

Tape Diagrams in US and Chinese Elementary Mathematics Classrooms

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Abstract

Tape diagrams are powerful tools for learning mathematics. This study compares US and Chinese teachers' tape diagram uses in natural classroom settings. Based on an examination of 64 videotaped mathematics lessons, we identified 109 diagram episodes. An analysis of these episodes shows that US classrooms contained more tape diagrams (pre-tapes, tapes, number lines) than Chinese lessons. However, while the Chinese lessons used tape diagrams only in story problems to model quantitative relationships, US lessons used these linear representations for both contextual and non-contextual problems but for a main purpose of finding computational answers. In general, US teachers rarely involved students in deep discussions of the tape diagrams, while Chinese teachers spent sufficient time engaging students in deep discussions of the quantitative relationships embedded in tape diagrams through activities such as co-construction, student gesturing, deep questioning, and comparisons.

Diagrams are important learning tools that can be used to model mathematics (Common Core State Standards Initiative [CCSSI], 2010; Lakin & Simon, 1987). Tape diagrams are tape-like representations that visually illustrate quantitative relationships (Murata, 2008). These diagrams are powerful because they can be used to learn varied topics across grades and they provide opportunities for class discussions about structural relationships (Ding & Li, 2014; Murata, 2008). The recently issued Common Core State Standards (CCSSI, 2010) expects U.S. students in sixth grade to be able to use the tape diagrams to solve ratio problems. However, tape diagrams are schematic diagrams that may not be very transparent and these diagrams are new to US mathematics education. In order for students to spontaneously use these diagrams for problem solving, students should have the opportunity to learn and get familiar with these diagrams in earlier grades. Prior studies on tape diagrams have mainly focused on textbook examinations (Ding & Li, 2014; Murata, 2008). However, how tape diagrams actually make an influence on student learning depends on how these diagrams are used in classrooms. No study has yet examined how US elementary teachers use tape diagrams, especially in a context of cross-cultural comparison. This study takes the first step by examining how US and Chinese expert teachers use tape diagrams in natural classroom settings. In particular, we ask: (a) What types of tape diagrams do teachers use? (b) What are the purposes of teachers' tape diagrams? And (c) How are students engaged in tape diagram uses? Below, we first review the relevant literature which serves as a conceptual framework for this study.

Literature Review

Why are Tape Diagrams Important?

The importance of tape diagrams lies in the feature of the diagrams. Larkin and Simon (1987) argued that a diagram is sometimes worth 10,000 words. This is because diagrams, in

comparison with sentences, can group all relevant information together using locations, thus avoiding the need to search large amount of information in a data structure. Diagrams also automatically offer perceptual support for problem solving, which also enhances the effectiveness of inference-making (Larkin & Simon, 1987). Tape diagrams share these critical features of diagrams and thus serve as mathematical tools to support students' cognitive development and classroom instruction (Murata, 2008). For instance, Murata argued that when used consistently, tape diagrams can mediate meaning by relating different quantities, allowing space to understand the problem situations, freezing the problems in time to clarify quantitative relationships, relating different mathematical operations, providing continuity across mathematical topics, experiencing mathematics as a subject of systematic relationships, and supporting classroom discussions through various aspects. Due to their influential nature, tape diagrams have been widely used in mathematically high-achieving countries like Singapore (Beckman, 2004; Ng & Lee, 2009), Japan (Murata, 2008), and China (Ding & Li, 2014).

The Types of Tape Diagrams

Tape diagrams are named differently in the literature. Some researchers call these type of representations strip diagrams (Beckman, 2004), some called them pictorial equations (Cai & Moyer, 2008), and still others call them the Singapore model method (Ng & Lee, 2009). In addition, some US textbooks (e.g., Go Math) termed them bar models. Regardless of the names, these diagrams share the common feature of “appropriately sized rectangles” (Cai & Moyer, 2008, p.284) to represent quantities. Murata (2008) identified various types of these diagrams in Japanese elementary mathematics textbooks, which included pre-tape, single tape, double tape, single tape with a number line, double tape with a number line, and double number line. In other words, Murata's operational definition of tapes included variations such as pre-tapes and number

lines. According to Murata (2008), pre-tapes are linear representations of objects. For example, a set of leaves arranged in a row can be considered as a pre-tape. Pre-tapes are relatively more concrete and may help to prepare students for learning tape diagrams. In contrast, replacing the rectangles with lines to create a number line, may be relatively more abstract for elementary students. The types of diagrams (pre-tapes, tapes, number lines) identified by Murata are not unique to Japanese textbooks. Ding and Li (2014) found that Chinese textbooks also used number lines in similar ways as tape diagrams. In the current study, we adopt Murata's operational definition and consider all linear representations (pre-tapes, tapes, and number lines) as tape diagrams. Since tapes and number lines were sometimes used together (Murata, 2008), we combined both into one category.

The Purpose of Tape Diagram Uses

As is the case with any diagrams, whether tape diagrams can be used to effectively support learning depends on how these diagrams are used (Larkin & Simon, 1987). Murata (2008) found that Japanese textbooks use tape diagrams for contextual problems while US textbooks also used tape diagrams for non-contextual problems. Consequently, Murata pointed out that the purposes of tape diagram use differed. While Japanese textbooks used tape diagrams to analyze the quantitative relationships embedded in a story problem, US textbooks often used these representations to complete a problem and find its answer. Such a difference in tape diagram uses is similar to what Ma (1999) reported about US and Chinese teachers' representation uses. For instance, to teach a problem like $25 - 6$, some US teachers would ask students to take away 6 objects (e.g., dinosaurs) from 25 to obtain the answer of 19, which was in contrast to Chinese teachers' focus on the regrouping process. The cross-cultural differences in representation uses may be attributed to teachers' different beliefs in representations. As Cai

(2005) reported, while US teachers value representations and strategies useful for finding answers, Chinese teachers treated concrete representations as tools to learn mathematics and thus they valued more generalized thinking. Cai concluded that if the central goal of classroom instruction is to foster the transfer of generation, students' learning should then be expected to go beyond concrete cases. In the same vein, Murata (2008) argues that the purpose of tape diagram uses should aim at quantitative relationships rather than finding specific answers because the former would better develop students' mathematical reasoning and critical thinking skills.

Students Engagement in Tape Diagram Uses

Prior studies on tape diagrams mainly analyzed textbooks (e.g., Cai & Moyer, 2008; Ding & Li, 2014; Murata, 2008). Very few studies have gone beyond static textbooks to explore how students are engaged in learning tape diagrams in complex classroom contexts. Since meaningful knowledge is not passively received but actively constructed (Piaget, 1952), an analysis of who actually uses tape diagrams during the learning process is of utmost importance. This is even truer considering tape diagrams are non-transparent schematic representations that are often hard for students to comprehend. As such, there is an urgent need to explore how students are engaged in classroom discourse on tape diagrams and how teachers may play a role in developing students' understanding of tape diagrams. When studying "analogy use" in mathematics lessons, Richland, Holyoak and Stigler (2004) classified the participant structure into teacher, students, and class. Following their logic, we are interested in exploring whether teachers used tape diagrams in interactive ways or whether they were completing and presenting complete tape diagrams without student engagement.

Even though there is little literature reporting teachers' tape diagram use in classroom, prior research on cross-cultural mathematics teaching presents general cultural images. Chinese

instructional lessons often result from careful lesson design with careful classroom delivery (Cai & Ding, 2017; Li & Huang, 2013). The downside of these well-structured lessons, as pointed out by some researchers (Mok, Cai, & Cheung, 2008), is that they may contain too much guidance (e.g., over-specific questions) which may restrict students' creative thinking. The current Chinese mathematics education field is fully aware of this type of instructional style that places an increasing emphasis on students' active engagement in learning (The Chinese Department of Education, 2011). With regards to US mathematics instruction contrasting images exist. While some US teachers employ traditional methods without engaging students in the process of learning, others employ student-centered approaches without using the teacher role appropriately (Ding, Li, Piccolo, & Kulm, 2007). In the current study, we were interested in exploring whether these cross-cultural images still apply when it comes to the use of tape diagram. In fact, even though Chinese teachers were reported to provide too much instruction that may limit students' learning opportunities (Mok et al., 2008), Ding and Li (2014) reported that recent Chinese textbooks appear to engage students in the process of co-constructing tape diagrams. This occurred when textbooks asked students to draw the second tape of a diagram, made students represent a quantity on a tape, or raised a deep question to promote students' further thinking about the relationships illustrated by a tape. Of course, opportunities are simply opportunities unless the intended curricula are transferred into classroom instruction. A recent case study (Chen & Ding, 2016) also reported that a Chinese expert teacher tended to ask comparison questions, request student gestures, and make variations on tape diagrams. This study continues this line of research (Chen & Ding, 2016; Murata, 2008; Ding & Li, 2014) by systematically exploring how US and Chinese teachers use tape diagrams differently in elementary classrooms.

Methods

Participants

This study is part of a five-year NSF funded project on early algebra. In the current study, we analyzed 8 US and 8 Chinese expert teachers' video data from the year 1 of the project. All teachers except for one US teacher were female. The criteria for selecting expert teachers included (a) at least 10-year of teaching experience, and (b) good teaching reputation. In China, all teachers have received teaching awards at a local and/or a national level. In the US, three of the eight teachers were national board certified teachers (NBCT). The rests were recommended by their principals and/or the school district. Since the project focuses on inverse relations between addition and subtraction (additive inverses) and between multiplication and division (multiplicative inverses), we recruited US teachers from grades 1-4 and Chinese teachers from grades 1-3. This is because multiplication is introduced in grade 2 in China, but not until grade 3 in the US. Consequently, four teachers in each county taught additive and multiplicative inverses, respectively. For convenience, we named these teachers as US-T1 to US-T8 and China-T1 to China-T8. Table 1 illustrates this detailed information.

Table 1. *US and Chinese Teacher Participants in This Study*

	US		China	
	Teacher	Grade	Teacher	Grade
Additive Inverse	T1, T2	G1	T1, T2	G1
	T3, T4	G2	T3, T4	G2
Multiplicative Inverse	T5, T6	G3	T5, T6	G2
	T7, T8	G4	T7, T8	G3

Data Sources

Each participating teacher taught 4 lessons directly or indirectly involving the targeted mathematical topic, resulting in a total of 64 lessons. All Chinese lessons were about 40 minutes

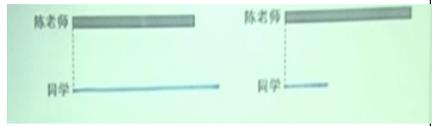
long. The US videos ranged from 34 to 84 minutes long with an average of 58 minutes. For each lesson, we used two cameras for videotaping, one following the teacher while the other followed the students. All lessons were transcribed before data analysis.

Coding and Data Analysis

We first identified diagram episodes using the 64 transcripts along with actual videos. A diagram episode is defined as one complete classroom discourse during which teachers sought to use tape diagrams (e.g., pre-tapes, tapes, and number lines). A new episode was considered if classroom discussion shifted to a new mathematical task or activity. Note that we only considered images of tapes but not actual objects as a diagram episode. If manipulatives (e.g., actual cubes) were used, we did not code them; however, if a teacher drew cubes in a row, this was considered a diagram.

To code each episode, we followed Chi's (1997) method of quantifying qualitative verbal data. In particular, we analyzed five aspects based on the conceptual framework including: type (pretape, tape, both), context (contextual, non-contextual), purpose (answer, relationship, both), who (teachers, students, class), and how (co-construction, direct telling, student gesturing, questioning, comparison, variation, and other). The codes for "how" were identified based on data screening of the US and Chinese videos. Table 2 illustrates a sample coding-sheet of a Chinese second grade teacher's lesson.

Table 2. *A Sample Coding-sheet*

Lesson	Diagram Episode	Time/Duration	Type/Context/Purpose	Who/How
2	1	3:08-8:22 (5'14'')	<p><i>Type:</i> Tape diagram</p> <p><i>Context/Purpose:</i> This is a real-world context about the students' and their teacher's favorite numbers (e.g., Ms. Chen's favorite number is 45. My favorite number is 3 more than Ms. Chen's). Tape diagrams were used to model the quantitative relationship (comparison). The screen shot was a student work selected by the teacher for class discussion.</p> 	<p><i>Who:</i> Class - Teacher provided the first tape; Students were asked to draw the second tape.</p> <p><i>How:</i> During the discussion of a selected student work, the teacher ask seemingly trivial questions that targeted on the key aspects:</p> <ol style="list-style-type: none"> (1) Why do you all draw the second tape longer? (2) What do you mean by “a bit longer”? (3) Why should it only be a little bit longer?

To check reliability, two authors independently coded 4 lessons of one US teacher and 4 lessons of one Chinese teacher. The codes were compared and reliability for both sides exceeded 90%. Disagreements were resolved through discussions among the authors, which enabled a refinement of several operational definitions. For instance, we discussed how “fine-grained” a diagram episode should be. To remain the “completeness” of a story, we decided a continuous classroom talk that involved sub-activities on the same tape diagram (e.g., discussion about how to draw a diagram, students' drawing of the diagram, comparisons of student products) would be coded as one episode. We also agreed that an array model (arranging objects in two rows for one quantity) is not a pre-tape diagram because it would not show a linear arrangement of objects. Moreover, we found that both US and Chinese teachers frequently used gestures. However, Chinese teachers tended to also ask students to gesture on the diagram. To capture the cross-cultural differences, we decided to code student gestures only as an instructional aid.

After all diagram episodes were coded, we used an excel spreadsheet to compile the different aspects and then computed relevant percentages for type of diagrams, instructional

purposes under each context, participant structures (who), and instructional aids (how) on the diagrams. Based on the results, we again inspected the diagram episodes and corresponding videos for enriched understanding of the emerging patterns. Typical experts and screen shots were identified, which will be reported along with our quantitative results.

Results

Types of Tape Diagrams in The US and Chinese Classrooms

A total of 109 tape diagram episodes were identified from the US and Chinese lessons, including 58 US episodes and 51 Chinese episodes. Each episode involved either pre-tape, tape, or both types of diagrams (US: $N_{\text{pre-tape}}=17$, $N_{\text{tape}}=39$, $N_{\text{both}}=2$; China: $N_{\text{pre-tape}}=31$, $N_{\text{tape}}=19$, $N_{\text{both}}=1$). Figure 1 indicates that US lessons contained a higher proportion of “tape” instances than pre-tapes, while Chinese lessons showed an opposite pattern. This seems to be counter-intuitive with the literature because prior studies reported that tape diagrams (not pre-tapes) have been widely used in Chinese textbook series across grades 1-6 (e.g., Ding & Li, 2014). This may be related to the fact that Chinese lower elementary grades (G1-G3) were involved in this study where pre-tapes may occur more frequently.

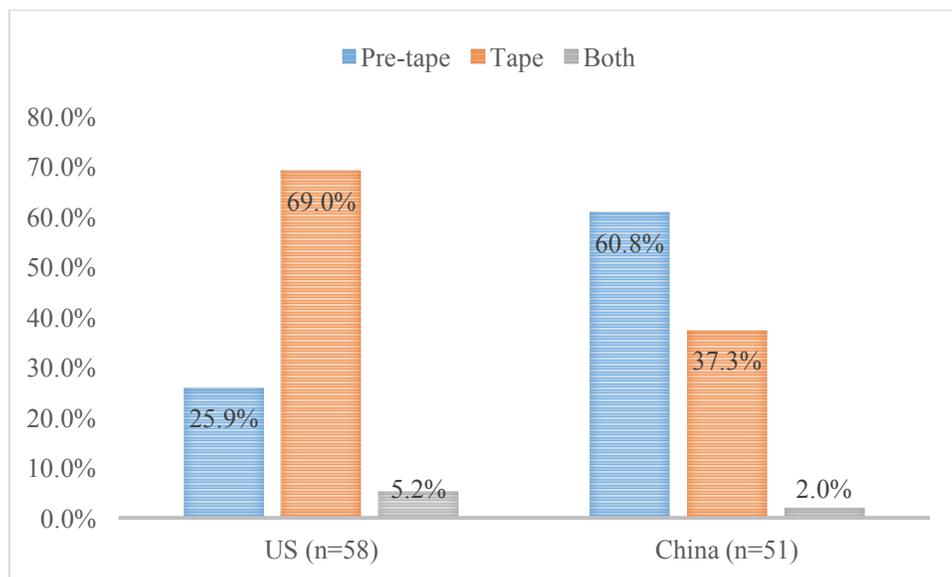


Figure 1. Episodes that involve types of tape diagrams in the US and Chinese lessons.

Table 3 lists the total number of episodes used by each teacher. As indicated, one US second grade teacher (US-T3) used the tape diagram with a high frequency. Her four lessons contained 24 diagram episodes (21 tapes and 3 both), which accounted for 41% ($24 \div 58$) of all US diagram episodes. Most of her diagrams came from her adopted CCSS-aligned textbook named *Go Math*. With regards to the Chinese lessons, the two second grade teachers used tape diagrams more frequently than the other teachers (China-T3: n=12, China-T4: n=13). Interestingly, while these two teachers taught the same four lessons, China-T3 used the pre-tapes suggested by the textbooks (n=8) while China-T4 changed some pre-tapes to tape diagrams based on her consideration of students actual learning level (as revealed by her interview). Another reason of the more frequent use of tape diagrams in the US classrooms may be related to the fact that many US lessons were much longer than the Chinese lessons, which may contain more tasks that involved the bar diagrams.

Table 3. *Number of Diagram Episodes in Each Teacher's Four Lessons*

		Pretape	Tape Diagram	Both	Total
US	T1	4	2	0	6
	T2	0	0	0	0
	T3	0	21	3	24
	T4	1	3	0	4
	T5	7	1	0	8
	T6	1	8	0	9
	T7	1	3	0	4
	T8	1	2	0	3
	Total	15	41	2	58
China	T1	3	0	0	3
	T2	3	0	0	3
	T3	8	3	1	12
	T4	4	9	0	13
	T5	1	1	0	2
	T6	4	0	0	4
	T7	3	5	0	8
	T8	5	1	0	6
	Total	31	19	1	51

A closer inspection of the actual diagram clips however, indicated important differences. While the average length of the US and Chinese diagram episodes were all around 5 minutes (US: 4'59''; China: 5'13''), the Chinese episodes spent a significant amount of time on discussion of the tape diagrams whereas many of the US episodes only contained a short discussion of the tape diagrams (often less than 1 minute) and then quickly shifted to other aspects of the task. For example, while both US-T3 and China-T4 spent about 15 minutes on one particular episode, Ch-T4 devoted almost all 15 minutes to discussion of the tape diagram which was in contrast to US-T3's episode that only contained 49 seconds of discussion focusing on the tape diagram.

Purpose of Tape Diagram Uses in The US and Chinese Classrooms

Overall, Chinese teachers used tape diagrams only in contextual contexts (e.g., story problems) with a clear purpose of modeling the quantitative relationships. In contrast, the tape diagrams in US classrooms are used in both contextual and non-contextual contexts (see Figure 2) with the purpose of finding answers and/or analyzing quantitative relationships. Figure 3 presents typical examples that illustrate cultural differences in the purposes of tape diagram uses. Elaborations are presented in the follow-up sections.

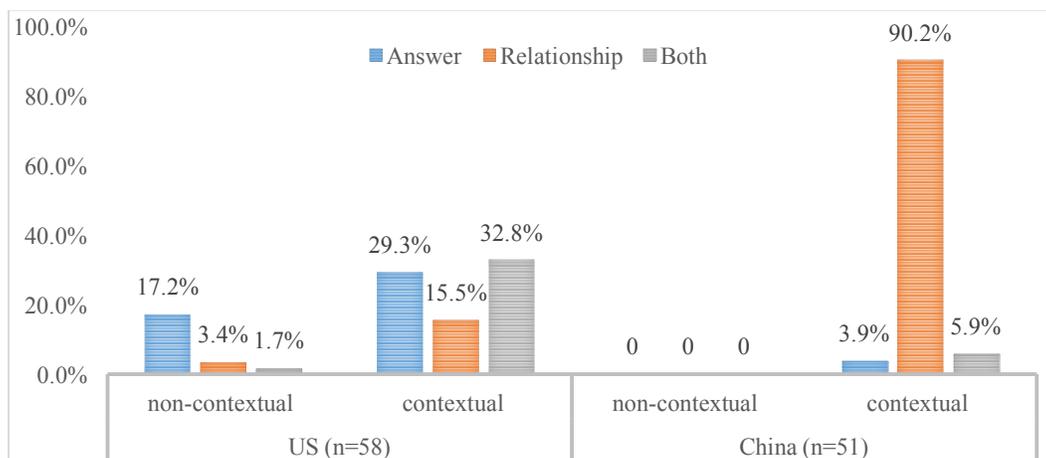


Figure 2. Different purposes of tape diagram use across contexts.

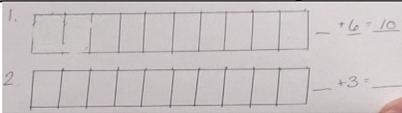
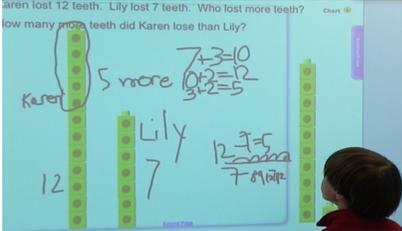
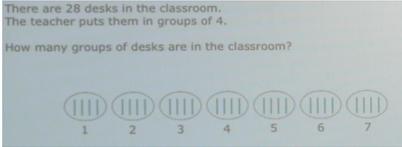
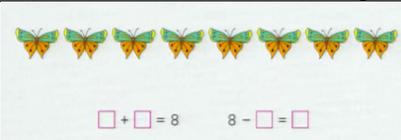
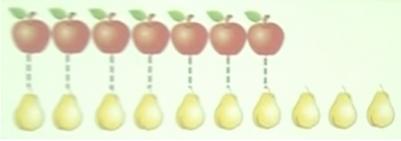
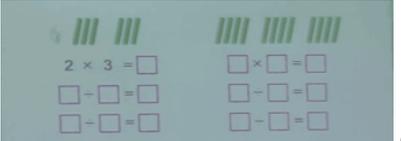
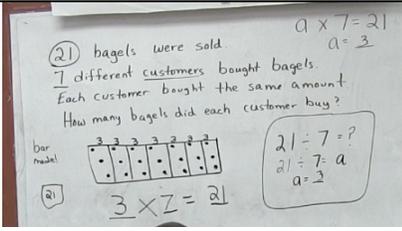
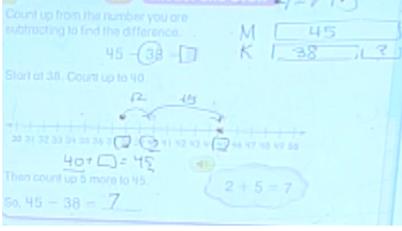
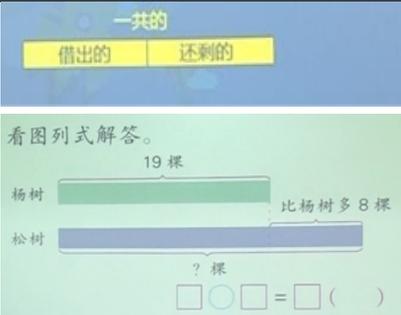
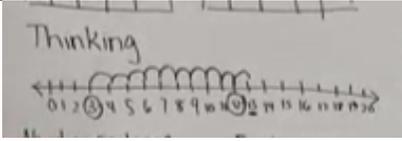
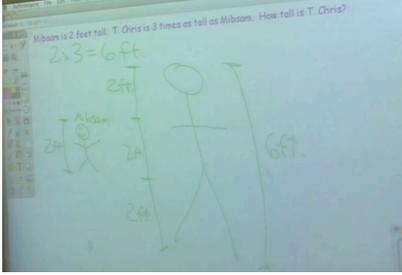
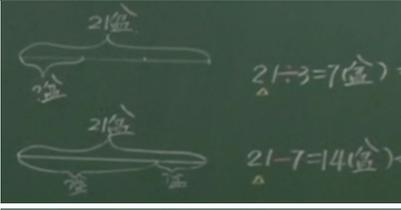
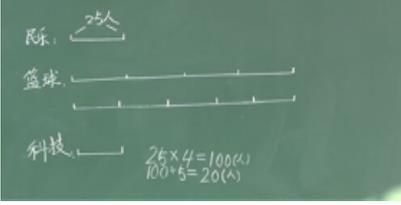
	US-Relationship/Answer	China-Relationship
Pre-tape	 <p>(US-T1)</p>  <p>(US-T4)</p>  <p>(US-T5)</p>	 <p>(Ch-T1)</p>  <p>(Ch-T3)</p>  <p>(Ch-T5)</p>
Tape	 <p>(US-T6)</p>  <p>(US-T3)</p>	 <p>(Ch-T4)</p>  <p>(Ch-T6)</p>
	 <p>(US-T1)</p>  <p>(US-T7)</p>	 <p>(Ch-T5)</p>  <p>(Ch-T6)</p>

Figure 3. Tape diagrams used in contextual and non-textual contexts in US lessons.

Purpose for using pre-tapes across contexts. As indicated by Figure 3, US-T1 asked students to use images of ten-sticks (pre-tapes) to solve non-contextual tasks such as $__ + 6 = 10$. This teacher suggested that students mark the known number on the model and count the left-over “boxes” to obtain the answer. This is different from Ch-T1’s use of the pre-tape (a set of

butterflies) to teach the part-whole relation and additive inverses through number composition of 8. Likewise, US-T4 used cubes for a contextual task comparing the loss of two children's teeth. Even though this teacher discussed the meaning of each cube stick, she only asked students to circle the part that showed "how many more" in order to obtain the answer of 5. In contrast, when teaching a similar comparison problem, Ch-T3 stressed the concept of "the same as" through a one-to-one correspondence and asked students to verbalize the quantitative relationships: the large quantity (pear) contains the same as part (apple, the small quantity) and the more than part. Similar differences were also visible in US-T5 and Ch-T5's pre-tape uses with multiplicative problems. Even though US-T5 drew 28 tallies with groups of 4 tallies circled to model the quantities relationships, she also labeled the resulting 7 groups to show the answer. In contrast, Ch-T5 expected students to generate a group of related number facts where "counting" cannot help obtain the answer if one did not understand multiplicative inverses.

Purpose for using tapes across contexts. Cross-cultural differences in the purposes of using tape diagrams, including number lines, are also evident. In Figure 3, US-T6 drew a bar model to illustrate equal sharing of 21 bagels with 7 customers. After the number sentence was generated ($21 \div 7 =$), she guided the class to distribute 21 dots evenly into 7 bar sections resulting in the answer of 3. In this case, the bar model was used to illustrate the quantitative relationships and to find the answer. Interestingly, US-T3 drew bar models to represent a comparative story problem (45 and 38 Pokémon cards). To find the answer of $45 - 38$, she drew a number line diagram teaching students to first count from 38 to 40 (2 steps) and then from 40 to 45 (5 steps), resulting in the answer of 7. Note that even though we considered US-T3's case as both analyzing quantitative relationships (bar model) and finding answers (number line), this class did not discuss the bar model. Rather, the bar model was used to simply organize the given

information in the story context. The use of bar models in the above US examples were quite different from the Chinese cases where tape diagrams were used to solely illustrate the quantitative relationships such as the part whole relation (“Total = lent-out + leftover”, Ch-T4) and relations about the additive and multiplicative comparisons (Ch-T4, Ch-T6).

The most obvious difference in tape diagram use was related to number lines. Similar to US-T3’s case above, many US classes used number lines to find answers for both contextual and non-contextual tasks (see US-T1 in Figure 3 for another example). This may explain why 46.5% of the diagram episodes (17.2% for non-contextual, 29.3% for contextual, see Figure 3) were coded as finding answers. There was only one case (US-T7) where a number line was used to illustrate a multiplicative comparison relationship (see Figure 3). US teachers’ use of number lines is in sharp contrast to the Chinese examples where number lines are flexibly used to represent quantitative relationships, including complex situations. For instance, Ch-T5 used the number line diagrams to compare two related, but different, story problems. One problem was solved by $21 \div 7$, while the other was solved by $21 - 7$. Similarly, Ch-T6 used number lines to illustrate a two-step word problem with the first solved by multiplication and the second step division.

Student Engagement with Tape Diagrams in The US and Chinese Classrooms

Tape diagrams are powerful but difficult schematic representations, which demands students to have an active involvement in the creation and discussion of the diagrams. Our finding show cross-cultural differences in the participant structure and teachers’ approaches to engaging students in diagram uses.

Overall findings. Table 4 indicates the situation of student involvement with tape diagrams and how teachers may support students in developing an understanding of tape diagrams.

Table 4. *Student Involvement in Tape Diagram Uses in the US and Chinese classrooms*

		US (n=58)	China (n=51)
Who	Teacher	55.2%	0
	Student	10.3%	0
	Class	34.5%	100%
How	Co-construction	19.0%	21.6%
	Direct telling	69.0%	0
	Questioning	22.4%	98.0%
	Comparison	13.8%	58.8%
	Variation	0	17.6%
	Student gesturing	1.7%	27.5%
	Other concrete aids	25.9%	0
	Other	10.3%	7.8%

As indicated by Table 4, while all Chinese episodes involved teacher-student conversation as a class, only 34.5% of the US episodes contain actual classroom talks centering on tape diagrams. More than half (55.2%) of the US episodes were presented by the teachers only. In these episodes, the US teacher may have asked questions, but those questions were not related to the tape diagrams. In addition, there were 10.3% of the US episodes coded as student only. In such cases, students were asked to work on tape diagrams (e.g., bar models) as independent practice without any teacher feedback provided to the class.

While zooming in on teachers' approaches to engaging students in the diagram uses, we noted that there were a similar portion of US and Chinese teachers (19.0% vs. 21.6%) who asked students to co-construct tape diagrams (see Table 4). However, an examination of the actual diagram episodes indicated that the general co-construction styles were very different. While the

US teachers tended to ask questions such as “how should I draw the tallies” or recorded students’ suggestions on the board (e.g., teachers drew out the diagrams), the Chinese teachers often asked students to draw part of the diagram themselves. In fact, even though co-construction occurred in both US and Chinese classrooms, the overall student involvement in US classrooms appeared to be much more passive. For instance, 69% of the US diagram episodes employed teachers’ direct telling, which is drastically different than the Chinese cases (0%). Instead, Chinese teachers almost always asked deep questions about the tape diagrams, which is in contrast with US cases (98% vs. 22.4%). Among these questions, many of them aimed to promote comparisons. In fact, comparison was another instructional technique frequently used by Chinese teachers (55.8%), but not US teachers (13.8%). Possibly, Chinese teachers’ unique use of variations (making changes on a tape diagram, 17.6%) may also have increased the opportunity to compare the tape diagrams. In addition to the above approaches, some Chinese teachers also asked students to come to the board to gesture on diagrams for the purpose of articulating and clarifying their thinking (27.5%). This strategy was used only once by only one US teacher (US-T4) when she asked a student to circle “how many more” in the cube trains (1.7%). Another interesting observation from the US lessons is that 25.9% of diagram episodes involved other concrete aids such as cubes, paper-folding, and fact triangles. For example, when US-T1 asked students to use the ten-sticks (pre-tapes) to find the answers (see Figure 3), she also used actual cubes to explain the procedures. Unfortunately, the mapping between these concrete aids and tape diagrams often lacked, leaving the effect of multiple representations in developing student understanding unknown.

Typical cases. To illustrate the differences of US and Chinese teachers’ approaches to engaging students in tape diagram uses, Figure 4 presents two Chinese cases (Ch-T4, Ch-T8),

which sharply differ from the approaches of US-T3 who held the highest frequency of tape diagram episodes.

	Ch-T4	Ch-T8
Co-construction		
Questioning/Comparison	<p>Why is the second bar longer/shorter?</p> <p>Why both diagrams have a "dot"?</p>	<p>Which one do you think is correct? Explain.</p> <p>Which one do you think is better? Why?</p>
Student gesturing		
Other Comparison		<p>学校里栽了92棵柳树, 栽柳树的棵数是杨树的4倍。栽杨树多少棵?</p> <p>杨树: ? 棵</p> <p>柳树: 92 棵</p>

Figure 4. Chinese ways to engage students in analyzing quantitative relationships.

In Figure 4, Ch-T4 presented a pair of example tasks on additive comparisons. The first problem (on the left) states, “Teacher Chen’s favorite number is 45. This student’s favorite

number is 3 bigger than 45. What is this student's favorite number?" The second problem (on the right) replaced "3 bigger" with "35 smaller" with the rests remaining. To understand the quantitative relationships, Ch-T4 presented the first tape for the first problem representing the teacher's favorite number 45. She then asked students to watch the movement of her computers' cursor when she pretend to draw the second tape. Students were asked to inform her of when and why she should stop. After this discussion, she asked her students to draw out the second tapes themselves. Typical student diagrams were then selected for class discussion during which the teacher compared different solutions and asked concept-specific questions (e.g., Why is the second tape longer/shorter? Why do both diagram have a "dot"?; See Figure 4). These seemingly trivial but deep questions elicited students' articulation of the quantitative relationships. For instance, multiple students explained that the "dot" indicated the part that was the same as teacher Chen's favorite number 45. To ensure students' understanding, Ch-T4 asked multiple students to come to the front of the room and had them gesture on the tape diagrams when explaining their thinking. Based on these discussions of the tape diagram, the class naturally generated the numerical solutions $45 + 3$ and $45 - 35$, which were further compared for general understanding.

Ch-T8's case in Figure 4 shares a similar instructional pattern. This teacher asked students to solve a practice problem about multiplicative comparisons, "There are 92 willow trees in a school. The number of willow trees is 4 times that of the poplar trees. How many poplar trees are in that school?" After presenting the first number line representing the 92 willow trees, Ch-T8 asked the class to draw the other number line diagram, which is a very similar co-construction technique as used by Ch-T4. Three student products (S1-correct vs. S2-wrong; S1-correct vs. S3-better) were compared and discussed (Which one do you think is correct?

Explain. / Which one do you think is better? Why?). Note that the “better” diagram that was encouraged by Ch-T8 was due to students’ adding the little tallies to separate the number of willows evenly into 4 sections, which indicates the relationship of “4 times” (Figure 4, right one). Similar to Ch-T4, students in Ch-T8’s class went to the board to gesture on the tape diagrams when explaining their thinking. It is worth mentioning that in addition to comparing different diagrams to the “same” problem as reported above, both Ch-T4 and Ch-T8 compared different diagrams to “different” problems. In Figure 4 (other comparisons), Ch-T4 used the variation technique to change the original representation of a practice problem into two related tape diagrams, which enabled comparison for structural differences. Ch-T8 used the same screen to present the practice problem of willow trees (number lines) together with the worked examples discussed earlier (pre-tapes), which enabled for a comparison of structural similarities.

Ch-T4 and Ch-T8’s cases illustrate how Chinese teachers may spend a significant amount of time focusing on a tape diagram and its embedded quantitative relationships. This is in contrast to many US episodes where discussion time was not devoted to the tape diagram. As reported in a previous section, US-T3 who had the most frequent diagram episodes tended to draw bar models for students. In most of her episodes, she did not ask students questions directly about the tape diagram. For instance, after drawing bar models, she often created a matching story problem (sometimes along with pre-tapes). She often then shifted discussion to solving the story problem by using different models (e.g., number line, fact triangle, see Figure 3 for US-T3’s example) other than the bar model. On the one hand, US-T3’s instructional style explains why a long US diagram episode may not really discuss the tape diagram. On the other hand, it indicates a unique feature of US lessons, that is, multiple concrete aids are used along with tape diagrams (25.9% of the episodes, see Table 4). While using multiple representations may be

beneficial for some students especially the novice ones, it should be noted that the connections between these concrete aids were often not explicit, resulting in missed opportunities for comparisons.

Discussion

This study is among the very first to examine how tape diagrams are used in US and Chinese elementary mathematics classrooms. Given that diagrams are powerful learning tools (Lakin & Simon, 1987) and the use of tape diagrams for learning mathematics is expected by the CCSS (CCSSI, 2010), a deep grasp of this type of model cannot be overemphasized. Our findings show that US classrooms do indeed present learning opportunities of tape diagrams, as indicated by the number of diagram episodes. However, issues revealed by the purpose and instructional approaches for using tape diagrams in US classrooms suggests room for improvement and raise questions for the field of mathematics education. Encouragingly, the Chinese instructional approach to tape diagrams provide insights that may be beneficial for US teachers. Meanwhile, US teachers' unique approaches may also shed light for Chinese teachers who are interested in enhancing the learning effect for diverse students.

Types of Tape Diagrams: Why surprising Findings?

In this study, we surprisingly found that the US lessons contain more diagram episodes including more “tapes” than the Chinese lessons. The US teacher who had the most diagram episodes is a second grade teacher using *Go Math*, a CCSS-based curriculum that introduced bar models. Even though tape diagrams are new to US teachers, it seems that when textbooks introduce this model, some teachers do try to present it to their students. The fewer number of episodes found in Chinese lessons may be related to the fact that many of the Chinese diagram episodes are long chunks with continuous mathematics talk devoted to the same diagram. In

contrast, many US episodes present tape diagrams with little or no depth, which allows time for more episodes. This is consistent with prior findings that US teachers tend to discuss more examples in quick paces in a lesson (Ding & Carlson, 2013). Moreover, Chinese teachers appear to use more pre-tapes than tapes. Given that Chinese teachers are usually loyal to textbook presentations, the diagrams occurred in classroom teaching mainly come from the textbooks (Ma, 1999). This seems to be inconsistent with the prior report that Chinese textbooks contained many tapes and number lines diagrams (Ding & Li, 2014). This inconsistency may be explained by the fact that the participating Chinese teachers in this study are lower elementary teachers (G1-3). Similar to Japanese curriculum, since the Chinese textbook series used by the teachers in this study introduce tape diagrams in a systematic fashion (Ding & Li, 2014; Murata, 2008), our observation that lower elementary teachers discuss more pre-tapes than tapes is reasonable because pre-tapes are less abstract and can lay a foundation for students' later learning of tape diagrams (Murata, 2008).

Purpose of Tape Diagrams: Computation or Quantitative Reasoning?

Regardless of the types of tape diagrams, a major difference in tape diagram uses appears to be the purpose behind why teachers use this type of representation. In this study, while Chinese teachers uniformly used tape diagrams to model quantitative relationships with contextual tasks, US teachers seemed to use these diagrams for answer-seeking. It should be acknowledged that some US teachers first used tape diagrams to represent story situations (see Figure 2 for T3, T4, T5, T6, T7), and then asked students to generate number sentences to solve the story problems. These findings are encouraging because this process is consistent with the recent research assertion of concreteness fading (Goldstone & Son, 2005). However, a closer inspection shows that most of these US teachers also used the tape diagrams to find or confirm

answers in addition to representing the problem situations. For instance, some US teachers distributed individual dots to each section of the bar model and then count the resulting dots in each section. In such a case, the bar model served as a place holder for manipulatives with the goal to find answer and not as a tool used to illustrate quantitative relationships. This is consistent with Ma's (1999) finding about US teachers' procedural use of manipulatives. In general, several of the US teachers used bar models to simply organize the given information of a story problem. Even though these teachers often asked questions about the meaning of each part of the diagram, they rarely asked deep questions about the interactions between quantities or the quantitative relationships embedded in a bar model. According to Chi and VanLehn, (2012), this is problematic because deep learning demands an understanding of interactions between quantities rather than a study of solely individual quantities.

In fact, the US teachers' treatment of number lines more clearly shows their goal of answer-seeking with both contextual and non-contextual tasks. In this study, except for one teacher (US-T7) who drew number lines to represent quantitative relationships, the other teachers (e.g., US-T1, US-T3, US-T4, US-T5) treated number lines as pre-tapes because they taught students to use number lines for counting each step on the line in order to obtain the answer. Even though knowing how to use number lines to find answers is an important computation skill, number lines as a type of tape diagram are more powerful than how we found them to be currently used in US classrooms. As illustrated by the Chinese teachers' diagram episodes, number lines can be used to model complex quantitative relationships including the ones involved in complex multiple-step problems, which is consistent with the Japanese and Singapore approaches to tape diagrams (Murata, 2008; Ng & Lee, 2012).

Two issues arise from US teachers' use of tape diagrams for solely finding answers. First, it decreases the power of tape diagrams which could have been better used to support students' quantitative reasoning and problem solving skills. Second, it decreases students' chances of mastering the basic facts because students may depend on these representations to obtain answers. These findings call for re-thinking the purpose behind using tape diagrams. Perhaps a better balance between using tape diagrams for computation and for quantitative reasoning skills is warranted. It seems that tape diagrams were invented for more than just computational purposes. In fact, using tape diagrams to find answers may not be helpful, and even perhaps harmful in developing students' computation and problem solving skills. Alternatively, US teachers may use tape diagrams with the goal of modeling quantitative relationships, as was illustrated by the Chinese teachers in this study.

Student Involvement in Tape Diagrams: What May We Learn from Each Other?

In this study, the Chinese teachers clearly engaged students in the process of constructing and discussion of tape diagrams, which is consistent with the direction of current Chinese educational reform, but different from prior observations of Chinese teachers' over-specific guidance that may restrict student thinking (Mok et al., 2008). Chinese teachers' approaches to tape diagrams reveal how a tape diagram can be sufficiently discussed for the purpose of focusing on quantitative relationships. Different from many US teachers who presented a completed tape diagram to students, Chinese teachers often employed various strategies to mentally and physically engage students in the construction of the tape diagrams. All observed strategies such as co-constructing, deep questioning, comparing, and student gesturing on tape diagrams are well supported by the cognitive literature and may be learned by US teachers. Elaboration follows.

First, Chinese teachers engage students in co-construction by asking them to draw part of the tape diagram, which is different from US teachers' requesting for oral suggestions. The process of actually drawing part of the diagram may foster students to think about the relationships embedded in the tape diagram. Such a co-construction process may also help shift students learning from passive receiving to active constructing (Piaget, 1952) because students become a co-owner of the tape diagram. In addition to co-construction, Chinese teachers uniformly asked questions about tape diagrams. Some questions are seemingly trivial but indeed critical to orienting students' attention to the quantitative relationships (Ma, 1999). Other questions request for students' comparisons, which is a powerful tool to extract underlying relationships and general problem solving methods (Goldstone, Day, & Son, 2010; Rittle-Johnson & Star, 2009). In this study, we observed Chinese teachers' comparisons of tape diagrams in different situations including within the same problem, between similar problems, and between completely different problems. Even though Rittle-Johnson and Star (2009) claimed that comparing different solutions to the same problem is most powerful, our findings indicate that types of comparisons (within the same problem, between similar problems, and between different problems) are widely used by Chinese teachers for different purposes. In fact, some Chinese teachers employed variation by altering part of the tape diagram to change it to the relevant ones. Variation is a cultural-based practice that is unique to Chinese classrooms (Sun, 2011), which provides more opportunities for comparisons. Finally, Chinese teachers uniquely asked students to come to the board to explain their thinking and provide gestures (e.g., point out a particular section or even a particular point) such as using their fingers or a teaching stick. Since gestures are embodied cognition (Alibali & Nathan, 2012), Chinese students' pointing

gestures on the tape diagram likely provide teachers timely assessment of students' understanding about the tape diagrams and embedded relationships.

As previously mentioned, we also observed US teachers' unique strategies of using other concrete aids along with tape diagrams, which is not evident in most Chinese diagram episodes. The use of multiple representations is consistent with the expectation of the mathematics education field (NCTM, 2000). In fact, instruction involving various representations has repetitively been shown to increase comprehension (Ainsworth, Bibby & Wood, 2002; Goldstone & Sakamoto 2003; Richland, Zur & Holyoak, 2007). In particular with tape diagrams that are by nature non-transparent, some students may need more help than their peers in order to fully understand them. For these students, other concrete aids such as cubes and paper folding may be helpful tools to assist students in understanding. Unfortunately, the US teachers often introduced multiple representations sequentially without making connections between the various representations. Chinese teachers who are skillful in using comparison techniques may refine this US approach to enhance the learning opportunities for more students. We are aware of the debate between multiple and single representations (NCTM, 2000; Murata, 2008). As such, while encouraging the use of other concrete aids along with tape diagrams when necessary, we stress the central focus on tape diagrams by relating the other aids to this representation.

Implementation and Future Direction

Findings from this study have implementation for both practice and research. In terms of practice, our findings suggest a need to re-evaluate the purpose and ways to engage students in tape diagram uses with US teachers. This may be done by teacher educators and curriculum designers through professional development, teacher education programs, and textbook revisions. In terms of research, our findings indicate several future research directions. One, it is

important to explore why many US teachers do not engage students in tape diagram discussion since the constructivism learning theory has been emphasized by the US mathematics education field for many years (e.g., NCTM, 2000). Is it because these teachers believe that tape diagrams are too challenging to involve students or is it because teacher themselves do not see the embedded rich quantitative relations inherent in tape diagrams? Future studies may explore these reasons through teacher interviews or knowledge surveys. Regardless of the former (belief) or the latter (knowledge), our Chinese diagram episodes may offer practical insights. As seen from the Chinese episodes, Chinese students also made mistakes when learning to draw the diagrams. However, if teachers used student mistakes as learning opportunities (e.g., comparing the correct and wrong diagrams), this may help students grasp this valuable type of representation. The second research direction is to test the identified approaches in peer countries to ensure the transferability. This can be investigated through design-based research. For instance, how do US teachers view any of the Chinese approaches to tape diagrams (co-constructing, deep questioning, comparison, student gesturing) and are any of these able to be learned by US teachers? Likewise, how do Chinese teacher view the strategies of using other concrete aids along with tape diagrams and what might be cultural barriers when employing this approach? Finally, our findings indicate inconsistency with the cognitive literature. For example, while Star and Ritter-Johnson argued the best effect occurred when comparing different solutions to the same problems, we found that Chinese teachers compared tape diagrams in all kinds of situations. Future research may conduct experiments to investigate the effects of different comparison of tape diagrams in mathematics lessons. In conclusion, with continuous research on tape diagrams and more broadly the uses of representations, we may expect to find better ways for helping students further develop mathematical quantitative reasoning skills.

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