Cesàro averaging and extension of functionals on $L^\infty(0,\infty)$

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Abstract

On the space of essentially bounded functions $L^\infty(0,\infty)$ we consider the Cesàro averaging operator $Jf(x) := \frac{1}{x}\int_0^x f(t)\,dt$. We then extend the concept of integer iterates of Cesàro averaging J^n, to an operator of the form $J^r f(x)$, where r is any positive real number and $f \in L^\infty(0,\infty)$. Our definition of fractional powers of Cesàro averaging is such that $(J^r)_{r>0}$ has the semigroup property.

Our paper contains the following result: $\big[$For any $f \in L^\infty(0,\infty)$, $J^r f(x)$ has a limit at infinity for some $r > 0$, if and only if $J^s f(x)$ has a limit at infinity for any $s > 0$. In this case, the limit values are all the same$\big]$. We present a strong quantitative version of the special case where $0 < r \leq 1$ and $s = 1 + r$.

We construct Banach limits Λ on $L^\infty(0,\infty)$ that are invariant under our continuous generalization of Cesàro iterates J^r. We also construct an example of a Banach limit Ψ on $L^\infty(0,\infty)$ that preserves Cesàro convergence, but is not Cesàro invariant.

Keywords — Functional Analysis, Cesàro averaging operators, invariant Banach limits, fractional powers

1 Introduction

On $L^\infty(0,\infty)$ we consider the Cesàro averaging operator $f \mapsto Jf$, where $Jf(x) := \frac{1}{x}\int_0^x f(t)\,dt$. We then extend the concept of integer iterates of Cesàro averaging J^n, to an operator of the form $J^r f(x)$, where r is any positive real number and $f \in L^\infty(0,\infty)$. Our definition of fractional powers of Cesàro averaging is such that $(J^r)_{r>0}$ has the semigroup property, that is, $J^r(J^s f) = J^{r+s}(f)$, for all $r, s > 0$, and for all $f \in L^\infty(0,\infty)$.

Our paper contains three main theorems. The second is: $\big[$For any $f \in L^\infty(0,\infty)$, $J^r f(x)$ has a limit at infinity for some $r > 0$, if and only if $J^s f(x)$ has a limit at infinity for any $s > 0$. In this case, the limit values are all the same (Theorem 3.4)$\big]$. Our first main theorem is a strong quantitative version of the special case where $0 < r \leq 1$ and $s = 1 + r$ (Theorem 3.3).

Our Theorem 3.3 is analogous to the quantitative result obtained in Sivek [17, theorem II.9] in the sequence space ℓ^∞, for the usual Cesàro averaging operator C and its square C^2. The qualitative version of the result in [17] follows from a theorem due to Frobenius [5], and a classical theorem of Hardy and Littlewood (see Theorem 7.3 of [9]).

Our third main result is the following (Theorem 4.2). We construct Banach limits Λ on $L^\infty(0,\infty)$ that are invariant under our continuous generalization of Cesàro iterates J^r; i.e., $\big[\Lambda(J^r f) = \Lambda(f)$, for all $f \in L^\infty$, for all $r > 0.\big]$

In Section 5, we construct an example of a Banach limit Ψ on $L^\infty(0,\infty)$ that preserves Cesàro convergence (i.e., $\Psi(w) = \lim_{x\to\infty} Jw(x)$, for all $w \in L^\infty(0,\infty)$), but is not Cesàro invariant (i.e., there exists $f \in L^\infty(0,\infty)$ with $\Psi(Jf) \neq \Psi(f)$). Hence, Cesàro invariance is a strictly stronger property than preserving Cesàro convergence.

The body of our paper includes a number of other interesting theorems concerning the Cesàro averaging map J: for example, Claim 2.5 and Theorem 2.6.

We also include an appendix, where we prove many supporting results for our theorems.

In Dodds, de Pagter, Sedaev, Semenov and Sukochev [3], Banach limits invariant under Cesàro averaging were first studied for the sequence space ℓ^∞. In [15], the authors E. Semenov and F. Sukochev, gave sufficient conditions for a linear operator on ℓ^∞, to guarantee the existence of Banach limits that would be invariant under the given operator. Also, they gave necessary and sufficient conditions for a sequence to have the same output under any Cesàro invariant Banach limit. In [16], the authors E. Semenov, F. Sukochev, A. Usachev, and D. Zanin, studied Banach limits on ℓ^∞ that are invariant under the Cesàro operator and the Dilation operator. Sukochev, Usachev, and Zanin [18] have also studied generalized limits on L^∞ invariant under the Cesàro operator and related operators. Moreover, their work has applications to non-commutative geometry. Also, their work does not discuss fractional powers of the Cesàro operator. A more recent paper related to [18] is [10], where convolution invariant linear functionals on L^∞ are studied, and Cesàro invariant functionals are also considered.

2 Cesàro averaging on $L^\infty(0,\infty)$

2.1 Notation and preliminaries

For all a and b in $\mathbb{R}$, $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$. Let $L^\infty = L^\infty(0,\infty)$ be the Banach space of (classes of) essentially bounded, real-valued Lebesgue measurable functions f on $(0,\infty)$, equipped with the uniform norm

$$\|f\|_\infty := \operatorname{ess\,sup} |f(t)|,$$

where the supremum is taken over all $t > 0$.

The integrals we consider in this document, are with respect to the Lebesgue measure on $(0,\infty)$.

The dual space of L^∞, $(L^\infty)^*$, consisting of all continuous linear functionals ψ on L^∞, is equipped with the norm

$$\|\psi\|_{(L^\infty)^*} := \sup |\psi(f)|,$$

where the supremum is taken over all $f \in L^\infty$ with $\|f\|_\infty \leq 1$.

$(L^\infty)^*$ is isometrically isomorphic to $fa(m)$, the space of finitely additive measures on the Borel subsets of $(0,\infty)$ that vanish on m-null sets, where m is Lebesgue measure (see Dunford and Schwartz [4], page 296). Moreover, $fa(m)$ is isometrically isomorphic to $L^1(0,\infty) \bigoplus_1 pfa(m)$, where $pfa(m)$ is the space of purely finitely additive measures on the Borel subsets of $(0,\infty)$ that vanish on m-null sets (see Yosida and Hewitt [20]).

The Hahn-Banach Theorem is often applied to obtain extensions of functionals. In particular, on $(L^\infty, \|\cdot\|_\infty)$ consider the closed vector subspaces

$$BC := \{f \in L^\infty : f \textit{ is continuous }\},$$
$$BC_L := \left\{f \in L^\infty : f \textit{ is continuous and } \lim_{x\to\infty} f(x) \textit{ exists in } \mathbb{R}\right\}, \text{ and}$$
$$Ces := \left\{g \in L^\infty : \psi(g) := \lim_{x\to\infty} \frac{1}{x}\int_0^x g(t)dt \textit{ exists in } \mathbb{R}\right\}.$$

Remark. For an element $f \in L^\infty$, when we write $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$, we mean the following: For every $\epsilon > 0$, there exists $N > 0$ such that $\operatorname{ess\,sup}_{x\geq N} |f(x) - L| \leq \epsilon$.

BC_L and Ces are Banach subspaces of L^∞, equipped with the induced norm. ψ is a continuous linear functional on Ces; and for all ϕ in Ces^* define

$$\|\phi\|_{Ces^*} := \sup |\phi(g)|,$$

where the supremum is taken over all $g \in Ces$ with $\|g\|_\infty \leq 1$.

By Hahn-Banach Extension Theorem, there exists $\Psi \in (L^\infty)^*$ such that

$$\|\Psi\|_{(L^\infty)^*} = \|\psi\|_{Ces^*} = 1 \text{ and } \Psi\mid_{Ces} = \psi.$$

We define the linear operator $J : L^\infty \to BC$ by

$$(Jf)(x) := \frac{1}{x}\int_0^x f(t)dt, \text{ for all } x \in (0,\infty), \text{ for all } f \in L^\infty.$$

Jf is called *the Cesàro average of f*. Jf is indeed in BC when f is bounded. It is bounded because $|Jf(x)| \leq \|f\|_\infty$ and continuous as the product of continuous functions.

Clearly, Ψ preserves Cesàro convergence. Also, it can be checked that if $f \in BC_L$, then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(Jf)(x)$, and so Ψ preserves classical convergence as well. (See, for example, Claim 3.1.1. of [2].)

We are interested in obtaining extensions of functionals that are also Banach limits. We define Banach limits on L^∞ next:

Definition 2.1. A continuous linear functional $\Lambda : L^\infty \to \mathbb{R}$ is a *Banach limit on* L^∞ if

1. $\|\Lambda\|_{(L^\infty)^*} = 1$.
2. $\Lambda(f) = \lim_{x\to\infty} f(x)$ for all $f \in BC_L$.
3. $\Lambda(S_r f) = \Lambda(f)$, for all $f \in L^\infty$, and for all $r \in (0,\infty)$; where $S_r f(x) = f(x+r)$.

The operator S_r is called *the left shift by* r *operator* on L^∞.
In [18], Banach limits are referred to as *translation invariant generalized limits*.

Lemma 2.1. *The linear functional* Ψ *defined above is a Banach limit on* L^∞.

Proof. Property 1 holds because $\|\Psi\|_{(L^\infty)^*} = \|\psi\|_{Ces^*} = 1$. Property 2 holds because we know that if $f \in BC_L$, then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (Jf)(x) = \psi(f)$.

The third property follows from the fact that for any $f \in L^\infty$,

$$f - S_r f \in Ces_0 := \{g \in Ces : \psi(g) = 0\}. \text{ (See, for example, Claim 3.1.2. of [2].)}$$

So, $\psi(f - S_r f) = 0$, which implies $\Psi(f - S_r f) = 0$, and therefore $\Psi(f) = \Psi(S_r f)$.

2.2 A stronger Banach limit on L^∞

Banach limits in general do not preserve Cesàro convergence, unlike our Banach limit Ψ. We will next construct a stronger Banach limit that is Cesàro invariant, a stronger condition than Cesàro-convergence preserving. First, we give the following definitions:

Definition 2.2. A linear functional $\Lambda : \ell^\infty \to \mathbb{R}$ is a *Banach limit on* ℓ^∞ if

1. Λ is continuous and $\|\Lambda\|_{(\ell^\infty)^*} = 1$.
2. $\Lambda(x) = \lim_{n\to\infty} x_n$, for all $x = (x_n)_{n\in\mathbb{N}} \in c$, where c is the subspace in ℓ^∞ of the convergent sequences.
3. $\Lambda(Sx) = \Lambda x$, for all $x \in \ell^\infty$. Here $S(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots)$ for all $(x_n)_{n\in\mathbb{N}} \in \ell^\infty$, and S is known as *the left shift operator*.

We see that definition 2.1 given above for Banach limits on the space L^∞ is completely analogous to definition 2.2 for Banach limits on the space ℓ^∞. It can be verified that the previous definition is equivalent to the one originally given by Banach (see [1], page 21), which we present next:

Definition 2.3. A *Banach limit* $\Lambda : \ell^\infty \to \mathbb{R}$ can also be defined as a linear functional such that

1. $\Lambda(x) \geq 0$ for all $x = (x_n)_n \in \ell^\infty$ such that $x_n \geq 0$ for all $n \in \mathbb{N}$.
2. $\Lambda(\mathbb{1}) = 1$.
3. $\Lambda(Sx) = \Lambda x$, for all $x \in \ell^\infty$.

We turn our attention back to L^∞, where we will next define Cesàro invariant Banach limits:

Definition 2.4. A Banach limit Λ on L^∞ is said to be Cesàro invariant if $\Lambda(Jf) = \Lambda(f)$ for every $f \in L^\infty$.

Remark. Using the terminology from [18], these linear functionals would fall under the definition of *translation invariant generalized limits* as well as *H-invariant generalized limits*.

Consider an arbitrary Banach limit σ on ℓ^∞, we will use σ to define a Banach limit on L^∞:

Claim 2.2. *Fix* σ *a Banach limit on* ℓ^∞. *Define* $\Delta : L^\infty \to \mathbb{R}$ *by*

$$\Delta(f) := \sigma(\Psi(f), \Psi(Jf), \Psi(J^2 f), \Psi(J^3 f), \ldots) \text{ for all } f \in L^\infty.$$

Δ *is a Cesàro invariant Banach limit on* L^∞.

Proof. The first and second properties from the definition of Banach limit are easy to check (see A.1). For the third property, we claim that for every $r > 0$ and $f \in L^\infty$ we have that $\Psi(J^n S_r f) = \Psi(J^n f)$, for all $n \in \mathbb{N} \cup \{0\}$, we will prove this claim later. And so,

$$\begin{aligned}\Delta(S_r f) &:= \sigma(\Psi(S_r f), \Psi(J S_r f), \Psi(J^2 S_r f), \Psi(J^3 S_r f), \ldots) \\ &= \sigma(\Psi(f), \Psi(Jf), \Psi(J^2 f), \Psi(J^3 f), \ldots) \\ &= \Delta(f).\end{aligned}$$

Finally, we notice that Δ is Cesàro averaging invariant since σ is a Banach limit and therefore it is left-shift invariant. Therefore, $\Delta(f) = \Delta(Jf)$.

It only remains to prove our earlier claim: *For every $r > 0$ and $f \in L^\infty$ we have that $\Psi(J^n S_r f) = \Psi(J^n f)$, for all $n \in \mathbb{N} \cup \{0\}$*: Fix $r > 0$. As stated before, $f - S_r f \in Ces_0$ for any $f \in L^\infty$, that is $\lim_{x\to\infty} J(S_r f - f) = 0$. We also know that if $g \in BC_L$ then $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} (Jg)(x)$, therefore $\lim_{x\to\infty} J^n(S_r f - f) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\Psi(J^n S_r f - J^n f) = 0$, for all $n \in \mathbb{N} \cup \{0\}$.

2.3 Defining the fractional Cesàro averaging operator on L^∞

In this subsection we define the operator J^r on L^∞, for each $r > 0$, which generalizes the concept of iterates of the J operator, and in such a way that the family of these operators indexed by $r > 0$, forms a commutative semigroup of operators on L^∞.

In the literature, the following well known operator [6] can be found:

Definition 2.5. For $\alpha > 0$, the *Hadamard fractional integral* is an operator that is defined by

$$\left(\mathcal{F}_+^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-\alpha}} f(t) \frac{dt}{t}, \text{ for } x > a.$$

This definition comes from generalizing the following formula:

$$\int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} \cdots \int_a^{t_{n-1}} \frac{f(t_n)}{t_n} dt_n = \frac{1}{(n-1)!} \int_a^x \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-n}} f(t) \frac{dt}{t}.$$

It is known that the operators $\mathcal{F}_+^\alpha$ admit the semigroup property, that is $\mathcal{F}_+^\alpha \mathcal{F}_+^\beta f = \mathcal{F}_+^{\alpha+\beta} f$, under appropriate assumptions of the function f and the exponents α and β.

In [8], the author introduced a new operator that generalizes both the Hadamard and the Riemann-Liouville fractional integral. The latter is another well known fractional integral obtained by generalizing the Cauchy's formula for repeated integration. An overview of fractional calculus can be found in the book [12].

Inspired by these fractional integrals, we obtained the following result (see A.2).

Claim 2.3. *For $n \in \mathbb{N}$, the nth iteration of applying the Cesàro averaging operator to a function $f \in L^\infty$ is given by the following formula:*

$$\int_0^x \frac{dt_1}{x} \int_0^{t_1} \frac{dt_2}{t_1} \cdots \int_0^{t_{n-1}} f(t_n) \frac{dt_n}{t_{n-1}} = \frac{1}{(n-1)!} \frac{1}{x} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{n-1} f(t) dt.$$

Based on the previous result, we give the following definition of the r power of the operator J, where $r > 0$.

Definition 2.6. For $r > 0$, define the operator J^r on L^∞ by

$$(J^r f)(x) := \frac{1}{\Gamma(r)} \frac{1}{x} \int_{t\in(0,x)} f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-r}} dt,$$

for $f \in L^\infty$, and $x \in (0, \infty)$.

We refer to this operator as *the fractional Cesàro averaging operator* on L^∞.

Next, we show that the family of operators $(J^r)_{r>0}$ has the semigroup property:

Theorem 2.4. *$J^r J^p f = J^{r+p} f$ for all $f \in L^\infty$, for all $p, r > 0$.*

Proof. Fix $f \in L^\infty$, and fix $p, r > 0$.

By making use of the identity $\Gamma(p)\Gamma(r) = B(p,r)\Gamma(p+r)$, we can express $\left(J^{p+r} f\right)(x)$ in the following way

$$\begin{aligned}\left(J^{p+r} f\right)(x) &= \frac{1}{\Gamma(p+r)} \frac{1}{x} \int_{s=0}^{x} f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds \\ &= \frac{1}{\Gamma(p)\Gamma(r)} \frac{1}{x} \int_{s=0}^{x} B(p,r) f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds \\ &= \frac{1}{\Gamma(p)} \frac{1}{\Gamma(r)} \frac{1}{x} \int_{s=0}^{x} \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(J^p (J^r f)\right)(x) &= \frac{1}{\Gamma(p)} \frac{1}{x} \int_{t=0}^{x} (J^r f)(t) \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt \\ &= \frac{1}{\Gamma(p)} \frac{1}{x} \int_{t=0}^{x} \frac{1}{\Gamma(r)} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt \\ &= \frac{1}{\Gamma(p)} \frac{1}{\Gamma(r)} \frac{1}{x} \int_{t=0}^{x} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt.\end{aligned}$$

So, we wish to show

$$\begin{aligned}&\int_{s=0}^{x} \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds \\ &= \int_{t=0}^{x} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt.\end{aligned}$$

And this can be proven by applying Fubini-Tonelli (see A.3).

Remark. It is worth noting that the definition for the operator J^r was originally derived using a different approach inspired by the following identity for real numbers:

For $t > 0$ and $r \in (0,1)$, it holds that $t^r = \frac{1}{K} \int_0^\infty q_t(\lambda) \frac{1}{\lambda^{r+1}} d\lambda$, where $K := K_r$ is a constant dependent on r, and the integrand $q := q_t$, dependent on t as well, is given by $q(\lambda) = \frac{t\lambda}{1+t\lambda}$.

Letting $\alpha := \frac{1}{\lambda}$, we have that $(\alpha + t)q = t$, and so, for a fixed $f \in L^\infty$ we posed the following integral equation: Find $q \in L^\infty$ such that $(\alpha I + J)q = Jf$.

We were able to solve the integral equation, and so, we defined the operator $T_{\frac{1}{\alpha}}$ on L^∞ by $T_{\frac{1}{\alpha}}(f) = q$, for $f \in L^\infty$. This let to the following definition of J^r for $r \in (0,1)$:

$$J^r(\cdot) := \frac{1}{K_r} \int_0^\infty T_{\frac{1}{\lambda}}(\cdot) \frac{1}{\lambda^{r+1}} d\lambda,$$

which we then extended to arbitrary $r > 0$ by setting $J^r := (J^{\lfloor r \rfloor})(J^{\langle r \rangle})$.

In Delgado [2, theorem 3.3.8], it is shown that the two different approaches described above to define fractional powers of J result in the same operator J^r.

2.4 Properties of the fractional Cesàro averaging operator J^r

By making the substitution $u = \ln\left(\frac{x}{t}\right)$, the fractional Cesàro averaging operator can also be written as

$$(J^p f)(x) = \frac{1}{\Gamma(p)} \int_{u \in (0,\infty)} [u]^{p-1} e^{-u} f(xe^{-u}) du.$$

It is easy to check that for any $p > 0$, we have that $\|J^p\|_{op} = 1$. (See, for example, Claim 3.4.1. of [2].)

The following Hardy-like inequality tells us that for any $r > 0$ the operator J^r maps $L^p(0,\infty)$ into $L^p(0,\infty)$, for $p > 1$. And so, for any $s > r$ and $f \in L^\infty$, if $J^r f \in L^p$ with $p > 1$, then $J^s f = J^{s-r}(J^r f) \in L^p$.

Claim 2.5. *If $f \in L^p(0,\infty)$ for some $p > 1$, then for every $r > 0$ we have that $J^r f \in L^p(0,\infty)$. In fact,*

$$\|J^r f\|_p \le \left(\frac{p}{p-1}\right)^r \|f\|_p.$$

Proof. Fix $p > 1$. Fix $r > 0$ and $f \in L^p(0,\infty)$.

$$\begin{aligned}
\|J^r f\|_p &\le \frac{1}{\Gamma(r)} \left(\int_0^\infty \left(\frac{1}{x} \int_0^x |f(t)| \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-r}} dt \right)^p dx \right)^{1/p} \\
&= \frac{1}{\Gamma(r)} \left(\int_0^\infty \left(\int_0^1 |f(ux)| \frac{1}{\left[\ln\left(\frac{1}{u}\right)\right]^{1-r}} du \right)^p dx \right)^{1/p}.
\end{aligned}$$

Where the last equality was obtained by making the substitution $u = t/x$ for the inner integral. We can do this change of variables since $g(u) := xu$ is a continuously differentiable, one-to-one mapping on $(0,1)$, which also implies the inner integrand in the last expression is a measurable function (see [11, theorem 3, section 9.3]).

Next, we apply Minkowski's integral inequality (see e.g. [7, theorem 202]), followed by the substitution $s = ux$ for the inner integral (which we are allow to do since $h(s) := s/u$ is continuously differentiable and one-to-one), to obtain:

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_0^1 |f(ux)| \frac{1}{\left[\ln\left(\frac{1}{u}\right)\right]^{1-r}} du \right)^p dx \right)^{1/p} \\
&\le \int_0^1 \left(\int_0^\infty \frac{|f(ux)|^p}{\left[\ln\left(\frac{1}{u}\right)\right]^{p(1-r)}} dx \right)^{1/p} du = \int_0^1 \left(\frac{1}{u} \int_0^\infty |f(s)|^p \frac{1}{|\ln(u)|^{p(1-r)}} ds \right)^{1/p} du \\
&= \int_0^1 \frac{1}{u^{1/p}} \frac{1}{|\ln(u)|^{(1-r)}} \left(\int_0^\infty |f(s)|^p ds \right)^{1/p} du = \|f\|_p \int_0^1 \frac{1}{u^{1/p}} \frac{1}{|\ln(u)|^{(1-r)}} du.
\end{aligned}$$

Finally, by denoting $q = 1 - 1/p = \frac{p-1}{p}$, and making the substitution $z = q \ln\left(\frac{1}{u}\right)$, we obtain:

$\frac{1}{\Gamma(r)} \|f\|_p \int_0^1 \frac{1}{u^{1/p}} \frac{1}{|\ln(u)|^{(1-r)}} du = \frac{1}{\Gamma(r)} \|f\|_p \frac{1}{q^r} \int_0^\infty z^{(r-1)} e^{-z} dz = \|f\|_p \frac{1}{q^r}$. (See, for example, proof of Claim 3.4.2. in [2].)

Therefore

$$\|J^r f\|_p \le \left(\frac{p}{p-1}\right)^r \|f\|_p.$$

Remark. Consider the function $f := \chi_{[0,1]}$. Clearly $f \in L^1(0,\infty)$. Nevertheless

$$Jf(x) = \begin{cases} 1, & 0 < x \le 1 \\ \frac{1}{x}, & 1 \le x \end{cases}$$

is not in $L^1(0,\infty)$. So, we see the previous Claim does not necessarily hold for $p = 1$.

The following Theorem will imply one more result concerning L^p spaces, for a particular type of function:

Theorem 2.6. *For every $s>0$ there exists a constant $K_s := K > 0$ such that, for every $f \in L^\infty$, we have that*

$$|(J^{s/2}f)(x)| \leq \left(\frac{\|f\|_\infty^2}{K}\right)^{\frac{1}{3}} \max\left\{((J^s f_1)(x))^{\frac{1}{3}}, ((J^s f_2)(x))^{\frac{1}{3}}\right\},$$

where

$$f_1 := max\{f, 0\} \text{ and } f_2 := max\{-f, 0\}.$$

Proof. Fix $s>0$. First assume that $f \geq 0$. Recall

$$(J^s f)(x) = \frac{1}{\Gamma(s)} \int_{u\in(0,\infty)} [u]^{s-1} e^{-u} f(xe^{-u}) du.$$

Clearly, the desired inequality holds when f is the constant function 0, for any value we choose for $K>0$. Therefore we may assume $f \in L^\infty \setminus \{0\}$. Let $g := \dfrac{f}{\|f\|_\infty}$, then $\|g\|_\infty = 1$ and $g \geq 0$. Fix $x>0$, and fix $\tau \in (0,s)$. Then

$$\begin{aligned}(J^s g)(x) &= \frac{1}{\Gamma(s)} \int_{u\in(0,\infty)} u^{s-1} e^{-u} g(xe^{-u}) du \\ &= \frac{1}{\Gamma(s)} \int_{u\in(0,\infty)} u^{\tau} u^{s-\tau-1} e^{-u} g(xe^{-u}) du.\end{aligned}$$

Observe that $\int_{u\in(0,\infty)} u^{s-\tau-1}e^{-u}du = \Gamma(s-\tau)$. So, we can re-write $J^s g$ as

$$(J^s g)(x) = \frac{\Gamma(s-\tau)}{\Gamma(s)} \int_{u\in(0,\infty)} u^\tau g(xe^{-u}) d\nu_\tau(u),$$

where $d\nu_\tau(u) := \dfrac{u^{s-\tau-1}e^{-u}du}{\Gamma(s-\tau)}$, with ν_τ being a probability measure on $\Delta_{[0,\infty)}$.

Fix $\sigma > 1$. Notice that that the integrand $u^\tau g(xe^{-u})$ is non-negative, and so we apply Jensen's inequality:

$$\begin{aligned}(J^s g)(x) &= \frac{\Gamma(s-\tau)}{\Gamma(s)} \int_{u\in(0,\infty)} \left(u^{\frac{\tau}{\sigma}} (g(xe^{-u}))^{\frac{1}{\sigma}}\right)^\sigma d\nu_\tau(u) \\ &\geq \frac{\Gamma(s-\tau)}{\Gamma(s)} \left(\int_{u\in(0,\infty)} u^{\frac{\tau}{\sigma}} (g(xe^{-u}))^{\frac{1}{\sigma}} d\nu_\tau(u)\right)^\sigma \\ &= \frac{\Gamma(s-\tau)}{\Gamma(s)} \left(\int_{u\in(0,\infty)} u^{\frac{\tau}{\sigma}} (g(xe^{-u}))^{\frac{1}{\sigma}} \frac{u^{s-\tau-1}e^{-u}du}{\Gamma(s-\tau)}\right)^\sigma.\end{aligned}$$

Since $0 \leq g(xe^{-u}) \leq 1$ and $0 < \frac{1}{\sigma} < 1$, we have that $(g(xe^{-u}))^{\frac{1}{\sigma}} \geq g(xe^{-u})$. Thus

$$\begin{aligned}(J^s g)(x) &\geq \frac{\Gamma(s-\tau)}{\Gamma(s)(\Gamma(s-\tau))^\sigma} \left(\int_{u\in(0,\infty)} u^{s-\tau-1+\frac{\tau}{\sigma}} g(xe^{-u}) e^{-u} du\right)^\sigma \\ &= \frac{\Gamma(s-\tau)\Gamma(s-\tau+\frac{\tau}{\sigma})^\sigma}{\Gamma(s)(\Gamma(s-\tau))^\sigma} \left(\left(J^{s-\tau+\frac{\tau}{\sigma}} g\right)(x)\right)^\sigma.\end{aligned}$$

So, we choose $\tau \in (0,s)$ and $\sigma > 1$ in such a way that we get the desired conclusion: take $\tau = \frac{3}{4}s$ and $\sigma = 3$. We get

$$(J^s g)(x) \geq \frac{\Gamma(s/4)\Gamma(s/2)^3}{\Gamma(s)\Gamma(s/4)^3} \left(\left(J^{s/2} g\right)(x)\right)^3.$$

This, after substituting $g = \dfrac{f}{\|f\|_\infty}$, gives us

$$\left(J^s \frac{f}{\|f\|_\infty}\right)(x) \geq \frac{\Gamma(s/2)^3}{\Gamma(s)\Gamma(s/4)^2} \left(\left(J^{s/2} \frac{f}{\|f\|_\infty}\right)(x)\right)^3.$$

Therefore, by letting $K := \dfrac{\Gamma(s/2)^3}{\Gamma(s)\Gamma(s/4)^2}$, we obtain

$$(J^s f)(x) \geq K \frac{\left(\left(J^{s/2} f\right)(x)\right)^3}{\|f\|_\infty^2}.$$

Next, consider an arbitrary $f \in L^\infty \setminus \{0\}$. Then f_1, f_2 as defined above, are elements of L^∞ and $f_1, f_2 \geq 0$. Consequently, we have that

$$0 \leq \left(J^{s/2} f_1\right)(x) \leq \left(\frac{\|f_1\|_\infty^2}{K}\right)^{\frac{1}{3}} \left(\left(J^s f_1\right)(x)\right)^{\frac{1}{3}},$$

and

$$-\left(\frac{\|f_2\|_\infty^2}{K}\right)^{\frac{1}{3}} \left(\left(J^s f_2\right)(x)\right)^{\frac{1}{3}} \leq -\left(J^{s/2} f_2\right)(x) \leq 0.$$

Notice that $\|f_1\|_\infty, \|f_2\|_\infty \leq \|f\|_\infty$, and $f = f_1 - f_2$. So, using the linearity of $J^{s/2}$ we obtain

$$-\left(\frac{\|f\|_\infty^2}{K}\right)^{\frac{1}{3}} \left(\left(J^s f_2\right)(x)\right)^{\frac{1}{3}} \leq (J^{s/2} f)(x) \leq \left(\frac{\|f\|_\infty^2}{K}\right)^{\frac{1}{3}} \left(\left(J^s f_1\right)(x)\right)^{\frac{1}{3}},$$

which implies the desired conclusion.

Lemma 2.7. *Let $f \in L^\infty$ such that $f \geq 0$ and $J^s f \in L^p(0,\infty)$, for some $p > 0$ and some $s > 0$. Then we have that $J^{s/2} f \in L^{3p}(0,\infty)$.*

Proof. We can assume f is not the constant function 0, since in this case the conclusion holds trivially.

Since we are assuming $f \geq 0$, then $f_1 = f$, and $f_2 = 0$, and so, by Theorem 2.6, we have that

$$\left(\frac{\|f\|_\infty^2}{K}\right)^{1/3} \left(\left(J^s f\right)(x)\right)^{1/3} \geq \left(J^{s/2} f\right)(x) \geq 0,$$

which implies

$$(J^s f)(x) \geq \frac{K}{\|f\|_\infty^2} \left(\left(J^{s/2} f\right)(x)\right)^3.$$

Therefore,

$$\infty > \int_0^\infty |\left(J^s f\right)(x)|^p dx \geq \left(\frac{K}{\|f\|_\infty^2}\right)^p \int_0^\infty |\left(J^{s/2} f\right)(x)|^{3p} dx.$$

Thus, $J^{s/2} \in L^{3p}(0,\infty)$.

Remark. We already mentioned that for the function $f := \chi_{[0,1]}$, we have that $Jf \notin L^1(0,\infty)$. Still

$$J^{1/2} f(x) = \begin{cases} 1, & 0 < x \leq 1 \\ \frac{1}{\Gamma(1/2)} \displaystyle\int_{\ln(x)}^\infty u^{-1/2} e^{-u} du, & 1 \leq x \end{cases}$$

is an element of $L^3(0,\infty)$, given that

$$\int_1^\infty \left| \int_{\ln(x)}^\infty u^{-1/2} e^{-u} du \right|^3 dx = -\sqrt{\pi} \left(6(-2+\sqrt{2}) + \pi\right).$$

Hence, we see the converse of the previous Lemma does not necessarily hold.

Remark. The previous Lemma can be modified to obtain the following result: *Let $f \in L^\infty$ such that $f \geq 0$ and $J^s f \in L^p(0,\infty)$, for some $p > 0$ and some $s > 0$. Then for any $n \in \mathbb{N}$ we have that $J^{\frac{2n+1}{n+1}\frac{s}{2}} f \in L^{\frac{n+1}{n}p}(0,\infty)$.*

We achieve this by modifying Theorem 2.6 and its proof: In the proof of Theorem 2.6,fix $n \in \mathbb{N}$ and let $\sigma := \dfrac{n+1}{n}$ and $\tau := s/2$. Then, the constant K in the statement of this Theorem would depend on s and n. And the inequality in the conclusion would be:

$$|(J^{\frac{2n+1}{n+1}\frac{s}{2}} f)(x)| \leq \left(\frac{\|f\|_\infty^{1/n}}{K}\right)^{\frac{n}{n+1}} \max\left\{((J^s f_1)(x))^{\frac{n}{n+1}}, ((J^s f_2)(x))^{\frac{n}{n+1}}\right\},$$

In the remainder of this section, we will discuss continuity results for the operator J^p.

Claim 2.8. *For all $f \in L^\infty$, we have that $\lim_{p\to 0+}(J^p f)(x) = f(x)$ for every $x > 0$ such that f is continuous at x.*

Proof. Fix $x > 0$ such that f is continuous at x. We want to show

$$\lim_{p\to 0+} \frac{1}{\Gamma(p)} \frac{1}{x} \int_{t\in(0,x)} f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}} dt = f(x).$$

Define $\varphi_\epsilon(t) := \dfrac{1}{x\,\Gamma(\epsilon)} \dfrac{\chi_{[0,x]}(x-t)}{\left[\ln\left(\dfrac{x}{x-t}\right)\right]^{1-\epsilon}}$. Then $\varphi_\epsilon(x-t) = \dfrac{1}{x\,\Gamma(\epsilon)} \dfrac{\chi_{[0,x]}(t)}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-\epsilon}}$.

Let $g(t) := f(t)\chi_{[0,x]}(t)$, then $g \in L^1(-\infty,\infty)$ since

$$\int_{-\infty}^{\infty} |g(t)| dt \leq x\|f\|_\infty < \infty.$$

We can rewrite our desired conclusion as

$$\int_{-\infty}^{\infty} g(t)\varphi_\epsilon(x-t) dt \to g(x) \text{ as } \epsilon \to 0^+.$$

Or equivalently, $g * \varphi_\epsilon(x) \to g(x)$ as $\epsilon \to 0^+$.

And this holds since φ_ϵ is a good kernel, according to the definition in [14, chapter 3, section 2] (see A.4).

Claim 2.9. *Let $f \in L^\infty$. Then*

$$\liminf_{x\to\infty} f(x) \leq \liminf_{x\to\infty} (J^p f)(x) \leq \limsup_{x\to\infty} (J^p f)(x) \leq \limsup_{x\to\infty} f(x),$$

for all $p > 0$.

(See A.5).

As an immediate corollary, we conclude that if $\lim_{x\to\infty} f(x) = L$ then $\lim_{x\to\infty}(J^p f)(x) = L$ for all $p > 0$.

Theorem 2.10. *For each $f \in L^\infty$, the map $p \mapsto J^p f : (0,\infty) \to L^\infty$ is norm-to-norm continuous.*

Proof. First consider $p \in (0,1)$. Consider a sequence of positive real numbers $(p_n)_n$ such that $p_n \to p$. We can assume that $p_n \in (0,1)$, for each $n \in \mathbb{N}$. So, for a fixed $f \in L^\infty$ we have that

$$\|J^{p_n} f - J^p f\|_\infty =$$
$$\operatorname*{ess\,sup}_{x\in(0,\infty)} \left| \frac{1}{x\Gamma(p_n)} \int_{t=0}^{x} f(t) \left[\ln\left(\frac{x}{t}\right)\right]^{p_n-1} dt - \frac{1}{x\Gamma(p)} \int_{t=0}^{x} f(t) \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt \right|.$$

We add and subtract the term $\frac{1}{x\Gamma(p)}\int_{t=0}^{x} f(t)\left[\ln\left(\frac{x}{t}\right)\right]^{p_n-1}dt$ and obtain

$$\|J^{p_n}f - J^p f\|_\infty \leq$$
$$\operatorname*{ess\,sup}_{x\in(0,\infty)}\left\{\left|\frac{1}{\Gamma(p_n)}-\frac{1}{\Gamma(p)}\right|\|f\|_\infty\frac{1}{x}\int_{t=0}^{x}\left[\ln\left(\frac{x}{t}\right)\right]^{p_n-1}dt+\right.$$
$$\left.\frac{1}{x\Gamma(p)}\int_{t=0}^{x}\left|f(t)\left(\left[\ln\left(\frac{x}{t}\right)\right]^{p_n-1}-\left[\ln\left(\frac{x}{t}\right)\right]^{p-1}\right)dt\right|\right\}.$$

Now, let $u=\ln\left(\frac{x}{t}\right)$. Then

$$\|J^{p_n}f - J^p f\|_\infty$$
$$\leq\left|\frac{1}{\Gamma(p_n)}-\frac{1}{\Gamma(p)}\right|\|f\|_\infty\Gamma(p_n)+\|f\|_\infty\frac{1}{\Gamma(p)}\int_{u=0}^{\infty}\left|[u]^{p_n-1}-[u]^{p-1}\right|e^{-u}du.$$

Since $\frac{1}{\Gamma(\cdot)}$ is continuous for positive values, it is enough to show that

$$\lim_{n\to\infty}\int_{u=0}^{\infty}\left|[u]^{p_n-1}-[u]^{p-1}\right|e^{-u}du=0.$$

And we can show this holds by applying Dominated Convergence Theorem (see A.6).

Now, for $p\geq 1$ such that $\langle p\rangle>0$, we know there exists $N\in\mathbb{N}$ such that $\lfloor p_n\rfloor=\lfloor p\rfloor$ for all $n\geq N$. Assume, without loss of generality, that $N=1$. Since $\langle p_n\rangle\to\langle p\rangle$, $\langle p_n\rangle,\langle p\rangle\in(0,1)$ for all n, we have that $(J^{\langle p_n\rangle}f)(x)\to(J^{\langle p\rangle}f)(x)$ uniformly in x as $n\to\infty$. Therefore $J^{\lfloor p_n\rfloor}(J^{\langle p_n\rangle}f)(x)\to J^{\lfloor p\rfloor}(J^{\langle p\rangle}f)(x)$ uniformly in x as $n\to\infty$.

Next, consider when $p\in\mathbb{N}$. If $s\to p$, then we can assume that $s>p-1$. So, we notice that

$$\|J^s f - J^p f\|_\infty \leq \|J^{p-1}\|_{op}\|J^{s-p+1}f - Jf\|_\infty,$$

where, in case $p=1$, we define $J^0 f := f$ for every $f\in L^\infty$. Then

$$\begin{aligned}\lim_{s\to p}\|J^s f - J^p f\| &\leq \|J^{p-1}\|_{op}\lim_{s\to p}\|(J^{s-p+1/2})(J^{1/2}f)-(J^{1/2})(J^{1/2}f)\|_\infty\\ &= \|J^{p-1}\|_{op}\lim_{r\to 1/2}\|(J^{r})(J^{1/2}f)-(J^{1/2})(J^{1/2}f)\|_\infty.\end{aligned}$$

Now, for $r\in(0,1)$ and fixed $g\in L^\infty$, we already checked that the mapping $r\mapsto J^r g:(0,1)\to L^\infty$ is norm-to-norm continuous. Since $J^{1/2}f\in L^\infty$, we can apply this result to the last expression, and get the desired result.

Theorem 2.11. *Fix $r>0$. Fix $f\in L^\infty$. Then $J^r f$ is continuous on $(0,\infty)$.*

Proof. First, we fix $r\in(0,1)$. Let $x,y\in(0,\infty)$. Without loss of generality, assume $y>x$. Then

$$|(J^r f)(x)-(J^r f)(y)|=\left|\frac{1}{\Gamma(r)}\left\{\frac{1}{x}\int_0^x\left[\ln\left(\frac{x}{t}\right)\right]^{r-1}f(t)dt-\frac{1}{y}\int_0^y\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}f(t)dt\right\}\right|.$$

Therefore

$$\Gamma(r)|(J^r f)(x)-(J^r f)(y)|=$$
$$\left|\frac{1}{x}\int_0^x\left[\ln\left(\frac{x}{t}\right)\right]^{r-1}f(t)dt-\left(\frac{1}{x}+\frac{x-y}{xy}\right)\left(\int_0^x\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}f(t)dt+\int_x^y\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}f(t)dt\right)\right|=$$
$$|I_1-I_2-I_3|\leq|I_1|+|I_2|+|I_3|;$$

where

$$\begin{aligned}I_1 &:= \int_0^x\left\{\frac{1}{x}\left[\ln\left(\frac{x}{t}\right)\right]^{r-1}-\frac{1}{x}\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}\right\}f(t)dt,\\ I_2 &:= \frac{x-y}{xy}\int_0^x\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}f(t)dt,\ \text{and}\\ I_3 &:= \frac{1}{y}\int_x^y\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}f(t)dt.\end{aligned}$$

For the second integral, we make the usual substitution $u=\ln\left(\frac{y}{t}\right)$ to get

$$\left|\frac{x-y}{xy}\int_0^x\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}f(t)dt\right|\leq\|f\|_\infty\left|\frac{x-y}{x}\right|\Gamma(r),$$

and this last expression goes to 0 as $y-x\to 0$.

For the last integral, again we make the substitution $u=\ln\left(\frac{y}{t}\right)$ to get

$$\begin{aligned}\frac{1}{y}\left|\int_x^y\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}f(t)dt\right|&\leq\|f\|_\infty\int_0^{\ln(y/x)}u^{r-1}e^{-u}du\\&\leq\|f\|_\infty\int_0^{\ln(y/x)}u^{r-1}du=\|f\|_\infty\frac{1}{r}\left[\ln\left(y/x\right)\right]^r,\end{aligned}$$

and this last expression goes to 0 as $y-x\to 0$.

For the first integral, since $r-1<0$ and $y>x$, then

$$\left|\frac{1}{x}\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}-\frac{1}{x}\left[\ln\left(\frac{x}{t}\right)\right]^{r-1}\right|\leq 2\frac{1}{x}\left[\ln\left(\frac{x}{t}\right)\right]^{r-1},$$

and recall that

$$2\int_0^x\frac{1}{x}\left[\ln\left(\frac{x}{t}\right)\right]^{r-1}dt=2\Gamma(r)<\infty.$$

Therefore, we can apply Dominated Convergence Theorem to get

$$\int_0^x\left\{\frac{1}{x}\left[\ln\left(\frac{x}{t}\right)\right]^{r-1}f(t)-\frac{1}{x}\left[\ln\left(\frac{y}{t}\right)\right]^{r-1}f(t)\right\}dt\to 0\text{ as }y-x\to 0.$$

Finally, for $r\geq 1$, we have that

$$|J^rf(y)-J^rf(x)|=|J^{\lfloor r\rfloor}J^{\langle r\rangle}f(y)-J^{\lfloor r\rfloor}J^{\langle r\rangle}f(x)|.$$

Let $h(x):=J^{\langle r\rangle}f(x)$, let $\lfloor r\rfloor=n$, and let $g:=J^{n-1}h$, where $J^0h:=h$. Then

$$\begin{aligned}|J^rf(y)-J^rf(x)|&=|J^nh(y)-J^nh(x)|=|Jg(y)-Jg(x)|\\&=\left|\frac{1}{y}\int_0^yg(t)dt-\frac{1}{x}\int_0^xg(t)dt\right|\\&=\left|\left(\frac{1}{x}+\frac{x-y}{xy}\right)\left(\int_0^xg(t)dt+\int_x^yg(t)dt\right)-\frac{1}{x}\int_0^xg(t)dt\right|\\&=\left|\frac{x-y}{xy}\int_0^xg(t)dt+\frac{1}{y}\int_x^yg(t)dt\right|\leq\left|\frac{x-y}{y}\right|\|g\|_\infty+\frac{y-x}{y}\|g\|_\infty,\end{aligned}$$

and this last expression goes to 0 as $y-x\to 0$.

3 The vector space Ces^r

Definition 3.1. For each $r>0$, we define the vector subspace of L^∞:

$$Ces^r:=\{f\in L^\infty:\psi_r(f):=\lim_{x\to\infty}(J^rf)(x)\in\mathbb{R}\}.$$

Remark. $\|\psi_r\|_{(Ces^r)^*}=1$ (see A.7).

The third author obtained a quantitative version [17, theorem II.9] of the following qualitative theorem:

For $x\in\ell^\infty$, Cx is convergent if and only if C^2x is convergent.

This qualitative result follows from a theorem due to Frobenius [5], and a classical theorem of Hardy and Littlewood (see Theorem 7.3 of [9]).

We followed the proof of Theorem II.9 from [17] closely in places, and obtained the exact analogue result for the space L^∞, including the same quantitative outcome. The qualitative version of our result tells us that $Ces^n=Ces^m$ for every $n,m\in\mathbb{N}$. We present our quantitative result next:

Theorem 3.1. *Let $f\in L^\infty$ with $Jf\notin BC_L$. Define*

$$q := \limsup_{x\to\infty} f(x) \text{ and } p := \liminf_{x\to\infty} f(x);$$

and also

$$b := \limsup_{x\to\infty} Jf(x) \text{ and } a := \liminf_{x\to\infty} Jf(x);$$

Let $d := b - a$ and $m := (a-p) \vee (q-b) := \max\{a-p, q-b\}$. Then

$$\limsup_{x\to\infty} J^2 f(x) - \liminf_{x\to\infty} J^2 f(x) \geq \frac{d^2}{10d + 8m + \sqrt{(10d+8m)^2 - 4d^2}}.$$

In particular, $h(x) := J^2 f(x) \notin BC_L$.

The proof of the previous theorem can be found in [2, theorem 3.5.1].

The following Corollary not only follows from Theorem 3.1, but also generalizes it:

Corollary 3.2. *For any $1 \leq r, s$ we have that $Ces^r = Ces^s$.*

Proof. Assume, without loss of generality, that $r < s$. First notice that we already know $Ces^r \subset Ces^s$. Indeed, from the semigroup property of $\{J^p\}_{p>0}$ we have that $J^s f(x) = J^{s-r}(J^r f(x))$. By the fact that the operator J^p preserves limits at infinity for every $p > 0$, and in particular for $p = s - r$, the result follows. Next, consider $f \in Ces^s$. This will imply that $f \in Ces^{\lfloor s \rfloor + 1}$ since $s < \lfloor s \rfloor + 1$. Then by Theorem 3.1 this implies that $f \in Ces^{\lfloor r \rfloor}$, which in turn implies $f \in Ces^r$.

To obtain the most general form of the previous results, we first present the following theorem. Its proof is a variation on, and a significant extension of, the proof of Theorem II.9 from Sivek [17]. Indeed, this is one of the main theorems of our paper (as mentioned in our Introduction).

Theorem 3.3. *Let $r \in (0,1]$. Let $f \in L^\infty$ with $J^r f \notin BC_L$. Define*

$$q := \limsup_{x\to\infty} f(x) \text{ and } p := \liminf_{x\to\infty} f(x);$$

and also

$$b := \limsup_{x\to\infty} J^r f(x) \text{ and } a := \liminf_{x\to\infty} J^r f(x);$$

Let $d := b - a$. Then for every $\tau \in (0, 1/2)$ there exists $\Theta := \Theta(r, d, \|f - \frac{a+b}{2}\|_\infty, \tau) > 0$ such that

$$\delta_{1+r} f := \limsup_{x\to\infty} J^{1+r} f(x) - \liminf_{x\to\infty} J^{1+r} f(x) \geq \Theta.$$

In particular, $J^{1+r} f(x) \notin BC_L$.

Moreover, we may choose Θ to be

$$\Theta := \left(\frac{r\gamma}{2V_r^{-1}(\gamma)}\right)^{1/r} \frac{(\tau/2)}{G_r^{-1}(\gamma)\|f - \frac{a+b}{2}\|_\infty^{1/r}} d^{1+1/r}.$$

Here,

$$\gamma := \gamma(\tau, r) := \frac{1/2 - \tau}{2\tau + \frac{1}{\Gamma(r)}}.$$

Also,

$$G_r(w) := w \int_0^{\ln(w)} u^{r-1} e^{-u} du, \text{ and } \ V_r(w) := w\frac{(\ln(w))^r}{r}, \text{ for all } w \in [1, \infty).$$

Proof. Without loss of generality, we may assume that $\dfrac{a+b}{2} = 0$. Since, the result holds for f if and only if it holds for $f_1 := f - \dfrac{a+b}{2}$. (See, for example, proof of Theorem 3.5.3. in [2].)

So, without loss of generality, we have that $a = -b$, and $d = 2b > 0$.

Recall, by Claim 2.9, we know that

$$-\infty < -\|f\|_\infty \le p \le a = \tfrac{-d}{2} < 0 \text{ and } 0 < b = \tfrac{d}{2} \le q \le \|f\|_\infty < \infty.$$

Fix real numbers θ, τ, $\theta^{'}$ and $\tau^{'}$ with $0 < \theta < \tau < 1/2$ and $0 < \theta^{'} < \tau^{'} < 1/2$, arbitrary. The relationships between $\tau, \theta, \tau^{'}$ and $\theta^{'}$ will be chosen later.

Fix $\epsilon \in (0, (1/2-\tau) \wedge (1/2-\tau^{'}))$. Then, there exists $K := K_\epsilon > 0$ such that for all $x \ge K$,

$$a - \epsilon d < J^r f(x) < b + \epsilon d.$$

Fix $x_0 > K$ arbitrary. We consider two cases: $J^{1+r} f(x_0) = \dfrac{1}{x_0}\displaystyle\int_0^{x_0} J^r f(t)dt \le 0$; and $J^{1+r} f(x_0) \ge 0$.

Case 1. $J^{1+r} f(x_0) \le 0$.

Since $b := \limsup\limits_{x\to\infty} J^r f(x)$, then there exists $x_1 > x_0$ such that

$$J^r f(x_1) > b - \epsilon d = \frac{a+b}{2} + \left(\tfrac{1}{2} - \epsilon\right) d = \left(\tfrac{1}{2} - \epsilon\right) d > \tau d.$$

Since $a := \liminf\limits_{x\to\infty} J^r f(x)$, then there exists $\tilde{x} > x_1$ such that

$$J^r f(\tilde{x}) < a + \epsilon d = \frac{a+b}{2} + \left(\epsilon - \tfrac{1}{2}\right) d = \left(\epsilon - \tfrac{1}{2}\right) d < -\tau d < \tau d.$$

We see then, that the set $A := \{x \in (0,\infty) : x > x_1 \textit{ and } J^r f(x) < \tau d\}$ is not empty, and is bounded below by x_1. So we let $x_2 := \inf A$.

Notice $x_1 \ne x_2$ since

$$J^r f(x_1) > \tau d \text{ and } J^r f(x_2) \le \tau d \text{ by continuity of } J^r f.$$

Therefore $x_1 < x_2$, and we have that

$$J^r f(x) \ge \tau d \textit{ for all } x \in [x_1, x_2], \textit{ and in fact } J^r f(x_2) = \tau d. \tag{$\diamond\,\diamond$}$$

Let $\Delta := x_2 - x_1 > 0$. Then,

$$\begin{aligned}
&\tau d - \frac{x_1}{x_2} J^r f(x_1)\\
&= J^r f(x_2) - \frac{x_1}{x_2} J^r f(x_1)\\
&= \frac{1}{x_2\Gamma(r)}\left(\int_0^{x_2} f(t)\left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1} dt - \int_0^{x_1} f(t)\left[\ln\left(\frac{x_1}{t}\right)\right]^{r-1} dt\right)\\
&= \frac{1}{x_2\Gamma(r)}\left(\int_0^{x_1} f(t)\left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1} dt + \int_{x_1}^{x_2} f(t)\left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1} dt - \int_0^{x_1} f(t)\left[\ln\left(\frac{x_1}{t}\right)\right]^{r-1} dt\right)\\
&= \frac{1}{x_2\Gamma(r)}\left(\int_{x_1}^{x_2} f(t)\left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1} dt - \int_0^{x_1} f(t)\left[\left[\ln\left(\frac{x_1}{t}\right)\right]^{r-1} - \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1}\right] dt\right)\\
&\ge \frac{1}{x_2\Gamma(r)}\left(-\|f\|_\infty \int_{x_1}^{x_2} \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1} dt - \|f\|_\infty \int_0^{x_1} \left[\left[\ln\left(\frac{x_1}{t}\right)\right]^{r-1} - \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1}\right] dt\right).
\end{aligned}$$

Therefore

$$\tau d - \frac{x_1}{x_2} J^r f(x_1) \ge \frac{-\|f\|_\infty}{x_2\Gamma(r)}\left(\int_{x_1}^{x_2} \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1} dt + \int_0^{x_1} \left[\left[\ln\left(\frac{x_1}{t}\right)\right]^{r-1} - \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1}\right] dt\right).$$

Denote by

$$\Phi_1(r, x_1, x_2) := \int_{x_1}^{x_2} \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1} dt > 0 \text{ and } \Phi_2(r, x_1, x_2) := \int_0^{x_1} \left[\left[\ln\left(\frac{x_1}{t}\right)\right]^{r-1} - \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1}\right] dt \ge 0.$$

Also denote

$$\Phi(r, x_1, x_2) := \Phi_1(r, x_1, x_2) + \Phi_2(r, x_1, x_2).$$

Notice that, by making the usual substitution $u = \ln\left(\frac{x_2}{t}\right)$ for $\Phi_1(r, x_1, x_2)$, we obtain

$$\Phi_1(r, x_1, x_2) = \int_{x_1}^{x_2} \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1} dt = x_2 \int_0^{\ln\left(\frac{x_2}{x_1}\right)} u^{r-1} e^{-u} du.$$

Also, by making the substitutions $u = \ln\left(\frac{x_1}{t}\right)$ and $v = \ln\left(\frac{x_2}{t}\right)$ for $\Phi_2(r, x_1, x_2)$,

$$\begin{aligned}
\Phi_2(r, x_1, x_2) &= \int_0^{x_1} \left[\left[\ln\left(\frac{x_1}{t}\right)\right]^{r-1} dt - \int_0^{x_1} \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1}\right] dt \\
&= x_1 \int_0^{\infty} u^{r-1} e^{-u} du - x_2 \int_{\ln\left(\frac{x_2}{x_1}\right)}^{\infty} v^{r-1} e^{-v} dv \\
&= (x_1 - x_2 + x_2)\Gamma(r) - x_2 \int_{\ln\left(\frac{x_2}{x_1}\right)}^{\infty} v^{r-1} e^{-v} dv \\
&= -\Delta\Gamma(r) + x_2 \int_0^{\ln\left(\frac{x_2}{x_1}\right)} v^{r-1} e^{-v} dv.
\end{aligned}$$

Therefore

$$\Phi(r, x_1, x_2) = -\Delta\Gamma(r) + 2x_2 \int_0^{\ln\left(\frac{x_2}{x_1}\right)} u^{r-1} e^{-u} du = -\Delta\Gamma(r) + 2\Phi_1(r, x_1, x_2).$$

Here, we make the observation that for $r = 1$, we have that $\Phi(1, x_1, x_2) = x_2 - x_1 = \Delta$.

Now, for $r \in (0, 1]$, we have the following inequality

$$\begin{aligned}
\tau d - \frac{x_1}{x_2} J^r f(x_1) &\geq \frac{-\|f\|_\infty}{x_2 \Gamma(r)} \Phi(r, x_1, x_2) \\
&= \frac{-\|f\|_\infty}{x_2 \Gamma(r)} \left(-\Delta\Gamma(r) + 2\Phi_1(r, x_1, x_2)\right),
\end{aligned}$$

or equivalently

$$\tau d x_2 \geq x_1 J^r f(x_1) + \frac{-\|f\|_\infty}{\Gamma(r)} (-\Delta\Gamma(r) + 2\Phi_1(r, x_1, x_2)).$$

Since $r \in (0, 1]$, then $\Gamma(r) \geq 1$, and so $\tau d x_2 = \tau d(x_1 + \Delta) \leq \tau d(x_1 + \Gamma(r)\Delta)$, given that $\tau d > 0$. Also recall that $J^r f(x_1) > (\frac{1}{2} - \epsilon)d$. Then,

$$\tau d(x_1 + \Gamma(r)\Delta) \geq x_1(\tfrac{1}{2} - \epsilon)d + \frac{-\|f\|_\infty}{\Gamma(r)} \Phi(r, x_1, x_2).$$

Next, since $\Phi(r, x_1, x_2) \geq 0$, and also $\Gamma(r)\Delta > 0$, we have that

$$\tau d(x_1 + \Gamma(r)\Delta + \Phi(r, x_1, x_2)) \geq x_1(\tfrac{1}{2} - \epsilon)d - \frac{\|f\|_\infty}{\Gamma(r)} (\Gamma(r)\Delta + \Phi(r, x_1, x_2)).$$

Recall that $\Gamma(r)\Delta + \Phi(r, x_1, x_2) = 2\Phi_1(r, x_1, x_2)$. Thus

$$\tau d(x_1 + 2\Phi_1(r, x_1, x_2)) \geq x_1(\tfrac{1}{2} - \epsilon)d - \frac{\|f\|_\infty}{\Gamma(r)} 2\Phi_1(r, x_1, x_2).$$

This implies

$$2\Phi_1(r, x_1, x_2) \left(\tau d + \frac{\|f\|_\infty}{\Gamma(r)}\right) \geq x_1(\tfrac{1}{2} - \epsilon - \tau)d.$$

Therefore

$$\Delta_r := \Phi_1(r, x_1, x_2) \geq \frac{x_1(\frac{1}{2} - \epsilon - \tau)d}{2\left(\tau d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

Equivalently

$$\frac{\Delta_r}{x_1} = \frac{1}{x_1}\Phi_1(r, x_1, x_2) = \frac{x_2}{x_1}\int_0^{\ln\left(\frac{x_2}{x_1}\right)} u^{r-1}e^{-u}du \geq \frac{(\frac{1}{2} - \epsilon - \tau)d}{2\left(\tau d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

Denote

$$P_\epsilon := P_{r,d,\|f\|_\infty,\tau,\epsilon} := \frac{(\frac{1}{2} - \epsilon - \tau)d}{2\left(\tau d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)},$$

and consider the function

$$G_r(w) := w\int_0^{\ln(w)} u^{r-1}e^{-u}du, \text{ for all } w \in [1, \infty).$$

Notice that G_r is a strictly increasing continuous function. Also $G_r(1) = 0$, and $\lim_{w\to\infty} G_r(w) = \infty$, thus $G_r : [1, \infty) \to [0, \infty)$ has an inverse. So, $(G_r)^{-1} : [0, \infty) \to [1, \infty)$ is well defined and it is also a strictly increasing continuous function. In particular $(G_r)^{-1}(s) > 1$ for all $s \in (0, \infty)$.

Also notice $G_r(\frac{x_2}{x_1}) = \dfrac{\Delta_r}{x_1} \geq P_\epsilon > 0$. Therefore

$$\tfrac{x_2}{x_1} \geq (G_r)^{-1}(P_\epsilon) > 1,$$

which implies

$$\Delta = x_2 - x_1 \geq x_1\left((G_r)^{-1}(P_\epsilon) - 1\right).$$

Here we make the following observation: $G_1(w) = w(1 - e^{-\ln(w)}) = w(1 - \frac{1}{w}) = w - 1$, for all $w \in [1, \infty)$. So $G_1^{-1}(s) = s + 1$, for all $s \in [0, \infty)$.

Now, recall that $0 < \theta < \tau < 1/2$.

Sub-case 1.a. $J^{1+r}f(x_1) \geq \theta d$.

Since $J^{1+r}f(x_0) \leq 0$, then

$$J^{1+r}f(x_1) - J^{1+r}f(x_0) \geq \theta d.$$

Sub-case 1.b. $J^{1+r}f(x_1) < \theta d$.

Then by fact $(\diamond\,\diamond)$ above,

$$\begin{aligned}
J^{1+r}f(x_1 + \Delta) - J^{1+r}f(x_1) &= \frac{1}{x_1 + \Delta}\int_0^{x_1+\Delta} J^r f(t)dt - J^{1+r}f(x_1) \\
&= \frac{1}{x_1 + \Delta}\int_0^{x_1} J^r f(t)dt + \frac{1}{x_1 + \Delta}\int_{x_1}^{x_1+\Delta} J^r f(t)dt - J^{1+r}f(x_1) \\
&\geq \frac{x_1}{x_1 + \Delta}J^{1+r}f(x_1) + \frac{\Delta}{x_1 + \Delta}\tau d - J^{1+r}f(x_1) \\
&= \frac{\Delta}{x_1 + \Delta}\tau d - \frac{\Delta}{x_1 + \Delta}J^{1+r}f(x_1) \\
&\geq \frac{\Delta}{x_1 + \Delta}(\tau - \theta)d \ = \left(1 - \frac{x_1}{x_1 + \Delta}\right)(\tau - \theta)d
\end{aligned}$$

$$\geq \left(1 - \frac{x_1}{x_1 + x_1[(G_r)^{-1}(P_\epsilon) - 1]}\right)(\tau - \theta)d$$
$$= \left(1 - \frac{1}{(G_r)^{-1}(P_\epsilon)}\right)(\tau - \theta)d$$
$$= \left(1 - \frac{1}{(G_r)^{-1}\left(\frac{(\frac{1}{2} - \epsilon - \tau)d}{2\left(\tau d + \frac{\|f\|_\infty}{\Gamma(r)}\right)}\right)}\right)(\tau - \theta)d.$$

It follows from Sub-cases 1.a and 1.b that

$$\sup_{x \geq x_0} J^{1+r} f(x) - \inf_{y \geq x_0} J^{1+r} f(y) \geq U_\epsilon,$$

where

$$U_\epsilon := \min\left\{\theta d, \left(1 - \frac{1}{(G_r)^{-1}(P_\epsilon)}\right)(\tau - \theta)d\right\}, \text{ and } P_\epsilon := \frac{(\frac{1}{2} - \epsilon - \tau)d}{2\left(\tau d + \frac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

Case 2. $J^{1+r} f(x_0) \geq 0$.

Let $g := -f$. Then $g \in L^\infty$, and $\|g\|_\infty = \|f\|_\infty$. Let $x_0 \geq K$ be fixed and arbitrary.

Notice

$$\liminf_{x \to \infty} J^r g(x) = \liminf_{x \to \infty} -(J^r f(x)) = \limsup_{x \to \infty} J^r f(x) = -b \text{ and } \limsup_{x \to \infty} J^r g(x) = -a.$$

Thus $-a - (-b) = b - a = d$. Also $\frac{-b + -a}{2} = 0$. Further $J^{1+r} g(x_0) = -(J^{1+r} f(x_0)) \leq 0$.

So, we are back in Case 1, with f replaced by $g := -f$. Also replace θ by θ', and τ by τ'. Then,

$$\sup_{x \geq x_0} J^{1+r} f(x) - \inf_{y \geq x_0} J^{1+r} f(y) = \sup_{y \geq x_0} J^{1+r} g(y) - \inf_{x \geq x_0} J^{1+r} g(x) \geq U'_\epsilon,$$

where

$$U'_\epsilon := \min\left\{\theta' d, \left(1 - \frac{1}{(G_r)^{-1}(P'_\epsilon)}\right)(\tau' - \theta')d\right\}, \text{ and } P'_\epsilon := \frac{(\frac{1}{2} - \epsilon - \tau')d}{2\left(\tau' d + \frac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

It follows from Case 1 and Case 2 that for all real numbers $x_0 \geq K$,

$$\sup_{x \geq x_0} J^{1+r} f(x) - \inf_{y \geq x_0} J^{1+r} f(y) \geq U_\epsilon \wedge U'_\epsilon.$$

Letting $x_0 \to \infty$ we get that

$$\delta_{1+r} f := \limsup_{x \to \infty} J^{1+r} f(x) - \liminf_{x \to \infty} J^{1+r} f(x) \geq U_\epsilon \wedge U'_\epsilon.$$

But $\epsilon \in (0, (1/2 - \tau) \wedge (1/2 - \tau'))$ is completely arbitrary. So we can let $\epsilon \to 0^+$, and by continuity of G_r^{-1}, we get

$$\delta_{1+r} f = \limsup_{x \to \infty} J^{1+r} f(x) - \liminf_{x \to \infty} J^{1+r} f(x) \geq U \wedge U',$$

where

$$U := \min\left\{\theta, \left(1 - \frac{1}{(G_r)^{-1}(P)}\right)(\tau - \theta)\right\} d,\ P := P_{\tau,d,r,f} := \frac{(\frac{1}{2} - \tau)d}{2\left(d\tau + \frac{\|f\|_\infty}{\Gamma(r)}\right)},$$

and

$$U^{'} := \min\left\{\theta^{'}, \left(1 - \frac{1}{(G_r)^{-1}(P^{'})}\right)(\tau^{'} - \theta^{'})\right\} d, \text{ and } P^{'} := P^{'}_{\tau^{'},d,r,f} := \frac{(\frac{1}{2} - \tau^{'})d}{2\left(d\tau^{'} + \frac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

Recall that the $0 < \theta < \tau < 1/2$ and $0 < \theta^{'} < \tau^{'} < 1/2$ are fixed and arbitrary. Let $\theta^{'} = \theta$ and $\tau^{'} = \tau$. Then $U^{'} = U$. Then $\delta_{1+r} f \geq U$, and $U > 0$.

Fix $\tau \in (0, 1/2)$ arbitrary. Let $\theta = \frac{\tau}{2}$. Then $\tau - \theta = \frac{\tau}{2}$ also. Thus

$$U = \left(1 - \frac{1}{G_r^{-1}(P)}\right)\frac{\tau}{2} d.$$

For example, for $r = 1$, we have $G_1^{-1}(s) = s + 1$, for all $s \in [0, \infty)$. So

$$U = \left(1 - \frac{1}{P + 1}\right)\frac{\tau}{2} d.$$

Let $\tau = 1/4$. Then in this case $P = \dfrac{d/4}{d/2 + 2\|f\|_\infty}$, and $U = \left(\dfrac{d/4}{3d/4 + 2\|f\|_\infty}\right)\dfrac{d}{8}$.

Therefore,

$$\delta_2 f \geq U = \frac{d^2/32}{3d/4 + 2\|f\|_\infty} = \frac{d^2}{24d + 64\|f\|_\infty} > 0.$$

This is similar to the optimized lower bound for $\delta_2 f$ given in Theorem 3.1.

Note also that $d = 2b \leq 2\|f\|_\infty$. So $\delta_2 f \geq \dfrac{d^2}{112\|f\|_\infty}$.

Now, going back to the general situation $r \in (0, 1]$, we have that

$$\delta_{1+r} f \geq U := \left(1 - \frac{1}{G_r^{-1}(P)}\right)\frac{\tau}{2} d, \text{ where } P := \frac{(1/2 - \tau)d}{2\tau d + \frac{2\|f\|_\infty}{\Gamma(r)}}.$$

Since $d \leq 2\|f\|_\infty$, we have that $P \geq \dfrac{(1/2 - \tau)d}{4\tau\|f\|_\infty + \frac{2\|f\|_\infty}{\Gamma(r)}}$.

Let $Q := Q_{r,d,\|f\|_\infty,\tau} := \dfrac{(1/2 - \tau)d}{4\tau\|f\|_\infty + \frac{2\|f\|_\infty}{\Gamma(r)}}$. Then

$$\delta_{1+r} f \geq \left(1 - \frac{1}{G_r^{-1}(Q)}\right)\frac{\tau}{2} d = \frac{G_r^{-1}(Q) - 1}{G_r^{-1}(Q)}\frac{\tau}{2} d.$$

Also notice that $Q \leq \dfrac{(1/2 - \tau)2\|f\|_\infty}{4\tau\|f\|_\infty + \frac{2\|f\|_\infty}{\Gamma(r)}} = \dfrac{2(1/2 - \tau)}{4\tau + \frac{2}{\Gamma(r)}} =: \gamma(r, \tau)$, thus $\delta_{1+r} f \geq \dfrac{G_r^{-1}(Q) - 1}{G_r^{-1}(\gamma(r, \tau))}\dfrac{\tau}{2} d$.

Also, $Q := Q_{r,d,\|f\|_\infty,\tau} = \dfrac{\gamma(r, \tau)}{2}\dfrac{d}{\|f\|_\infty} \leq \gamma(r, \tau)$.

Fix $w_0 > 1$. For all $w \in (1, w_0]$,

$$G_r(w) := w\int_0^{\ln(w)} u^{r-1}e^{-u}du \leq w_0\int_0^{\ln(w)} u^{r-1}du = w_0\frac{(\ln(w))^r}{r}.$$

Consider the mapping $s = H_r(w) := w_0 \dfrac{(\ln(w))^r}{r}$, for $w \geq 1$. H_r is strictly increasing and continuous, and such that $H_r(1) = 0$. Therefore $H_r : [1, w_0] \to [0, H_r(w_0)]$ has an inverse, and $w = H_r^{-1}(s) = \exp\left(\left(\dfrac{rs}{w_0}\right)^{1/r}\right)$. So, we have that for $w \in (1, w_0]$,

$$w \leq G_r^{-1}(H_r(w)).$$

Now, for all $s \in (0, H_r(w_0)]$,

$$G_r^{-1}(s) - 1 \geq w - 1 = \exp\left(\left(\frac{rs}{w_0}\right)^{1/r}\right) - 1 \geq \left(\frac{rs}{w_0}\right)^{1/r}.$$

Recall that $0 < Q \leq \gamma := \gamma(r, \tau)$, and also $H_r^{-1} : [0, H_r(w_0)] \to [1, w_0]$. We wish to find $w_0 > 1$ such that $H_r(w_0) = \gamma$. If we define $V_r(w) := w \dfrac{(\ln(w))^r}{r}$, for all $w \in [1, \infty)$, then we know its inverse V_r^{-1} is well defined on $[0, \infty)$, since V_r strictly increasing to infinity. Then $H_r(w_0) = \gamma$ is equivalent to $w_0 = V_r^{-1}(\gamma)$. In this case, since we have that $Q \in (0, \gamma]$, then

$$G_r^{-1}(Q) - 1 \geq \left(\frac{rQ}{V_r^{-1}(\gamma)}\right)^{1/r} = \left(\frac{r\gamma d}{2V_r^{-1}(\gamma)\|f\|_\infty}\right)^{1/r}.$$

Thus

$$\delta_{1+r} f \geq \left(\frac{r\gamma}{2V_r^{-1}(\gamma)\|f\|_\infty}\right)^{1/r} \frac{1}{G_r^{-1}(\gamma)} \frac{\tau}{2} d^{1+1/r} > 0.$$

Now we have all the tools to prove the summarizing theorem of this section, which is the second main theorem of our paper. (Recall that Theorem 3.3 is the first of our main theorems.)

Theorem 3.4. *For any $0 < r, s$ we have that $Ces^r = Ces^s$.*

Proof. Without loss of generality assume $0 < r < s$. If $1 \leq r$, then by Corollary 3.2 we already have the conclusion. So, assume $0 < r \leq 1$ and $r < s$. We wish to show $Ces^s \subset Ces^r$, since the other inclusion is already known. Let $f \in Ces^s$, then $f \in Ces^{\lfloor s \rfloor + 1 + r}$, since $s < \lfloor s \rfloor + 1 + r$. This is equivalent to $J^r f \in Ces^{\lfloor s \rfloor + 1}$. By Theorem 3.1 this implies $J^r f \in Ces^1$, that is $f \in Ces^{1+r}$. Thus, by Theorem 3.3 we have that $f \in Ces^r$.

Remark. Notice that the previous result not only says that $Ces^r = Ces^s$ for any $s, r > 0$ but also $\lim_{x\to\infty} J^s f(x) = \lim_{x\to\infty} J^r f(x)$ for any $0 < r < s$ whenever one of these two limits exist. This is clear when we assume $f \in Ces^r$, since the operator J^{s-r} preserves limit at infinity. But also, notice that if we assume $\lim_{x\to\infty} J^s f(x) = L \in \mathbb{R}$, then $\lim_{x\to\infty} J^r f(x)$ exists, which implies $\lim_{x\to\infty} J^r f(x) = \lim_{x\to\infty} J^s f(x) = L$.

Remark. Notice that the function sine is an element of Ces^1, but sine $\notin BC_L$. So, $BC_L \subsetneq Ces^1$.

4 Banach limits on L^∞ invariant under J^r

In this section, we construct Banach limits on L^∞ that are invariant under our definition of fractional powers of the Cesàro averaging operator. That is, Banach limits Λ invariant under J^r, for each $r > 0$; and therefore these Banach limits preserve J^r-convergence. Given the results from the previous section, for $f \in L^\infty$ such that $f \in Ces^r$, for some $r > 0$, there exists $L \in \mathbb{R}$ such that $\lim_{x\to\infty} J^s f(x) = L$ for all $s > 0$. In this case, for any Banach limit Λ that is fractional-power-Cesàro-invariant, we have that $\Lambda(f) = L$.

We follow the same approach used in Claim 2.2, to construct the desired J^r-invariant Banach limits. We will make use of Ψ, the extension of the Cesàro limit functional, obtained through Hahn-Banach Theorem:

Recall the closed vector subspace of L^∞

$$Ces := \Big\{ g \in L^\infty : \psi(g) := \lim_{x\to\infty} Jg(x) \text{ exists in } \mathbb{R} \Big\}.$$

By Hahn-Banach Extension Theorem, there exists $\Psi \in (L^\infty)^*$ such that

$$\|\Psi\|_{(L^\infty)^*} = \|\psi\|_{Ces^*} = 1 \text{ and } \Psi\mid_{Ces} = \psi.$$

Definition 4.1. For each $f \in L^\infty$, we define the map $\Gamma_f : (0,\infty) \to \mathbb{R}$ by

$$\Gamma_f(r) := \Psi(J^r f), \text{ for all } r \in (0,\infty)\ .$$

Claim 4.1. *For each $f \in L^\infty$, $\Gamma_f \in BC(0,\infty) \subseteq L^\infty(0,\infty)$.*

Proof. Fix $f \in L^\infty$. We already establish on Theorem 2.10 that the map $r \mapsto J^r f$ is continuous, therefore the composition $\Psi(J^r f)$ is also a continuous function of r, since Ψ is a continuous functional.

We see then $\Gamma_f \in BC(0,\infty)$ since for all $r \in (0,\infty)$, we have that

$$|\Gamma_f(r)| \leq \|\Psi\|_{(L^\infty)^*} \|J^r\|_{op} \|f\|_\infty = \|f\|_\infty < \infty.$$

In analogy with Claim 2.2 we present the following result, which is the third main theorem of our paper. (Recall that Theorems 3.3 and 3.4 are our other main theorems.)

Theorem 4.2. *Define $\Lambda : L^\infty \to \mathbb{R}$ by*

$$\Lambda(f) := \Psi\Gamma_f, \text{ for every } f \in L^\infty.$$

Λ is a Banach limit in L^∞ that is invariant under fractional powers of the Cesàro operator, i.e. $\Lambda(J^r f) = \Lambda(f)$, for all $f \in L^\infty$, for all $r > 0$.

Proof. Fix $f \in L^\infty$. Fix $r_0 > 0$. We first check that Λ in fact is invariant under fractional powers of the Cesàro averaging operator:

Recall that for every $r \in [0,\infty)$, we define the left shift operator by r on L^∞, by $S_r f(\cdot) := f(\cdot + r) \in L^\infty$, for every $f \in L^\infty$. Also recall Ψ is left-shift invariant, since it is a Banach limit. So,

$$\begin{aligned}
\Lambda(J^{r_0} f) &= \Psi(r \mapsto \Psi(J^r J^{r_0} f)) \ = \Psi(r \mapsto \Psi(J^{r_0+r} f)) \\
&= \Psi(r \mapsto \Gamma_f(r_0 + r)) \ = \Psi(r \mapsto (S_{r_0}\Gamma_f)(r)) \\
&= \Psi(S_{r_0}\Gamma_f) \ = \Psi(\Gamma_f) \ = \Lambda(f).
\end{aligned}$$

It is easy to check that $\|\Lambda\|_{(L^\infty)^*} = 1$. It can also be checked that if $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$ then $\Lambda(f) = L$. (See A.8.)

Next, we verify that $\Lambda(f) = \Lambda(S_r f)$ for all $r > 0$: Notice that $\Lambda(S_r f) = \Psi(s \mapsto \Psi(J^s S_r f))$. We claim that for any $f \in L^\infty$, we have that $\Psi(J^s S_r f) = \Psi(J^s f)$ for all $r > 0$ and all $s > 0$. This holds since we know that $\lim_{x\to\infty} J(S_r f - f) = 0$, and by the remark made after Theorem 3.4, this implies that $\lim_{x\to\infty} J^s(S_r f - f) = 0$ for any $s > 0$. Therefore,

$$\Psi(J^s S_r f) = \Psi(J^s f), \text{ for all } r > 0 \text{ and all } s > 0.$$

And so,

$$\begin{aligned}
\Lambda(S_r f) &= \Psi(s \mapsto \Psi(J^s S_r f)) \ = \Psi(s \mapsto \Psi(J^s f)) \ = \Psi(s \mapsto \Gamma_f(s)) \\
&= \Psi(\Gamma_f) \ = \Lambda(f).
\end{aligned}$$

Remark. We can modify Theorem 4.2 to get a more general one, using the definition of Λ below:

Theorem 4.2* Fix Δ and T any two Banach limits in L^∞. Then $\Lambda : L^\infty \to \mathbb{R}$ defined by

$$\Lambda(f) := T(r \mapsto \Delta(J^r f)).$$

is a Banach limit that is invariant under fractional powers of the Cesàro averaging operator.

Open Question 4.3. *It was proven in [17, lemma II.27], that the Pascal Operator defined on ℓ^∞ commutes with the Cesàro averaging operator. In fact, along with the left shift operator and the identity operator, they generate an abelian semigroup of linear operators on $\ell^\infty/_{c_0}$. Then, using a generalization of the Hahn Banach Extension Theorem [13, proposition 5, chapter 10, section 3], it was shown the existence of a Banach limit invariant under any number of compositions of these operators ([17, lemma II.29]).*

What could be an analogue definition of the Pascal Operator P for the space $L^\infty(0,\infty)$? Can continuous iterates of this operator be defined? If this new operator, and its iterations, commute with our definition of iterates of Cesàro averaging, one could use the generalization of the Hahn Banach Extension Theorem to get the existence of Banach limits invariant under compositions of J^r and P^s, for any $r, s > 0$.

Open Question 4.4. *The quantitative result from Theorem 3.3 could be improved by choosing τ in an appropriate way. What value of τ would give us an optimal lower bound for the inequality of this theorem?*

5 A Banach limit in L^∞ that preserves Cesàro convergence, but is not Cesàro invariant

We start by stating the following result derived from the Bohnenblust-Sobczyk version of The Hahn-Banach extension Theorem for seminorms [19, chapter IV, section 4]:

Theorem 5.1. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let V be a vector subspace of X, and $a, z \in X$. Consider the affine subspace M generated by a and z, that is, $M := \{(1-t)z + ta : t \in \mathbb{K}\}$. Assume there is a positive distance from M to V, that is, $\eta := d(M, V) := \inf\{\|(1-t)z + ta - v\| : t \in \mathbb{K}, v \in V\} > 0$. Then, there exists $\Psi \in X^*$ such that $\|\Psi\|_{X^*} = 1$, $\Psi|_V = 0$, and $\Psi(z) = \Psi(a) = \eta$.*

The proof of this theorem can be found in [2, theorem A.0.2].

To show the existence of a Banach limit Ψ in L^∞, such that Ψ is not Cesàro invariant, but preserves Cesàro convergence, we will first construct a special element $f \in L^\infty$. Later on, a constant times the function $Jf - f$ will play the role of a when applying Theorem 5.1, to obtain this desired Banach limit.

We construct $f \in L^\infty$ the following way: Choose $N_1 > 1$, choose N_2 such that $N_2 > 2N_1$, and choose N_3 such that $N_3 > 2N_2$, and such that $\frac{N_1}{N_3} < \frac{1}{2}$. For $k \in \mathbb{N}$, $k > 1$, inductively we choose N_{2k} such that $N_{2k} > 2N_{2k-1}$, and N_{2k+1} such that $N_{2k+1} > 2N_{2k}$, and such that $\dfrac{N_{2k-1}}{N_{2k+1}} < \dfrac{1}{2k}$ (e.g. define $N_j := (j+1)!$ for every $j \in \mathbb{N}$). Then, we have that

$$1 < N_1 < 2N_1 < N_2 < 2N_2 < N_3 < 2N_3 < N_4 < 2N_4 < N_5 < \ldots.$$

Define f by

$$f(x) := \begin{cases} 1, & x \in (0, N_1] \cup (2N_1, N_3] \cup (2N_3, N_5] \cup \ldots \\ 0, & x \in (N_1, 2N_1] \cup (N_3, 2N_3] \cup (N_5, 2N_5] \cup \ldots. \end{cases}$$

We notice that, for any $x > 0$, $Jf(x) \leq 1$, since $\|f\|_\infty = 1$. Consequently,

$$\frac{1}{x}\int_0^x f(t)dt - f(x) = \frac{1}{x}\int_0^x f(t)dt - 1 \leq 0, \text{ for all } x \in (N_{2k}, 2N_{2k}], \text{ for any } k \in \mathbb{N}.$$

For all $x \in (N_{2k+1}, 2N_{2k+1})$ for all $k \in \mathbb{N}$, we have that $x = N_{2k+1} + y$, for $y = x - N_{2k+1} \in (0, N_{2k+1})$. Hence

$$\begin{aligned}\frac{1}{x}\int_0^x f(t)dt - f(x) &= \frac{N_{2k+1} - N_{2k-1} - \cdots - N_1}{N_{2k+1} + y} - 0 \\ &> \frac{N_{2k+1} - N_{2k-1} - \cdots - N_1}{2N_{2k+1}} \\ &= \frac{1}{2} - \frac{1}{2}\frac{N_{2k-1} + \cdots + N_1}{N_{2k+1}} \\ &> \frac{1}{2} - \frac{1}{2}\frac{kN_{2k-1}}{N_{2k+1}} \\ &> \frac{1}{4}.\end{aligned}$$

Now, we apply Theorem 5.1:

Let $g(x) := 4\left(\frac{1}{x}\int_0^x f(t)dt - f(x)\right) \in L^\infty$. Let $h(x) := \mathbb{1}(x) \in L^\infty$. Let $V := Ces_0 := \{f \in L^\infty : \lim_{x\to\infty} Jf(x) = 0\}$. Let $M := \{(1-t)h + tg : t \in \mathbb{R}\}$.

Claim 5.2. *$d(M, V) := \inf\{\|(1-t)h + tg - j\|_\infty : t \in \mathbb{R}$ and $j \in Ces_0\} = 1$.*

Proof. First notice that, for the particular function $j(x) := 0$ for all $x > 0$, and for the particular value $t = 0$, we have that

$$d(M, V) := \inf\{\|(1-t)h + tg - j\|_\infty : t \in \mathbb{R} \text{ and } j \in Ces_0\} \leq \|h\|_\infty = 1.$$

On the other hand, fix arbitrary $t \geq 0$ and $j \in Ces_0$. Also fix $k \in \mathbb{N}$ and $x \in (N_{2k+1}, 2N_{2k+1})$.

$$\begin{aligned}\Delta := \|(1-t)h + tg - j\|_\infty &\geq |(1-t)h(x) + tg(x) - j(x)| \\ &\geq (1-t)\cdot 1 + t\cdot 4(Jf(x) - f(x)) - j(x) \\ &\geq 1 - t + t\cdot 1 - j(x) = 1 - j(x).\end{aligned}$$

Therefore

$$\begin{aligned}\Delta = \frac{1}{N_{2k+1}}\int_{x=N_{2k+1}}^{x=2N_{2k+1}} \Delta dx &\geq \frac{1}{N_{2k+1}}\int_{x=N_{2k+1}}^{x=2N_{2k+1}} (1 - j(x))dx \\ &= 1 - \frac{1}{N_{2k+1}}\int_{x=N_{2k+1}}^{x=2N_{2k+1}} j(x)dx \\ &= 1 - \frac{1}{N_{2k+1}}\left(\int_{x=0}^{x=2N_{2k+1}} j(x)dx - \int_{x=0}^{x=N_{2k+1}} j(x)dx\right).\end{aligned}$$

This implies

$$\Delta \geq 1 - 2Jj(2N_{2k+1}) + Jj(N_{2k+1}) \to 1 - 0 + 0, \text{ as } k \to \infty, \text{ since } j \in Ces_0.$$

Thus, for any $t \geq 0$, for any $j \in Ces_0$,

$$\|(1-t)h + tg - j\|_\infty \geq 1.$$

Next, fix $t < 0$ and $j \in Ces_0$. Also, fix $k \in \mathbb{N}$ and $x \in (N_{2k}, 2N_{2k})$.

$$\begin{aligned}\Delta := \|(1-t)h + tg - j\|_\infty &\geq (1-t)\cdot 1 + t\cdot 4(Jf(x) - f(x)) - j(x) \\ &\geq 1 - t - j(x) \geq 1 - j(x).\end{aligned}$$

Consequently, similarly to above, we get that

$$\Delta \geq 1 - 2Jj(2N_{2k}) + Jj(N_{2k}) \to 1, \text{ as } k \to \infty, \text{ since } j \in Ces_0.$$

Hence, for all $t < 0$ and for all $j \in Ces_0$,

$$\|(1-t)h + tg - j\|_\infty \geq 1.$$

So, we see that

$$\inf\{\|(1-t)h + tg - j\|_\infty : t \in \mathbb{R} \textit{ and } j \in Ces_0\} \geq 1.$$

This implies that $d(M, V) = 1$.

Thus, by Theorem 5.1, we know there exists Ψ in $(L^\infty)^*$, with $\|\Psi\|_{(L^\infty)^*} = 1$, and: (i) $\Psi|_{Ces_0} = 0$, (ii) $\Psi(h) = \Psi(\mathbb{1}) = 1$, and (iii) $\Psi(g) = 1$.

Notice that (iii) implies Ψ is not Cesàro invariant, since for the function f constructed above, we have that

$$\Psi(Jf - f) = \tfrac{1}{4}\Psi(g) = \tfrac{1}{4} \neq 0.$$

On the other hand, (i) and (ii) imply that Ψ is Cesàro convergence preserving, since for any $g \in Ces$, we have that $g = L\mathbb{1} + j$, where $L := \lim_{x\to\infty} Jg(x)$, and $j \in Ces_0$ (simply define $j(x) := g(x) - L\mathbb{1}(x)$, for all $x > 0$). Therefore

$$\Psi(g) = \Psi(L\mathbb{1} + j) = L.$$

Finally, to verify Ψ is a Banach limit, first notice that for any $f \in BC_L$, we have that $f \in Ces$, and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} Jf(x)$. So we see that Ψ preserves classical convergence. Also recall that $\|\Psi\|_{(L^\infty)^*} = 1$. We also have that Ψ is left-shift invariant, since we know that for any $r > 0$ we have $\{f - S_r f : f \in L^\infty\} \subseteq Ces_0$. Hence $\Psi(f - S_r f) = 0$, for any $r > 0$ and $f \in L^\infty$.

A Appendix

Claim A.1. *Δ satisfies property 1 and 2 from the definition a Banach limit.*

Proof. Take an arbitrary $f \in L^\infty$.

$$\begin{aligned}
|\Delta(f)| &\leq \|\sigma\|_{(\ell^\infty)^*}\|(\Psi(f), \Psi(Jf), \Psi(J^2 f), \Psi(J^3 f), \ldots)\|_\infty \\
&= 1\|(\Psi(f), \Psi(Jf), \Psi(J^2 f), \Psi(J^3 f), \ldots)\|_\infty \\
&= \sup\{|\Psi(f)|, |\Psi(Jf)|, |\Psi(J^2 f)|, |\Psi(J^3 f)|, \ldots\} \\
&\leq \sup\{\|\Psi\|_{(L^\infty)^*}\|f\|_\infty, \|\Psi\|_{(L^\infty)^*}\|Jf\|_\infty, \|\Psi\|_{(L^\infty)^*}\|J^2 f\|_\infty, \ldots\} \\
&= \sup\{\|f\|_\infty, \|Jf\|_\infty, \|J^2 f\|_\infty, \ldots\}.
\end{aligned}$$

Since $\|Jf\|_\infty \leq \|f\|_\infty$ for every $f \in L^\infty$, we have that $\|J^n f\|_\infty \leq \|f\|_\infty$, for all $n \in \mathbb{N}$ and all $f \in L^\infty$. Therefore

$$|\Delta(f)| \leq \|f\|_\infty \text{ for all } f \in L^\infty,$$

which implies $\|\Delta\|_{(L^\infty)^*} \leq 1$. On the other hand, let $\mathbb{1}(x) = 1$, for all $x \in (0, \infty)$. Then,

$$\Delta(\mathbb{1}) = \sigma(\Psi(\mathbb{1}), \Psi(\mathbb{1}), \Psi(\mathbb{1}), \Psi(\mathbb{1}), \ldots) = \sigma(1, 1, 1, \ldots) = 1.$$

Therefore $\|\Delta\|_{(L^\infty)^*} \geq 1$. Thus $\|\Delta\|_{(L^\infty)^*} = 1$.

Next, let $f \in BC_L$ and let $L := \lim_{x\to\infty} f(x)$, then $\lim_{x\to\infty} Jf(x) = L$, and therefore $\lim_{x\to\infty} J^n f(x) = L$, for each $n \in \mathbb{N}$. This implies that $\Psi(J^n f) = L$, for each $n \in \mathbb{N}$. So, we get that

$$\Delta(f) = \sigma(L, L, L, \ldots) = L.$$

Claim A.2. *For $n \in \mathbb{N}$, the nth iteration of applying the Cesàro averaging operator to a function $f \in L^\infty$ is given by the following formula:*

$$\int_0^x \frac{dt_1}{x}\int_0^{t_1}\frac{dt_2}{t_1}\cdots\int_0^{t_{n-1}} f(t_n)\frac{dt_n}{t_{n-1}} = \frac{1}{(n-1)!}\frac{1}{x}\int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{n-1} f(t)dt.$$

Proof. We proceed by induction on n:
If $n = 1$, then

$$\int_0^x f(t_1)\frac{dt_1}{x} = \frac{1}{0!}\frac{1}{x}\int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^0 f(t)dt.$$

Next, fix $n \in \mathbb{N}$, $n > 1$, and let

$$J^{(n-1)}f(x) := \int_0^x \frac{dt_1}{x} \int_0^{t_1} \frac{dt_2}{t_1} \cdots \int_0^{t_{n-2}} f(t_{n-1}) \frac{dt_{n-1}}{t_{n-2}}.$$

Assume

$$J^{(n-1)}f(x) = \frac{1}{(n-2)!} \frac{1}{x} \int_0^x \left[\ln\left(\frac{x}{t}\right) \right]^{n-2} f(t) dt.$$

Then

$$\begin{aligned} J^n f(x) &:= J(J^{(n-1)}f)(x) \\ &= \frac{1}{x} \int_0^x \frac{1}{(n-2)!} \frac{1}{u} \int_0^u \left[\ln\left(\frac{u}{t}\right) \right]^{n-2} f(t) dt du \\ &= \frac{1}{x} \frac{1}{(n-2)!} \int_0^x \int_t^x \frac{1}{u} \left[\ln\left(\frac{u}{t}\right) \right]^{n-2} f(t) du dt. \end{aligned}$$

In this last step, we were able to apply Fubini-Tonelli to change the order of integration since the integrand is a measurable function, and

$$\begin{aligned} \frac{1}{x} \int_0^x \int_0^u \frac{1}{u} \left| \left[\ln\left(\frac{u}{t}\right) \right]^{n-2} f(t) \right| dt du &\leq \|f\|_\infty \frac{1}{x} \int_0^x \int_0^u \frac{1}{u} \left[\ln\left(\frac{u}{t}\right) \right]^{n-2} dt du \\ &= \|f\|_\infty \frac{1}{x} \int_0^x \int_0^\infty \frac{1}{u} y^{n-2} u e^{-y} dy du \\ &= \|f\|_\infty \frac{1}{x} \int_0^x \Gamma(n-1) du \\ &= \|f\|_\infty \Gamma(n-1) < \infty. \end{aligned}$$

Here, we made the substitution $y = \ln\left(\frac{u}{t}\right)$. We will continue to make use of this substitution throughout this document. We also note that the expression $\Gamma(n-1)$ is the Gamma function evaluated at $n-1$, which equals $(n-2)!$, and therefore is finite.

So, after applying Fubini-Tonelli we obtain

$$\begin{aligned} J^n f(x) &:= J(J^{(n-1)}f)(x) \\ &= \frac{1}{x} \frac{1}{(n-2)!} \int_0^x \int_t^x \frac{1}{u} \left[\ln\left(\frac{u}{t}\right) \right]^{n-2} f(t) du dt \\ &= \frac{1}{x} \frac{1}{(n-2)!} \int_0^x \int_0^{\ln\left(\frac{x}{t}\right)} [y]^{n-2} f(t) dy dt \\ &= \frac{1}{x} \frac{1}{(n-1)!} \int_0^x \left[\ln\left(\frac{x}{t}\right) \right]^{n-1} f(t) dt. \end{aligned}$$

Where again we made the substitution $y = \ln\left(\frac{u}{t}\right)$.

Claim A.3. *Fix $x > 0$. Fix $p, r > 0$. Then*

$$\int_{s=0}^x \int_{u=0}^1 u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right) \right]^{p+r-1} ds = \int_{t=0}^x \frac{1}{t} \int_{s=0}^t f(s) \left[\ln\left(\frac{t}{s}\right) \right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right) \right]^{p-1} dt.$$

Proof. This can be proven by applying Fubini-Tonelli:

Denote

$$I_1 := \int_{s=0}^x \int_{u=0}^1 u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right) \right]^{p+r-1} ds,$$

and

$$I_2 := \int_{t=0}^x \frac{1}{t} \int_{s=0}^t f(s) \left[\ln\left(\frac{t}{s}\right) \right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right) \right]^{p-1} dt.$$

For I_2, we make the substitution $u = \frac{t}{x}$, to get

$$\begin{aligned}
&\int_{t=0}^{x} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt \\
&= \int_{u=0}^{1} \frac{1}{u} \int_{s=0}^{xu} f(s) \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} \left[\ln\left(\frac{1}{u}\right)\right]^{p-1} dsdu \\
&= \int_{s=0}^{x} f(s) \int_{u=\frac{s}{x}}^{1} \frac{1}{u} \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} \left[\ln\left(\frac{1}{u}\right)\right]^{p-1} duds.
\end{aligned}$$

We were able to change the order of integration since the function

$$g(s,u) := \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} \frac{(-\ln(u))^{p-1}}{u} \chi_{[0,ux]}(s)$$

is continuous a.e. on $[0,\infty) \times [0,1]$, f is measurable, therefore $g(s,u)f(s)$ is measurable. Also, we have that

$$\begin{aligned}
&\int_{u=0}^{1} \int_{s=0}^{ux} |f(s)| \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} (-\ln(u))^{p-1} ds \frac{1}{u} du \\
&\leq \|f\|_\infty \int_{u=0}^{1} \int_{s=0}^{ux} \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} ds (-\ln(u))^{p-1} \frac{1}{u} du \\
&= \|f\|_\infty \int_{u=0}^{1} xu\Gamma(r)(-\ln(u))^{p-1} \frac{1}{u} du \\
&= \|f\|_\infty x\Gamma(r) \int_{u=0}^{1} (-\ln(u))^{p-1} du \\
&= \|f\|_\infty x\Gamma(r)\Gamma(p) < \infty;
\end{aligned}$$

where the Gamma function evaluated at r, $\Gamma(r)$, is obtained by making the substitution $y = \ln\left(\frac{xu}{s}\right)$ and the Gamma function evaluated at p, $\Gamma(p)$, is obtained by making the substitution $y = \ln\left(\frac{1}{u}\right)$.

Next, we rewrite I_1 as

$$\begin{aligned}
&\int_{s=0}^{x} \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds \\
&= \int_{s=0}^{x} f(s) \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} duds.
\end{aligned}$$

Let

$$Q(s,x) := \int_{u=\frac{s}{x}}^{1} \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} \frac{(-\ln(u))^{p-1}}{u} du,$$

which is the inner integral on I_2. Therefore, it is enough to show that

$$Q(s,x) = \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} du.$$

Notice

$$\begin{aligned}
Q(s,x) &= \int_{u=\frac{s}{x}}^{1} \left[\ln\left(\frac{x}{s}\right) - (-\ln(u))\right]^{r-1} \frac{(-\ln(u))^{p-1}}{u} du \\
&= \int_{u=\frac{s}{x}}^{1} \left[\ln\left(\frac{x}{s}\right)\right]^{r-1+p} \left[1 - \frac{(-\ln(u))}{\ln\left(\frac{x}{s}\right)}\right]^{r-1} \left(\frac{-\ln(u)}{\ln\left(\frac{x}{s}\right)}\right)^{p} \frac{1}{-\ln(u)} \frac{du}{u} \\
&= \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} \int_{1}^{0} (1-q)^{r-1} q^{p} \frac{-1}{q} dq.
\end{aligned}$$

Where $q = \dfrac{-\ln(u)}{\ln\left(\frac{x}{s}\right)}$, therefore $dq = \dfrac{-1}{u} \dfrac{1}{\ln\left(\frac{x}{s}\right)} du$ and so $\dfrac{1}{u(-\ln(u))} du = \dfrac{-1}{q} dq$. Thus we get the desired result.

Claim A.4. *Fix $x > 0$, and define*

$$\varphi_\epsilon(t) := \tfrac{1}{x}\tfrac{1}{\Gamma(\epsilon)} \frac{1}{\left[\ln\left(\dfrac{x}{x-t}\right)\right]^{1-\epsilon}} \chi_{[0,x]}(x-t), \text{ for } t \in \mathbb{R}.$$

φ_ϵ is a good kernel, according to the definition in [14, chapter 3, section 2].

Proof. To prove φ_ϵ is a good kernel according to the definition in [14], we need to show the following:

1. $\displaystyle\int_{-\infty}^{\infty} \varphi_\epsilon(t)dt = 1.$
2. $\displaystyle\int_{-\infty}^{\infty} |\varphi_\epsilon(t)|dt \leq A$, for some constant A independent of ϵ.
3. for any $\nu > 0$, $\displaystyle\int_{|x|\geq\nu} \varphi_\epsilon(t)dt$ tends to 0 as $\epsilon \to 0^+$.

This will imply that

$$g * \varphi_\epsilon(x) \to g(x) = f(x) \text{ as } \epsilon \to 0^+, \text{ at every point } x \text{ of continuity of } f.$$

We first check $\displaystyle\int_{-\infty}^{\infty} \varphi_\epsilon(t)dt = 1$:

$$\int_{-\infty}^{\infty} \varphi_\epsilon(t)dt = \frac{1}{x\Gamma(\epsilon)} \int_0^x \frac{1}{\left[\ln\left(\dfrac{x}{x-t}\right)\right]^{1-\epsilon}} dt.$$

Let $s = \ln\left(\frac{x}{x-t}\right)$, then

$$\int_{-\infty}^{\infty} \varphi_\epsilon(t)dt = \frac{1}{\Gamma(\epsilon)} \int_0^\infty s^{\epsilon-1}e^{-s}ds = \frac{\Gamma(\epsilon)}{\Gamma(\epsilon)} = 1.$$

Thus the first condition holds.

Next notice that $\varphi_\epsilon(t) \geq 0$, and so $\displaystyle\int_{-\infty}^{\infty} |\varphi_\epsilon(t)|dt = 1.$

Finally, notice that for any $\nu \in (0, x)$, by making again the substitution $s = \ln\left(\frac{x}{x-t}\right)$ we obtain

$$\frac{1}{x\Gamma(\epsilon)} \int_\nu^x \frac{1}{\left[\ln\left(\dfrac{x}{x-t}\right)\right]^{1-\epsilon}} dt = \frac{1}{\Gamma(\epsilon)} \int_{\ln\left(\frac{x}{x-\nu}\right)}^\infty s^{\epsilon-1}e^{-s}ds.$$

Let $R := \ln\left(\frac{x}{x-\nu}\right)$, notice $R > 0$. Since we are going to let ϵ tend to 0, we may assume $\epsilon < 1$. Then for $s \in (R, \infty)$ we have that $s^{\epsilon-1} \leq R^{\epsilon-1}$. Therefore

$$\int_{\ln\left(\frac{x}{x-\nu}\right)}^\infty s^{\epsilon-1}e^{-s}ds \leq R^{\epsilon-1} \int_R^\infty e^{-s}ds = R^{\epsilon-1}e^{-R}.$$

Thus,

$$\int_{(-\infty,-\nu)\cup(\nu,\infty)} \varphi_\epsilon(t)dm(t) \text{ tends to 0 as } \epsilon \to 0^+, \text{ since } \tfrac{1}{\Gamma(\epsilon)} \to 0.$$

Therefore the desired conclusion holds.

Claim A.5. *Let $f \in L^\infty$. Then*

$$\liminf_{x\to\infty} f(x) \leq \liminf_{x\to\infty} (J^p f)(x) \leq \limsup_{x\to\infty} (J^p f)(x) \leq \limsup_{x\to\infty} f(x),$$

for all $p > 0$.

Proof. Let $f \in L^\infty$. Let $\ell := \liminf_{x\to\infty} f(x)$. This means the following: For every $\epsilon > 0$, there exists $N > 0$ such that $\ell - \epsilon \leq f(x)$ for almost all $x > N$.

Fix $p > 0$. For each $x > 0$ we have that

$$\begin{aligned}
\ell - (J^p f)(x) &= \ell - \frac{1}{x\Gamma(p)} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} f(t)dt \\
&= \frac{1}{x\Gamma(p)} \left(\ell x\Gamma(p) - \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} f(t)dt\right) \\
&= \frac{1}{x\Gamma(p)} \left(\ell x \int_0^\infty u^{p-1}e^{-u}du - \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} f(t)dt\right) \\
&= \frac{1}{\Gamma(p)} \left(\ell \int_0^\infty u^{p-1}e^{-u}du - \int_0^\infty [u]^{p-1} f\left(\frac{x}{e^u}\right) e^{-u}du\right) \\
&= \frac{1}{\Gamma(p)} \int_0^\infty \left(\ell - f\left(\frac{x}{e^u}\right)\right) u^{p-1}e^{-u}du.
\end{aligned}$$

Where the previous to last expression comes from the usual substitution $u = \ln\left(\frac{x}{t}\right)$.

Now, fix $\epsilon > 0$. Since $\Gamma(p) < \infty$, there is $S > 0$ such that $\int_S^\infty u^{p-1}e^{-u}du < \frac{\Gamma(p)}{2\|f\|_\infty}\epsilon$. In fact, for any $R > S$, we have that

$$\int_R^\infty u^{p-1}e^{-u}du < \int_S^\infty u^{p-1}e^{-u}du < \frac{\Gamma(p)}{2\|f\|_\infty}\epsilon.$$

Let $x > Ne^S$, then $S < \ln\left(\frac{x}{N}\right)$. Next, choose R such that $R \in (S, \ln(\frac{x}{N}))$. Notice that, if $u < R$, then

$$\frac{x}{e^u} > \frac{x}{e^R} > \frac{x}{e^{\ln(x/N)}} = N, \text{ and so } \ell - \epsilon \leq f\left(\frac{x}{e^u}\right).$$

Consequently,

$$\begin{aligned}
\ell - (J^p f)(x) &= \frac{1}{\Gamma(p)} \left(\int_0^R \left(\ell - f\left(\frac{x}{e^u}\right)\right) u^{p-1}e^{-u}du + \int_R^\infty \left(\ell - f\left(\frac{x}{e^u}\right)\right) u^{p-1}e^{-u}du\right) \\
&\leq \frac{1}{\Gamma(p)}\epsilon \int_0^R u^{p-1}e^{-u}du + \frac{2\|f\|_\infty}{\Gamma(p)} \int_R^\infty u^{p-1}e^{-u}du \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

As $\epsilon \to 0^+$, we obtain

$$\ell \leq \liminf_{x\to\infty} (J^p f)(x).$$

Similarly we can verify that

$$\limsup_{x\to\infty} (J^p f)(x) \leq \limsup_{x\to\infty} f(x).$$

Claim A.6. *Let $p \in (0,1)$. Consider a sequence of positive real numbers $(p_n)_n$ such that $p_n \in (0,1)$ for each $n \in \mathbb{N}$, and $p_n \to p$. Then*

$$\lim_{n\to\infty} \int_{u=0}^\infty \left|[u]^{p_n-1} - [u]^{p-1}\right| e^{-u}du = 0.$$

Proof. We apply Dominated Convergence Theorem:

Since $p_n - 1 \to p - 1$ and $p - 1 < 0$, there exists $N \in \mathbb{N}$ such that $p - 1 - \frac{p}{2} \leq p_n - 1 < 0$ for all $n \geq N$.

If $u \geq 1$ then we have that $\left|u^{p_n-1} - u^{p-1}\right| e^{-u} \leq (u^{p_n-1} + u^{p-1})e^{-u} \leq 2e^{-u}$ for all $n \geq N$.

If $u \in (0,1)$, then u^y is decreasing as function of y. Then $u^{p_n-1} \leq u^{\frac{p}{2}-1}$ for all $n \geq N$, and so

$$\left|u^{p_n-1} - u^{p-1}\right| e^{-u} \leq \left|u^{p_n-1} - u^{p-1}\right| \leq u^{p_n-1} + u^{p-1} \leq u^{\frac{p}{2}-1} + u^{p-1}.$$

Define

$$g(u) = \begin{cases} u^{\frac{p}{2}-1} + u^{p-1}, & 0 < u < 1 \\ 2e^{-u}, & 1 \le u \end{cases}$$

Notice

$$\int_0^\infty g(u)du = \int_0^1 \left(u^{\frac{p}{2}-1} + u^{p-1}\right) du + 2\int_1^\infty e^{-u} du.$$

We know this integral is finite since $p-1,\ \frac{p}{2}-1 \in (-1,0)$. Therefore we can apply Dominated Convergence Theorem and reach the desired conclusion.

Claim A.7. $\|\psi_r\|_{(Ces^r)^*} = 1$

Proof. Recall that we already know $\|J^r\|_{op} = 1$, therefore for any $f \in L^\infty$ we have that

$$|\psi_r(f)| = |\lim_{x\to\infty}(J^r f)(x)| = \lim_{x\to\infty}|(J^r f)(x)| \le \|f\|_\infty.$$

So $\|\psi_r\|_{(Ces^r)^*} \le 1$. Also, $\psi_r(\mathbb{1}) = \lim_{x\to\infty}(J^r\mathbb{1})(x) = \lim_{x\to\infty}\mathbb{1}(x) = 1$. Thus $\|\psi_r\|_{(Ces^r)^*} = 1$.

Claim A.8. Λ *satisfies property 1 and property 2 of the definition of Banach limit.*

Proof. Notice that for any $f \in L^\infty$ we have that $|\Lambda(f)| \le \|\Psi\|_{(L^\infty)^*}\|\Gamma_f\|_\infty \le \|f\|_\infty$. Therefore, $\|\Lambda\|_{(L^\infty)^*} \le 1$. On the other hand,

$$\Lambda(\mathbb{1}) = \Psi(r \mapsto \Psi(J^r\mathbb{1})) = \Psi(r \mapsto \Psi(\mathbb{1})) = \Psi(r \mapsto 1) = \Psi(\mathbb{1}) = 1.$$

Thus $\|\Lambda\|_{(L^\infty)^*} = 1$.

Next, we check that if $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$ then $\Lambda(f) = L$. Recall that if $\lim_{x\to\infty} f(x) = L$ then $\lim_{x\to\infty}(J^r f)(x) = L$ for all $r > 0$. Therefore,

$$\begin{aligned}\Lambda(f) &= \Psi(r \mapsto \Psi(J^r f)) \ = \Psi(r \mapsto L)\\ &= \Psi(L\mathbb{1}) \ = L.\end{aligned}$$

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