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Understanding the properties of operations: a cross-cultural analysis

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ABSTRACT

This study examines how sampled Chinese and U.S. third and fourth grade students ($N_{\text{China}} = 167$, $N_{\text{US}} = 97$) understand the commutative, associative, and distributive properties. These students took both pre- and post-tests conducted at the beginning and end of a school year. Comparisons between students' pre- and post-tests within and across countries indicate different learning patterns. Overall, Chinese students demonstrate a much better understanding than their U.S. counterparts. Among these properties, the associative and distributive properties appear to be most challenging, especially for the U.S. students. By the end of grade 4, some Chinese students demonstrate explicit understanding of the associative and distributive properties across tasks; almost no U.S. students achieve a comparable level of understanding on these properties. Student understanding in different contexts also reveals cross-cultural differences. Chinese students tend to reason upon concrete contexts for sense-making, which is rare with U.S. students. Finally, the Chinese student sample shows clear growth of understanding across grades, but this is not seen in the U.S. sample. This understanding gap between the two countries is found to dramatically increase over time. Implications are discussed.

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KEYWORDS
Properties of operations; student understanding; cross-cultural analysis

1. Introduction

The commutative, associative, and distributive properties (CP, AP, and DP, respectively) are at the heart of mathematics because these basic properties allow tremendous flexibility in doing arithmetic, serve as fundamentals when working with equations, and provide a foundation for generalizations and proofs (Bruner, 1977; Carpenter, Franke, & Levi, 2003; National Research Council [NRC], 2001; Schifter, Monk, Russell, & Bastable, 2008). Thus, an extensive use of these properties can serve as a good introduction to algebra (Wu, 2009). Many U.S. students however, including even undergraduates, conflate the CP and the AP (Ding, Li, & Capraro, 2013; Larsen, 2010) and cannot apply the DP to solve equations (Koedinger, Alibali, & Nathan, 2008). Given that the learning and understanding of

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these basic properties should take place in elementary school as expected by the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), it is necessary to explore the current status of elementary students' understanding of these properties. To date, few studies have systematically explored how elementary students understand these basic properties. Given that international studies indicate general learning differences between U.S. students and their counterparts in mathematically high-achieving countries such as China (Program for International Student Assessment, 2009, 2012, 2015), this study takes an initial step from a cross-cultural perspective, examining how U.S. and Chinese third and fourth graders perform on tasks that assess understanding of the basic properties. We specifically ask three questions: (1) In general, how do sampled U.S. and Chinese elementary students understand the CP, AP, and DP? (2) How do sampled U.S. and Chinese elementary students demonstrate understanding of the basic properties in particular contexts? And (3) How do sampled U.S. and Chinese elementary students gain understanding of the basic properties over time?

2. Literature review

To situate the study, we first introduce the basic properties and their significance, followed by a review of students' learning difficulties of these properties. Next, we review the literature on how students' understanding of these properties may be developed. Taking together, these reviews provide a conceptual framework to explore the proposed research questions.

2.1. Importance of the basic properties

The commutative, associative, and distributive properties undergird the arithmetic operations (NRC, 2001). Considering a, b, and c as any arbitrary numbers in a given number system, the commutative property of addition (CP+) states that $a + b = b + a$ while the commutative property of multiplication (CP×) states that $a \times b = b \times a$. In other words, CP deals with the changing of order of numbers with the results invariant. Distinct from CP, the associative property of addition (AP+) states that $(a + b) + c = a + (b + c)$, while the associative property of multiplication (AP×) states that $(a \times b) \times c = a \times (b \times c)$. As such, instead of changing the order of numbers, AP deals with changing the order of operations. Finally, the distributive property of multiplication over addition (DP) states that $a \times (b + c) = a \times b + a \times c$, which involves the interaction between the two different operations.

The above properties of operations are of fundamental importance because they serve as the basic laws of arithmetic, as rules for algebraic manipulation, and as foundations for reasoning, generalization and proof (NRC, 2001; Wu, 2009). These basic properties are also crucial for developing a structural notion of operations – an 'operation sense' (Slavit, 1998) – in which arithmetic operations are conceptualized as mental objects. For instance, to compute 8×6, students in early grades may use the known facts $5 \times 6 = 30$ and $3 \times 6 = 18$ to compute the answer. With a structural guidance on more arithmetic examples such as $8 \times 6 = (5 + 3) \times 6 = 5 \times 6 + 3 \times 6$, students may be prompted to see the distributive property, $a \times (b + c) = a \times b + a \times c$. A transition towards this 'structural' thinking is of great importance to the development of mathematical understanding (Sfard, 1991) and to the development of students' success in algebra (Slavit, 1998). Explicit understanding of these basic properties is necessary for students to grasp algebra as a generalization of arithmetic (Tent, 2006).

2.2. Student learning difficulties with the properties

Despite the importance of these basic properties, past studies reveal that students have demonstrated learning difficulties and poor understanding of these properties. Among the three properties, the CP (especially of addition) is relatively straightforward and used intuitively and extensively by children from early grades (Baroody, Ginsburg, & Waxman, 1983; Slavit, 1998). With the learning of more properties, however, many students, including undergraduates, struggle to differentiate between the CP and the AP (Ding et al., 2013; Fletcher, 1972; Larsen, 2010; Tent, 2006). This may be partly due to the fact that the two properties often co-occur in the same problem (Tent, 2006). For instance, ubiquitous facts such as '$a + b + c = c + b + a$' involve both the CP and the AP. Additionally, the CP and the AP both relate to the notion of reordering – while the CP refers to the reordering of *operands*, the AP refers to the reordering of *operations* – and student difficulty in understanding this distinction may be the cause of much of their difficulty in distinguishing the two properties (Larsen, 2010). Similar difficulties were observed with many students when learning and applying the DP. Koedinger et al. (2008) reported that 71% of the U.S. undergraduates in their study could not solve the equation x–$0.15x = 38.24$, even though most of them could solve a single variable equation like $(1–0.15)\ x = 38.24$. These students' failure in retrieving the DP to simplify 'x - $0.15x$' into '(1 - 0.15) x' indicates their poor understanding of this property. The above findings call for a review of why students struggle in the learning of these properties.

2.3. Developing student understanding of the basic properties

Prior research indicates common limitations of students' initial learning environments of these properties, which may be hindering students' development of relevant understanding. First, textbooks and instruction are often limited to strategies rather than explicit understanding of the basic properties that undergird those strategies. For instance, Ding & Li (2010) reported that current U.S. elementary textbooks present many computation strategies such as using a known fact (e.g. $3 \times 8 = 2 \times 8 + 8$), double strategy (e.g. $8 \times 5 = 4 \times 5 + 4 \times 5$), and breaking apart a number to multiply (e.g. $18 \times 12 = 18 \times 10 + 18 \times 2$); yet, the underlying DP was rarely made explicit. In addition, Schifter et al. (2008) reported classroom scenarios that involved U.S. third and fourth graders' implicit use of the DP and the AP during problem solving, yet the teachers in these scenarios did not promote explicit understanding of the properties based on students' implicit knowledge. Since explicit understanding is a necessary condition for the transfer of learning (Goldstone & Son, 2005; Greeno & Riley, 1987), a focus on implicit application of computational strategies rather than explicit understanding of those generalizable, structural-based properties of operations may not only increase student cognitive load, but also hinder students' transfer of learning to new contexts.

In addition to a lack of explicit understanding, prior research also indicates that elementary students mainly learn the basic properties through computational tasks (Baroody et al., 1983; Schifter et al., 2008) without contextual support for sense-making. While these non-contextual tasks are important learning opportunities, contextual support enables students to make sense of the to-be-learned property, thus creating a path for retrieval which helps students access the properties when needed (Koedinger et al., 2008). NRC (2001) discussed examples of using concrete contexts to make sense of the basic properties of operations. For instance, cube trains with different colours can be used to illustrate the CP+ and the AP+, whereas the CP× and the AP× can be modelled by using an array or volume model. NRC also suggests that contextual tasks such as solving for the perimeter of a rectangle in two different ways, $2L + 2W$ and $2(L + W)$, may help students make sense of the DP. In a similar vein, Ding and Li (2010) found that Chinese textbooks situate the initial learning of the DP in story situations, which were also solved in two ways to illustrate the property. In contrast, existing U.S. textbooks rarely take advantage of concrete contexts to support students' sense-making. For example, even though U.S. textbooks contain similar word problems as the Chinese textbooks, the U.S. word problems were used primarily as a pretext for computation (Ding & Li, 2010). A similar observation was also noted for the AP (Ding, 2016). This, perhaps, at least partially explains students' difficulties in learning and understanding the basic properties.

Students' meaningful and explicit understanding of the basic properties is not an all-or-nothing phenomenon. Rather, it progressively develops through varied contexts over time, often moving from implicit to explicit (Greeno & Riley, 1987). Being able to 'apply' a property for computation or 'evaluate' the legitimacy of a strategy may only indicate students' implicit understanding. In contrast, students' explicit understanding demands their 'recognition' and 'explanation' of the underlying property in a more general/structural sense (Greeno & Riley, 1987). For instance, students clearly identifying and naming a target property or explaining a computational strategy based on the relevant property may indicate their explicit understanding of a certain property. Note that student explanations often contain different levels of understanding (Ding & Auxter, 2017). For instance, providing a specific example of the CP+ (e.g. $3 + 7 = 7 + 3$) may indicate an implicit understanding of a single case, whereas providing an algebraic formula (e.g. $a + b = b + a$) may reveal a students' more general understanding of the CP + . Overall, the transition from implicit to explicit understanding is in some sense similar to the notion of shifting from an operational (process-oriented) to a structural (object-oriented) conception, which is developed through the reification of the processes (Sfard, 1991; Sfard & Linchevski, 1994). In fact, with continuous application of the properties through varied contexts (both implicitly and explicitly), students' understanding may grow both spirally and hierarchically into a more advanced level (Mason, 1998). In other words, as grade level increases, students are expected to increase their understanding of the basic properties. However, not all empirical studies have supported this prediction – for instance, Canobi (2005) found that as grade level increased, students' computation accuracy improved; yet, their explanations did not necessarily improve.

2.4. The current study

Motivated by the research assertions described above, we take a cross-cultural perspective to explore to what extent sampled U.S. and Chinese elementary students possess explicit understanding of the basic properties. In addition, given the importance of both contextual and non-contextual tasks in developing students' understanding, we evaluate how students demonstrate understanding of the basic properties in different contexts. In fact, prior studies show that students' understanding depends on the contexts to which they are exposed and therefore, we agree that it is necessary to assess students' understanding using both contextual and non-contextual tasks (Bisanz & LeFevre, 1992; Bisanz, Watchorn, Piatt, & Sherman, 2009) through which students can be asked to evaluate, apply, explain, and recognize a certain property (elaborated upon later). Taken together, these varied tasks provide an assessment of students' procedural and conceptual understanding (Bisanz & LeFevre, 1992; Bisanz et al., 2009). Finally, analysing students' paper-and-pencil responses on these tasks across two elementary grade levels allows us to examine any developmental trends in understanding over time.

3. Method

3.1. Participants

A total of 97 U.S. and 167 Chinese third and fourth grade students who took both pre- and post-tests participated in this study. We use the terms of 'U.S.' and 'Chinese' to differentiate between our cross-cultural samples. Neither of the terms are intended for broad generalizations about either country. According to the textbooks used by our student samples, the Chinese students formally learn the basic properties in fourth grade, over the span of two focused chapters (an entire chapter devoted to the DP is taught in the second semester; the rest of the properties are all included in one chapter taught during the first semester). Prior to fourth grade, Chinese students only learn the basic properties in implicit ways. In contrast, the U.S. student samples are formally introduced to the basic properties much earlier (e.g. learning the CP in first grade). They also have several opportunities to revisit and relearn these properties across several grades. Regardless of these differences, both U.S. and Chinese students are formally and explicitly taught the CP, AP, and DP by the end of fourth grade. As such, we are interested in exploring the current status of third and fourth graders' understanding of these properties. The U.S. student sample was recruited from 5 classrooms across 4 different schools in one large East Coast urban school district. Similar to other typical large urban schools, the student population is quite diverse. Regardless of student diversity, state standardized test results indicate that all of these schools were above the overall school district average. In addition, one of the schools received the blue ribbon recognition from the U.S. Department of Education for its overall academic excellence and progress in closing achievement gaps among student subgroups. The Chinese students were also sampled from an urban school district, and were from 4 different classrooms across three different schools in the Southeast part of China. Students from both countries were taught by teachers who all had more than 10 years of teaching experience and who had good teaching reputations (e.g. earned teaching awards or were highly recommended by school principals). Table 1 shows more detailed information about the samples.

3.2. Instrument

The pre- and post-tests used the same instrument and were conducted at the beginning and the end of a school year, respectively. The development of the student instrument was

Table 1. The sample size for both pre- and post-tests.

U.S. (N = 97)				China (N = 167)			
Grade 3 (N = 32)	Class 1	N = 10	School 1	Grade 3 (N = 84)	Class 1	N = 48	**School 1**
	Class 2	N = 22	**School 2**		Class 2	N = 36	School 2
Grade 4 (N = 65)	Class 3	N = 17	School 3	Grade 4 (N = 83)	Class 3	N = 44	**School 1**
	Class 4	N = 27	School 4		Class 4	N = 39	School 3
	Class 5	N = 21	**School 2**				

Q1: If you know 7 + 5= 12 does that help you solve 5 + 7? Why?	*Non-contextual evaluation/explanation task: CP+*
Q7a: Please use efficient strategies to solve. Show your strategy and explain why it works. (3 × 25) × 4.	*Non-contextual application/explanation task: AP×*
Q10: The length of a rectangular playground is 118 m and the width is 82 m. What is the perimeter? *John solved it with*: 2 × 118 + 2 × 82 *Mary solved it with:* 2 × (118 + 82) Both are correct. Compare the two strategies, what do you find?	*Contextual recognition task: DP*

Figure 1. Sample tasks used in the instrument.

guided by Bisanz and colleagues (Bisanz & LeFevre, 1992; Bisanz et al., 2009). Items used to measure each property (CP, AP, and DP) included both contextual and non-contextual tasks that demand evaluation, application, recognition, and explanation of the properties. Evaluation tasks expect students to judge whether a computation strategy works, application tasks invite students to use properties to do computation; explanation tasks ask students why a strategy or a computation procedure works and recognition tasks expect students to recognize a certain property that is illustrated by a concrete situation. Figure 1 illustrates sample tasks taken from the instrument. The first two items are non-contextual tasks that require students' evaluation/application and explanation based on the CP or the AP. The last item is a contextual task that requires students' recognition of the DP.

Overall, there were 10 items (16 sub-tasks) gleaned from the literature (e.g. Baroody, 1999) and from elementary textbooks in both countries. Table 2 shows the structure of the instrument (the full instrument is provided in the appendix).

After this instrument had been reviewed by two well-known experts in the field (one mathematician and one mathematics educator), it was validated through pilot testing in actual classrooms. Note that for item 2 (Q2), one teacher of the pilot class suggested removing the words 'instead of counting from 3' due to concerns of children's reading ability. However, when revised in this way, the task seemed to misguide students' attention away from the CP+ and towards a focus on the counting process. As such, we excluded Q2 and used the remaining 15 subtasks (5 each for the CP, AP and DP) to assess students' understanding of the properties (referred to as Q1, Q3a, Q3b, Q3c, Q4, etc.).

3.3. Data coding and procedures

We first developed a coding rubric based on two authors' collaborative coding of student responses from one U.S. and one Chinese class. It was decided that subtasks would be

Table 2. Structure of the instrument.

Property			Item	Context	Nature	Points
CP	(+)	Q1	If you know $7+5=12$ does that help you solve $5+7$? Why?	Non-contextual	Evaluation, Explanation	2
		Q3a	$2+7+8$ (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		Q4	Story problem about 8 boys and 5 girls. Solved by: $8+5$ and $5+8$	Contextual	Recognition	2
	(×)	Q6	To solve 3×28, Mary wrote: $\begin{array}{r} 28 \\ \times\ \ 3 \\ \hline \end{array}$	Non-contextual	Evaluation, Explanation	2
		Q8c	8×6 is solved by: Since $6\times 8=48$, $8\times 6=48$.	Non-contextual	Explanation	2
AP	(+)	Q3b	$(7+19)+1$ (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		Q3c	$2+(98+17)$ (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		Q5	Story problem about three book shelfs with 7, 8, and 5 books on each. Solved by: $(7+8)+5$ and $7+(8+5)$	Contextual	Recognition	2
	(×)	Q7a	$(3\times 25)\times 4$ (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		Q9	Story problem about 3 tables of 2 plates of 5 mangos. Solved by: $(3\times 2)\times 5$ and $3\times(2\times 5)$	Contextual	Recognition	2
	DP	Q7b	102×7 (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		Q7c	$98\times 7+2\times 7$ (Use efficient strategies to solve)	Non-contextual	Application, Explanation	2
		Q8a	8×6 is solved by: $3\times 6=18$, $5\times 6=30$, and $18+30=48$	Non-contextual	Explanation	2
		Q8b	8×6 is solved by: $10\times 6=60$, $2\times 6=12$, and $60-12=48$	Non-contextual	Explanation	2
		Q10	Story problem about a playground perimeter with the length of118 m and width of 82 m. Solved by: $2\times 118+2\times 82$ and $2\times(118+82)$	Contextual	Recognition	2

Note: CP, AP, and DP refer to the commutative property, the associative property, and the distributive property, respectively. Full instrument is available in Appendix 1.

assigned 2 points, resulting in a total of 10 points for each of the CP, AP, and DP. For an item that contains both evaluation/application and explanation (Q1, Q3a, Q3b, Q3c, Q6, Q7a, Q7b, and Q7c), we assigned 1 point to evaluation/application and the other 1 point to the explanation. Student explanations were further classified into three levels: explicit (1 point), implicit (0.5 points), or no/wrong (0 points). The rest of the explanation/recognition tasks (Q4, Q5, Q8a, Q8b, Q8c, Q9, and Q10) were each assigned 2 points, which were again classified into three levels: explicit (2 points), implicit (1 point), and no/wrong (0 points). As suggested by the literature, indicators of explicit understanding included explicit recognition of a property in a given context or general description of the relevant property (as opposed to a specific example). Consequently, any instance between explicit and no understanding was classified into implicit understanding. While this classification may result in a coding of 'implicit' for students who do have explicit understanding yet provide only an implicit response, in the absence of a more-rigorous testing protocol (such as student interviews) this limitation was deemed acceptable for the purposes of this study.

To achieve shared understanding, we supplemented the coding rubric with typical Chinese and U.S. responses for each item. Figure 2 illustrates the coding rubric for Q1 (non-contextual) and Q4 (contextual), both of which were related to the CP. Note that we considered U.S. students' use of the term 'turn around facts' as explicit-understanding of the CP because the U.S. textbooks (e.g. *Everyday Mathematics*) used by our student sample in their previous grades call the CP the 'turn-around property.' After the rubric was defined, we trained the other two coders who coded part of the U.S. and Chinese data for reliability checking. Our reliability (the number of common codes/the number of total codes) was about 94% for the U.S. data and 97% for the Chinese data.

3.4. Data analysis

Both quantitative and qualitative methods were used for data analysis. The quantitative analysis provides a relatively accurate picture of the levels of understanding across grades and countries, while the qualitative analysis suggests interesting patterns that enrich the cross-cultural quantitative findings. To answer the first research question about students' general understanding of the CP, AP, and DP, we first computed the overall pre- and post-test scores for each property within each country, regardless of grade level. Independent t-tests were conducted to determine the significance of differences between different groups. For all tests, the type 1 error was controlled with a Bonferoni post-hoc analysis. To obtain a clearer picture of students' understanding by the end of grade 4 (by then, the Chinese students have formally been taught all properties), we conducted a further inspection on grade 4 students' post-tests. This included calculating the percentage of explicit, implicit, and no understanding responses for each property. Students' implicit understanding in each question refers to the assigned score of 1 or 1.5 points (see Figure 2 for an example). In these situations, students may have correctly evaluated a situation or applied a correct property for computation (score of 1), but their explanations were implicit (score of 0.5) or even no/wrong (score of 0).

To answer the second research question about students' understanding of the basic properties across contexts, student performance on both contextual and non-contextual tasks were analysed. Cross-cultural differences were identified. Likewise, we further compared the fourth graders' post-tests in terms of explicit, implicit, and no understanding under different contexts. Typical student examples and associated patterns under different contexts (and different types of problems) were identified.

Finally, to answer the third research question about students' learning gains of the basic properties over time, we examined U.S. and Chinese third and fourth graders' responses to the pre- and post-tests from different angles by using matched-pair t-tests and independent t-tests. In addition, we examined the trends over several time spots to obtain a general sense of student understanding gains: the beginning and end of G3, and the beginning and end of G4. We are cautious of the limitation that the two student groups (G3 and G4) were different. Nevertheless, given that these students were selected from the same

<table>
<tr><td></td><td>Q1:
Evaluation/Explanation Task of CP+
(Non-contextual)

If you know 7+5=12 does that help you solve 5+7? Why?</td><td>Q4:
Recognition Task of CP+
(contextual)

There are 8 boys and 5 girls in a swimming pool. How many children are there altogether?
John solved it with: 8+5; Mary solved it with: 5+8. Both are correct, comparing the two strategies, what do you find?</td></tr>
<tr><td>Explicit understanding</td><td>Correct evaluation (1)
Explicit explanation (1)

Chinese example:
有，因为学过加法交换律的同学知道"a+b=b+a"，变成现在的数字就是"5+7=7+5"。
Yes. If one has learned the commutative property of addition, he knows a+b=b+a. In this case, It is 5+7=7+5.

US example:
Yes because it is communative property you will get the same anwser on both of them</td><td>Explicit recognition (2)

Chinese example:
我发现"8+5=5+8"，还发现了我们学过的加法交换律。
I found that 8+5=5+8 and I discovered the commutative property of addition that we have learned.

US example:
Both are correct. Compare the two strategies, what do you find?
I realized that the strategies are turn arou problems. John added 8+5 and mary added 5+8.</td></tr>
<tr><td rowspan="2">Implicit understanding</td><td>Correct evaluation (1)
Implicit explanation (0.5)

Chinese example:
答、有帮助因为5+7=7+5。
Answer: Yes, helpful, because 5+7=7+5

US example:
Knowing 7+5=12 does help me solve 5+7 because it is the same problem but it is backwards.</td><td rowspan="2">Implicit recognition (1)

Chinese example:
答：我发现这两道算式都一样，只是调了个头。
Answer: I found the two number sentences are the same, only the numbers are flipped.

Chinese example:
答：比较这两种方法，我发现了小明用的是先算男孩在算女孩，小芳的方法是先算女孩在算男孩。
Answer: Comparing these two methods, I found Xiaoming's computation first considered boys and then girls while Xiaofang first considered girls and then boys.

US example:
I found out both strategies are the same there Just backwards.</td></tr>
<tr><td>Correct evaluation (1)
No/wrong explanation (0)

Chinese example:
答：是对的。
Answer: Yes, it is correct.</td></tr>
<tr><td>No Understanding</td><td>No/Wrong evaluation (0)
No/wrong explanation (0)

Chinese example:
答：没有帮助因为这两道算式一样。
Answer: Not helpful, because the two number sentences are the same.

US example:
No response at all.</td><td>No/wrong recognition (0)

Chinese example:
答：它们的答案一样。
Answer: They have the same answer.</td></tr>
</table>

Figure 2. Example coding rubric for Q1 and Q4.

school district in each country, we believe that these comparative results provide informative findings. In fact, to address the above limitation, we further analysed the responses between the third and fourth graders from the same schools (see Table 1). Below we report findings in alignment with the three research questions.

4. Results

4.1. Student overall understanding of the basic properties

Table 3 indicates the overall mean difference for the CP, AP, and DP, treating all students across grades in each country as one single group. The mean score for each property was obtained by averaging the scores of the corresponding five items for that property (see Table 2). Even though students in both countries showed improvement from the pre- to post-tests, independent t-tests revealed that Chinese students performed better than U.S. students for each property in both pre- and post-tests.

As indicated by Table 3, students in both countries performed better in the CP than in the AP and the DP, where cross-cultural gaps in student understanding were most evident. Many U.S. students conflated the AP and the CP. For instance, students in Q5 were expected to identify the AP+ from the story situation. A typical U.S. response was, 'The strategy I found was both Mary and John used the commutative property. It is just like the turn-around fact.' (Many U.S. students called the CP the 'turn-around property' or 'turn-around facts'). Similar responses were found with Q9, a contextual task for the AP×, '... all they did was changed the numbers order. Just like the commutative property.' Such conflation between the CP and the AP was rare with the Chinese students. For the DP, many U.S. students referred to it as the 'breaking down' strategy or simply mentioned the level of easiness of a strategy, which did not show explicit understanding. For example, when asked to explain Mary's strategy ($3 \times 6 = 18$, $5 \times 6 = 30$, $18 + 30 = 48$) to solve 8×6 in Q8, a student responded, 'Mary's strategy works because she is breaking down the problem to make it easier.' Similarly, a typical U.S. response to Q10 was, 'One is making it harder to do and the other one is doing it the simple.' In summary, U.S. students typically responded by explaining strategies (as opposed to the underlying properties) they used. In contrast, many Chinese students who responded to these same questions explicitly pointed out the DP in either words or by showing a formula.

It should be noted that even though the Chinese students performed better in each property, their overall mean scores for the third and fourth graders were relatively low (e.g. $M_{\mathrm{CP-pst}} = 5.1$, $M_{\mathrm{AP-pst}} = 4.2$, and $M_{\mathrm{DP-pst}} = 3.4$; out of 10). This may be due to the fact that Chinese students are not formally taught these properties until fourth grade. As such, we further examined the U.S. and Chinese fourth grade post-tests to obtain a truer picture about students' understanding by the end of grade 4. We found that Chinese fourth graders have improved their understanding of the AP and the DP to almost the same level as their understanding of the CP, which is quite different from the case

Table 3. U.S. and Chinese students' overall performance on the CP, AP, and DP.

	U.S. ($n = 97$)		China ($n = 167$)				
Test	Mean	SD	Mean	SD	t	p	df
CP_Pre	2.8	1.5	4.5	1.5	8.88	< .001	200.78
CP_Post	3.9	1.5	5.1	2.0	5.20	< .001	247.89
AP_Pre	1.1	1.2	3.3	1.8	12.10	< .001	254.7
AP_Post	1.6	1.5	4.2	2.9	9.60	< .001	258.83
DP_Pre	0.3	0.8	2.2	2.0	10.97	< .001	235.6
DP_Post	0.9	1.2	3.4	2.8	10.10	< .001	248.71

Note: CP, AP, and DP refer to the commutative, associative, and distributive properties, respectively.

of their U.S. counterparts (China: $M_{\mathrm{CP-pst}} = 5.9$, $M_{\mathrm{AP-pst}} = 6.1$, and $M_{\mathrm{DP-pst}} = 5.1$; U.S.: $M_{\mathrm{CP-pst}} = 3.7$, $M_{\mathrm{AP-pst}} = 1.5$, and $M_{\mathrm{DP-pst}} = 0.8$). In other words, Chinese fourth graders, on average, can correctly answer about 59% of CP, 61% of AP and 51% of DP tasks while U.S. fourth graders can only answer 37% of CP, 15% of AP and 8% of DP tasks. In fact, there were 7 Chinese fourth graders who correctly answered 90–97% of all tasks. Table 4 provides further information about the extent to which students demonstrated explicit, implicit, and no understanding of each property through each relevant task.

As explained in Methods, students' explicit understanding of a property refers to their explicit explanations or recognition of the target property using a correct term, a formula (e.g. $a + b = b + a$), or verbal statement (see Figure 2 for examples). We acknowledge that being able to name a property or suggest a formula may not show full understanding. However, we argue that students' explicit naming of the property based on a given task at least shows their explicit awareness of this underlying property. This indicates a higher level of understanding than simply applying computational procedures without awareness of the mathematics behind the procedures. In Table 4, about 22% of U.S. students in Q1 and 15% in Q4 demonstrated explicit understanding of the CP$+$. The demonstrated understanding of the CP$+$ was inconsistent across tasks (e.g. no students demonstrated understanding in Q3). Overall, far fewer sampled U.S. students (2%-3%) demonstrated explicit understanding of the CP$\times$. With regards to the AP and the DP, almost no sampled U.S. students demonstrated explicit understanding of these two properties across tasks (note: only one student correctly mentioned the DP in one task: Q10). In contrast, more Chinese fourth graders demonstrated explicit understanding of these three properties across tasks (CP: 16%-59%; AP: 24%-51%; and DP: 16%-63%). In particular, while 58%-85% of U.S. students completely lacked understanding of the AP across tasks, only 2%-34% of Chinese students demonstrated no understanding.

4.2. Student understanding of the basic properties in different contexts

Combining all contextual items (8 points) and all non-contextual items (22 points) respectively, Figure 3 illustrates the average percentages of total points earned from sampled U.S. and Chinese students. Table 5 indicates detailed mean scores and standard deviation. Independent t-tests reveal that in both types of tasks, Chinese students significantly outperformed their U.S. counterparts in all pre- and post-tests, except for the G3-post-test non-contextual tasks (see Table 5, the case of Chinese G3 will be elaborated upon later).

Table 6 below provides further information about the sampled U.S. and Chinese students' levels of understanding across contexts by the end of grade 4. Overall, student understanding within each context is fairly consistent. Across contexts, results show interesting cross-cultural similarities and differences. Elaboration follows.

Non-contextual evaluation tasks. Q1 and Q6 asked students to evaluate the computational strategies where the CP$+$ or the CP$\times$ were involved. Even though Chinese fourth graders performed better than their U.S. counterparts under this context, students in both countries understood the CP$+$ much better than the CP$\times$. The percentages of students' explicit, implicit, and no understanding responses support this conclusion. Specifically, in Q6 (CP$\times$) when students were asked to explain whether Mary could write $\begin{array}{r} 2\ 8 \\ \times\ \ \ 3 \\ \hline \end{array}$ to

Table 4. Students' levels of understanding of CP, AP, and DP by the end of grade 4.

		CP					AP					DP				
		+	+	+	×	×	+	+	+	×	×					
	Under-standing	Q1 Eva, Exp	Q3a App, Exp	Q4 Rec	Q6 Eva, Exp	Q8c Exp	Q3b App, Exp	Q3c App, Exp	Q5 Rec	Q7a App, Exp	Q9 Rec	Q7b App, Exp	Q7c App, Exp	Q8a Exp	Q8b Exp	Q10 Rec
U.S. (n = 65)	Explicit	22%	0%	15%	2%	3%	0%	0%	0%	0%	0%	0%	0%	0%	0%	2%
	Implicit	74%	25%	46%	75%	26%	28%	38%	42%	15%	29%	23%	6%	23%	20%	9%
	No	5%	75%	38%	23%	71%	72%	62%	58%	85%	71%	77%	94%	77%	80%	89%
China (n = 83)	Explicit	46%	16%	59%	16%	30%	22%	25%	49%	24%	51%	20%	43%	16%	16%	63%
	Implicit	46%	81%	17%	82%	8%	75%	71%	22%	73%	16%	70%	51%	31%	33%	8%
	No	8%	4%	24%	2%	61%	4%	4%	29%	2%	34%	10%	6%	53%	52%	29%

Note: CP, AP, and DP refer to the commutative, associative, and distributive properties, respectively. Due to rounding totals of percentages may not equal 100%.

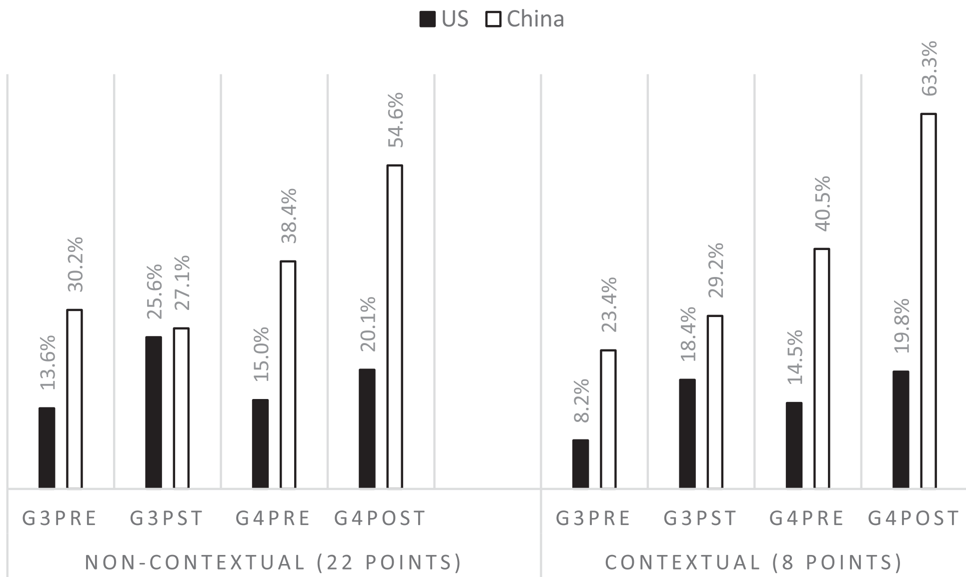


Figure 3. Percentage of points students earned in different contexts.

Table 5. U.S. and Chinese students' scores on non-contextual and contextual tasks.

Test	U.S. (n_{G3} = 32, n_{G4} = 65)		China (n_{G3} = 84, n_{G4} = 83)				
	Mean	SD	Mean	SD	*t*	*p*	*df*
Non-Contextual							
G3-Pre	3.0	2.4	6.6	3.4	6.61	< .001	79.1
G3-Post	5.6	3.2	5.9	3.5	0.45	0.65	62
G4-Pre	3.3	1.9	8.5	3.7	10.92	< .001	129.2
G4-Post	4.4	2.4	12.0	4.5	13.14	< .001	131.54
Contextual							
G3-Pre	0.7	0.9	1.9	1.6	5.10	< .001	92.8
G3-Post	1.4	1.1	2.3	1.8	3.13	< .01	94.5
G4 Pre	1.2	1.3	3.2	1.5	9.08	< .001	145.4
G4 Post	1.6	1.4	5.1	3.0	9.41	< .001	124.3

Note: CP, AP, and DP refer to the commutative, associative, and distributive properties, respectively.

solve 3 × 28, many students provided procedural explanations such as 'because the number with more digits should be placed above' (a Chinese example) and 'because you always do big number first' (a U.S. example).

Non-contextual application tasks. Q3 and Q7 asked students to apply the basic properties (CP+, AP+, AP×, and DP) and to explain their strategies for computing tasks such as $2 + 7 + 8$, $(7 + 19) + 1$, $(3 \times 25) \times 4$, and $98 \times 7 + 2 \times 7$ (see Table 5). Across countries, there were students who demonstrated no understanding (U.S.: 62%-94%, China: 2%-10%) or showed implicit understanding (U.S.: 6–38%, China: 51–81%). Figure 4 illustrates typical examples. As indicated, the first common response was that students in both countries simply followed the order of operations without using properties, and thus no understanding was detected (see Figure 4 for examples). This observation is particularly interesting with Chinese G3 students who completed these tasks using the correct properties in the

Table 6. Fourth graders' understanding of the basic property across contexts.

		Non-contextual (22 points)											Contextual (8points)			
		Eva/Exp		App/Exp						Exp			Rec			
		Q1 CP+	Q6 CP×	Q3a CP+	Q3b AP+	Q3c AP+	Q7a AP×	Q7b DP	Q7c DP	Q8a DP	Q8b DP	Q8c CP×	Q4 CP+	Q5 AP+	Q9 AP×	Q10 DP
U.S. ($n = 65$)	Explicit	22%	2%	0%	0%	0%	0%	0%	0%	0%	0%	3%	15%	0%	0%	2%
	Implicit	74%	75%	25%	28%	38%	15%	23%	6%	23%	20%	26%	46%	42%	28%	9%
	No	5%	23%	75%	72%	62%	85%	77%	94%	77%	80%	71%	38%	58%	72%	89%
China ($n = 83$)	Explicit	46%	16%	16%	22%	25%	24%	20%	43%	16%	16%	30%	59%	49%	51%	63%
	Implicit	46%	82%	81%	75%	71%	73%	70%	51%	31%	33%	8%	17%	22%	16%	8%
	No	8%	2%	4%	4%	4%	2%	10%	6%	53%	52%	61%	24%	29%	34%	29%

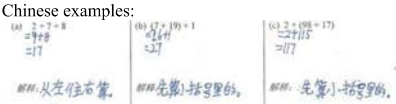
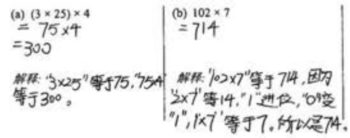
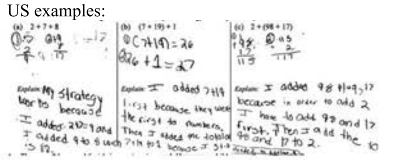
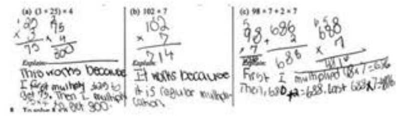
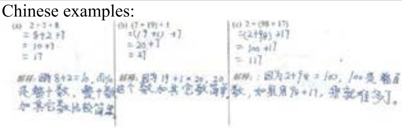
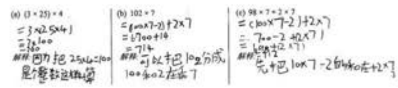
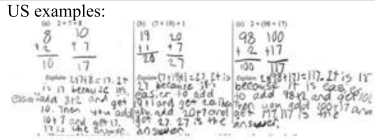
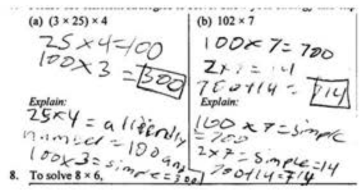
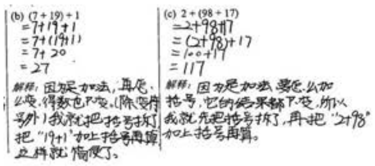
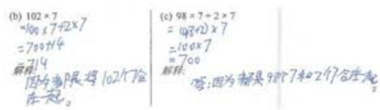
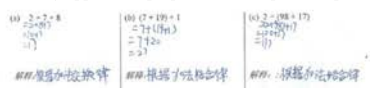
Note: CP, AP, and DP refer to the commutative, associative, and distributive properties, respectively.

pre-test, but merely followed the order of operations in the post-test. In their explanations, these students stressed that one should first perform the computation inside the parenthesis. It seems that Chinese third graders' formal learning of the order of operations overshadowed their intuitive/implicit knowledge of the basic properties. The second cross-cultural commonality found in students' response was that some students correctly applied the properties for computations but only explained their strategies by describing procedures (see Figure 4 for examples). For instance, students from both countries often described that they first attempted to make a 10 or a 100, which only showed their implicit understanding of the respective property.

Figure 4 also indicates student responses that were unique to Chinese students. In the implicit understanding situation, a few Chinese students (10 students in Q3a, 4 in Q7c, and a few in other tasks) provided explanations by referring to the meaning of operations, which shows sense-making and an overall better understanding than the procedural descriptions reported above. For instance, to explain their strategy for $98 \times 7 + 2 \times 7 = (98 + 2) \times 7 = 100 \times 7 = 700$ (Q7c), some Chinese students stated, 'combining 98 groups of 7 and 2 groups of 7' (see Figure 4). None of the sampled U.S. students provided similar explanations. In fact, many U.S. students particularly struggled with Q7c. While 94% of the Chinese fourth graders applied DP to solve this problem, only 6% of their U.S. counterparts did so. This may be due to the fact that the DP in this item needs to be applied in an 'opposite' direction $ab + ac = a\,(b + c)$, a task which rarely appears in U.S. textbooks but is frequently seen in Chinese textbooks (Ding & Li, 2010). Indeed, across all non-contextual application tasks, none of the sampled U.S. fourth graders demonstrated explicit understanding of any of these properties, while between 16% and 43% of the sampled Chinese fourth graders could explicitly and correctly point out the CP, AP and DP (see Figure 4 for examples).

Non-contextual explanation task. Q8 asked students to explain how the properties were used in solving 8×6, a task that was modified from U.S. textbooks. In particular, Q8a and Q8b used the DP while Q8c used the CP$\times$. By nature, this question is similar to Q3 and Q7 except that the numbers are smaller. Surprisingly, this question turned out to be challenging for Chinese students. For instance, only 16% of the sampled Chinese students demonstrated explicit understanding of the DP, which is less than the percentage in both Q3 (20%) and Q7 (43%, see Table 5). Q8c, the part of the question that involved the CP$\times$, revealed more positive results (30%). Figure 5 indicates typical examples of Chinese students' explicit understanding of Q8. Even though Chinese students did not perform well on this item, no sampled U.S. students (0%) could explicitly identify the DP (Q8a, Q8b) and only 3% pointed out the CP (Q8c, see Table 5). Some U.S. students referred to the DP as the 'breaking down strategy' and thus received partial credit. Overall, a much higher proportion of U.S. students demonstrated no understanding (77%, 80%, and 71% for Q8a, Q8b, Q8c, respectively) than did the Chinese students (53%, 52%, and 61%, see Table 5).

Contextual-recognition. In this study, students were expected to 'recognize' the basic properties (CP+, AP+, AP$\times$, DP) illustrated by the story problem solutions (Q4, Q5, Q9, and Q10). As indicated by Table 5, the sampled Chinese students consistently performed the best on this type of task (59%, 49%, 51%, 63% show explicit understanding of the above question). This is in stark contrast to only 15% of the U.S. students who recognized the CP and even fewer who recognized the AP+, AP$\times$, or the DP (0%, 0%, and 2%,

<table>
<tr><th colspan="3">Common Responses between US and China</th></tr>
<tr><td rowspan="2">No understanding</td><td colspan="2">Computation does not apply the properties. Explanation confirms the procedures. (A total of 0 point)</td></tr>
<tr><td>Chinese examples:

Translation: (a) from left to right. (b) and (c) first compute what is inside of the parenthesis.

Translation: (a) 3 × 25 equals to 75, 75 × 4 equals to 300. (b) 102 × 7 = 714, because 2 × 7 = 14, carrying 1, then 0 is changed to 1, 1 × 7 = 7. Thus, it is 714.</td><td>US examples:

</td></tr>
<tr><td rowspan="2">Implicit understanding</td><td colspan="2">Correctly applies the properties to compute. Explanation repeats the computation procedures such as making a 10 or 100 first. (A total of 1 point)</td></tr>
<tr><td>Chinese examples:

Translation: (a) Because 8+2=10, 10 is tens. Tens is easier to compute than the other number. (b) Because 19+1=20, Adding 20 is easier to compute than the other number. (c) Because 2+98=100, 100 is hundreds. If you add 98+17, that is be harder.

Translation: (a) because 25 × 4 = 100 which is a whole number. (b) You can separate 102 as 100 and 2, and then multiplying by 7. (c) First compute 100 × 7 and then add 2 × 7.</td><td>US examples:

</td></tr>
<tr><th colspan="3">Unique Responses of China</th></tr>
<tr><td rowspan="4">Implicit Understanding</td><td colspan="2">(1) Correctly applies the properties to compute. Explanation refers to surface features of a property (1.5 points)</td></tr>
<tr><td></td><td>Translation
(b) Because for addition, however you change, the answer will not be changed (expect for changing the sign). I removed the parenthesis and then added it to 19+1. This makes computation easy. (c) Because for addition, however you add the parenthesis, the answer will not be changed. As such, I first removed the parenthesis and then added it to 2+98.</td></tr>
<tr><td colspan="2">(2) Correctly applies the properties to compute. Explanation referring to meaning of an operation (1.5 points).</td></tr>
<tr><td></td><td>Translation
(b) because combines 102 groups of 7.
(c) because it combines 98 groups of 7 and 2 groups of 7.</td></tr>
<tr><td rowspan="2">Explicit understanding</td><td colspan="2">Correctly applies the properties to compute. Explanation explicitly pointed out the properties involved (2 points).</td></tr>
<tr><td></td><td>Translation
(a) according to the CP of addition, (b) and (c) according to AP of addition.

Translation: (a) using the AP of multiplication, (b)</td></tr>
</table>

Figure 4. Examples of students work with non-contextual application/explanation tasks.

8. To solve 8×6,

(a) Mary thought:	(b) John thought:	(c) Kate thought:
$3 \times 6 = 18$, $5 \times 6 = 30$, $18 + 30 = 48$	$10 \times 6 = 60$, $2 \times 6 = 12$, $60 - 12 = 48$	Since $6 \times 8 = 48$, $8 \times 6 = 48$

Please explain why each strategy works.

8. 计算 8×6：

(a) 小红算法是：	(b) 小华算法是：	(c) 小丽算法是：
$3 \times 6 = 18$, $5 \times 6 = 30$, $18 + 30 = 48$	$10 \times 6 = 60$, $2 \times 6 = 12$, $60 - 12 = 48$	因为 $6 \times 8 = 48$ 所以 $8 \times 6 = 48$

请解释为什么每种算法都正确。

(Note: The English names Mary, John, and Kate were replaced with popular Chinese names, Xiaohong, Xiaohua, and XiaoLi)

Ex1

答：小红把8拆成3和5,运用了乘法分配律,小华把8算成10-2,再运用乘法分配律。小丽运用了乘法交换律. $6\times8=8\times6(a\times b=b\times a)$.

Translation: Xiaohong (Mary) broke down 8 into 3 and 5 and used the DP. XiaoHua (John) viewed 8 as 10-2 and then used the DP. XiaoLi (Kate) used the CP of multiplication.

Ex2

答红：用了乘法分配律，$8\times6=(3+5)\times6=3\times6+5\times6=18+30=48$。
华：用了乘法分配律，$8\times6=(10-2)\times6=10\times6-2\times6=60-12=48$。
丽：用了乘法交换律，$8\times6=6\times8=48$。

Translation:
Hong (Mary): Used the DP. $8\times 6=(3+5)\times6=3\times6 + 5\times6 = 18 + 30= 48$.
Hua (John): Used the DP. $8\times 6=(10-2) \times 6=10\times6 – 2\times6 = 60 – 12= 48$.
Li (Kate): Used the CP of multiplication. $8\times 6=6 \times 8=48$.

Figure 5. Typical Chinese student responses that show explicit understanding in Q8.

respectively). Across countries, descriptions of surface patterns that at most show implicit understanding were commonly found (e.g. 'the parenthesis moved place,' 'the order of the parenthesis is changed'). Interesting cross-cultural differences also lie in students' implicit understanding, in that many Chinese students but no U.S. students reasoned in terms of the meaning of the story situations (see Figure 2, Q4). Below are unique Chinese examples that show connection-making between number sentences and contextual information. In some regards, this is similar to students' referring to the meaning of multiplication in the non-contextual application tasks as reported above.

> (*Q5, CP*+) I found that Xiaoming first figured out the number of books on the first and second bookshelves while Xiaofang first figured out the books on the second and third bookshelves. They both get the same answer.
>
> (*Q9, AP*×) I found that Xiaoming first figured out the total of six plates and then the total of 30 mangos; Xiaofaing first figured out that there were 10 mangos on each table and then a total of 30 mangos. They got the same answer.
>
> (*Q10, DP*) I found that Xiaoming first computed the (total) length of the playground and then the (total) width of the playground; Xiaofaing first compute "length + width" of the playground, and then find two "length + width." Both answers are correct.

4.3. Student understanding of the basic properties over time

To examine the progression of student learning, we analysed student performance across grades from different angles. Figure 6 illustrates students' average scores for each property in the pre- and post-tests. Matched-pair t-tests were conducted to examine the performance difference between pre- and post-tests at each grade and in each country.

For Chinese third grade students, there is no statistically significant increase between pre- and post-tests of total scores, $t(83)$ = -.82, p = .79. This may be due to the fact

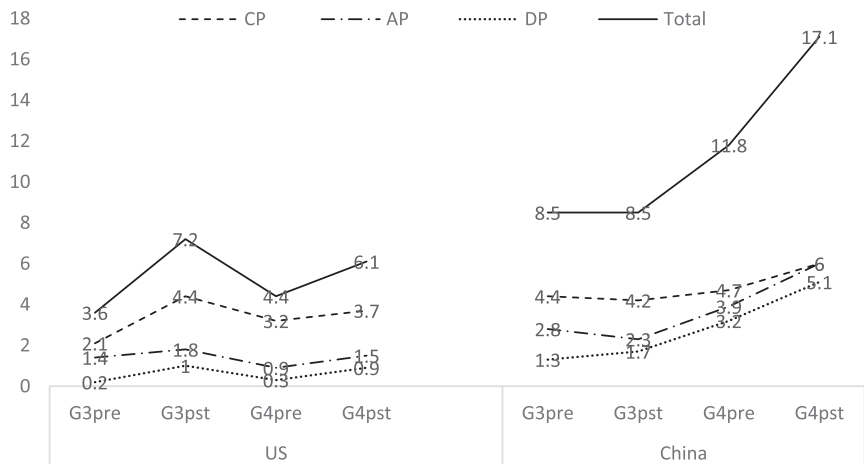


Figure 6. U.S. and Chinese students' understanding of each property over time.

that Chinese third graders were not formally taught the basic properties until grade 4. Results for Chinese fourth graders however did show significant progress from the pre- to post-tests, $t(82) = 7.6$, $p < 0.01$. For the U.S. students, both the third and fourth graders scored significantly better on their post-tests than pre-tests, $t_{G3}(31) = 4.99$, $p_{G3} < 0.01$; $t_{G4}(64) = 4.54$, $p_{G4} < 0.01$, indicating significant progress made over the course of both corresponding grades.

Independent t-tests for each property were conducted to compare the sampled Chinese and U.S. students' performances within each grade. As seen in Table 7, for the third graders, it was found that the Chinese students performed significantly better than their U.S. counterparts in each property on the pre-tests. However, on the post-tests, there was no difference in understanding the CP and AP but a slight difference in understanding the DP. It seems that through formal learning of the basic properties in third grade, the U.S. students have closed the achievement gap, especially with the CP and the AP. This may be also due to previously mentioned little-progress of Chinese third graders whose application of properties was hindered by their learning of order of operations. A comparison of fourth graders' performance, however, appears to follow a different pattern: the sampled Chinese students performed significantly better than the sampled U.S. counterparts in each property in all pre- and post-tests (see Table 7). This shows that once the Chinese fourth graders have formally learned the basic properties, the cultural understanding gap reappears.

Finally, we inspected the progress that students made from G3 to G4. We are cognizant of the fact that the third and fourth graders are not the same group of students and thus suggest interpreting the following results with caution. Regardless of this limitation, the results are informative because students in each country were selected from the same school district. Combining all three properties, progress in the Chinese students' understanding was very evident after formal instruction on the properties occurred in fourth grade (see Figure 6). In contrast, even though the sampled U.S. students were formally introduced to all properties by grade 3 and then re-visited them during grade 4, they seemed to make little progress overall. In particular, the understanding gap between the sampled Chinese

Table 7. U.S. and Chinese 3rd and 4th graders performance on CP, AP, and DP.

	U.S. ($n_{G3} = 32, n_{G4} = 65$)		China ($n_{G3} = 84, n_{G4} = 83$)				
Test	Mean	SD	Mean	SD	t	p	df
Grade 3							
CP-Pre	2.0	1.6	4.4	1.5	7.13	< .001	53.6
AP-Pre	1.4	1.3	2.8	1.7	4.71	< .001	71.5
DP-Pre	0.2	0.8	1.3	1.6	5.09	< .001	109.4
CP-Post	4.4	1.8	4.2	1.6	0.62	0.54	51.7
AP-Post	1.8	1.5	2.3	2.2	1.63	0.11	85.6
DP-Post	1.0	1.2	1.7	1.9	2.44	0.02	85.5
Grade 4							
CP-Pre	3.2	1.3	4.7	1.5	6.29	< .001	144.19
AP-Pre	0.9	1.1	3.9	1.7	12.49	< .001	142.87
DP-Pre	0.3	0.8	3.2	2.0	11.52	< .001	112.86
CP-Post	3.7	1.3	5.9	2.0	8.31	< .001	140.02
AP-Post	1.5	1.5	6.1	2.2	14.92	< .001	143.8
DP-Post	0.8	1.3	5.1	2.5	13.64	< .001	127.1

Note: CP, AP, and DP refer to the commutative, associative, and distributive properties, respectively.

Table 8. The learning progress of students from the same school.

		AP-pre	AP-post	Progress	DP-pre	DP-post	Progress
U.S._ School 2	G3	1.73	1.36	−0.37	0.32	1.23	0.91
	G4	1.48	1.33	−0.15	0.81	1.62	0.81
China_School 1	G3	3.82	2.89	−0.93	1.73	2.16	0.43
	G4	3.77	6.92	**3.15**	1.95	5.05	**3.1**

and U.S. students dramatically increased from the beginning of grade 3 to the end of grade 4 for the AP and the DP. The initial gaps for AP and DP at the beginning of the third grade were 1.4 and 1.1 points respectively (AP: $M_{\text{US}} = 1.4, M_{\text{China}} = 2.8$; DP: $M_{\text{US}} = 0.2$, $M_{\text{China}} = 1.3$), yet by the end of the fourth grade, the gaps were increased to 4.5 and 4.2 points (AP: $M_{\text{US}} = 1.5$, $M_{\text{China}} = 6$; DP: $M_{\text{US}} = 0.9$, $M_{\text{China}} = 5.1$). In other words, the sampled U.S. and Chinese participating students' understanding gap was magnified three- to four-fold once the Chinese students were formally exposed to all properties.

Being cognizant of the limitation that the G3 and G4 students were different in each country, we further examined the responses of the third and fourth graders within the same school in each country. Results indicate similar observations as reported above. As seen in Table 8, for both the AP and the DP, only Chinese fourth graders demonstrated unique learning progress. A comparison of students' pre-test in grade 3 and their post-test in grade 4 confirmed that the cross-cultural understanding gap is increasing over time (see the highlights in Table 8). For instance, the initial cross-cultural gaps in grade 3 for AP and DP were 2.09 and 1.41 points, respectively (AP: $M_{\text{US}} = 1.73$, $M_{\text{China}} = 3.82$; DP: $M_{\text{US}} = 0.32$, $M_{\text{China}} = 1.73$). Similarly to before, by the end of the fourth grade, the understanding gaps were increased to 5.59 and 3.42 points (AP: $M_{\text{US}} = 1.33$, $M_{\text{China}} = 6.92$; DP: $M_{\text{US}} = 1.62$, $M_{\text{China}} = 5.05$).

5. Discussion

An emphasis on the understanding of the basic properties of operations (CP, AP, and DP) in elementary school should never be overstated because these properties are the foundations for future learning of more advanced topics such as algebra (Carpenter et al., 2003; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Mathematics Advisory Panel [NMAP], 2008; Schifter et al., 2008; Wu, 2009). From a cross-cultural perspective, this study systematically examines students' understanding of these basic properties. Findings reveal students' successes and difficulties in learning these properties and provide suggestions for helping students better develop a meaningful and explicit understanding of these basic properties.

5.1. Develop explicit understanding of the properties behind computational strategies

In this study, many students (especially in the U.S. sample) lacked explicit understanding of the basic properties, as evident by their attention to 'strategies' rather than the undergirding properties of those strategies. As such, these students lacked the flexibility needed for applying the basic properties for computation. Despite students' relatively decent performance on the CP, their understanding of the AP and the DP was limited. As reported, there were almost no sampled U.S. students who demonstrated explicit understanding of these two properties. Instead, students tended to view the AP as merely the use of parenthesis and the DP as simply breaking-down a factor. This level of understanding illustrates superficial features and procedural descriptions but not the essence of these properties. While most Chinese students who demonstrated implicit understanding could flexibly use the DP in the direction of $ab + ac = a\ (b + c)$ to perform a computation, very few U.S. students could do so. Note that such a use of the DP is in an opposite direction from $a\ (b + c) = ab + ac$, and appears frequently in the Chinese but not the U.S. textbooks (Ding & Li, 2010). Given that an understanding of the DP in an opposite direction is critical for solving algebraic equations such as x–$0.15x = 38.24$ (Koedinger et al., 2008), our findings about U.S. students' lack of flexibility in using the DP in the opposite direction calls for attention. Perhaps, future textbook design and revision as well as teacher education programmes should take this issue into consideration. As opposed to the U.S. students, more than half of the sampled Chinese students in the current study demonstrated explicit understanding once they had been exposed to these properties in fourth grade. In fact, there were seven Chinese fourth graders who scored above 90% for all properties across all tasks. This indicates that all of the basic properties are learnable if taught correctly and appropriately. In contrast, even though U.S. students are repeatedly taught these properties in both grades 3 and 4, almost none of them obtained explicit understanding. This calls for examination of the instructional approaches that U.S. classroom teachers use for teaching these properties.

Across grades, the U.S. students did not progress towards explicit understanding of the basic properties, which is different from the observed Chinese pattern. Is this different learning pattern caused by different curriculum designs? For instance, the Chinese students are informally taught the properties in grade 3, while the U.S. students are explicitly taught and retaught both properties (AP, DP) in grades 3 and 4. In fact, even though the U.S. students have formally been taught the properties in two different years, their explicit understanding of the AP and the DP by the end of grade 4 was quite low. Since explicit understanding of structural knowledge (e.g. the basic properties) is a necessary level of understanding that enables transfer of learning to new contexts (Goldstone & Son, 2005; Greeno & Riley, 1987), U.S. students' systematic lack of explicit understanding of these basic properties might partially explain why so many students could not transfer their learning from arithmetic to algebra (Carpenter et al., 2003; NMAP, 2008). In addition, we observed that many U.S. third and fourth graders called the CP a 'turn-around fact.' Even though we coded this as 'explicit understanding,' we question why these students do not know or do not use the actual terminology such as 'the commutative property.' It is understandable that when the CP is initially introduced to U.S. first graders that the vivid metaphor of 'turn-around' may be helpful; however, why does students' formal understanding or use of terminology not evolve over time? It is important to note that this is not merely a pedantic quibble about word-usage; the term 'turn-around fact' places instructional emphasis on the specific case of the expression itself at the expense of the underlying generalizable rule. Thus, lacking the accurate terminology for this basic property may inhibit students' future learning of more advanced mathematics.

Encouragingly, we notice that many U.S. students possess implicit understanding of these properties, which can serve as a springboard for developing students' explicit understanding (Ding & Auxter, 2017). In fact, prior research argues for a learning progression from implicit to explicit (Fyfe, McNeil, & Borias, 2015; Greeno & Riley, 1987; Pirie & Kieren, 1994; Sfard & Linchevski, 1994). However, students in this study demonstrated difficulties with transforming implicit knowledge of the properties into explicit understanding. Many U.S. third and fourth graders only focused on superficial features such as 'moving a parenthesis' or computational strategies such as 'breaking down a factor.' This seems to be consistent with Ding and Li's (2010) textbook report where U.S. textbooks presented many strategies, but failed to help the teacher and students perceive and comprehend the underlying properties. In fact, U.S. textbooks sometimes treat the properties as one of the computation strategies. For instance, 'using distributive property' was viewed as a parallel strategy, rather than the undergirding reason of 'breaking a number apart to multiply' (Ding & Li, 2010). This may explain why U.S. students attended to strategies, but not the basic properties, even though they had repeatedly been taught them across multiple grades.

Across grades, the Chinese students' learning pattern is quite different. Although there is room for improvement, many Chinese students have shifted their understanding of the basic properties from implicit to explicit by the end of fourth grade. This suggests that an exploration of Chinese formal lessons on the basic properties may be fruitful. Of course, Chinese students' relative underperformance on the easy task 8×6, also calls for re-thinking of how to deepen students' understanding by drawing their attention back to familiar knowledge of basic multiplication facts. Probably, a more informative research direction is to explore U.S. and Chinese lessons with attention to both implicit and explicit teaching. For instance, what are the important textbook and instructional factors that contribute to the shifting of student thinking from implicit to explicit?

Overall, based on the above findings, we propose that students need to be better supported in developing explicit understanding of the basic properties in elementary grades.

Once students are formally introduced to these properties, they should be prompted through later practices to understand the underlying concepts rather than only computational strategies and features such as 'break down numbers' or 'turn around fact.' In fact, inaccurate interpretations of the basic properties may be hard to be replaced with relearning of these properties at later grades.

5.2. Developing meaningful understanding of the properties through varied contexts

Students in this study demonstrated different patterns in reasoning and making sense of the computational strategies or solutions that involved the basic properties. In particular, with contextual tasks where the basic properties were illustrated through a comparison of multiple solutions, the Chinese students could explain the meaning of each step in the given solutions and explain why both solutions were correct in solving the same word problems (especially before formal instruction of the properties). Further, their explanations often drew on the basic meaning of operations (e.g. addition, multiplication). Similar reasoning skills of the Chinese students also occurred with non-contextual tasks (e.g. explaining '$98 \times 7 + 2 \times 7 = (98 + 2) \times 7$' as 'combining 98 groups of 7 and 2 groups of 7'). Such reasoning skills are likely to enable students' sense-making of the abstract properties once they are formally introduced to them. For instance, students may interpret $a \times c + b \times c = (a + b) \times c$ as combining a groups of c and b groups of c, resulting in $a + b$ groups of c.

In contrast, U.S. students' responses to the contextual tasks were in nature different from their Chinese counterparts because their reasoning focused on numerical relationships that did not need contextual support. For instance, many U.S. students re-computed the given expressions to ensure the answers were indeed the same. No U.S. student explained the given two solutions based on the story situation. Given that the use of concrete support can help students make sense of the basic properties (NRC, 2001), perhaps developing U.S. students' meaningful understanding of the basic properties through a more focused use of concrete contexts may better facilitate students' deep initial learning. As Chi and Van-Lehn (2012) argued, student deep initial learning lies in their understanding of quantitative interactions, which goes beyond numerical relationships.

Why is there a different pattern in U.S. and Chinese students' sense-making with these basic property oriented tasks? Prior textbook studies (e.g. Ding & Li, 2010) indicate one of the possible reasons. For instance, to introduce the DP, Chinese textbooks situate the initial teaching of this property in a word problem context. Through solving a word problem in two different ways based on the meaning of multiplication, students are asked to compare the two number sentences and then must pose additional similar examples. Based on these exercises, the textbook then formally reveals the DP and its algebraic formula. This approach - from concrete to abstract and from specific to general – is consistent with the concreteness fading method (Goldstone & Son, 2005) that has been shown to effectively develop students' mathematical understanding (Fyfe et al., 2015; McNeil & Fyfe, 2012). Such contextual support for making sense of the basic properties was completely missing in the U.S. textbooks (Ding & Li, 2010). Of course textbooks are only opportunities to learn – the praxis of implementing textbook content in actual classroom teaching deserves further exploration.

The development of students' meaningful understanding of the basic properties also demands the use of varied tasks, including both contextual and non-contextual ones that connections between current and prior knowledge. In this study, students in both countries demonstrated inconsistent understanding across tasks. For instance, even though many Chinese students could use the terminologies of the basic properties in contextual tasks, they did not perform equally well with non-contextual tasks, especially when explaining $8 \times 6 = 5 \times 6 + 3 \times 6$. In fact, most Chinese students were able to apply the basic properties to solve problems with larger numbers (e.g. 102×7) and some even explicitly pointed out the underlying properties. However, when facing a similar task with smaller numbers, more Chinese students failed to explain the involved properties. This may be due to the fact that Chinese students in second grade are expected to master the multiplication facts like 8×6 based on conceptual initial learning and they do not need 'breaking up a factor' to find the answer in later grades, or else their lack of explanation of the given computation strategies may reflect a lack of flexibility in understanding DP. This confirms the importance of assessing students' understanding using a variety of tasks – to develop students' meaningful understanding of fundamental concepts, varied contexts and tasks should be utilized to learn new concepts. This might especially be true for forming connections to prior knowledge.

5.3. Conclusion

This study examines U.S. and Chinese third and fourth graders' understanding of the basic properties. We are cognizant of the limitations in this study. First, we only assessed students' understanding through paper-and-pencil tests and did not conduct interviews to confirm our interpretations. As such, we caution against equating our scoring students' explicit understanding with their full understanding; students may possess more-explicit understanding than indicated by their responses. Second, as previously acknowledged, the third and fourth graders are different groups of students, which may affect the observed learning patterns over time. Third, there are various factors that affect student learning, yet our study cannot provide an answer for causal factors. Nevertheless, our cross-cultural analysis sheds light on student learning of the basic properties, which otherwise may be neglected. We call for an increased attention to the AP and the DP and a better use of varied contexts during instruction (especially contextual ones) to support student learning. We also argue for explicit understanding of the basic properties after students are formally exposed to the properties. The performance of the Chinese sample shows students' capacity to reach this level of understanding if appropriate instructional support is provided. These findings have practical implications for classroom instruction, teacher education, and textbook design. For instance, the basic properties should be viewed as mathematics principles that go beyond computational strategies. In other words, instead of teaching students various computational strategies, instruction should lead them to understand the undergirding properties behind those strategies. Further, textbooks and classroom instruction should help students develop meaningful initial learning through contextual support and should include more varied contexts to prompt a shift from students' implicit to explicit understanding over time. With joint efforts on improving current teaching and learning environments, it is expected that students will be able to better develop an understanding of the basic properties, which in turn may benefit their future study of algebra and more advanced mathematical topics.

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Appendix. Items in the full instrument.

Name ____________ Grade_______ School ___________ Teacher __________Date

1. If you know $7 + 5 = 12$ does that help you solve $5 + 7$? Why?
2. When solving $3 + 8$, Mary's strategy was to start with 8 and count 9, 10, 11. She ended up with her answer as 11. Is this strategy correct? Why?
3. Please use efficient strategies to solve. Show your strategy and explain why it works.

(a) $2 + 7 + 8$	(b) $(7 + 19) + 1$	(c) $2 + (98 + 17)$
Explain:	*Explain:*	*Explain:*

4. There are 8 boys and 5 girls in a swimming pool. How many children are there altogether? Both are correct. Compare the two strategies, what do you find?

John solved it with: $8+5$	Mary solved it with: $5+8$

5. There is a bookcase with three shelves. The first shelf has 7 books, the second shelf has 8 books, and the third shelf has 5 books. How many books are there in all?

John solved it with: $(7+8)+5$	*Mary solved it with*: $7+(8+5)$

Both are correct. Compare the two strategies, what do you find?

6. When solving 3×28, Mary wrote it as $\begin{array}{r} 28 \\ \times \quad 3 \\ \hline \end{array}$ Is this order correct? Why?
7. Please use efficient strategies to solve. Show your strategy and explain why it works.

(a) $(3 \times 25) \times 4$	(b) 102×7	(c) $98 \times 7 + 2 \times 7$
Explain:	*Explain*	*Explain:*

8. To solve 8×6,

(a) Mary thought:	(b) John thought:	(c) Kate thought:
$3 \times 6 = 18$, $5 \times 6 = 30$, $18 + 30 = 48$	$10 \times 6 = 60$, $2 \times 6 = 12$, $60 - 12 = 48$	Since $6 \times 8 = 48$, $8 \times 6 = 48$

Please explain why each strategy works.

9. Mr. Levin's students are tasting foods grown in rainforests. He put 5 pieces of mango on each plate and put 2 plates on each table. There are 3 tables. How many pieces of mango are there?

John solved it with: $(3 \times 2) \times 5$	*Mary solved it with: $3 \times (2 \times 5)$*

Both are correct. Compare the two strategies, what do you find?

10. The length of a rectangular playground is 118 m and the width is 82 m. What is the perimeter?

John solved it with: $2 \times 118 + 2 \times 82$	*Mary solved it with*: $2 \times (118 + 82)$

Both are correct. Compare the two strategies, what do you find?