f_0 = rf_0$. Thus, $rW_r(Tf_0) = rf_0$. So this construction shows promise.

$$W_r : C_{1/2} \to C_{1/2} : f \mapsto (1-r)f + (1-r)\sum_{n=1}^{\infty} r^n T^n f$$

To prove that W_r is contractive we will use the following lemma.

Lemma 2. Let $\{c_n\}_{n=0}^{\infty} \subseteq (0,1)$ be such that $\sum_{n=0}^{\infty} c_n = 1$. Then $G : C_{1/2} \to C_{1/2}$ given by $G = \sum_{n=0}^{\infty} c_n T^n$ is a contractive map.

Proof: We note first that rearrangement of terms preserves the value of $(Gf)x = \sum_{n=0}^{\infty} c_n (T^n f)(x)$ for any $f \in C_{1/2}$ and any $x \in [0,1]$ because the convergence of this series is absolute. We will decompose the series as $G = E + \sum_{n=1}^{\infty} d_n \left(\frac{I + T^n}{2}\right)$. We will call E the "extra terms". The terms with $\frac{I+T^n}{2}$ make G contractive and the extra terms keep G a convex combination of T^n.

If $c_0 \le \sum_{n=1}^{\infty} c_n$, then (noting the sum on the right hand side equals $1 - c_0$ and also that $c_0 \le 1/2$) define $d_n = \frac{2c_0 c_n}{1 - c_0}$. In this case the extra terms are $E = \frac{1 - 2c_0}{1 - c_0}\sum_{n=1}^{\infty} c_n T^n$.

To see that G is contractive in this case, let $f, g \in C_{1/2}$ be given with $\|f - g\| > 0$. Because $\|T^n f - T^n g\| = \|f - g\|$ for every n, it follows that

$$\begin{aligned} \|Ef - Eg\| &\le \frac{1 - 2c_0}{1 - c_0}\sum_{n=1}^{\infty} c_n \|T^n f - T^n g\| \\ &= \frac{1 - 2c_0}{1 - c_0}\sum_{n=1}^{\infty} c_n \|f - g\| = (1 - 2c_0)\|f - g\|. \end{aligned}$$

And also,

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} d_n\left[\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right]\right\| &\leq \sum_{n=1}^{\infty} d_n\left\|\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right\| \\ &< \sum_{n=1}^{\infty} d_n\|f-g\| \\ &= \frac{2c_0\|f-g\|}{1-c_0}\sum_{n=1}^{\infty} c_n = 2c_0\|f-g\|.\end{aligned}$$

The strict inequality above is provided by Lemma 1. Putting these together, we have

$$\begin{aligned}\|Gf-Gg\| &= \left\|Ef-Eg+\sum_{n=1}^{\infty} d_n\left[\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right]\right\| \\ &< (1-2c_0)\|f-g\|+2c_0\|f-g\| = \|f-g\|.\end{aligned}$$

In the remaining case, we have $c_0 > \sum_{n=1}^{\infty} c_n$. In this case, we can let $d_n = 2c_n$ and $E = (2c_0-1)I$. Then we have

$$\begin{aligned}\|Gf-Gg\| &= \left\|Ef-Eg+\sum_{n=1}^{\infty} d_n\left[\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right]\right\| \\ &= \left\|(2c_0-1)(f-g)+\sum_{n=1}^{\infty} 2c_n\left[\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right]\right\| \\ &\leq (2c_0-1)\|f-g\|+\sum_{n=1}^{\infty} 2c_n\left\|\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right\| \\ &< (2c_0-1)\|f-g\|+2\sum_{n=1}^{\infty} c_n\|f-g\| \\ &= (2c_0-1)\|f-g\|+2(1-c_0)\|f-g\| = \|f-g\|.\end{aligned}$$

Once again, the strict inequality above comes from Lemma 1. □

Lemma 3. If the map W_r is one-to-one for a given $r \in (0,1)$, then it is fixed point free.

Proof: Let $r \in (0,1)$ be given and suppose that W_r is one-to-one. By way of contradiction, suppose that $f_0 \in C_{1/2}$ is such that $W_r(f_0) = f_0$. This means $f_0 = (1-r)f_0 + r(1-r)Tf_0 + r^2(1-r)T^2f_0 + \cdots$. Subtracting $(1-r)f_0$ from both sides of this equation and then dividing by r gives the following.

$$\begin{aligned}f_0 &= (1-r)f_0 + r(1-r)Tf_0 + r^2(1-r)T^2f_0 + \cdots \\ rf_0 &= r(1-r)Tf_0 + r^2(1-r)T^2f_0 + r^3(1-r)T^3f_0 + \cdots \\ f_0 &= (1-r)Tf_0 + r(1-r)T^2f_0 + r^2(1-r)T^3f_0 + \cdots = W_r(Tf_0).\end{aligned}$$

We have shown that $W_r(Tf_0) = f_0 = W_r(f_0)$. Supposing that W_r is one-to-one, we would have $Tf_0 = f_0$. Since T is fixed point free, this would be a contradiction. □

Lemma 4. For any $r \in (0,1/2]$, W_r is one-to-one on $C_{1/2}$.

Proof: Let $f,g \in C_{1/2}$ be given so that $\|f-g\| > 0$. In the following calculation, we use that W_r is contractive.

$$\begin{aligned}\|W_rf-W_rg\| &= \|(1-r)(f-g)+r(1-r)(Tf-Tg)+r^2(1-r)(T^2f-T^2g)+\cdots\| \\ &\geq (1-r)\|f-g\| - \|r(1-r)(Tf-Tg)+r^2(1-r)(T^2f-T^2g)+\cdots\| \\ &= (1-r)\|f-g\| - r\|W_rTf-W_rTg\| \\ &> (1-r)\|f-g\| - r\|Tf-Tg\| = (1-r)\|f-g\| - r\|f-g\| \geq 0. \quad \square\end{aligned}$$

Putting these lemmas together, we have proven the following theorem.

Theorem 5. For any $r \in (0, 1/2]$, W_r is contractive and fixed point free on $C_{1/2}$.

The question remains open when $r > 1/2$. By Lemma 2, the maps are all contractive. Lemma 4 breaks down when $r > 1/2$. Considering a map like $A_r = (I + W_r)/2$ (which is fixed point free if and only if W_r is fixed point free) extends the values of r to which a result like Lemma 4 applies, however it is unclear if the appropriate variation of Lemma 3 holds.

We note another natural way to generalize R is the family of maps $G_r : C_{1/2} \to C_{1/2} : f \mapsto \frac{1-2r}{1-r} f + \sum_{n=1}^{\infty} r^n T^n f$. These maps are contractive and we can see that $G_{1/2}$ is R. But it is unclear if these maps are fixed point free.

4. Minimal Invariant Sets

Question: The minimal invariant set for T has already been characterized [4]. Can we do the same for R and W_r?

What is a minimal invariant set?

Definition. A set, D, is **T-invariant** if $T(D) \subseteq D$. We call a closed, bounded, and convex set, C, a **minimal invariant set** for T if it is invariant and no closed, bounded, and convex set $K \subsetneq C$ is T-invariant.

Kazimierz Goebel has noted that minimal invariant sets are "bizarre objects" surrounded by much interest [5]. They were studied and some of their properties were catalogued before Alspach's example proved that they existed non-trivially.

With new fixed point free maps identified above, it raises the natural question about their minimal invariant sets. Are these new minimal invariant sets similar to those characterized by Day and Lennard [4]. Would iterating the fixed point free maps identified above produce the same weak limits?

These questions informed the next investigations of this study. If a map is fixed point free, the minimal invariant set is the "next best thing". It is the closest we can get to a fixed point for a fixed point free map. The way we know that these minimal invariant sets must exist is by application of Zorn's Lemma. A simpler understanding of Zorn's Lemma can be achieved via the Nested Interval Theorem.

Nested Interval Theorem

We begin with a proof of the Monotone Convergence theorem:

Theorem 6. If a sequence of real numbers is monotone and bounded, then it converges.

Proof. Let (a_n) be an increasing sequence of real numbers that is bounded. Then $\exists A \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, a_n \le A$. Consider $\{a_n\}$ which is bounded since (a_n) is bounded. Denote the supremum of the set as $U = \sup\{a_n\}$. Let $\epsilon > 0$. $U - \epsilon$ cannot be an upper bound and hence, there exists some a_N such that $a_N > U - \epsilon$. Thus, because (a_n) is increasing, we have for any $n > N$

$$\circledast \quad U - \epsilon < a_n < U + \epsilon$$

which implies $\|a_n - U\| < \epsilon$ for any $n > N$. Hence, $\lim_{n\to\infty} a_n = U$ and (a_n) is convergent. A similar proof holds for decreasing sequences.

Proof of $\circledast$**.**
Claim. For $U = \sup\{a_n\}, U - \epsilon < a_n < U + \epsilon$ holds for n sufficiently large.

Proof. We want to show that $\forall \epsilon > 0, \exists a_0 \in \{a_n\}$ such that $a_0 > U - \epsilon$. Suppose that $a_0 < U - \epsilon$ and that no element of $\{a_n\}$ exists and is greater than $(U - \epsilon)$. Then, $(U - \epsilon)$ would be an upper bound of $\{a_n\}$ by definition, contradicting $U = \sup\{a_n\}$.
The Nested Interval Theorem (N.I.T.) is as follows:

Theorem. Let $(C_k)_{k=1}^{\infty}$ be a collection of closed intervals, $C_k = [x_k, y_k]$, such that $C_{k+1} \subseteq C_k \forall k$. Then, $\cap_{k=1}^{\infty} C_k \neq \emptyset$. Additionally, if $\lim_{k\to\infty}(y_k - x_k) = 0$, then $\cap_{k=1}^{\infty} C_k$ consists of a single point.

Below, we prove the Nested Interval Theorem:

Proof. Let $C_k = [x_k, y_k]$ and $C_{k+1} = [x_{k+1}, y_{k+1}]$. Since $C_{k+1} = [x_{k+1}, y_{k+1}] \subseteq C_k = [x_k, y_k]$, $x_{k+1} \geq x_k$ and $y_{k+1} \leq y_k$. Hence, $\{x_k\}$ is an increasing sequence and is bounded below. By the Monotone Convergence Theorem above, $\{x_k\}$ converges. That is to say, $\exists \lim_{k\to\infty}\{x_k\} = x$. From this, we know that $\sup\{x_k\} = x$. Note that $\{y_k\}$ is bounded below by $x_1 \leq x_k \leq y_k \ \forall k$.

Similarly, $\{y_k\}$ is a decreasing sequence and is also bounded. Again, by the Monotone Convergence Theorem, there is a limit $y = \lim_{k\to\infty}\{y_k\}$. This implies that $\inf\{y_k\} = y$. Since $x_k \leq y_k \ \forall k$, any arbitrary y_j of the sequence $\{y_k\}$ is an upper bound of $\{x_k\}$ and any arbitrary x_j of the sequence $\{x_k\}$ is a lower bound of $\{y_k\}$. Hence $x \leq y_k \forall k$ and $x \leq y$. Thus, we obtain $\forall k$

$$x_k \leq x \leq y \leq y_k.$$

This demonstrates that $[x, y] \subseteq [x_k, y_k]$. Thus, $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$.

Cantor's Intersection Theorem

The Nested Interval Theorem can also be generalized to higher finite dimensions. And an even more general result is demonstrated in Cantor's Intersection Theorem:

Theorem. Let S be a topological space. A decreasing nested sequence of non-empty compact, closed subsets of S has a non-empty intersection. That is to say, if we suppose that $(C_k)_{k\geq 0}$ is a sequence of non-empty compact, closed subsets of S such that $C_0 \supset C_1 \supset C_2 \supset ...C_n \supset C_{n+1} \supset ...$, then $\bigcap_{k=0}^{\infty} C_k \neq \emptyset$.

Proof. By way of contradiction, suppose $\bigcap_{k=0}^{\infty} C_k = \emptyset$. For all $k \in \mathbb{N}$, let $U_k = C_0 \setminus C_k$. Then, $\bigcap_{k=0}^{\infty} U_k = C_0 \setminus \bigcup_{k=0}^{\infty} C_k$. Thus, since $\bigcap_{k=0}^{\infty} C_k = \emptyset$, we have $\bigcup_{k=0}^{\infty} U_k = C_0$. Since the C_k are closed relative to S, the C_k are also closed relative to C_0 and the U_k are open relative to C_0 since the U_k are the set complements of the C_k.

Now, since $C_0 \subset S$ is compact and $\{U_k | k \geq 0\}$ is an open cover of C_0, we can extract a finite cover $\{U_{k_1}, U_{k_2}, U_{k_3}, ..., U_{k_m}\}$. Denote $M = \max_{1\leq i\leq m} k_i$. Then, $\bigcup_{i=1}^{m} U_{k_i} = U_M$ because $U_1 \subset U_2 \subset ... \subset U_n \subset U_{n+1} \subset ...$ by our nesting hypothesis.

Consequently, $C_0 = \bigcup_{i=1}^{m} U_{k_i} = U_M$. But then $C_M = C_0 \setminus U_M = \emptyset$, which is a contradiction.

Zorn's Lemma

Proved in the early 20th century, Zorn's Lemma is a proposition of set theory. The lemma has multiple applications and is a foundational element throughout numerous branches of mathematics, including analysis, topology, and ring theory. Before any discussion of the lemma or its proof, here are a few foundational definitions:

Definition. Let B be a partially ordered set. A subset $A \subseteq B$ is called an **initial segment** if for $\forall a \in A$ and $\forall b \in B$, $a \leq b$ implies that $a \in B$.

Definition. A subset is called the **greatest common segment** of two or more sets if it is the largest subset that is contained within all of the sets.

Definition. A subset or segment $A \subset B$ is called **proper** if all elements of A are in B but $A \neq B$.

Zorn's Lemma is as follows:

Lemma. Let P be a partially ordered set $(P, \leq)$ with the property that every chain $C \subseteq P$ is bounded. Then P has a maximal element.

Proof. By way of contradiction, suppose P is a partially ordered set such that every chain in P has an upper bound in P and P has no maximal elements. Take any nonempty chain $C \subseteq P$. If C is a chain in P, then, by assumption, the set of upper bounds of C in P is nonempty. Let us denote the set of all upper bounds of C that do not lie in C as $f(C)$.

Let us define a function $f : \mathcal{P}(P) \to \mathcal{P}(P)$ such that $\forall \{p_j\} \in P, f(\{p_j\}) \supseteq \{p_j\}$. Now, take $p_1 \in P$. If p_1 is not maximal, let $p_1 < p_2 \in f(\{p_1\})$. If p_2 is not maximal, let $p_2 < p_3 \in f(\{p_1, p_2\})$. If p_3 is not maximal, let $p_3 < p_4 \in f(\{p_1, p_2, p_3\})$. After finitely many steps, if we do not have a maximal element, let $p_{n-1} < p_n \in f(\{p_1, p_2, ..., p_{n-1})\}$. This process continues infinitely, only terminating if P has a maximal element.

Now, suppose we fix an element $p \in P$ and let B denote the set of subsets $C \subseteq P$ where

(i) C is a well-ordered chain in P

(ii) p is the least element of C

(iii) For ever nonempty initial segment $\exists t \in T \subset C$ such that the least element of (C/T) is $f(T)$

Clearly, B is nonempty since it contains $\{p\}$. We note that if C_1, $C_2 \in B$ are nonempty, then either C_1 or C_2 is an initial segment of the other. To see that this is true, let R be the union of all sets that are initial segments of C_1 and C_2. Thus, R is the greatest common segment of C_1 and C_2. If R is a proper initial segment, then by (iii), $f(R)$ is the least element of $(C_1 - R)$ and $(C_2 - R)$ but then $R \cup f(R)$ would be an initial segment of both C_1 and C_2, which contradicts our definition of R since $f(R) \cap R = \{\varnothing\}$. Hence, this implies that R cannot be a proper subset of both C_1 and C_2 and thus, R must be equal to either C_1 or C_2.

Denote U to be the union of all members of B. By (ii), U is well-ordered. Moreover, all members of B are initial segments of U and the least element of U is $\{p\}$.

We now want to show that (iii) holds for U: If T is a proper nonempty initial segment of U, $\exists u \in (U/T)$. By construction, u belongs to some $C \in B$, and T is a proper initial segment of C by construction. Thus, by (iii), $f(t)$ is the least element of (C/T) which implies that $f(t)$ is the least element of (U/T). Hence, U is an element of B.

Now, suppose U does not contain a maximal element of P. Then, $U \cup f(U) \subseteq B$ but clearly, $U \cup f(U) \notin U$, since $\exists f(u) \in f(U) \cap U^C$. This contradicts our definition of U, which states that U is the union of all members of B. Therefore, U must contain a maximal element of P. Thus, a maximal element exists, thereby proving our lemma. A similar proof shows the existence of minimal elements.

4.1. **Advanced Properties of Minimal Invariant Sets.** In 1998, Goebel provided a property of minimal invariant sets [5]. In what follows $\mathrm{co}(S)$ will refer to the closed convex hull of S. Note that $\mathrm{co}(S) = \{\sum_{k=1}^{n} t_k s_k : s_k \in S\ \forall k, \sum_{k=1}^{n} t_k = 1\}$.

Property 1. If K is a minimal invariant set, then $K = \mathrm{co}(T(K))$

Proof. By definition of T-invariant, we know that $T(K) \subseteq K$ and that T refers to Alspach's map. We also know that $\mathrm{co}(T(K))$ is the set of all convex combinations of points in $T(K)$. Thus, $T(K) \subseteq \mathrm{co}(T(K))$. Now, to show set inclusion, by way of contradiction, suppose that $x \in K$ but $x \notin \mathrm{co}(T(K))$. Then, $\mathrm{co}(T(K))$ is "smaller" than K, contradicting the minimality of K. Therefore if $x \in K, x \in \mathrm{co}(T(K))$ and thus, $K \subseteq \mathrm{co}(T(K))$. Now, we know that $T(K) \subseteq K$. Take the convex hull of both sides of the inequality. $\mathrm{co}(T(K)) \subseteq \mathrm{co}(K) = K$ since K is convex. Thus, $\mathrm{co}(T(K)) \subseteq K$. Therefore, $K = \mathrm{co}(T(K))$.

Definition. For any $F : D \to D$ and any $f \in D$, define $M_f(F)$ to be the minimal invariant set for F that contains f.

Consider R the contractive, fixed point free map defined above. Note that because R is contractive, it has only one minimal invariant set [5]. Therefore we will use the notation $M(R)$ to denote this set. It's also known that in $C_{1/2}$, T has a unique minimal invariant set which we will denote $M(T)$.

Claim: It holds that $M(R) \subseteq M(T)$.

Proof. Let $f \in M(T)$. We want to show that $M(T)$ is R-invariant which would follow from $R(f) \in M(T)$. Recall that $M(T)$ is convex and closed. Recall that $R(f) = \sum_{n=0}^{\infty} \frac{T^n f}{2^{n+1}}$.
Now, $T : M(T) \to M(T)$. Thus, $Tf \in M(T)$, $T^2 f \in M(T)$, and so on. Hence, all convex combinations must also be in $M(T)$, including $R(f)$. Therefore, $R : M(T) \to M(T)$, which is to say that $M(T)$ is R-invariant. Thus $M(R) \subseteq M(T)$.

4.2. **Outputs of R.** Previous discoveries of minimal invariant sets have hinged on computing the weak limits of iterates of the fixed point free map [4]. We aim to do the same for W_r starting with $R = W_{1/2}$. Define $g(x) = \frac{1}{2}$. We begin by proving that $Rg = \frac{3}{4} - \frac{1}{2}x$. The following lemma is closely related to a lemma in [3]. We prove the version we need here.

Lemma 7. For any $n \geq 1, T^n g = \sum_{0 \leq k < 2^{n-1}} \chi_{[\frac{2k}{2^n}, \frac{2k+1}{2^n}]}$.

Proof: We proceed by induction.

Let $n = 1$ be our base case.
For $x \in [0, \frac{1}{2})$, $Tg(x) = 2g(2x) \wedge 1 = 1$
For $x \in [\frac{1}{2}, 1]$, $Tg(x) = (2g(2x-1) - 1) \vee 0 = 0$

Assume by way of induction that $T^n g(x) = \sum_{0 \leq k < 2^{n-1}} \chi_{[\frac{2k}{2^n}, \frac{2k+1}{2^n}]}$. Then, throughout, let $f = T^n g$, allowing the abbreviation Tf for the next iterate. Below, we check the following cases. These are appropriate dyadic intervals starting with either an even or odd numerator.

We have four cases:
Case 1: $x \in [\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}) \cap [0, \frac{1}{2})$
Then, $2x \in [\frac{2k}{2^n}, \frac{2k+1}{2^n})$ so $Tf(x) = 2f(2x) \wedge 1 = 2 * 1 \wedge 1 = 1$

Case 2: $x \in [0, \frac{1}{2}) \cap [\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}})$
Then, $2x \in [1,2] \Rightarrow (2x-1) \in [0,1]$ so $Tf(x) = 2f(2x) \wedge 1 = 2 * 0 \wedge 1 = 0$

Case 3: $x \in [\frac{1}{2}, 1] \cap [\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}})$
Then, $2x \in [1,2] \Rightarrow (2x-1) \in [0,1]$ so $Tf(x) = (2f(2x-1)-1) \vee 0 = 1 \vee 0 = 1$

Case 4: $x \in [\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}) \cap [\frac{1}{2}, 1)$
Then, $2x \in [\frac{2k}{2^n}, \frac{2k+1}{2^n}) \Rightarrow (2x-1) \in [\frac{2k+1-2^n}{2^n}, \frac{2k+2-2^n}{2^n})$ so $Tf(x) = (2f(2x-1)-1) \vee 0 = (2*0-1) \vee 0 =$
$= -1 \vee 0 = 0$

We have shown that $Tf(x) = 1$ exactly when x is in an appropriate dyadic interval starting with an even numerator. Therefore, $T^{n+1}g(x) = \sum_{0 \le k < 2^n} \chi_{[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}]}$.

Let h_N denote the N-the Rademacher function. Defined below, the function has the domain [0,1].

$$h_n(x) = \begin{cases} 1 & \text{if } x \in [\frac{2k}{2^n}, \frac{2k+1}{2^n}) \text{ for } 0 \le k < 2^{n-1} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 8. With $s_N = \sum_{n=0}^{N} \frac{T^n g}{2^{n+1}}$ and $f(x) = \frac{3}{4} - \frac{1}{2}x$, we have $\|f - s_{n+1}\| = \frac{1}{2}\|f - s_n\|$ for every $n \in \mathbb{N}$.

Proof: We will prove this by showing that the following three criteria hold for the graph of $f - s_n$ for any n:

- The graph consists of 2^n triangles
- The triangles are sitting over intervals $[\frac{2k}{2^{n+1}}, \frac{2k+2}{2^{n+1}}] = [\frac{k}{2^n}, \frac{k+1}{2^n}]$ $\forall k \in \{0, ..., 2^n - 1\}$
- Height of the triangles is $\frac{1}{2^{n+1}} = \frac{1}{2} \cdot \frac{1}{2^n}$ attained at the left endpoints

Note that the total area of the triangles would be

$$(\text{number of triangles}) \cdot (\tfrac{1}{2} \text{ (height} \cdot \text{interval length)}) = 2^n(\tfrac{1}{2}(\tfrac{1}{2^n} \cdot \tfrac{1}{2^{n+1}})) = \tfrac{1}{2^{n+2}}$$

Base Case: n=0

$$f - s_0 = f - \frac{g}{2} = (\frac{3}{4} - \frac{1}{2}x) - \frac{1}{4} = \frac{1}{2} - \frac{1}{2}x$$

We have three criteria:

- $2^0 = 1$ amount of triangles
- This triangle is sitting over intervals $[\frac{k}{2^0}, \frac{k+1}{2^0}]$ for $k \in \{0\}$ which is the interval $[0,1]$ in this case
- The height of each triangle is $\frac{1}{2^{0+1}} = \frac{1}{2}$ attained at the left endpoints

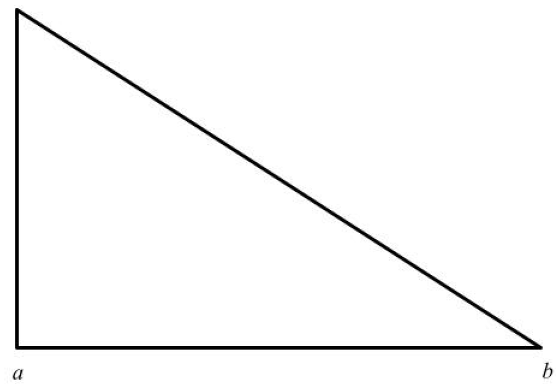
Thus, our total area is $1(\frac{1}{2}(\frac{1}{2} \cdot 1)) = \frac{1}{4}$.

Now suppose by way of induction that $f - s_n$ satisfies the criteria listed above.

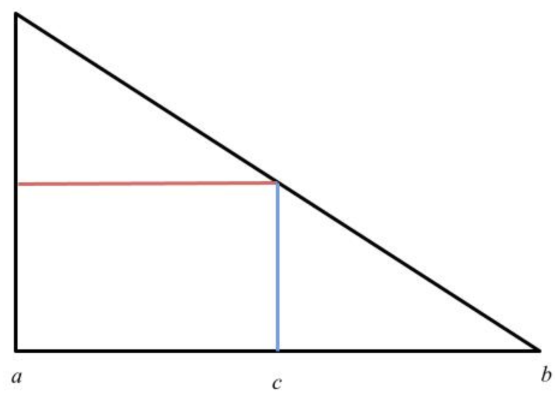
We want to show $f - s_{n+1}$ satisfies our criteria:

- There are 2^{n+1} triangles.
- The triangles are sitting over intervals $[\frac{2k}{2^{n+2}}, \frac{2k+2}{2^{n+2}}] = [\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}] \forall k \in \{0, ..., 2^{n+1} - 1\}$
- The height of the triangles in $f - s_{n+1}$ is half of the previous iterate's height

The inductive hypothesis says that there are 2^n triangles, each of which looks as follows



Here, $a = \frac{j}{2^n} = \frac{2j}{2^{n+1}}$ and $b = \frac{j+1}{2^n} = \frac{2j+2}{2^{n+1}}$. The height of the triangle is $\frac{1}{2^{n+1}}$. Now, consider subtracting $\frac{T^{n+1}g}{2^{n+2}}$. The effect on $[\frac{2j}{2^{n+1}}, \frac{2j+2}{2^{n+1}}]$ is demonstrated below, with $c = \dfrac{\frac{2j}{2^{n+1}} + \frac{2j+2}{2^{n+1}}}{2} = \dfrac{2j+1}{2}$.



Recall that $s_{n+1} - s_n = \frac{T^{n+1}g}{2^{n+2}} \iff s_n - s_{n+1} = -\frac{T^{n+1}g}{2^{n+2}}$. We can use this to re-write

$$\|f - s_{n+1}\| = \|f - s_n + s_n - s_{n+1}\| = \left\| f - s_n - \left(\frac{T^{n+1}g}{2^{n+2}}\right)\right\|$$

By Lemma 7, to subtract $\frac{T^{n+1}g}{2^{n+2}}$ from $f - s_n$ is to subtract $\frac{1}{2^{n+2}}$ on the interval $[a, c]$. This results in the red line, which bisects the height of the original triangle. Lemma 1 also says to subtract 0 on the interval $[c, b]$ which results in no change. Overall, we are left with two smaller triangles, both of which have a height equal to half of that of the original triangle.

Thus, the total area is $2^{n+1}(\frac{1}{2}(\frac{1}{2^n} \cdot \frac{1}{2^{n+1}})) = \frac{1}{2^{n+3}} = \frac{1}{2}(\frac{1}{2^{n+2}})$, which is half of the area of our n-th triangle.

Thus, we've demonstrated that for all n, the area of each set of triangles decreases by a factor of $\frac{1}{2}$ each iteration. That is to say, $\|f - s_{n+1}\| = \frac{1}{2}\|f - s_n\|$.

Theorem 9. With $g = \frac{1}{2}\chi_{[0,1]}$, $Rg = \frac{3}{4} - \frac{1}{2}x$.

Proof: Again, define $f = \frac{3}{4} - \frac{1}{2}x$ and $s_n = \sum_{n=0}^{N} \frac{T^n g}{2^{n+1}}$. Now, by Lemmas 7 and 8, $\|f - s_{n+1}\| = \frac{1}{2^{n+2}} \xrightarrow[n\to\infty]{} 0$, which is to say that the series Rg converges to f.

We hope to continue understanding $\{R^n g\}$ or $\{R^n f\}$ for other starting functions f. Understanding such sequences of iterates enabled the characterization in [4]. The irregularity of R may make this more difficult, but the result above shows that progress is possible.

5. Applying R to Rg

Many of the results in this section and the next have been submitted for publication as [9]. Here $C_{1/2}$ and R are as previously defined.

Lemma 10. $R(f) \neq h_n \forall f \in C_{1/2}$ and $\forall n$.

Proof: Assume $h_n(x) = Rf(x) = \frac{f(x)}{2} + \frac{Tf(x)}{4} + \frac{T^2 f(x)}{8} + \frac{T^3 f(x)}{16} + ...$ Let x be such that $h_n(x) = 1$. By assumption, $1 = h_n(x) = \frac{f(x)}{2} + \frac{Tf(x)}{4} + \frac{T^2 f(x)}{8} + \frac{T^3 f(x)}{16} + ...$ Since the RHS is a convex combination of numbers in $[0,1]$ that add up to 1, it holds that $T^n(x) = 1 \forall n \geq 0$. In particular, $f(x) = 1$. The same reasoning shows that $f(x) = 0$ when $h_n(x) = 0$. Therefore, $f = h_n$. This then implies that $R(h_n) = h_n$, which is a contradiction since R is a fixed point free map.

Claim: $2R(h_m) - h_m = R(h_{m+1})$
Proof: Recall that $Th_n = h_{n+1}$. Now,

$$R(h_m) = \frac{h_m}{2} + \frac{Th_m}{4} + \frac{T^2 h_m}{8} + \frac{T^3 h_m}{16} + ...$$

$$R(h_m) - \frac{h_m}{2} = \frac{Th_m}{4} + \frac{T^2 h_m}{8} + \frac{T^3 h_m}{16} + ... = \frac{h_{m+1}}{4} + \frac{T^2 h_{m+1}}{8} + \frac{T^3 h_{m+1}}{16} + ...$$

$$2R(h_m) - h_m = \frac{h_{m+1}}{2} + \frac{T^2 h_{m+1}}{4} + \frac{T^3 h_{m+1}}{8} + ... = R(h_{m+1}).$$

Claim: $R(h_n)(x) = 2^{n-1} - 2^{n-1}x - \sum_{k=1}^{n-1} 2^{n-k-1} h_k(x)$
Proof: We proceed by induction.
Base case: $n = 2$

$R(h_2)(x) = 2 - 2x - h_1 x$ as proven above.

Suppose, by way of induction that the claim holds for $n = m$. Then

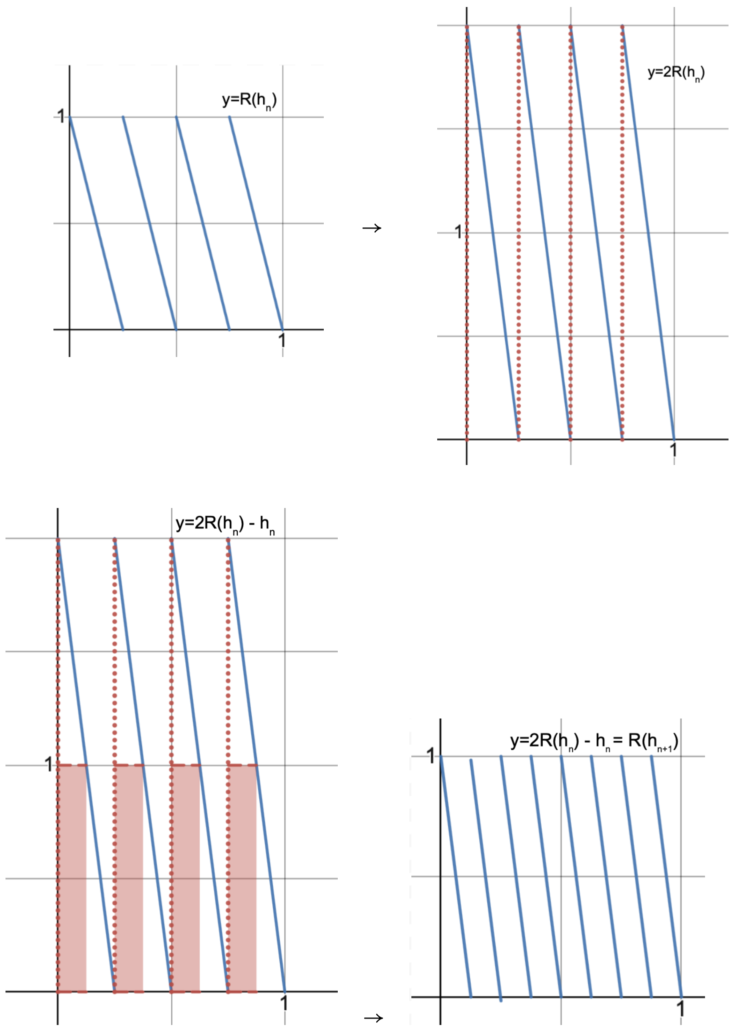
$$\begin{aligned} 2R(h_{m+1})(x) &= 2R(h_m)(x) - h_m(x) \\ &= 2(2^{m-1}(1-x) - \sum_{k=1}^{m-1} 2^{m-k-1} h_k(x)) - h_m(x) \\ &= 2^m(1-x) - (\sum_{k=1}^{m-1} 2^{m-k} h_k(x)) - h_m(x) = 2^m(1-x) - \sum_{k=1}^{m} 2^{m-k} h_k(x) \end{aligned}$$

Thus proving our claim for $n = m+1$. Hence this claim holds $\forall n \in \mathbb{Z}$.

Claim: The graph of $R(h_n)$ consists of 2^{n-1} triangles sitting above intervals$[\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}]$ for $k \in \{0, 1, ..., 2^{n-1} - 1\}$ with a height of 1 attained at $\frac{k}{2^{n-1}}$.

Proof:
Base Case: The graph of $R(h_2)$ produces $2^{2-1} = 2$ triangles as displayed below. The height of the triangles is 1 and they exist over the intervals $[0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$.

Induction Step: Fix $j \in \mathbb{N}$. Assume the claim holds for j. Now consider $j+1$. By a previously established claim, $R(h_{j+1}) = 2R(h_j) - h_j$. Taking $2R(h_j)$ doubles the height of the graph. This results in the same amount of triangles but twice as tall, namely 2, then subtracting h_j causes the height to become 1. By the images below, you can see how when subtracting h_j, the amount of triangles doubles, similar to the process in Lemma 8. We apply Lemma 7 here by taking $n = j - 2$. Thus, the rest of the proof holds as it did in Lemma 8.



6. The Sizes of Minimal Invariant Set

Recalling that $M(T)$ is closed and convex and that W_r is a convex series of iterates of T, it must be the case that $M(W_r) \subseteq M(T)$ for every $r \in (0, 1/2]$. In this section, we discuss the possibility that $M(R) \subsetneq M(T)$. This would be achieved by proving that $h_n \notin M(R)$. In this document, we begin work in that direction and indicate the remaining open question.

Proposition A: $h_n \notin co(R(C_{1/2}))$.

Proof: Assume $h_n = \sum_{k=1}^{\infty} c_k R(f_k)$, where $f_k \in C_{1/2}$ and $\sum_{k=1}^{\infty} c_k = 1$. Recall that the only outputs of h_n are 1's and 0's. Recall also that the outputs of $R(f_k)$ are in [0,1]. Whenever h_n produces a 1, it must be the case that $R(f_k)$ produces 1 also, just as in the Lemma 10 above. By the same reasoning, whenever h_n produces a 0, it must be the case that $R(f_k)$ produces a 0. This then implies that the outputs of $R(f_k)$ are 1's and 0's on the same intervals as h_n. But, this is impossible by the Lemma 10. Thus, we have a contradiction. Therefore, $h_n \notin co(R(C_{1/2}))$.

Proposition B: $h_n \notin \overline{co}^{wk}(R(C_{1/2}))$.

We do not have a proof of this proposition currently. Our work indicates that it is true. Supposing it were true we would have the following affirmative proof of the main question.

Argument: $M(R)$ is a c.b.c (closed, bounded, and convex) set by definition of minimal invariant sets. $\overline{co}^{wk}(R(C_{1/2})) \subseteq C_{1/2}$ and is c.b.c. as well. Therefore, by finite intersection $S := \overline{co}^{wk}(R(C_{1/2})) \cap M(R)$ is c.b.c. and thus, $S = M(R)$. By Goebel 4.1, $M(R) = \overline{co}^{wk}(R(M(R)))$ which then implies that $M(R) \subseteq \overline{co}^{wk}(R(M(R)))$. But, $M(R) \subseteq C_{1/2}$ which implies that $M(R) \subseteq \overline{co}^{wk}(R(M(C_{1/2})))$. Since $M(R) \subseteq \overline{co}^{wk}(R(C_{1/2})), h_n \notin M(R)$.

Open Question: Is it the case that $h_n \notin \overline{co}^{wk}(R(C_{1/2}))$? As mentioned, it would follow that $h_n \notin M(T)$ and $M(R)$ is strictly contained in $M(T)$.
Open Question: $M(T)$ is a canonical example of a small domain that admits a fixed point free map. Is $M(R)$ smaller in a structural sense than $M(T)$? No such examples are known. Specifically, does $M(R)$ span a space that contains a copy of L^1 as $M(T)$ does?

7. Applications

Above, we saw a proof of the Banach Contraction Mapping Theorem. In order to illustrate the possible applications of fixed point theory, we have included three exercises below.

Question 11. Let K be a continuous function on the unit square $\{(x,y) : 0 \le x, y \le 1\}$ satisfying $|K(x,y)| < 1$ for all x and y. Show that there is a unique continuous function $f(x)$ on $[0,1]$ such that

$$f(x) + \int_0^1 K(x,y) f(y)\, dy = e^{x^2}.$$

Answer: Re-arranging, note that our equation is solved if and only if

$$e^{x^2} - \int_0^1 K(x,y) f(y)\, dy = f(x).$$

We define the left hand side of this to be $T(f) = e^{x^2} - \int_0^1 K(x,y) f(y)\, dy$. Note that the question is answered if and only if T has a unique fixed point, f.

As a final preliminary note, since $|K|$ is continuous on a compact set, it attains its max. Since $|K| < 1$ it must be the case that this max (call it $\tilde{k}$) satisfies $\tilde{k} < 1$.

Let us prove that T is a strict contraction on the set of continuous functions on the unit square equipped with the "sup"-norm: $\|\cdot\|_\infty$.

$$\begin{aligned}\|T(f_1)-T(f_2)\|_\infty &= \sup_x \left\{\left|e^{x^2}-\int_0^1 K(x,y)f_1(y)\,dy-\left(e^{x^2}-\int_0^1 K(x,y)f_2(y)\,dy\right)\right|\right\}\\ &= \sup_x \left\{\left|\int_0^1 K(x,y)f_2(y)\,dy-\int_0^1 K(x,y)f_1(y)\,dy\right|\right\}\\ &= \sup_x \left\{\left|\int_0^1 K(x,y)(f_2(y)-f_1(y))\,dy\right|\right\}\\ &\le \sup_x \left\{\int_0^1 |K(x,y)||f_2(y)-f_1(y)|dy\right\}\end{aligned}$$

But $|K|\le\tilde{k}$ and $|f_2(y)-f_1(y)|\le\|f_2-f_1\|_\infty$. So the last quantity above is less than or equal to $\int_0^1 \tilde{k}\cdot\|f_2-f_1\|_\infty dy=\tilde{k}\cdot\|f_2-f_1\|_\infty$.
By transitivity of inequality, $\|T(f_1)-T(f_2)\|_\infty\le\tilde{k}\cdot\|f_2-f_1\|_\infty$, proving that T is a strict contraction. Then the Banach Contraction Mapping Theorem gives that $T(f)=f$ for exactly one f.

A similar problem. [1]

Question 12. Let g be a continuous real-valued function on $[0,1]$. Prove that there exists a continuous real-valued function f on $[0,1]$ satisfying the equation

$$f(x)-\int_0^x f(x-t)e^{-t^2}dt=g(x)$$

Rearranging, we find $f(x)=g(x)+\int_0^x f(x-t)e^{-t^2}dt$. Define this to be $T(f)=g(x)+\int_0^x f(x-t)e^{-t^2}dt$. Note that there exists a unique continuous function if and only if T has a unique fixed point, f.

Now, we want to show that T is a strict contraction on the set of continuous functions on the unit interval equipped with the "sup"-norm: $\|\cdot\|_\infty$. Let x be arbitrary in $[0,1]$. Then

$$\begin{aligned}|T(f_2)x-T(f_1)x| &= \left|g(x)-\int_0^x f_2(x-t)e^{-t^2}dt-g(x)-\int_0^x f_1(x-t)e^{-t^2}dt\right|\\ &= \left|\int_0^x f_2(x-t)e^{-t^2}dt-\int_0^x f_1(x-t)e^{-t^2}dt\right|\\ &\le \int_0^x |e^{-t^2}||f_2(x-t)-f_1(x-t)|dt\\ &\le \int_0^x e^{-t^2}\|f_2-f_1\|_\infty dt=\spadesuit\end{aligned}$$

$f_1(y)-f_2(y)$ is bounded by $\|f_1-f_2\|_\infty$. Now, we want to show that $\int_0^1 e^{-t^2}dy=k<1$. Exponential is a positive and increasing function and $t\in[0,1]$. Thus, $t\ge0$ implies that $e^{t^2}\ge e^0$. Therefore, $e^{-t^2}=\frac{1}{e^{t^2}}\le\frac{1}{e^0}=1$ and $k<1$.

And so we see that $|T(f_2)x-T(f_1)x|=\spadesuit<k\|f_2-f_1\|_\infty$.

This demonstrates that T is a strict contraction and, by Banach Contraction Mapping Theorem, $T(f)=f$ for exactly one f. Therefore, since we've shown that T has a unique fixed point, there exists a unique continuous, real-valued function that satisfies $f(x)-\int_0^x f(x-t)e^{-t^2}dt=g(x)$.

Question 13. Show that there is a unique continuous function $f:[0,1]\to\mathbb{R}$ such that

$$f(x)=\sin x+\int_0^1 \frac{f(y)}{e^{x+y+1}}dy.$$

Define this to be $T(f) = \sin(x) + \int \frac{f(y)}{e^{x+y+1}} dy$. Note that there exists a unique continuous function if and only if T has a unique fixed point, f. Now, we want to show that T is a strict contraction. Let x be arbitrary in $[0,1]$. Then,

$$\begin{aligned} |T(f_2)x - T(f_1)x| &= \left|\sin(x) - \int \frac{f_2(y)}{e^{x+y+1}} dy - \sin(x) + \int \frac{f_1(y)}{e^{x+y+1}} dy\right| \\ &= \left|\int \frac{f_1(y)}{e^{x+y+1}} dy - \int \frac{f_2(y)}{e^{x+y+1}} dy\right| = \left|\int \frac{f_1(y) - f_2(y)}{e^{x+y+1}} dy\right| \\ &\le \int \left|\frac{f_1(y) - f_2(y)}{e^{x+y+1}}\right| dy = \clubsuit \end{aligned}$$

$f_1(y) - f_2(y)$ is bounded by $\|f_1 - f_2\|_\infty$. Now, we want to show that $\int_0^1 \frac{1}{e^{x+y+1}} dy \le \frac{1}{e} < 1$. The exponential function is increasing and $x, y \in [0,1]$. Thus, for $x, y \ge 0$, $e^{x+y} \ge 1$ implies $e^{x+y+1} \ge e$. Hence, $\frac{1}{e^{x+y+1}} \le \frac{1}{e}$. Therefore, $\clubsuit < \frac{1}{e} < 1$.

This thereby demonstrates that T is a strict contraction and, by Banach Contraction Mapping Theorem, $T(f) = f$ for exactly one f. Therefore, since we've shown that T has a unique fixed point, there exists a unique continuous function that is a solution to $f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}} dy$.

8. Appendix

Appendix A. Measure Zero

Definition. Measure Zero
We say that $A \subseteq R$ has **measure zero** if $\forall \varepsilon > 0$, $\exists \{I_n\}_{n\in\mathbb{N}}$ such that $A \subseteq \bigcup I_n$ and $\sum l(I_n) < \varepsilon$.

Lemma 14. Let $A \subseteq B$. Then, if B has measure zero, A has measure zero.

Proof. By definition, $B \subseteq \bigcup\{I_n\}$. Then $A \subseteq B \subseteq \bigcup\{I_n\}$ implies $A \subseteq \bigcup\{I_n\}$.

Lemma 15. Let $A \cup B$. Suppose A and B have measure zero. Then, $A \cup B$ has measure zero.

Proof. Let $\varepsilon > 0$. Take $\{I_k\}$ with $A \subseteq \bigcup\{I_k\}$ such that $\sum l(I_k) < \frac{\varepsilon}{2}$ and $\{\tilde{I_k}\}$ with $B \subseteq \bigcup\{\tilde{I_k}\}$ such that $\sum l(\tilde{I_k}) < \frac{\varepsilon}{2}$. Note $A \cup B \subseteq (\bigcup\{I_k\}) \cup (\bigcup\{\tilde{I_k}\})$. We can then denote $C := \{I_k\} \cup \{\tilde{I_k}\}$. Then, $\sum l(I_k) + \sum l(\tilde{I_k}) < \varepsilon$. Hence, C has measure zero. Thus, by Lemma 1, $A \cup B \subseteq C$ has measure zero.

Consider the following functions: $f, g : \mathbb{R} \to \mathbb{R}$. Let's define $\sim$ by $f \sim g$ if $\{x : f(x) \ne g(x)\}$ has measure zero.

1. $\sim$ is reflexive. We want to show $\forall f : \mathbb{R} \to \mathbb{R}$, $f \sim f$.

   $\{x : f(x) \ne f(x)\} = \varnothing$ which has measure zero.

2. $\sim$ is symmetric. We want to show $\forall f, g : \mathbb{R} \to \mathbb{R}$, if $f \sim g$ then $g \sim f$.

$\{x : g(x) \ne f(x)\} = \{x : f(x) \ne g(x)\}$, which has measure zero. Therefore, the other set has measure zero.

3. $\sim$ is transitive. We want to show $\forall f, g, h : \mathbb{R} \to \mathbb{R}$, if $f \sim g$ and $g \sim h$, then $f \sim h$.

Consider $A = \{x : f(x) \neq g(x)\}$ and $B = \{x : g(x) \neq h(x)\}$. A and B have measure zero by definition. Now, consider the set $C = \{x : f(x) \neq h(x)\}$.

Consider x where $f(x) = g(x)$. If $x \in A^C \cup B^C, x \in C^C$. If $x \notin C^C, x \notin A^C \cup B^C$. By DeMorgan's Laws, if $x \in C, x \in A \cup B$. Thus, $C \subseteq A \cup B$. Therefore, C also has measure zero.

Appendix B. Reflexive and Normal Structure

Definition. Let X be a normed space and $X^{**} = (X^*)^*$ denote the second dual vector space of X. The canonical map $x \mapsto x'$ defined by $x'(f) = f(x)$, $f \in X^*$ gives an isometric linear isomorphism from X into X^{**}. The space X is a **reflexive space** if this map is surjective.

Before defining normal structure, here are a few foundational definitions:

Definition. We say that the **radius** of D relative to the point $u \in D$ is $r_u(D) = \sup\{\|u - v\| : v \in D\}$.

Definition. The **diameter** of D is defined as $diam(D) = \sup\{\|x - y\| : x, y \in D\}$. A point p is considered **diametral** if $r_p(D) = diam(D)$.

In fact, if our set is c.b.c., then its diametral points must be on the boundary and not in the interior of the set.
Proof. By way of contradiction, assume the claim to be true. Let D be c.b.c. and let x be diametral and an interior point of D. Then $x \in V_\varepsilon(x)$ with $V_\varepsilon(x) \subseteq D$.

Because x is diametral, $\exists v \in D$ with $\|v - x\| = d > diamD - \frac{2\varepsilon}{3}$.

Take $p = \frac{2\varepsilon}{3}\frac{x-v}{\|x-v\|} + x$. Then $\|v - p\| = d + \frac{2\varepsilon}{3} > diamD - \frac{\varepsilon}{2} + \frac{2\varepsilon}{3} = diamD + \frac{\varepsilon}{6}$. However, this contradicts the fact that the diameter of D by having v and p farther apart than allowed.

Definition. A bounded, convex subset K of a Banach space X is said to have **normal structure** if it contains any non-diametral points. That is to say, if a set lacks normal structure, all of its points are diametral.

Appendix C. Baker Transform

The **Baker Transform** is a map $T : L^1[0,1] \to L^1[0,1]$ where $L^1[0,1]$ the set of all real-valued Lebesque-integrable functions with domain [0,1]. Note that every Riemann-integrable function is Lebesque-integrable. Each iteration of the transform makes the function half as wide and twice as tall, as can be seen in the graphs below. This results in Haar Wavelet-like graphs when f is a constant function. We know the following about T:

1. $\lim_{n\to\infty} \int_a^b T^n f = \frac{1}{2}(b-a)$ if and only if the weak limit of $\{T^n f\}$ is $h(x) = \frac{1}{2}$
2. T preserves area: $\int_0^1 f = \int_0^1 Tf$
3. T is fixed point free on certain domains, such as $C_{1/2} = \{f : [0,1] \to [0,1] \text{ s.t. } \int_0^1 f = \frac{1}{2}\}$

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