NEW EXAMPLES OF FIXED POINT FREE MAPS AND GENERAL RESULTS CONCERNING CONTRACTIVE SERIES

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Abstract. In this paper, we construct new examples of fixed point free (fpf) maps. We construct a fpf isometry of a weakly compact and convex subset of a complex function space, inspired by Alspach's famous 1981 example on a real function space. We also use a known fpf nonexpansive map and previously established technique involving series to construct a new fpf contractive map. In this second case, we hope that the necessary details in the lemma regarding contractiveness shed light on a natural general question about such series. We generalize the contractive series construction in another direction, showing that the coefficients can vary over a certain interval.

In addition, we prove that once a map is fpf and contractive on an appropriate domain, the previously mentioned series technique will always produce a fpf and contractive map.

1. Introduction

For any map $f : X \to X$ on a set, we say $x \in X$ is a fixed point if $f(x) = x$. Moreover, f is *fixed point free* (fpf) if there is no such $x \in X$.

Let X be a Banach space and C be a closed, bounded, and convex (cbc) subset of X. Let $T : C \to C$ be a map. We say that T is *nonexpansive* if for all $x, y \in C$ we have $\|Tx - Ty\| \leq \|x - y\|$, *contractive* if for all $x, y \in C$ we have $\|Tx - Ty\| < \|x - y\|$, and an *isometry* if for all $x, y \in C$ we have $\|Tx - Ty\| = \|x - y\|$. We note the difference between contractive as defined here and the stronger notion of a strict contraction. The map T is a (strict) *contraction* on C if $\exists k \in (0, 1)$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in C$.

Fixed point theory studies the mix of conditions on maps and domains that guarantee that all such maps on all such domains have a fixed point. Three well-known results of this type follow.

Brouwer's Theorem: Let $C \subseteq \mathbb{R}^n$ be cbc. Then any continuous function $f : C \to C$ has a fixed point.

Banach's Contraction Mapping Theorem: If (X, d) is a complete metric space, then every contraction mapping $f : X \to X$ has a unique fixed point.

The next result [5] applies to reflexive spaces and their subsets with normal structure. We do not define these properties here. We do note that the label "normal structure" is apt. These domains share some geometric properties with their simpler analogues, such as the fixed point property established by the theorem.

Kirk's Theorem: Let C be a nonempty cbc subset of a reflexive Banach space X, and assume C has normal structure. If $f : C \to C$ is a nonexpansive map, then it has a fixed point.

Beyond these positive results, counterexamples show certain conditions that are not sufficient to guarantee the existence of a fixed point. Taken together, these two types of results establish the contours of fixed point theory. The new results derived in this paper are all counterexamples.

In [1], Dale Alspach gave such a counterexample. His example of a fixed point free map acts on a weakly compact and convex subset of a Banach space. This map is defined (with $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$) as:

$$(Tf)(t) = \begin{cases} 2f(2t) \wedge 1, & \text{if } t \in [0, \frac{1}{2}) \\ (2f(2t-1)) - 1) \vee 0, & \text{if } t \in [\frac{1}{2}, 1) \end{cases}$$

The convex iterate series of T was considered in [2], where the map $R := \sum_{n=0}^{\infty} \frac{T^n}{2^{n+1}}$ was shown to be a fixed point free and contractive map on a weakly compact and convex subset of a Banach space. In section 2, we generalize this construction to series with coefficients r^n where $r \in (0, 1/2]$.

In section 3, we consider a convex iterate series of R, which leads to a generalization about maps of this kind. We will prove that the standard iterate series of any fpf and contractive map on a cbc subset of a Banach space is itself fpf and contractive.

In Section 4, we define a complex analogue, V, of Alspach's original map T. While analogous to T in the sense of doubling outputs and "slicing" them, V is not built from T. V does mimic T's behavior of pushing outputs of transformed functions to "extreme" values. We prove that V is a fixed point free isometry of a domain of complex-valued integrable functions.

In Section 5, we construct an iterate series of $T\Delta$, a variation of Alspach's map ([1]) as defined in [4]. We prove that this series is fpf and contractive even though $T\Delta$ is merely nonexpansive. This proof uses similar techniques to [2], where it is proven that the convex series of iterates for Alspach's map is fpf and contractive. This raises another natural general question about iterate series of nonexpansive fpf maps which we are unable to fully resolve.

2. An Interval of Fixed Point Free Contractive Maps

In this section we will prove that for any $r \in (0, 1/2]$, the following functions are fixed point free and contractive on $C_{1/2}$. Note that $W_{1/2}$ is the map R.

$$W_r : C_{1/2} \to C_{1/2} : f \mapsto (1-r)f + (1-r)\sum_{n=1}^{\infty} r^n T^n f$$

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To prove that W_r is contractive we will use the following lemmas. Lemma 1 here is Lemma 3.3 from [2]. We will use $\|\cdot\|$ for $\|\cdot\|_1$ on elements of $C_{1/2} \subseteq L^1$.

Lemma 1. For any $f, g \in C_{1/2}$ with $\|f - g\| > 0$ there is an $N \in \mathbb{N}$ such that

$$\left\| \frac{I + T^N}{2} f - \frac{I + T^n}{2} g \right\| < \|f - g\|.$$

Lemma 2. Let $\{c_n\}_{n=0}^{\infty} \subseteq (0,1)$ be such that $\sum_{n=0}^{\infty} c_n = 1$. Then $G : C_{1/2} \to C_{1/2}$ given by $G = \sum_{n=0}^{\infty} c_n T^n$ is a contractive map.

Proof: We note first that rearrangement of terms preserves the value of $(Gf)x = \sum_{n=0}^{\infty} c_n (T^n f)(x)$ for any $f \in C_{1/2}$ and any $x \in [0,1]$ because the convergence of this series is absolute. We will decompose the series as $G = E + \sum_{n=1}^{\infty} d_n \left(\frac{I + T^n}{2} \right)$. We will call E the "extra terms". The terms with $\frac{I+T^n}{2}$ make G contractive and the extra terms keep G a convex combination of T^n.

If $c_0 \leq \sum_{n=1}^{\infty} c_n$, then (noting the sum on the right hand side equals $1 - c_0$ and also that $c_0 \leq 1/2$) define $d_n = \frac{2c_0 c_n}{1 - c_0}$. In this case the extra terms are $E = \frac{1 - 2c_0}{1 - c_0} \sum_{n=1}^{\infty} c_n T^n$.

To see that G is contractive in this case, let $f, g \in C_{1/2}$ be given with $\|f - g\| > 0$. Because $\|T^n f - T^n g\| = \|f - g\|$ for every n, it follows that

$$\begin{aligned} \|Ef - Eg\| &\leq \frac{1 - 2c_0}{1 - c_0} \sum_{n=1}^{\infty} c_n \|T^n f - T^n g\| \\ &= \frac{1 - 2c_0}{1 - c_0} \sum_{n=1}^{\infty} c_n \|f - g\| = (1 - 2c_0)\|f - g\|. \end{aligned}$$

And also,

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} d_n\left[\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right]\right\| &\le \sum_{n=1}^{\infty} d_n\left\|\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right\| \\ &< \sum_{n=1}^{\infty} d_n\|f-g\| \\ &= \frac{2c_0\|f-g\|}{1-c_0}\sum_{n=1}^{\infty} c_n = 2c_0\|f-g\|.\end{aligned}$$

The strict inequality above is provided by Lemma 1. Putting these together, we have

$$\begin{aligned}\|Gf-Gg\| &= \left\|Ef-Eg+\sum_{n=1}^{\infty} d_n\left[\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right]\right\| \\ &< (1-2c_0)\|f-g\|+2c_0\|f-g\| = \|f-g\|.\end{aligned}$$

In the remaining case, we have $c_0 > \sum_{n=1}^{\infty} c_n$. In this case, we can let $d_n = 2c_n$ and $E = (2c_0-1)I$. Then we have

$$\begin{aligned}\|Gf-Gg\| &= \left\|Ef-Eg+\sum_{n=1}^{\infty} d_n\left[\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right]\right\| \\ &= \left\|(2c_0-1)(f-g)+\sum_{n=1}^{\infty} 2c_n\left[\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right]\right\| \\ &\le (2c_0-1)\|f-g\|+\sum_{n=1}^{\infty} 2c_n\left\|\left(\frac{I+T^n}{2}\right)f-\left(\frac{I+T^n}{2}\right)g\right\| \\ &< (2c_0-1)\|f-g\|+2\sum_{n=1}^{\infty} c_n\|f-g\| \\ &= (2c_0-1)\|f-g\|+2(1-c_0)\|f-g\| = \|f-g\|.\end{aligned}$$

Once again, the strict inequality above comes from Lemma 1. □

Lemma 3. If the map W_r is one-to-one for a given $r \in (0,1)$, then it is fixed point free.

Proof: Let $r \in (0,1)$ be given and suppose that W_r is one-to-one. By way of contradiction, suppose that $f_0 \in C_{1/2}$ is such that $W_r(f_0) = f_0$. This means $f_0 = (1-r)f_0 + r(1-r)Tf_0 + r^2(1-r)T^2f_0 + \cdots$. Subtracting $(1-r)f_0$ from both sides of this equation and then dividing by r gives the following.

$$\begin{aligned}f_0 &= (1-r)f_0 + r(1-r)Tf_0 + r^2(1-r)T^2f_0 + \cdots \\ rf_0 &= r(1-r)Tf_0 + r^2(1-r)T^2f_0 + r^3(1-r)T^3f_0 + \cdots \\ f_0 &= (1-r)Tf_0 + r(1-r)T^2f_0 + r^2(1-r)T^3f_0 + \cdots = W_r(Tf_0).\end{aligned}$$

We have shown that $W_r(Tf_0) = f_0 = W_r(f_0)$. Supposing that W_r is one-to-one, we would have $Tf_0 = f_0$. Since T is fixed point free, this would be a contradiction. □

Lemma 4. For any $r \in (0,1/2]$, W_r is one-to-one.

Proof: Let $f,g \in C_{1/2}$ be given so that $\|f-g\| > 0$. In the following calculation, we use that W_r is contractive.

$$\begin{aligned}\|W_rf-W_rg\| &= \|(1-r)(f-g)+r(1-r)(Tf-Tg)+r^2(1-r)(T^2f-T^2g)+\cdots\| \\ &\ge (1-r)\|f-g\|-\|r(1-r)(Tf-Tg)+r^2(1-r)(T^2f-T^2g)+\cdots\| \\ &= (1-r)\|f-g\|-r\|W_rTf-W_rTg\| \\ &> (1-r)\|f-g\|-r\|Tf-Tg\| = (1-r)\|f-g\|-r\|f-g\| \ge 0. \quad \square\end{aligned}$$

Putting these lemmas together, we have proven the following theorem.

Theorem 5. For any $r \in (0,1/2]$, W_r is contractive and fixed point free.

The question remains open when $r > 1/2$. By Lemma 2, the maps are all contractive. Lemma 4 breaks down when $r > 1/2$. Considering a map like $A_r = (I + W_r)/2$ (which is fixed point free if and only if W_r is fixed point free) extends the values of r to which a result like Lemma 4 applies, however it is unclear if the appropriate variation of Lemma 3 holds.

We note another natural way to generalize R is the family of maps $F_r : C_{1/2} \to C_{1/2} : f \mapsto \frac{1-2r}{1-r} f + \sum_{n=1}^{\infty} r^n T^n f$. These maps are contractive. $F_{1/2} = R$. But it is unclear if these maps are fixed point free.

3. Iterate Series of Fixed Point Free and Contractive Maps

In this section, we prove a theorem about fpf and contractive maps. The iterate series (following the construction in [2]) of such maps are always contractive and fpf. The following example motivated the details of the proof of this general theorem.

Example 1: Let $F := \sum_{n=0}^{\infty} \frac{R^n}{2^{n+1}}$ with R as in [2]. Then F is fpf and contractive.

Upon writing down the details of the proof of the above statement, it became clear that there were sufficient conditions for general iterate series to be fpf and contractive. We prove below that the iterate series of any fpf and contractive mapping on a cbc subset of a Banach space is itself fpf and contractive. Moreover, Theorem 6 gives a new class of fpf and contractive maps.

Theorem 6. Let D be any cbc subset of a Banach space. If H is any map $H : D \mapsto D$ that is fixed point free and contractive, then the map $J := \sum_{n=0}^{\infty} \frac{H^n}{2^{n+1}}$ is also fixed point free and contractive.

Proof: Let D be a cbc subset of a Banach space. To see that J is contractive, consider $\|Jf - Jg\|$ for $f, g \in D$. Grouping like coefficients,

$$\|Jf - Jg\| = \left\| \frac{1}{2}(f - g) + \frac{1}{4}(Hf - Hg) + \frac{1}{8}(H^2 f - H^2 g) + \cdots \right\|$$

Using the triangle inequality, we get

$$\|Jf - Jg\| \leq \frac{1}{2}\|f - g\| + \frac{1}{4}\|Hf - Hg\| + \frac{1}{8}\|H^2 f - H^2 g\| + \cdots.$$

Since H is contractive, for every $j \in \mathbb{N}$ it follows that $\|H^j f - H^j g\| < \|f - g\|$. So

$$\|Jf - Jg\| < \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \|f - g\| = \|f - g\|,$$

proving that J is contractive.

Now, we will show that J is fpf. To do this, we will first show that J is 1-1. Let $f, g \in D$ with $\|f - g\| > 0$. Grouping $\|Jf - Jg\|$ as above and using the reverse triangle inequality, we get

$$\|Jf - Jg\| \geq \frac{1}{2}\|f - g\| - \left\| \frac{1}{4}(Hf - Hg) + \frac{1}{8}(H^2 f - H^2 g) + \cdots \right\| = \frac{1}{2}\|f - g\| - \frac{1}{2}\|JHf - JHg\| = **$$

Since J and H are contractive, $** > \frac{1}{2}\|f - g\| - \frac{1}{2}\|Hf - Hg\| > \frac{1}{2}\|f - g\| - \frac{1}{2}\|f - g\| = 0$. Hence $\|Jf - Jg\| > 0$ when $\|f - g\| > 0$, proving that J is 1-1.

Finally, assume by way of contradiction that $Jf = f$ for some $f \in D$. By definition,

$$f = \frac{f}{2} + \frac{Hf}{4} + \frac{H^2 f}{8} + \cdots.$$

Subtracting the $\frac{f}{2}$ from both sides, we get $\frac{f}{2} = \frac{Hf}{4} + \frac{H^2 f}{8} + \cdots$. Multiplying by 2 then gives

$$Jf = f = \frac{Hf}{2} + \frac{H^2 f}{4} + \cdots = JHf$$

Since J is 1-1, this implies $Hf = f$, a contradiction to H being fpf. □

4. A Complex Version of Alspach's Map

In this section we define a complex analogue of Alspach's famous map T. The map V, defined below, was named in honor of Vladimir Visnjic. In this section, we prove that V is a fixed point free isometry. Letting $S = [0,1] \times [0,1] \subseteq \mathbb{C}$, we take V to be acting on a domain (specified in a later subsection) of functions $g : S \to S$.

$$Vg(z) = \begin{cases} h(z) = 2g(2z) \wedge 1 \wedge i, & \text{if } z \in [0,1/2] \times [0,1/2] = D_3 \\ b(z) = (2g(2z-i) - i) \wedge 1 \vee 0i, & \text{if } z \in [0,1/2] \times (1/2,1] = D_2 \\ u(z) = (2g(2z-1) - 1) \vee 0 \wedge i, & \text{if } z \in (1/2,1] \times [0,1/2] = D_4 \\ d(z) = (2g(2z-1-i) - 1 - i) \vee 0 \vee 0i, & \text{if } z \in (1/2,1] \times (1/2,1] = D_1 \end{cases} \tag{1}$$

Here, we have taken some liberties with $\wedge$ and $\vee$ notation. We define $(a+bi) \wedge c \vee di = \min\{a,c\} + i\max\{b,d\}$, $(a+bi) \vee c \wedge di = \max\{a,c\} + i\min\{b,d\}$, etc. In every case we have a complex number $(a+bi)$ on the left of the first operator ($\wedge$ or $\vee$), then a purely real number (0 or 1) to the right of the first operator, and then a purely imaginary number ($0i$ or i) to the right of the second operator. This notation is simply used to take $\wedge = \min$ or $\vee = \max$ one coordinate at a time.

In order to more clearly denote the relationship between the values that g acts on as we iterate V and the inputs to $V^n g$, we define the correspondence $\phi : S \to S$ as

$$\phi(z) = \begin{cases} 2z, & \text{if } z \in D_3 \\ 2z - i, & \text{if } z \in D_2 \\ 2z - 1, & \text{if } z \in D_4 \\ 2z - 1 - i, & \text{if } z \in D_1 \end{cases} \tag{2}$$

and note that we can re-write V in the following way

$$Vg(z) = \begin{cases} h(z) = 2g(\phi(z)) \wedge 1 \wedge i, & \text{if } z \in D_3 \\ b(z) = (2g(\phi(z)) - i) \wedge 1 \vee 0i, & \text{if } z \in D_2 \\ u(z) = (2g(\phi(z)) - 1) \vee 0 \wedge i, & \text{if } z \in D_4 \\ d(z) = (2g(\phi(z)) - 1 - i) \vee 0 \vee 0i, & \text{if } z \in D_1 \end{cases} \tag{3}$$

4.1. **V is Fixed Point Free.** Consider the domain $D = \{f \in L^1(S) : f(z) \in S \ \forall z \in S$ and $\iint_S f = \frac{1}{2} + \frac{i}{2}\}$. Suppose that $g \in D$ with $Vg = g$.

In this subsection, we will define $A = \{z \in S : Re(g(z)) = 1\}$ and $B_n = \{z \in S : 1/2^{n-1} > Re(g(z)) \geq 1/2^n\}$. As before $S = [0,1] \times [0,1]$. Then

$$\begin{aligned} A &= \{z : Re(g(z)) = 1\} \\ &= \{z : Re(Vg(z)) = 1\} \\ &= \{z \in D_1 \bigcup D_4 : Re(g(\phi(z))) = 1\} \bigcup \{z \in D_2 \bigcup D_3 : \frac{1}{2} \leq Re(g(\phi(z)))\} \end{aligned}$$

Recall that in D_1, $\phi(z) = 2z - 1 - i$ (and therefore $z = \frac{\phi}{2} + \frac{1}{2} + \frac{i}{2}$). Consider

$$\begin{aligned} A \bigcap D_1 &= \{z \in D_1 : Re(Vg(z)) = 1\} \\ &= \{z \in D_1 : Re(g(\phi(z))) = 1\} \\ &= \left\{\frac{\phi}{2} + \frac{1}{2} + \frac{i}{2} : \phi \in S \text{ and } Re(g(\phi)) = 1\right\} = \frac{1}{2}A + \frac{1}{2} + \frac{i}{2}. \end{aligned}$$

Similarly, $A \bigcap D_4 = \frac{1}{2}A + \frac{1}{2}$.

Also recall that in D_2, $\phi(z) = 2z - i$ and consider

$$A \bigcap D_2 = \left\{z \in D_2 : Re(g(\phi(z))) \geq \frac{1}{2}\right\} = \{z \in D_2 : Re(g(\phi(z))) = 1\} \bigcup \left\{z \in D_2 : 1 > Re(g(\phi(z))) \geq \frac{1}{2}\right\}.$$

As above, $\{z \in D_2 : Re(Vg(z)) = 1\} = \frac{1}{2}A + \frac{i}{2}$. And similarly, $\left\{z \in D_2 : 1 > Re(Vg(z)) \geq \frac{1}{2}\right\} = \frac{1}{2}B_1 + \frac{i}{2}$.

Similarly, $A \bigcap D_3 = \{z \in D_3 : Re(g(\phi(z))) = 1\} \bigcup \{z \in D_3 : Re(g(\phi(z))) \geq \frac{1}{2}\}$ where the first part is equal to $\frac{1}{2}A$ and the second part is equal to $\frac{1}{2}B_1$.

Summarizing, we have that

$$\begin{aligned} A &= (A \bigcap D_1) \bigcup (A \bigcap D_2) \bigcup (A \bigcap D_3) \bigcup (A \bigcap D_4) \\ &= \left(\frac{1}{2}A + \frac{1}{2} + \frac{i}{2}\right) \bigcup \left((\frac{1}{2}A + \frac{i}{2}) \bigcup (\frac{1}{2}B_1 + \frac{i}{2}) \right) \bigcup \left(\frac{1}{2}A \bigcup \frac{1}{2}B_1 \right) \bigcup \left(\frac{1}{2}A + \frac{1}{2} \right) \\ &= \left((\frac{1}{2}A + \frac{1}{2} + \frac{i}{2}) \bigcup (\frac{1}{2}A + \frac{i}{2}) \bigcup \frac{1}{2}A \bigcup (\frac{1}{2}A + \frac{1}{2}) \right) \bigcup \left((\frac{1}{2}B_1 + \frac{i}{2}) \bigcup \frac{1}{2}B_1 \right). \end{aligned}$$

And if we take the measures of both sides of this (recalling that in 2 dimensions, scaling down by a factor of 2 decreases area by a factor of 4) we get $m(A) = 4 * \dfrac{1}{4}m(A) + 2 * \dfrac{1}{4}m(B_1)$. Subtracting $m(A)$ from both sides of this equation reveals that $m(B_1) = 0$.

Continuing in this way, we can see that $m(B_n) = 0$ for every $n \in \mathbb{N}$. The omitted proof follows the above structure using that $V^n g = g$ and therefore scaled-down copies of B_n can be seen to land in A for every n just as with B_1. From this we can conclude that (up to a set of measure zero in the domain), the outputs of $Re(g)$ are only 0 and 1.

And focusing on imaginary parts of the outputs of $V^n g$ shows that $m(\{z : Im(g(z)) \in (0,1)\}) = 0$ just as with the real part above. In fact, the rest of the proof only needs these facts for the real part, and proceeds much as the proof in [8] does.

Using the decomposition of A from above - and discarding the copies of B_1 which we now know have measure zero - produces the following.

$$A = \frac{1}{2}A \bigcup (\frac{1}{2}A + \frac{i}{2}) \bigcup (\frac{1}{2}A + \frac{1}{2}) \bigcup (\frac{1}{2}A + \frac{1}{2} + \frac{i}{2})$$

Substituting again gives

$$\begin{aligned} A = \quad & \frac{1}{4}A \bigcup (\frac{1}{4}A + \frac{1}{4}) \bigcup (\frac{1}{4}A + \frac{i}{4}) \bigcup (\frac{1}{4}A + \frac{1}{4} + \frac{i}{4}) \\ & \bigcup (\frac{1}{4}A + \frac{1}{2}) \bigcup (\frac{1}{4}A + \frac{3}{4}) \bigcup (\frac{1}{4}A + \frac{1}{2} + \frac{i}{4}) \bigcup (\frac{1}{4}A + \frac{3}{4} + \frac{i}{4}) \\ & \bigcup (\frac{1}{4}A + \frac{i}{2}) \bigcup (\frac{1}{4}A + \frac{1}{4} + \frac{i}{2}) \bigcup (\frac{1}{4}A + \frac{3i}{4}) \bigcup (\frac{1}{4}A + \frac{1}{4} + \frac{3i}{4}) \\ & \bigcup (\frac{1}{4}A + \frac{1}{2} + \frac{i}{2}) \bigcup (\frac{1}{4}A + \frac{3}{4} + \frac{i}{2}) \bigcup (\frac{1}{4}A + \frac{1}{2} + \frac{3i}{4}) \bigcup (\frac{1}{4}A + \frac{3}{4} + \frac{3i}{4}) \end{aligned}$$

This process continues infinitely, showing that the intersection of A with any dyadic rectangle (i.e. any set of the form $[k/2^n, (k+1)/2^n] \times [j/2^m, (j+1)/2^m] \in S$ where k, j, m, and n are in $\mathbb{N}$) contains such scaled down copies of A. Recalling that $Re(\iint_S g) = 1/2$, we can also now compute

$$m(A) = \iint_A 1 = \iint_A Re(g) = \iint_S Re(g) = Re(\iint_S g) = 1/2.$$

From this it follows that the intersection of A with those dyadic rectangles is exactly half the measure of those rectangles, contradicting the measurability (and not full measure) of A. This finishes the proof of the following theorem.

Theorem 7. The map $V : D \to D$ is fixed point free.

4.2. **V Preserves Integrals.** Consider any integrable $g : S \to S$.

By definition $\iint_S Vg(z)\,dA = \iint_{D_1} Vg(z)dA + \iint_{D_2} Vg(z)dA + \iint_{D_3} Vg(z)dA + \iint_{D_4} Vg(z)dA$. Consider $\iint_{D_1} Vg(z)\,dA = \iint_{D_1} ((2g(\phi(z)) - 1 - i) \vee 0 \vee 0i)\,dA$. Separating real and imaginary coordinates, we have

$$\iint_{D_1} Vg(z)\,dA = \int_{1/2}^{1}\int_{1/2}^{1} [(2g(2x-1+i(2y-1)) - 1 - i) \vee 0 \vee 0i]\,dx\,dy$$

and with the substitutions $u = 2x - 1$ and $w = 2y - 1$, this becomes

$$\iint_{D_1} Vg(z)\,dA = \int_0^1\int_0^1 [(2g(u+iw) - 1 - i) \vee 0 \vee 0i]\,\frac{1}{2}du\,\frac{1}{2}dw.$$

Similarly,

$$\iint_{D_2} Vg(z)\,dA = \iint_S [(2g(u+iw) - i) \wedge 1 \vee 0i]\,\frac{1}{4}du\,dw$$

$$\iint_{D_3} Vg(z)\,dA = \iint_S [(2g(u+iw)) \wedge 1 \wedge i)]\frac{1}{4}du\,dw$$

$$\iint_{D_4} Vg(z)\,dA = \iint_S [(2g(u+iw) - 1) \vee 0 \wedge i]\frac{1}{4}du\,dw$$

And so, $\iint_S Vg(z)dA = \iint_S [[(2g(u+iw) - 1 - i) \vee (0+0i)] + [(2g(u+iw) - i) \wedge 1 \vee 0i] + [(2g(u+iw)) \wedge (1+i)] + [(2g(u+iw) - 1) \vee 0 \wedge i]]\,\frac{1}{4}dudw$.

Let $x = Re(g(u+iw))$. Then the real component of the sum of the 4 integrands (notated as $Re(Int)$) can be written as $Re(Int) = \frac{1}{4}[((2x-1) \vee 0) + (2x \wedge 1) + (2x \wedge 1) + ((2x-1) \vee 0)] = \frac{1}{2}[((2x-1) \vee 0) + (2x \wedge 1)]$. At this point, there are two cases we have to consider: $x \le \frac{1}{2}$ and $x > \frac{1}{2}$.

Case 1: If $x \le \frac{1}{2}$, consider $Re(Int) = \frac{1}{2}[((2x-1) \vee 0) + (2x \wedge 1)]$. The first term will be 0 and $2x$ will be less than or equal to 1, making the second term $2x$. This gives $Re(Int) = \frac{1}{2}(0 + 2x) = x$.

Case 2: If $x > \frac{1}{2}$, then we get $Re(Int) = \frac{1}{2}[((2x-1) \vee 0) + (1 \wedge 1)]$. The second term equals 1 and the first term is $2x - 1$ because $x > 1/2$. The two ones will cancel. As a result, we get that $Re(Int) = x$, and

$$Re\left(\iint_S Vg(z)\right)dA = \iint_S Re(Int)du\,dw = \iint_S x\,du\,dw = \iint_S Re(g(u+iw))du\,dw.$$

The imaginary part of g acts in the same way, with the imaginary component of the sum of the 4 integrands equaling $\frac{1}{2}[((2y-1) \vee 0) + (2y \wedge 1)]$, where $y = Im(g(u+iw))$. Again, for both $y \le \frac{1}{2}$ and $y > \frac{1}{2}$, the sum equals y, and thus

$$Im\left(\iint_S Vg(z)\right)dA = \iint_S Im(Int)du\,dw = \iint_S y\,du\,dw = \iint_S Im(g(u+iw))du\,dw$$

Putting these two parts of the integral together, we have proven the following theorem.

Lemma 8. For any integrable $g : S \to S$, $\iint_S Vg(z)\,dA = \iint_S g(u+iw)du\,dw$. That is, V is integral preserving (just as T is).

Putting this together with the result of subsection 4.1, we get the following.

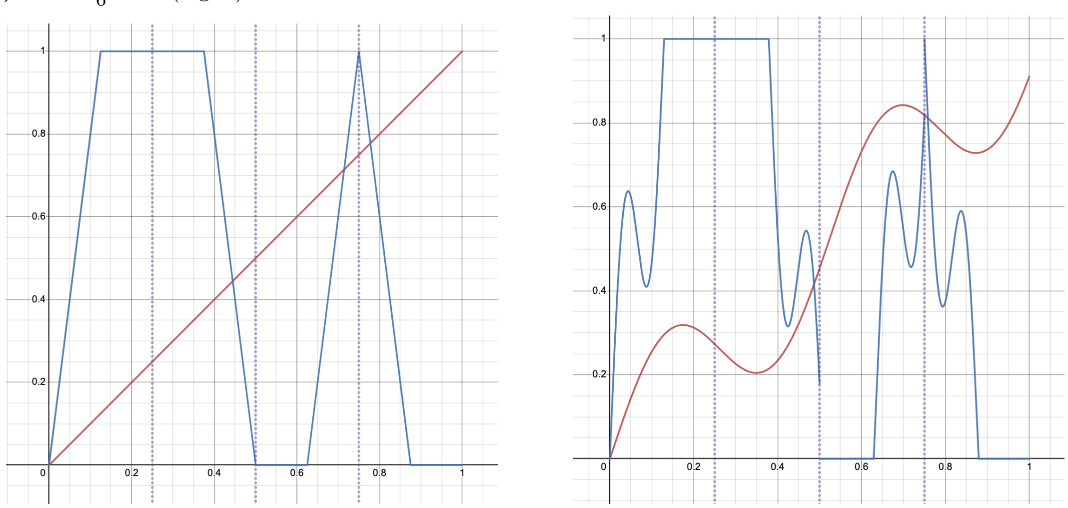
Theorem 9. The map $V : D \to D$ is a fixed point free isometry.

5. Contractiveness of TΔ Series

A reasonable question is whether the iterate series of variations of Alspach's map become contractive and remain fpf as seen with the map R. In this section, we consider the isometry $T\Delta$ from [4]. This map is similar to T but has distinct behavior on four intervals of $[0,1]$, leading to new questions about its behavior when iterated. $T\Delta : C_{\frac{1}{2}} \to C_{\frac{1}{2}}$ is defined by:

$$(T\Delta f)(t) = \begin{cases} 2f(4t) \wedge 1, & \text{if } t \in [0, \frac{1}{4}) \\ 2(1 - f(4t-1)) \wedge 1, & \text{if } t \in [\frac{1}{4}, \frac{1}{2}) \\ (2f(4t-2) - 1) \vee 0, & \text{if } t \in [\frac{1}{2}, \frac{3}{4}) \\ (1 - 2f(4t-3)) \vee 0, & \text{if } t \in [\frac{3}{4}, 1] \end{cases} \tag{4}$$

5.1. **Graphs of TΔ.** Below are the graphs of $T\Delta$ applied to the functions $f(x) = x$ (left) and $f(x) = \frac{\sin(12x)+6x}{6}$ (right).



5.2. **Iterate Series of TΔ.** Using ideas from [2], we will show the map $R_\Delta := \sum_{n=0}^{\infty} \frac{(T\Delta)^n}{2^{n+1}}$ is contractive and fpf on $C_{\frac{1}{2}}$. We begin by showing the outputs of $(T\Delta)^n$ tend to the extreme values 0 and 1 as n increases.

Lemma 10. Let $A_n(f) = \{x \in [0,1] : (T\Delta)^n f(x) \in (0,1)\}$. For every $f \in C_{\frac{1}{2}}$, $\lim_{n\to\infty} m(A_n(f)) = 0$.

Proof: Let $f \in C_{\frac{1}{2}}$ and consider the set $A_1(f)$. We can write this as the disjoint union

$$A_1(f) = \left(A_1(f) \bigcap \left[0, \frac{1}{4}\right)\right) \bigcup \left(A_1(f) \bigcap \left[\frac{1}{4}, \frac{1}{2}\right)\right) \bigcup \left(A_1(f) \bigcap \left[\frac{1}{2}, \frac{3}{4}\right)\right) \bigcup \left(A_1(f) \bigcap \left[\frac{3}{4}, 1\right]\right)$$

If $x \in A_1(f) \cap [0, \frac{1}{4})$, then $x \in [0, \frac{1}{4})$ and $(T\Delta)f(x) \in (0,1)$. By definition, $(T\Delta)f(x) = 2f(4x) \wedge 1$, so this implies $f(4x) \in (0, \frac{1}{2})$. Hence

$$x \in A_1(f) \bigcap \left[0, \frac{1}{4}\right) \iff f(4x) \in \left(0, \frac{1}{2}\right)$$

Similar arguments show that if $x \in A_1(f) \cap [\frac{1}{4}, 1]$, then $f(4x-1) \in \left(\frac{1}{2}, 1\right)$, $f(4x-2) \in \left(\frac{1}{2}, 1\right)$, or $f(4x-3) \in \left(0, \frac{1}{2}\right)$. This gives:

$$m\left(A_1(f) \bigcap \left[0, \frac{1}{4}\right)\right) = \frac{1}{4} m\{x \in (0,1) : f(x) \in \left(0, \frac{1}{2}\right)\}$$

$$m\left(A_1(f) \bigcap \left[\frac{1}{4}, \frac{1}{2}\right)\right) = \frac{1}{4} m\{x \in (0,1) : f(x) \in \left(\frac{1}{2}, 1\right)\}$$

$$m\left(A_1(f)\bigcap\left[\frac{1}{2},\frac{3}{4}\right)\right)=\frac{1}{4}m\{x\in(0,1):f(x)\in\left(\frac{1}{2},1\right)\}$$

$$m\left(A_1(f)\bigcap\left[\frac{3}{4},1\right]\right)=\frac{1}{4}m\{x\in(0,1):f(x)\in\left(0,\frac{1}{2}\right)\}$$

Since $m(A_0(f))\geq m[0<f<\frac{1}{2}]+m[\frac{1}{2}<f<1]$, we get:

$$\begin{aligned}\frac{1}{2}m(A_0(f)) &\geq \frac{1}{4}m\{x\in(0,1):f(x)\in\left(0,\frac{1}{2}\right)\}+\frac{1}{4}m\{x\in(0,1):f(x)\in\left(\frac{1}{2},1\right)\}\\ &+ \frac{1}{4}m\{x\in(0,1):f(x)\in\left(0,\frac{1}{2}\right)\}+\frac{1}{4}m\{x\in(0,1):f(x)\in\left(\frac{1}{2},1\right)\}\\ &= m\left(A_1(f)\bigcap\left[0,\frac{1}{4}\right)\right)+m\left(A_1(f)\bigcap\left[\frac{1}{4},\frac{1}{2}\right)\right)\\ &+ m\left(A_1(f)\bigcap\left[\frac{1}{2},\frac{3}{4}\right)\right)+m\left(A_1(f)\bigcap\left[\frac{3}{4},1\right]\right)\\ &= m(A_1(f)).\end{aligned}$$

This gives $\frac{1}{2}m(A_0(f))\geq m(A_1(f))$ for every $f\in C_{1/2}$. We can apply this to $(T\Delta)^{n-1}f$ and see that

$$m(A_n(f))=m(A_1((T\Delta)^{n-1}f))\leq\frac{1}{2}m(A_0((T\Delta)^{n-1}f))=\frac{1}{2}m(A_{n-1}(f))$$

which shows

$$m(A_n(f))\leq\frac{1}{2^n}m(A_0(f))\leq\frac{1}{2^n}\xrightarrow[n\to\infty]{}0.$$

□

Lemma 11. Let $f\in C_{\frac{1}{2}}$ and let y be nondyadic in $[0,1]$. Let $n\in\mathbb{N}$. Then for every $j\in\{0,\cdots,4^n-1\}$, when $f(y)\in\{0,1\}$, $(T\Delta)^n f(\frac{y+j}{4^n})\in\{0,1\}$.

Proof: Base case: n=1

We need to show the above for $j\in\{0,1,2,3\}$.
If $j=0$, $T\Delta f(\frac{y}{4})=2f(y)\wedge 1$ since $\frac{y}{4}\in[0,\frac{1}{4})$. If $f(y)=0$, $T\Delta f(\frac{y}{4})=0$ and if $f(y)=1$, $T\Delta f(\frac{y}{4})=1$.
If $j=1$, $T\Delta f(\frac{y+1}{4})=2(1-f(y))\wedge 1$ since $\frac{y+1}{4}\in[\frac{1}{4},\frac{1}{2})$. If $f(y)=1$, $T\Delta f(\frac{y+1}{4})=0$ and if $f(y)=1$, $T\Delta f(\frac{y+1}{4})=0$.
If $j=2$, $T\Delta f(\frac{y+2}{4})=(2f(y)-1)\vee 0$ since $\frac{y+2}{4}\in[\frac{1}{2},\frac{3}{4})$. If $f(y)=0$, $T\Delta f(\frac{y+2}{4})=0$ and if $f(y)=1$, $T\Delta f(\frac{y+2}{4})=1$.
If $j=3$, $T\Delta f(\frac{y+3}{4})=(1-2f(y))\vee 0$ since $\frac{y+3}{4}\in[\frac{3}{4},1]$. If $f(y)=0$, $T\Delta f(\frac{y+3}{4})=1$ and if $f(y)=1$, $T\Delta f(\frac{y+3}{4})=0$.

Hence the claim holds for $n=1$ and all $f\in C_{1/2}$.

Now, by way of induction, suppose the claim holds for some $m\in\mathbb{N}$ and some $f\in C_{1/2}$. Then for each $j\in\{0,\ldots,4^m-1\}$, we have that $(T\Delta)^m f\left(\frac{y+j}{4^m}\right)\in\{0,1\}$ when $f(y)\in\{0,1\}$.

By applying the base case (n=1) to $(T\Delta)^m f$, we then have that

$$(T\Delta)^{m+1}f\left(\frac{\frac{y+j}{4^m}+k}{4}\right)\in\{0,1\}.$$

for each $k\in\{0,1,2,3\}$. By rewriting the above, we get that:

$$T\Delta^{m+1} f\left(\frac{\frac{y+j}{4^m}+k}{4}\right) = T\Delta^{m+1} f\left(\frac{y+j+4^m k}{4^{m+1}}\right) \in \{0,1\}$$

This holds for each $j \in \{0,\dots,4^m-1\}$ and $k \in \{0,1,2,3\}$, and so it holds for each $\ell = j+4^m k \in \{0,\dots,4^m-1,4^m,\dots,2\cdot 4^m-1,2\cdot 4^m,\dots,3\cdot 4^m-1,3\cdot 4^m,\dots,4\cdot 4^m-1 = 4^{m+1}-1\}$ which proves the lemma.

□

We have shown that $T\Delta$ copies the extreme values of functions $f \in C_{\frac{1}{2}}$ (excluding dyadic numbers constituting a set of measure 0) by either preserving 0's and 1's or exchanging them.

5.3. **Locations of Extreme Values Upon Iterations of TΔ.**

Now, for any $f \in C_{\frac{1}{2}}$ we define

$$B_n(f) = \{x \in [0,1] : (T\Delta)^n f(x) = 1\},$$
$$C_n(f) = \{x \in [0,1] : (T\Delta)^n f(x) = 0\}.$$

Note that $A_n(f) \cup B_n(f) \cup C_n(f) = [0,1]$ for any f. In the following claim, we write $C_{n+k} = C_{n+k}(f)$ and so on for brevity.

Claim: For all $k \in \mathbb{N}$,

$$C_{n+k} \supseteq \bigcup_{j=0}^{4^k-1} \left(\frac{j}{4^k} + \frac{1}{4^k} X_{n,j}\right)$$

and

$$B_{n+k} \supseteq \bigcup_{j=0}^{4^k-1} \left(\frac{j}{4^k} + \frac{1}{4^k} Y_{n,j}\right)$$

where, considering j in base 4 as $j = (j_0 j_1 \cdots j_t)_4$ for some $t \geq 0$, $X_{n,j}, Y_{n,j}$ are defined as:

$$X_{n,j} = \begin{cases} C_n, & \text{if } (j_0 + j_1 + \cdots + j_t) \equiv 0 \bmod 2 \\ B_n, & \text{if } (j_0 + j_1 + \cdots + j_t) \not\equiv 0 \bmod 2 \end{cases}$$

and

$$Y_{n,j} = \begin{cases} B_n, & \text{if } (j_0 + j_1 + \cdots + j_t) \equiv 0 \bmod 2 \\ C_n, & \text{if } (j_0 + j_1 + \cdots + j_t) \not\equiv 0 \bmod 2 \end{cases}$$

Proof: Base case: k=1

$$C_{n+1} \supseteq \left(\frac{1}{4}C_n \bigcup (\frac{1}{4} + \frac{1}{4}B_n) \bigcup (\frac{2}{4} + \frac{1}{4}C_n) \bigcup (\frac{3}{4} + \frac{1}{4}B_n)\right)$$
$$B_{n+1} \supseteq \left(\frac{1}{4}B_n \bigcup (\frac{1}{4} + \frac{1}{4}C_n) \bigcup (\frac{2}{4} + \frac{1}{4}B_n) \bigcup (\frac{3}{4} + \frac{1}{4}C_n)\right)$$

These inclusions follow from the base case in Lemma 11 and by a simple check that the B_n and C_n terms correspond with the even and odd sums of j's base 4 digits.

Now let $k \in \mathbb{N}$ be given. By way of induction, suppose it holds that

$$C_{n+k-1} \supseteq \bigcup_{j=0}^{4^{k-1}-1} \left(\frac{j}{4^{k-1}} + \frac{1}{4^{k-1}} X_{n,j}\right)$$

and

$$B_{n+k-1} \supseteq \bigcup_{j=0}^{4^{k-1}-1} \left(\frac{j}{4^{k-1}} + \frac{1}{4^{k-1}} Y_{n,j}\right)$$

If we consider $C_{n+k} = C_{(n+k-1)+1}$ and $B_{n+k} = B_{(n+k-1)+1}$ and apply our base case to each, we see that

$$C_{n+k} \supseteq \Big(\frac{1}{4}C_{n+k-1} \bigcup (\frac{1}{4} + \frac{1}{4}B_{n+k-1}) \bigcup (\frac{2}{4} + \frac{1}{4}C_{n+k-1}) \bigcup (\frac{3}{4} + \frac{1}{4}B_{n+k-1})\Big)$$
$$B_{n+k} \supseteq \Big(\frac{1}{4}B_{n+k-1} \bigcup (\frac{1}{4} + \frac{1}{4}C_{n+k-1}) \bigcup (\frac{2}{4} + \frac{1}{4}B_{n+k-1}) \bigcup (\frac{3}{4} + \frac{1}{4}C_{n+k-1})\Big)$$

We will show that these unions are equivalent to the inclusions for B_{n+k} and C_{n+k} stated in the claim. Specifically, we will prove the inclusion for C_{n+k}, as the proof for B_{n+k} is similar.

First, by our inductive hypothesis, we have

$$\frac{1}{4}C_{n+k-1} \supseteq \frac{1}{4}\left(\bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^{k-1}} + \frac{1}{4^{k-1}}X_{n,j}\right)\right) = \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}X_{n,j}\right)$$
$$\left(\frac{1}{4} + \frac{1}{4}B_{n+k-1}\right) \supseteq \frac{1}{4} + \frac{1}{4}\left(\bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^{k-1}} + \frac{1}{4^{k-1}}Y_{n,j}\right)\right) = \frac{1}{4} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}Y_{n,j}\right)$$
$$\left(\frac{2}{4} + \frac{1}{4}C_{n+k-1}\right) \supseteq \frac{2}{4} + \frac{1}{4}\left(\bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^{k-1}} + \frac{1}{4^{k-1}}X_{n,j}\right)\right) = \frac{2}{4} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}X_{n,j}\right)$$
$$\left(\frac{3}{4} + \frac{1}{4}B_{n+k-1}\right) \supseteq \frac{3}{4} + \frac{1}{4}\left(\bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^{k-1}} + \frac{1}{4^{k-1}}Y_{n,j}\right)\right) = \frac{3}{4} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}Y_{n,j}\right)$$

$\frac{1}{4}C_{n+k-1}$ gives us a portion on the union contained in C_{n+k} in the claim (with the correct X term).

For $\left(\frac{1}{4} + \frac{1}{4}B_{n+k-1}\right)$, we write:

$$\frac{1}{4} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}Y_{n,j}\right) = \frac{4^{k-1}}{4^k} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}Y_{n,j}\right) = \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j+4^{k-1}}{4^k} + \frac{1}{4^k}Y_{n,j}\right)$$

Let $\ell_1 := j + 4^{k-1}$. Note that the base 4 addition of 4^{k-1} and j is $(10\cdots\cdots0)_4 + (j_0\cdots j_t)_4$, where $t \le (k-2)$ since $j < 4^{k-1}$. Therefore $\ell_1 = (1j_0\cdots j_t)_4$ has digits adding to $1 + j_0 + \cdots + j_t$, flipping the parity of ℓ_1's sum of digits in base 4 from that of j. This means $Y_{n,j} = X_{x,\ell_1}$ and hence gives

$$\left(\frac{1}{4} + \frac{1}{4}B_{n+k-1}\right) \supseteq \bigcup_{\ell_1=4^{k-1}}^{2\cdot 4^{k-1}-1}\left(\frac{\ell_1}{4^k} + \frac{1}{4^k}X_{n,\ell_1}\right)$$

We now have another portion of the union in our claim (with the correct X term) that continues where the previous one left off.

Next, for $\left(\frac{2}{4} + \frac{1}{4}C_{n+k-1}\right)$, we write:

$$\frac{2}{4} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}X_{n,j}\right) = \frac{2\cdot 4^{k-1}}{4^k} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}X_{n,j}\right) = \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j+2\cdot 4^{k-1}}{4^k} + \frac{1}{4^k}X_{n,j}\right)$$

Let $\ell_2 := j + 2\cdot 4^{k-1}$. Note that the base 4 addition of $2\cdot 4^{k-1}$ and j is $(20\cdots\cdots0)_4 + (j_0\cdots j_t)_4$, where $t \le (k-2)$ since $j < 4^{k-1}$. Therefore $\ell_1 = (2j_0\cdots j_t)_4$ has digits adding to $2 + j_0 + \cdots + j_t$, preserving the parity of ℓ_2's sum of digits in base 4 from that of j. This means $X_{n,j} = X_{x,\ell_2}$ and hence gives

$$\left(\frac{2}{4} + \frac{1}{4}C_{n+k-1}\right) \supseteq \bigcup_{\ell_2=4^{k-1}}^{2\cdot 4^{k-1}-1}\left(\frac{\ell_2}{4^k} + \frac{1}{4^k}X_{n,\ell_2}\right)$$

Again, we have a portion of the union in our claim (with the correct X term) that continues where the previous one left off.

Finally, for $\left(\frac{3}{4} + \frac{1}{4}B_{n+k-1}\right)$, we write:

$$\frac{3}{4} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}Y_{n,j}\right) = \frac{3\cdot 4^{k-1}}{4^k} + \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j}{4^k} + \frac{1}{4^k}Y_{n,j}\right) = \bigcup_{j=0}^{4^{k-1}-1}\left(\frac{j+3\cdot 4^{k-1}}{4^k} + \frac{1}{4^k}Y_{n,j}\right)$$

Let $\ell_3 := j + 3\cdot 4^{k-1}$. Note that the base 4 addition of $3\cdot 4^{k-1}$ and j is $(30\cdots\cdots 0)_4 + (j_0\cdots j_t)_4$, where $t \le (k-2)$ since $j < 4^{k-1}$. Therefore $\ell_3 = (3j_0\cdots j_t)_4$ has digits adding to $3 + j_0 + \cdots + j_t$, preserving the parity of ℓ_3's sum of digits in base 4 from that of j. This means $Y_{n,j} = X_{x,\ell_3}$ and hence gives

$$\left(\frac{3}{4} + \frac{1}{4}B_{n+k-1}\right) \supseteq \bigcup_{\ell_3 = 3\cdot 4^{k-1}}^{4\cdot 4^{k-1}-1} \left(\frac{\ell_3}{4^k} + \frac{1}{4^k}X_{n,\ell_3}\right) = \bigcup_{\ell_3 = 3\cdot 4^{k-1}}^{4^k-1} \left(\frac{\ell_3}{4^k} + \frac{1}{4^k}X_{n,\ell_3}\right)$$

Together, these unions span 0 to $4^k - 1$, which gives

$$C_{n+k} \supseteq \bigcup_{j=0}^{4^k-1} \left(\frac{j}{4^k} + \frac{1}{4^k}X_{n,j}\right)$$

as claimed. The proof for the B_{n+k} inclusion follows the same argument but swaps any B-sets for C-sets and vice-versa.

□

5.4. **Lemma for Contractivness of TΔ.**

We now define the following for any $f, g \in C_{\frac{1}{2}}$:

$$D_n = B_n(f) \bigcap C_n(g)$$

$$E_n = B_n(g) \bigcap C_n(f)$$

$$F_n = (B_n(f) \bigcap B_n(g)) \bigcup (C_n(f) \bigcap C_n(g))$$

$$G_n = A_n(f) \bigcup A_n(g)$$

Note that the disjoint union $D_n \cup E_n \cup F_n \cup G_n = [0,1]$ for any f, g.

Let $f, g \in C_{\frac{1}{2}}$ be such that $f \ne g$. By Lemma 10, $m(G_n) \longrightarrow 0$. Since $\|f - g\| > 0$ and $\int_0^1 f = \int_0^1 g = \frac{1}{2}$, $m[f > g]$ and $m[g > f]$ are both positive. We will now show $m(D_n)$ and $m(E_n)$ are both positive. Since $T\Delta$ is an isometry, we have

$$\begin{aligned} \|f - g\| = \|(T\Delta)^n f - (T\Delta)^n g\| &= \int_{G_n} |(T\Delta)^n f - (T\Delta)^n g| + \int_{F_n} 0 + \int_{D_n} 1 + \int_{E_n} 1 \\ &= \int_{G_n} |(T\Delta)^n f - (T\Delta)^n g| + m(D_n) + m(E_n) \end{aligned}$$

This shows $m(D_n) + m(E_n) = \|f - g\| - \int_{G_n} |(T\Delta)^n f - (T\Delta)^n g|$. Also, since $\int_0^1 (T\Delta)^n f = \int_0^1 (T\Delta)^n g = \frac{1}{2}$,

$$\int_{G_n} ((T\Delta)^n f - (T\Delta)^n g) + \int_{D_n} ((T\Delta)^n f - (T\Delta)^n g) + \int_{E_n} ((T\Delta)^n f - (T\Delta)^n g) + \int_{F_n} ((T\Delta)^n f - (T\Delta)^n g) = 0$$

so

$$\int_{G_n} ((T\Delta)^n f - (T\Delta)^n g) + \int_{D_n} 1 + \int_{E_n} (-1) = 0$$

and hence

$$m(E_n) - m(D_n) = \int_{G_n} ((T\Delta)^n f - (T\Delta)^n g).$$

From the above and the fact that $\int_{G_n} |(T\Delta)^n f - (T\Delta)^n g| \le m(G_n)$ (because the integrand is bounded above by 1), we conclude that

$$\|f - g\| \ge m(D_n) + m(E_n) \ge \|f - g\| - m(G_n)$$

and

$$|m(E_n) - m(D_n)| \le m(G_n)$$

Since $m(G_n) \longrightarrow 0$, it holds that $m(D_n) \longrightarrow \frac{1}{2}\|f - g\|$ and $m(E_n) \longrightarrow \frac{1}{2}\|f - g\|$ as n gets large. Hence we can choose n large enough such that $m(D_n)$ and $m(E_n)$ are both greater than $\frac{1}{4}\|f - g\|$.

(♠)**Claim:** Choosing n as above, there exists $k \in \mathbb{N}$ such that the sets

$$S_1 = E_{n+k} \bigcap [f > g]$$

$$S_2 = D_{n+k} \bigcap [f < g]$$

are both of positive measure.

(♠)**Proof:** Let $W := [f < g]$. Fix $\epsilon > 0$. By Proposition 15 in [7], there exists a finite sequence of open intervals $(I_l)_{l=1}^{\nu}$ such that $m(W \backslash \Gamma \cup \Gamma \backslash W) = m(W \Delta \Gamma) < \epsilon$ where $\Gamma := \bigcup_{l=1}^{\nu} I_l$. Without loss of generality, we assume that the intervals I_l are pairwise disjoint and dyadic intervals of the form $(\frac{j_l}{4^k}, \frac{j_l+1}{4^k})$ for some $j_l \in \{0, \cdots, 4^k - 1\}$ and $k \in \mathbb{N}$. We write:

$$\chi_\Gamma = \sum_{j=0}^{4^k-1} \beta_j \chi_{(\frac{j}{4^k}, \frac{j+1}{4^k})},$$

where each $\beta_j \in \{0,1\}$. Let $p_n = \min\{m(D_n), m(E_n)\}$. Note that the measures of D_n and E_n both exceed $\frac{1}{4}\|f-g\|$ and that $(\frac{j}{4^k} + \frac{1}{4^k} X_{n,j}(g)) \cap (\frac{j}{4^k} + \frac{1}{4^k} Y_{n,j}(f)))$ is $\frac{j}{4^k} + \frac{1}{4^k} D_n$ for certain values of j and $\frac{j}{4^k} + \frac{1}{4^k} E_n$ for others. Then

$$\begin{aligned}
m(D_{n+k} \cap W) &\geq m\left(\left(\bigcup_{j=0}^{4^k-1} \frac{j}{4^k} + \frac{1}{4^k} X_{n,j}(g)\right) \cap \left(\bigcup_{j=0}^{4^k-1} \frac{j}{4^k} + \frac{1}{4^k} Y_{n,j}(f)\right) \cap W \cap \Gamma\right) \\
&= m\left(\left(\bigcup_{j=0}^{4^k-1} \left(\frac{j}{4^k} + \frac{1}{4^k} X_{n,j}(g)\right) \cap \left(\frac{j}{4^k} + \frac{1}{4^k} Y_{n,j}(f)\right)\right) \cap W \cap \Gamma\right) \\
&= m\left(\left(\bigcup_{j=0}^{4^k-1} \left(\frac{j}{4^k} + \frac{1}{4^k} X_{n,j}(g)\right) \cap \left(\frac{j}{4^k} + \frac{1}{4^k} Y_{n,j}(f)\right)\right) \cap \Gamma\right) \\
&- m\left(\left(\bigcup_{j=0}^{4^k-1} \left(\frac{j}{4^k} + \frac{1}{4^k} X_{n,j}(g)\right) \cap \left(\frac{j}{4^k} + \frac{1}{4^k} Y_{n,j}(f)\right)\right) \cap \Gamma \backslash W\right) \\
&\geq \int_0^1 \sum_{j=0}^{4^k-1} \chi_{((\frac{j}{4^k} + \frac{1}{4^k} X_{n,j}(g)) \cap (\frac{j}{4^k} + \frac{1}{4^k} Y_{n,j}(f)))} \sum_{s=0}^{4^k-1} \beta_s \chi_{(\frac{s}{4^k}, \frac{s+1}{4^k})} dm - m(\Gamma \backslash W) \\
&> \int_0^1 \sum_{j=0}^{4^k-1} \beta_j \chi_{((\frac{j}{4^k} + \frac{1}{4^k} X_{n,j}(g)) \cap (\frac{j}{4^k} + \frac{1}{4^k} Y_{n,j}(f)))} dm - \epsilon \geq p_n \frac{1}{4^k} \sum_{j=0}^{4^k-1} \beta_j - \epsilon \\
&= p_n m(\Gamma) - \epsilon > p_n(m(W) - \epsilon) - \epsilon = p_n m(W) - p_n \epsilon - \epsilon \geq p_n m(W) - 2\epsilon \\
&\geq \frac{\|f-g\|}{4} m(W) - 2\epsilon > \frac{\|f-g\|}{8} m(W) > 0
\end{aligned}$$

for $\epsilon < \frac{\|f-g\|}{16} m(W)$. This holds for all $k \geq k_1$ for some $k_1 \in \mathbb{N}$.
A similar argument shows there exists $k_2 \in \mathbb{N}$ such that $m(E_{n+k_2} \cap [f > g]) > 0$ for small enough ϵ. Hence we can choose $k \geq \max\{k_1, k_2\}$, and we have proven the claim.

□

We can now prove the following lemma:

Lemma 12. For every $f, g \in C_{\frac{1}{2}}$ with $\|f - g\| > 0$ there is some $N \in \mathbb{N}$ such that

$$\left\| \frac{I + (T\Delta)^N}{2} f - \frac{I + (T\Delta)^N}{2} g \right\| < \|f - g\|.$$

Proof: With $n \in \mathbb{N}$ and k as in claim ♠, let $N := n + k$. Define the set $S_3 := [0,1] \backslash (S_1 \cup S_2)$. We will achieve the desired inequality by breaking down $[0,1]$ into S_1, S_2, and S_3:

$$\begin{aligned}
\left\| \frac{I+(T\Delta)^N}{2} f - \frac{I+(T\Delta)^N}{2} g \right\| &= \int_0^1 \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
= \int_{S_1} \left| \frac{f-g+((T\Delta)^N f-(T\Delta)^N g)}{2} \right| &+ \int_{S_2} \left| \frac{f-g+((T\Delta)^N f-(T\Delta)^N g)}{2} \right| \\
&+ \int_{S_3} \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
= \int_{S_1} \left| \frac{f-g-1}{2} \right| + \int_{S_2} \left| \frac{f-g+1}{2} \right| &+ \int_{S_3} \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
= \int_{S_1} \frac{1+g-f}{2} + \int_{S_2} \frac{1+f-g}{2} &+ \int_{S_3} \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
< \int_{S_1} \frac{1+f-g}{2} + \int_{S_2} \frac{1+g-f}{2} &+ \int_{S_3} \left| \frac{f+(T\Delta)^N f}{2} - \frac{g+(T\Delta)^N g}{2} \right| \\
= \int_{S_1} \left(\left| \frac{(T\Delta)^N f-(T\Delta)^N g}{2} \right| + \left| \frac{f-g}{2} \right| \right) &+ \int_{S_2} \left(\left| \frac{(T\Delta)^N f-(T\Delta)^N g}{2} \right| + \left| \frac{f-g}{2} \right| \right) \\
&+ \int_{S_3} \left| \frac{f-g}{2} + \frac{(T\Delta)^N f-(T\Delta)^N g}{2} \right| \\
\leq \int_0^1 \left(\left| \frac{f-g}{2} \right| + \left| \frac{(T\Delta)^N f-(T\Delta)^N g}{2} \right| \right) &= \int_0^1 \left(\left| \frac{f-g}{2} \right| + \left| \frac{f-g}{2} \right| \right) \\
&= \int_0^1 |f-g| = \|f-g\|
\end{aligned}$$

□

Theorem 13. The map $R_\Delta := \sum_{n=0}^{\infty} \frac{(T\Delta)^n}{2^{n+1}}$ is fpf and contractive.

Proof: Let $f \in C_{\frac{1}{2}}$.

$$R_\Delta f = \left(\frac{I}{2} + \frac{T\Delta}{4} + \frac{(T\Delta)^2}{8} + \cdots \right) f$$

Distributing the identity term throughout the sum and accounting for the correct coefficients on each power of $T\Delta$, we get

$$R_\Delta f = \frac{1}{2}\left(\frac{I+T\Delta}{2}\right) f + \frac{1}{4}\left(\frac{I+(T\Delta)^2}{2}\right) f + \frac{1}{8}\left(\frac{I+(T\Delta)^3}{2}\right) f + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left(\frac{I+(T\Delta)^n}{2}\right) f.$$

Hence for any $f, g \in C_{\frac{1}{2}}$ such that $\|f-g\| > 0$, we have

$$\|R_\Delta f - R_\Delta g\| = \left\| \frac{1}{2}\left(\left(\frac{I+T\Delta}{2}\right) f - \left(\frac{I+T\Delta}{2}\right) g \right) + \frac{1}{4}\left(\left(\frac{I+(T\Delta)^2}{2}\right) f - \left(\frac{I+(T\Delta)^2}{2}\right) g \right) + \cdots \right\|$$

Using the triangle inequality, we get

$$\|R_\Delta f - R_\Delta g\| \leq \frac{1}{2} \left\| \left(\frac{I+T\Delta}{2}\right) f - \left(\frac{I+T\Delta}{2}\right) g \right\| + \frac{1}{4} \left\| \left(\frac{I+(T\Delta)^2}{2}\right) f - \left(\frac{I+(T\Delta)^2}{2}\right) g \right\| + \cdots$$

Each $\frac{I+(T\Delta)^n}{2}$ is nonexpansive, and by Lemma 12 there exists an $N \in \mathbb{N}$ such that $\frac{I+(T\Delta)^N}{2}$ is contractive. Therefore

$$\|R_\Delta f - R_\Delta g\| < \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \|f-g\| = \|f-g\|$$

and we conclude that $R\Delta$ is contractive.

To show R_Δ is fpf, we use the same techniques from the proof of Theorem 6. Note that in this case $T\Delta$ is an isometry and hence we get

$$\|R_\Delta f - R_\Delta g\| > \frac{1}{2}\|f-g\| - \frac{1}{2}\|T\Delta f - T\Delta g\| = \frac{1}{2}\|f-g\| - \frac{1}{2}\|f-g\| = 0$$

which shows R_Δ is 1-1. The rest of the proof follows exactly as in Theorem 6 and uses the fact from [4] that $T\Delta$ is fpf.

□

As a result of Theorem 6, we immediately have the following.

Corollary 1: The map $J_{R_\Delta} := \sum_{n=0}^{\infty} \frac{(R_\Delta)^n}{2^{n+1}}$ is fpf and contractive.

6. Conclusion

Proving that V is fixed point free and nonexpansive raises a natural question about whether $\sum_{n=0}^{\infty} \frac{V^n}{2^{n+1}}$ is contractive and fpf. If this were the case, in light of Theorem 6 and previous results, we might ask a more general question about starting with any nonexpansive and fpf map F: Is the series $\sum_{n=0}^{\infty} \frac{F^n}{2^{n+1}}$ fpf and contractive? Since 2014, there have been other formulations of fixed point free and contractive maps ([6], [3]) coming from such series, possibly pointing to an affirmative result. Although, we have only seen the details of the analogue of Lemma 12 for the original series R and the series R_Δ and W_r in this paper.

And the question about W_r being contractive and fpf remains open (to us) for $r > 1/2$. For $r > 1/2$, we have tried both the formula given above for W_r and re-formulations without resolving the question.

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