MATHEMATICAL COMPREHENSION FACILITATED BY SITUATION MODELS: LEARNING OPPORTUNITIES FOR INVERSE RELATIONS IN ELEMENTARY SCHOOL

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ABSTRACT

The Common Core State Standards call for more rigorous, focused, and coherent curriculum and instruction, has resulted in students being faced with more cognitively high-demanding tasks which involve forming connections within and between fundamental mathematical concepts. Because mathematical comprehension generally relates back to one's ability to form connections to prior knowledge, this study sought to examine the extent to which current learning environments expose students to connection-making opportunities that may help facilitate mathematical understanding of elementary multiplicative inverses. As part of an embedded mixed-methods design, I analyzed curriculum materials, classroom instruction, and student assessments from four elementary mathematics teachers' classrooms. A situation model perspective of comprehension was used for analysis. The aim of this study was thus to determine how instructional tasks, representations, and deep questions are used for connection-making, which is the foundation of a situation model that can be used for inference-making. Results suggest that student comprehension depends more on connection-making opportunities afforded by classroom teachers, rather than on learning opportunities found solely within a curriculum. This included instruction that focused on deeply unpacking side-by-side comparison type examples, situated examples in personal concrete contexts, used semi-concrete representations to illustrate structural relationships, promoted efficiency through the sequence of presented representations, and posed deep questions which supported students' sense-making and emphasized the interconnectedness of mathematics. By analyzing these key aspects, this study contributes to research on

mathematical understanding and provides a foundation for helping students facilitate transfer of prior knowledge into novel mathematical situation.

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CHAPTER 1

INTRODUCTION TO STUDY

The goal of mathematics instruction has arguably always been to facilitate learning for understanding (Hiebert & Carpenter, 1992; Hiebert et al., 1997; Stylianides & Stylianides, 2007; Silver, Mesa, Morris, Star & Benken, 2009). In the current field of mathematics education, the ever-increasing emphasis on students' comprehension of fundamental mathematical ideas reflects this goal. How to achieve this goal, however, does not seem to be simplistic. Learning for understanding has been a central theme in mathematics education research since at least the 1930's (Greeno, 1978); but, it does "not have a single referent" (Greeno & Riley, 1987, p. 289) and the question of what exactly mathematical comprehension entails seems not to have a simple answer (Nickerson, 1985). This may be in large part due to the debate that has historically centered on whether what to learn (content) or how to learn (pedagogy) is most important for developing mathematical comprehension (Baroody, 1992; Resnick & Ford, 1981; Rittle-Johnson, Siegler, & Alibali, 2001). Today's mathematicians, mathematics educators and cognitive psychologists still do not seem to have a consensus on what constitutes mathematical understanding/comprehension (used interchangeably from this point forward; Cai & Ding, 2015).

The recent Common Core State Standards (The Common Core State Standards Initiatives [CCSSI], 2010) however, call for more rigorous, focused, and coherent curriculum and instruction. This has resulted in students being faced with more cognitively high-demanding tasks, which according to Stein and colleagues (2000) includes mathematical processes that involve building connections to prior knowledge.

When faced with unknown quantitative situations, Johnson-Laird (1983) suggest that students have a better chance of increasing comprehension when they are able to form connections among various relationships in order to create coherent mental models. Known as *situation models* in the reading comprehension research of van Dijk and Kintsch (1983), these mental models are an internal network of connections that form a "cognitive representation of the events, actions, persons and in general" that is to be learned in current and future situations. Whereas the process of forming these connections is known as connection-making, the reasoning process involved in using these connections to make conclusions in unfamiliar situations is known as inference-making. A situation model therefore acts as a catalyst for converting connections into the inferences that lead to increased comprehension. Recent empirical evidence indicates that comprehension does in fact improve when students use conceptually relevant connections to prior knowledge (Sidney & Alibali, 2015) in order to make inferences.

Although past researchers have explored mathematical understanding from both a processes lens (what leads to understanding) and a product lens (the outcome of understanding, Cai & Ding, 2015), overall understanding generally relates back to one's ability to form connections to prior knowledge (Hiebert & Carpenter, 1992). Achieving understanding by means of forming connections is therefore a common theme found across most current educational research on mathematical comprehension (Anthony & Walshaw, 2009; Barmby, Harries, Higgins & Suggate, 2009; Blum, Galbraith, Henn & Niss, 2007; Businskas, 2008; Sidney & Alibali, 2015). Few researchers however, have explored a cognitive construct for which to facilitate mathematical connection-making. How current learning environments (i.e., curriculum materials, classroom instruction)

provide learners with opportunities to develop mathematical understanding thus remains largely unknown. In this study, I explore opportunities that facilitate the creation of situation models in order to support the development of students' mathematical comprehension of multiplicative inverse relations.

Inverse relations is a fundamental concept that transcends various mathematical contexts (Baroody, Torbeyns, & Verschaffel, 2009; Carpenter, Franke, & Levi, 2003; Nunes, Bryant, & Watson, 2009; Piaget, 1952; Resnick, 1983, 1992), and therefore provides a promising domain for investigating how students develop mathematical comprehension. Even though the importance of this fundamental mathematical idea has been acknowledged (CCSSI, 2010), today's students often find inverse relations hard to comprehend (Nunes et al., 2009). This lack of comprehension may be related to the fact that in most U.S. mathematics classrooms, "instructional tasks tend to emphasize low-level rather than high-level cognitive processes" (Silver et al., 2009, p. 503), as teaching students how to synthesize or evaluate knowledge in order to develop connections rarely occurs (McKenna & Robinson, 1990). Further, U.S. instructional preference for procedural focused learning (Baroody, 1999; DeSmedt et al., 2010; Torbeyns et al., 2009) with few references to tasks that assess targeted concepts (Crooks & Alibali, 2014), may be limiting connection-making opportunities during classroom instruction. In other words, expecting students to carry out procedures that are not conceptually connected, represents a low-level cognitive demand. Students' inability to comprehend fundamental mathematical ideas may therefore be a consequence of not being able to make connections within concepts such as inverse relations that transcend across various contexts. To position the current study around the comprehension of multiplicative

inverse relations, I will first present a historical look at the topic of mathematical understanding, followed by the current status of this problem, both of which indicate that there has never been a consensus on what mathematical understanding entails and how one may achieve it (Cai & Ding, 2015).

The Problem of Mathematical Comprehension: A Historical Perspective

Prior to the 20th century, teaching mathematics in the U.S. was considered a formal means by which mental discipline was practiced (Grouws & Cebulla, 2000). This traditional view of mathematics education was based solely on student's ability to arrive at the correct answer (Walmsley, 2007). Mathematical content primarily consisted of procedures based computational arithmetic whereas pedagogy entailed root memorization. Assessments often involved drill-and-skill type activities and thus students were rarely asked to work with mathematical propositions or implied content. As a result, learning environments lacked connection-making opportunities. By the early 1900's however, calls for a more progressive style education system began to "cast doubt on the value of mental discipline" (Klein, 2003, p. 177). Brownell and Chazal (1935) are often credited with providing the first empirical evidence to support this notion when they found that arithmetic drills within a third grade classroom did not guarantee recall nor contribute to "growth in quantitative thinking" (p. 26). In essence, memorization of procedures, without understanding, resulted in "fragile learning" (Stylianides & Stylianides, 2007, p. 104). Specifically, the use of basic skills to assess understanding was found to inhibit transfer, the ability to apply skills learned in one situation to another situation (Klein, 2003). Thus, in order to help facilitate connections between learning situations, the reformed view of mathematics education placed an emphasis on

understanding. The student was no longer passive in a teacher-directed classroom; rather, instruction consisted of problem solving and discovery-based learning.

Although the idea of teaching and learning for understanding has largely dominated American schools since the early 20th century (Klein, 2003), Walmsley (2007) uses the analogy of a pendulum swing to explain how mathematics education in the U.S. has historically oscillated between the traditional and the reform ideas surrounding content and pedagogy. For example, the inception of "New Math" in the 1950's saw mathematics instruction take on a meaning-centered approach that stressed comprehension over computation. In hopes of helping the U.S. win the race to the moon, mathematicians for the first time became involved with school curriculum, and as a result, problem solving involving difficult abstract concepts replaced basic skills instruction (Klein, 2003). By the 1970's however, educators realized that "New Math" resulted in children having weaker computational skills, and thus procedural memorization once again became important as the pendulum swung "Back to the Basics" (Walmsley, 2007, p. 35). This swing back to the traditional curriculum was met by a huge push to create a conceptual framework that could identify specific characteristics related to the acquisition of mathematical understanding. Among this research, Erlwanger (1973) reiterated the idea that arriving at a correct answer did not imply comprehension, and Skemp (1976; 1978) proposed a theory of relational understanding (knowing what to do and why) versus instrumental understanding (executing mathematical rules and procedures) in attempt to increase knowledge transfer in new situations.

By 1989, the National Council of Teachers of Mathematics (NCTM) published *Curriculum and Evaluation Standards for School Mathematics*, a list of mainly

procedural skills that all students were expected to master. Even though most schools adopted NCTM's standards based curriculum during the 1990's, a decade of "math wars" raged on between those who favored drill-and-skill versus those who favored a problem solving approach for implementation of the standards (Walmsley, 2007). In an attempt to resolve these debates, researchers turned toward finding a middle ground between the traditional and reform views on mathematics education. In one of the most robust findings surrounding mathematical understanding, Bransford, Brown, and Cocking (1999) concluded that along with factual knowledge, conceptual understanding and procedural capabilities were both critical components of mathematical comprehension. Wu (1999) also argued for a common ground by stating that in the discipline of mathematics, "skills and understanding are completely intertwined…precision and fluency in the execution of the skills are the requisite vehicles to convey the conceptual understanding" (p. 14).

Instead of a benchmark that could identify specific characteristics related to the acquisition of mathematical understanding, the 1989 NCTM standards promoted an American mathematics curriculum that was a mile wide but only an inch deep (Schmidt et al., 2001). Even after the release of curriculum focal points (NCTM, 2006), individual state's curriculum expectations were still requiring teachers to cover anywhere from 26 to 89 procedural-based content topics per grade level (Rey, et al., 2006). This resulted in little time being spent on helping students build the connections necessary for conceptual understanding (Schmidt et al., 2001). Moreover, even if time was not an issue, the standards based curriculum provided no guidance on how to structure learning

environments in order to maximize students' learning opportunities (Stylianides & Stylianides, 2007).

The Problem of Mathematical Comprehension: The Current Status

Realizing the importance of making connections, the Learning Principle released by NCTM (2000) suggested that "students must learn mathematics with understanding, [while] actively building new knowledge from experience and prior knowledge" (p. 2). Initializing the inference-making process in order to make connections to prior knowledge is a critical component of mathematical comprehension, especially when students must solve problems in unknown situations. Providing students with learning opportunities to make inferences and create connections is aligned with van Dijk and Kintch's (1983) notion of *situation models*, and therefore should increase mathematical comprehension. In the view of the situation model perspective, comprehension refers to the construction of a mental representation of what the to-be-learned content is about and predicts that learners are influenced by the nature of how they form connections between the current situation and prior knowledge (Zwaan & Radvansky, 1998). Not only do learners "consistently form causal connections during comprehension, but these connections have also been shown to facilitate the retrieval of information from long-term memory" (p. 178).

The Common Core State Standards (CCSSI, 2010) provide the most current and comprehensive view on mathematical understanding. They were built from components of previous standards (i.e., NCTM 2000; 2006), and they call for three key shifts in mathematics education: rigor, focus and coherence. In terms of *rigor*, this latest round of curriculum reform emphasizes that both "mathematical understanding and procedural

skill are equally important, and both are assessable using mathematical tasks of sufficient richness" (CCSSI, 2010, p. 4). This has led to a call for K-12 mathematics curricula to include "more coverage of higher levels of cognitive demand" (Polikoff, 2015, p. 1194), such as connection-making. In contrast to previous curriculum frameworks (Porter et al., 2011), the CCSS is also more *focused*, providing descriptions of only the most significant mathematics concepts at each grade level, while also identifying important connections that teachers should help their students make. For instance, the inverse relations between addition and subtraction and between multiplication and division are systematically emphasized across all elementary grade levels. In an effort to create a greater emphasis on *coherence* of mathematics education, the CCSS suggest a need to link topics and thinking both within and across grade levels. This call for well-connected and conceptually grounded mathematical ideas which may help facilitate transfer of learning, occurs when students "build logical progressions of statements" (CCSSI, 2010, p. 6) in order to form connections for the purpose of applying "the mathematics they know to solve problems arising in everyday life" (p. 7). Forming deeper connections among mathematical experiences in order to make inferences is the essential component of a situation model (van Dijk & Kintsch, 1983).

With the publication of the CCSS, a national curriculum aimed at improving mathematical understanding finally highlights the importance of making connections between ideas in mathematics in order to facilitate the transfer of prior knowledge to novel situations. This current belief on connection-making is perhaps best summarized by the following definition of mathematical understanding provided by Hiebert and Carpenter (1992):

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections (p. 67).

If understanding results from the process of creating mental representations and generating connections within internal networks, then teaching for understanding "should consider the mental activities that a student must perform in order to understand" (Greeno, 1978, p. 267). Further, with the Common Core's call for more *rigor*, greater *focus*, and stronger *coherence*, the need to examine the quantity and nature of learning opportunities for connection-making, has become increasingly more relevant.

Purpose of the Study

Recent empirical studies have indicated that U.S. curriculum materials and mathematics instruction generally lack connections within and across topics (Ding, 2016; Ding & Li, 2010; Schmidt, Wang, & McKnight, 2005), which most likely is detrimental to the development of mathematical understanding. Even in the CCSS era, teaching and learning for mathematical understanding therefore remains problematic. Perhaps the foundational components of the Common Core have not yet reached current curriculum and instruction practices, or perhaps they have not had positive influences on today's learning environments. Examining the extent to which current learning environments expose students to connection-making opportunities should therefore be at the forefront of current research on mathematical comprehension.

The purpose of this study is to examine how current learning environments provide opportunities for connection-making in order to develop students' mathematical comprehension. Specifically, this study analyzes the learning opportunities afforded to elementary students in regards to multiplicative inverse relations. As part of this study, I analyzed both curriculum materials and classroom instruction for the purpose of identifying how best to structure learning environments in order to maximize students' mathematical comprehension. From a situation model perspective of comprehension, this study explores how reformed CCSS textbooks and classroom instruction facilitate connection-making. In particular, this study explores the following research questions:

(1) How do reformed elementary CCSS textbooks facilitate connection-making through the presentation of instructional tasks, representations, and deep questions in order to promote students' comprehension of multiplicative inverses?

(2) How do expert elementary mathematics teachers facilitate connection-making through the use of instructional tasks, representations and deep questions in order to promote students' comprehension of multiplicative inverses?

(3) How do connection-making opportunities afforded by reformed CCSS textbooks and provided by expert teachers' classroom instruction relate to elementary students' comprehension of multiplicative inverse relations?

Significance of the Study

Current research suggests that comprehension improves when conceptually relevant connections to prior knowledge are formed (Sidney & Alibali, 2015), yet few

have explored how to facilitate mathematical connection-making from the situation model perspective of comprehension. Even though the Common Core State Standards (CCSSI, 2010) include a call for students to make connections between fundamental mathematical concepts such as inverse relations, rarely studied, are the ways to facilitate these relationships. This has resulted in a limited understanding about how reformed CCSS mathematics textbooks and current classroom instruction facilitate connection-making. Because text comprehension shares features with problem solving in that learners are presented with information and must try to form connections through explanatory inferences, this study takes an integrated comprehension perspective in which mathematical understanding is explored through connection-making, the foundation of a situation model that can be used for inference-making. This study is important because few have integrated a situation model perspective with research outside of the reading comprehension domain. Even rarer, are studies that have explored how textbooks and classroom instruction facilitate the creation of situation models to support the learning of fundamental mathematical ideas.

The second importance of this study is to explore how students are exposed to fundamental mathematical concepts that are emphasized by the Common Core (CCSSI, 2010). This study involves inverse relations, a fundamental mathematical idea that transcends various mathematical contexts (Bruner, 1960; CCSSI, 2010), and is critical for comprehension across all levels of mathematics (Baroody, Torbeyns, & Verschaffel, 2009; Carpenter, Franke, & Levi, 2003; Nunes, Bryant, & Watson, 2009; Piaget, 1952; Resnick, 1983, 1992). I expect that the findings from this study will contribute to improving curriculum design and will help to enhance classroom teaching of inverse

relations. The coding framework developed for this study may also be useful for future studies surrounding the comprehension of other fundamental mathematical concepts.

This chapter has provided both a historical perspective and the current status surrounding the problem of mathematical comprehension. I have outlined the research questions that will be explored in this study and have linked them to both the purpose and overall significance of this project. Chapter 2 will provide more details on the situation model theoretical framework used in this study and will review current literature on how to facilitate connection-making. This review of literature will also include the current state of knowledge surrounding children's comprehension of inverse relations. An examination of prior studies involving U.S. mathematics textbook analyses, as well as a review of past research that has studied expert teachers will also be included in Chapter 2. The third chapter of this dissertation will explain the research design of this study and will include a detailed description of the data sources and procedures that were used to analyze each research question. Chapter 4 will present the findings surrounding the three research questions involving connection-making opportunities afforded by textbooks and elementary expert teachers. The final chapter of this dissertation, Chapter 5, will provide conclusions and implications for practice and future research involving mathematical comprehension.

CHAPTER 2

LITERATURE REVIEW

This chapter begins by outlining the situation model theoretical framework. This includes a discussion on the theory of mental modeling, a review of seminal research in the domain of reading comprehension, and a description of how reading comprehension viewed through the lens of mental modeling leads to the situation model perspective of comprehension. Next, how to facilitate situation models is explored through a review of literature that involves analyzing connection-making opportunities that can be afforded to learners through the use of instructional tasks, representations and deep questions. A discussion on the current knowledge surrounding elementary school children's comprehension of inverse relations will follow. Finally, prior research on expert elementary teacher's classroom instruction and prior findings of studies involving U.S. mathematics textbook analyses will lead to the justification and research questions for this study.

Situation Model: What is it?

As described in the previous chapter, mathematical comprehension has evolved to require conceptual understanding. Conceptual understanding is a "deep understanding of the subject matter, so that the information acquired can be used productively in novel environments" (Kintsch, 1998, p. 294). In contrast to the act of simply doing mathematics, comprehension involves the ability to make connections and easily transfer knowledge between different aspects of mathematics, in order to form coherent mental representations of quantitative situations (Langer, 1984; Shepherd, Selden & Selden, 2012). Known as *situation models* (van Dijk & Kintsch, 1983) these mental

representations are a critical component of comprehension. Ultimately, they determine one's ability to make connections among various relationships in the surrounding world, which leads to improved mathematical comprehension (Johnson-Laird, 1983).

Kintch (1986) suggests that a mental model serves as both a representation of what is to be learned and a tool that learners use to develop and assess their own comprehension. This suggests that comprehension requires both a process and a product. Specifically, Kintch (1988) claimed that in order for the deepest level of comprehension to occur, a situation model must be formed for propositions to be transformed into understanding. Other researchers agree that the most influential factor of comprehension is a learner's ability to construct a coherent situation model (Glenberg, Kruley, & Langston, 1994; Graesser, Millis & Zwaan, 1997; Perfetti, 1989; Zwaan, Magliano, & Graesser, 1995). In order to determine how specific learning opportunities can lead to the construction of situation models, it is important to analyze the mental processes involved in making connections within mental representations.

Mental Modeling

The concept of a mental model, an internal representation of the thought process that occurs when an individual attempts to create meaning from external experiences encountered with the world (Bruner, 1990), developed as a result of the epistemological paradigm shift from behaviorism towards constructivism. First introduced in 1943, cognitive psychologist Kenneth Craik hypothesized that an internal model is created when someone has a restricted knowledge of some unknown phenomena. In order to provide rational explanations about that phenomena, an individual creates practicable methods that consist of forming integrated and connected internal models of both

semantic and situation specific knowledge (Seel, 2006). Craik's hypothesis was centered on the notion that the mind processes information similar to the way that mechanical devices (i.e. calculating machine, anti-aircraft predictor, Kelvin's machine for predicting tides) were used to aid thought and calculation (Craik, 1943). In essence, the mind acts as a machine in the process of creating small-scale models of reality for drawing testable inferences about future situations (Johnson-Laird, 1983). One might relate this to current research on conceptual change (see, for example, Dole & Sinatra, 1998; Lombardi, Sinatra & Nussbaum, 2013), which is based on learners' modifying and restructuring internal mental models through the process of assessing whether knowledge presented in current external situations is more plausible and convincing than their prior knowledge.

Bransford, Barclay, and Franks (1972) provided the early evidence that comprehension was significantly influenced by the nature of a situation when they argued that linguistic input of text merely acted as a cue which people used to recreate and modify previous knowledge of the world whereas, comprehension involved making connections in order to construct a model of a described situation. Using undergraduate psychology students as subjects, they examined the difference between an interpretive versus a constructive view of sentence memory. They looked at sentences that provoked only simple recall of facts versus sentences that provoked students to draw inferences. Subjects were given either a non-inference (NI) scenario (e.g., Three turtles are sitting beside a log. A fish swam under them.), or a potential-inference (PI) scenario (e.g., Three turtles are sitting on a log. A fish swam under them). In PI, but not in NI, it is possible to infer that the fish had swum *under the log* as well as under the turtles. The results of asking the subjects whether the fish swam *under the log*, revealed that those given NI

tended to not recall the wording that the fish swam *under the log*, whereas those given PI, were confident that they had received the *under the log* wording. This false recognition suggested that when given PI scenarios, the subjects related proposition as integrated concepts and went beyond the semantic structure of the sentences to construct an internal visual image of the entire scenario. By 1983, Johnson-Laird had coined the term *mental model* to describe this assumption that individuals create internal representations of external situations.

Mental models serve as an important tool for reasoning and making connections in that learners must interact between prior knowledge and current stimuli in order to determine how current or future actions might change their thought process (Long, Seely, Oppy & Golding, 1996). Through means of making connections, it is believed that individuals learn from the reasoning process of their "subjective experiences, ideas, thoughts and feelings" (Seel, 2006, p. 86). The idea that reasoning depends on mental models and not on logical form is the basis of Johnson-Laird and Byrne's (1991) theory of mental models. According to their theory, the reasoning that occurs as a result of the creation of mental models is not based on formal rules of reasoning, but on the structure of the external situation that the models are formed to represent. They argued that mental models are based on a principle of truth, in that they represent only situations that are possible. This idea of truth however does not leave out the possibility that the model could represent counterfactual beliefs (Byrne, 2005). Instead, they proposed that reasoning occurs in a recursive manner that involves making connections to "thought, meaning, grammar, discourse, and consciousness" (Johnson-Laird, 1983, p. xi). The recursive process involved in establishing valid conclusions therefore includes relying on

counter-examples to refute or refine prior inferences. This in turn strengthens and creates numerous connections. Without a counter-example or external stimuli to challenge current knowledge, individuals deem their mental models as valid representations of reality to the point that no further connections are made and comprehension is therefore not enhanced (Johnson-Laird, 1983).

Reading Comprehension

One important application of being able to construct meaning in an externally located situation is when a learner is trying to comprehend written language. Cognitive and educational psychologists have extensively explored reading comprehension as a means by which to explore both the process and the ability to cognitively interact with external stimuli in order to create meaning (Lorch & van den Broek, 1997). While it is widely believed that comprehension can take the form of either a shallow or a deep level of meaning (Craik & Lockhart, 1972), reading comprehension literature illustrates that comprehension should not be considered a binary phenomenon. Instead, the connections and the level of understanding that an individual forms can fall across a wide spectrum of possibilities and are shaped by the context and learning opportunities presented within the learning environment. Like all research on understanding, reading comprehension research initially focused primarily on memorization.

Bartlett (1932) challenged this passive and receptive view of understanding, by identifying reading as an active and constructive process. In one of his most famous experiments, Bartlett showed that subjects who were asked to recall a Native American reading passage about ghosts often drastically changed details and the overall context of the story. He concluded that the reconstructive nature of human memory is often

influenced by cultural background and one's prior knowledge about the world. By distinguishing between a text's surface representation and a reader's mental representation, he found that "readers' memories for textual information were systematically distorted to fit their own factual and cultural knowledge" (Lorch & van den Broek, 1997, p. 214). Barrlett's findings represented the first empirical reading comprehension evidence that people's memory depends highly on prior knowledge and the ability to reconstruct instead of merely duplicate previous experiences. Thus, early reading comprehension literature often focused on advanced organizers (Ausubel, 1960) and adjunct questions (Frase, 1968; Rothkoph, 1966), tools used to facilitate connections to prior knowledge.

Advancements in linguistic and artificial intelligence research during the 1970's (Lorch & van den Broek, 1997) shifted the focus of reading comprehension toward examining mental processes that occur during reading. Topics such as encoding, the use of representations, and the way in which forming connections helped to retrieve knowledge, became the backbone of new reading comprehension theories that attempted to serve as models for knowledge acquisition. Among these noteworthy cognitive theories, Kintsch's (1974) theory of reading comprehension along with the mental modeling theory of Johnson-Laird and Miller (1976) suggests a text processing theory that views reading comprehension as mental modeling. This theory views reading as the process of constructing mental models of written text and comprehension refers to the coherence of the situation model that a reader constructs.

Reading Comprehension as Mental Modeling

The Construction-Integration (CI) reading comprehension model presented by Walter Kintsch (1988) suggests that a learner creates a mental model when reading text. Not only does this mental model serve as a representation of the text, but also it is used by the reader to develop and assess comprehension. According to Kintsch (1986), mental models are created as a result of the inferential process of evaluating propositions in relationship to three components of mental representations: surface components, textbase representations, and situation models. The surface component includes a verbatim representation of the text in which the words and phrases themselves are encoded into memory. A textbase component represents the semantic structure of the text in that it captures the linguistic relations among propositions represented in the text. In short, the textbase describes the meaning of the text. The final component, the situation model, involves a reader's drawing on prior knowledge to create a more complete mental representation that is based upon making connections between the situation the text represents and other contexts to which that text may be applied.

The process of forming these mental representations begins with the reader creating an initial list of propositions based solely on the words that they are reading (surface component). These propositions turn into a network of propositions (textbase component) as entire sentences are read and the reader attempts to begin making meaning of the text. Indications of understanding based on the surface and textbase components include a learner's ability to verify and recall statements from the text, answer questions about explicit content discussed within the text, and form a summary of what they have read (McNamara, Kintsch, Songer & Kintsch, 1996). Because these first two components

mainly involve direct translation of what is explicitly written in the text, learners are required to make few inferences and therefore limited connections to prior knowledge are needed. The first two components constitute procedural understanding, whereas the final component of a situation model represents the conceptual understanding process of connection-making. In order for the deepest level of comprehension to occur, a situation model must be formed so that propositions can be transformed into understanding that can be applied in future situations (Glenberg et al., 1994; Graesser et al., 1997; Zwaan et al., 1995).

A situation model is a catalyst that gives students the ability to make strong connections within mental representations, for the purpose of solving problems, making inferences, and drawing conclusions in unfamiliar situations. Figure 1 illustrates the relationship between the three mental representations of Kintch's (1988) CI model, concerning the depth level of comprehension. The figure indicates that as the learner interacts with propositional representations, the surface component has the potential to evolve into the textbase component and/or the situation model, which in turn indicates the depth level of comprehension that is achievable. Stronger connections formed within propositional representations lead to a deeper level of comprehension, as indicated by the downward comprehension arrow.

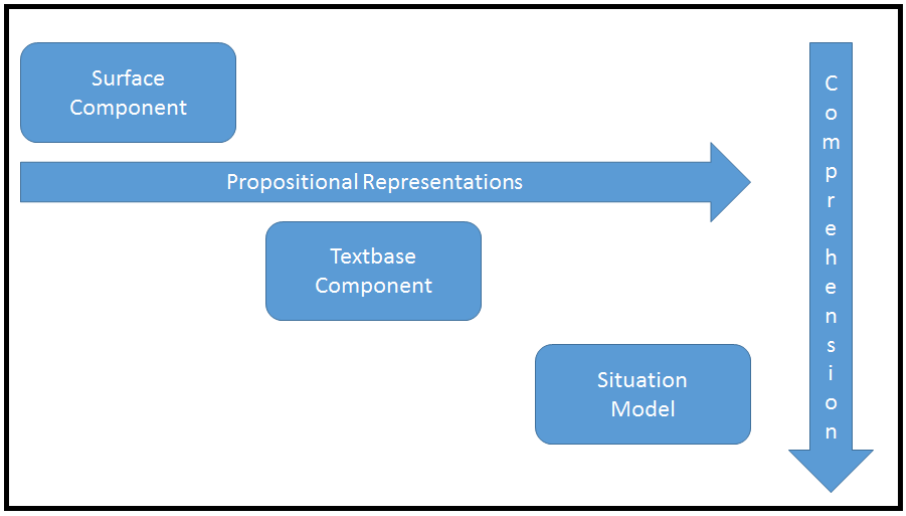


Figure 1. Depth level of comprehension within Kintch's (1988) CI model.

Because "text comprehension is defined as the process of constructing a connected memory representation" (Lorch & van den Broek, 1997, p. 219), Kintsch's CI model is based upon an inferential process that evaluates propositions in relationship to the three types of mental representations that a reader forms during the act of reading. The emphasis that the situation model component places on inference allows it to be transferable to the process of acquiring mathematical comprehension, a context where making inferences about unknown quantitative situations is highly dependent on one's ability to form connections. Further, whereas reading comprehension involves more of an internal reasoning process, mathematical comprehension most often requires students to articulate their reasoning process step-by-step explanations of written solutions. Integrating reading comprehension and mental modeling research to create a situation model framework for mathematical comprehension is therefore in direct alignment with the CCSS suggestion that students must be able to form better connections among

mathematical experiences in order to make better inferences involving their quantitative reasoning process.

Creating an effective situation model entails that a learner must implement a deep level of inference making that demands connecting explicit and implicit information to one's prior knowledge (Zwaan & Madden, 2004). Because the three components of mental representations compete for a limited amount of working memory resources, the degree to which a situation model is formed is likely hindered by the capacity of the working memory (van Dijk & Kintch, 1983; Schmalhofer & Glavnov, 1986). When learners encounter pieces of content that do not agree with their current mental representations, they must activate their long-term memory and begin searching for similar prior experiences (Pinker & Bloom, 1990). Inferences may be limited if one cannot create or find connections to a prior experience related to the current learning situation. Without a well-developed situation model, propositions remain propositions and a deep level of comprehension is difficult to achieve. Instruction that provides numerous opportunities for connection-making will help students construct and strengthen their situation models and will ultimately lead to an increase in mathematical comprehension.

Situation Models

To form mental models, Rumelhart, Hinton and Williams (1986) claimed that the cognitive process consists of two parts. The first part is concerned with the activation of schema. A schema, first introduced by Piaget (1928), is a generic cognitive data structure that an individual forms in order to organize, interpret, and store knowledge. When a schema is activated, an individual recalls a past action and is able to manipulate that

action in order to apply it to a new situation. The second part of the cognitive process is concerned with the construction of a situation model that represents the external world. The construction of this model is based on intended actions present in the schema and the interpretations of what would happen if those actions were executed (Rumelhart et al., 1986). In other words, as described by Zwaan and Radvansky (1998), schema are mental models of stereotypical situations whereas a situation model is a mental representation of a specific real life experience. Whereas, the first part of the cognitive process takes input from the world and through interpretation produces reactions; the second part predicts how the input would change in response to those reactions in future situations. This illustrates that a critical component of the learning process involves cognitively constructed situational representations.

Greeno (1991) explained that a situation model "works because operations with mental objects in the model have effects that are like the effects of that operation on the objects that the model represents" (p. 178). By stating that understanding involves "knowledge that results from extensive activity in a domain through which people learn to interact successfully with various resources of the domain" (p. 170), Greeno further argued that mathematical comprehension involves the ability to find and use concepts that have been triggered by an external situation. Take for example if $a + b = c$ then $c – a = b$, the arithmetic conjecture that illustrates the complement principle of the inverse relation between addition and subtraction (Baroody, 1999). When encountering this statement in text, students first need to be well versed with the symbolic notation of mathematics. If the student's surface and textbase components have been established, all symbols are properly interpreted and mathematical meaning is developed. However, if

the construction of meaning ceases at this point, although students might be able to recall this conjecture or even replicate the mathematical procedures needed to simplify this inverse expression, they are often unable to transfer their knowledge to new situations. According to Bloom, Krathwohl and Massia (1984), when transfer does not occur, comprehension does not occur. This suggests that the construction of a situation model is needed for full comprehension of this mathematical conjecture. Figure 2 presents a possible situation model that a student might form for this mathematical conjecture.

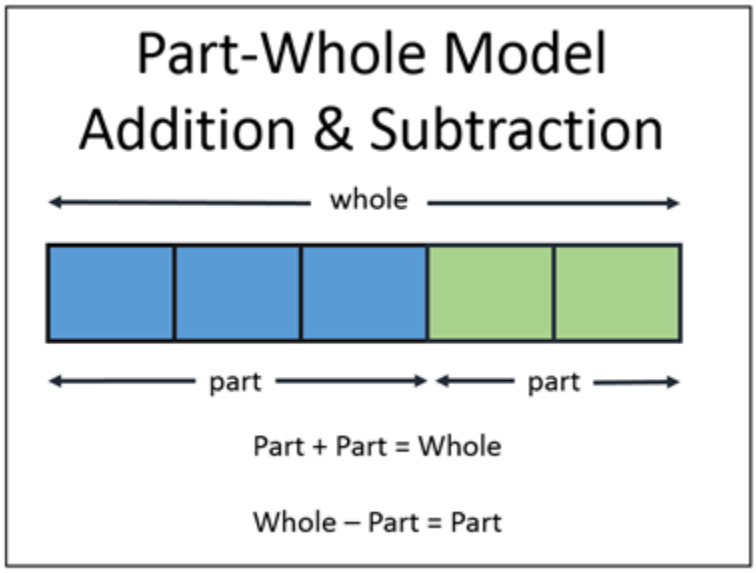


Figure 2. Part-Whole Model (Ginsburg, Leinwand, Angstrom & Pollok, 2005).

Although this is an example of an external model of the specific situation representing the fact that if 3 + 2 = 5 then 5 – 2 = 3, if given adequate connection-making opportunities to develop a schema that is based on deep structural knowledge (i.e., part + part = whole, whole – part = part), students will be able to transfer the principle-knowledge to novice situations (Goldstone & Son, 2005). In other words, if presented with the equation

7 + 4 = 11, a student with a well-defined part-whole situation model may activate connections that allow for the updating of new model parameters and in turn this will most likely lead to the inference that 11 – 4 must equal 7.

A well-developed mathematical situation model is therefore connected to prior mathematical knowledge and allows a learner to use new content knowledge in "novel environments and for unanticipated problem solving tasks" (McNamara et al., 1996, p.4). Multiple studies (e.g., Kintsch, 1994; Osterholm, 2006; Weaver, Bryant & Burns, 1995) have shown the important role that situation models have in altering the definition of learning from not what is simply to be remembered, but rather what conclusion can be drawn based on the inference making process. Kintch (1986) noted that in both a first grade and a college setting, comprehension increased once a situation model was formed for a mathematics based word problem. This occurred because learners were able to make inferences based on prior mathematical knowledge and were able to reconstruct the problem using their situation model instead of simply recalling the problem by use of the textbase component. The mental representation on which recall is based differs from the representation on which inference is based, and thus additional factors other than the ability to form a coherent textbase are needed for the creation of a situation model. Analyzing experimental variables within learning opportunities that affect the ability for readers to make connections and draw inferences is essential in the pursuit of helping students enhance their ability to create mathematical situation models.

Viewing the cognitive process as a reasoning process built on creating a mental model of an external situation, illustrates the key connection that must be made between a learner's available schema (internal) and the presented external input. Namely, as a

learner is presented with new information, a continuous updating of his or her mental representation occurs (Zwaan & Madden, 2004). This suggests that the development of schemas lays the foundation for the construction of a situation model (Rumelhart et al., 1986). Although no literature was found specific to developing situation models for comprehension of inverse relations, Kintsch and colleagues (Cummins, Kintsch, Reusser & Weimer, 1988; Greeno & Kintsch, 1985; Weaver & Kintsch, 1992) performed a series of studies illustrating the role that situation models have in simulating the construction of cognitive representations needed for enhancing children's comprehension surrounding the conceptual structure of both arithmetic and algebraic word problems. Similarly, Leiss and colleagues (2010) investigated the overall role that situation models played in comprehension of the Pythagorean Theorem and linear functions among a sample of ninth graders. Their results showed that strategies for constructing an adequate situation model have a significant influence on mathematical comprehension. Further, how much a situation model influences comprehension is related to intra-mathematical competencies. Procedural skills alone will therefore not suffice to convey complex competencies such as connection-making that are required for conceptual understanding of mathematics. Viewing mathematical comprehension from a situation model perspective provides "an important support in the diagnosis of student learning processes" (p. 139), which is evident by the fact that when students are asked to "consciously construct and thus externalize their situation model…students' difficulties become visible to the teacher at an early phase of development" (p. 139).

Situation Model: How to facilitate it?

Cognitive and educational psychologists, as well as mathematics educators, have found that providing students with worked examples facilitates the development of schemas that ultimately cause an increase in later transfer of solving new problems (Chi & VanLehn, 2012; Sweller, & Cooper, 1985). It has been shown that an increase of variability of examples helps to better support a learner's ability to make connections to the abstract principles underlying these example tasks (Renkl, Atkinson, Maier & Staley, 2002). In addition, literature suggests that during initial learning, activation of a learner's current schema is best done through concrete representations (Resnick & Omanson, 1987); however, fading from concreteness into abstract representations has been shown to deepen comprehension (Goldstone & Son, 2005). Rubin (2009) argues that comprehension is also increased when students are asked questions aimed at developing deep connections between and within concepts. Taken together, instructional tasks, types of representations, and deep questions used during instruction, have all been shown to affect a student's ability to form connections and make inferences, the critical components of how mathematical comprehension can be facilitated by situation models. In fact, these components have been recommended as key principles to organize instruction to improve student learning (Pashler et al., 2007).

Instructional Tasks

Using a greater variability of instructional problems can establish deep connections that result in an increase of encoding and extraction of core underlying mathematical principles (Renkl et al., 2002). The creation of a situation model is thus facilitated by empirical evidence that students' mathematical comprehension is enhanced

by forming connections within instructional tasks. For example, Rittle-Johnson and Star (2007) found that seventh graders learning to solve algebra equations gained more procedural and more conceptual knowledge when the use of contrasting alternative solution methods (drawing connections between methods) were used during instruction, as opposed to when solution methods were analyzed one at a time. Moreover, Booth and colleagues (2013) revealed that when students form connections between correct and incorrect examples, both procedural and conceptual understanding improves in Algebra 1 students. Specifically, their results suggest, "incorrect examples, either alone or in combination with correct examples, may be especially beneficial for fostering conceptual understanding" (p. 24). This means that not only does the use of incorrect examples help reinforce students' surface and textbase components of comprehension (procedural), but their use also helps to strengthen situation models (conceptual) by forcing students to make inferences involving the similarities and differences between quantitative situations. Further empirical evidence of instructional tasks suggest that students perform better when teachers alternate between examples and practice problems, as opposed to all examples occurring at the beginning of instruction (Trafton & Resier, 1993).

Literature in mathematics education suggests that providing students with worked examples, "a step-by-step demonstration of how to perform a task or how to solve a problem" (Clark, Nguyen & Sweller, 2006, p. 190), is one way to increase initial comprehension within cognitively high demanding tasks (van Merriënboer, 1997; Renkl, 1997). Several empirical studies have supported the use of worked examples. For example, Sweller and Cooper (1985) conducted one of the first empirical studies of worked examples in mathematics when they studied how students solved algebraic

manipulation problems (e.g., express b in terms of the other variables in an equation similar to $b = a \times b + g$). Their study consisted of comparing performance between two groups of ninth graders, one that was given worked examples during knowledge acquisition and one that was not. The results indicated that the use of worked examples improved test performance, but only on questions that were similar to the worked examples. In other words, the experimental group had no advantage when presented with dissimilar problems, which suggested that transfer was not enhanced by the use of worked examples. As a follow up, Sweller and Cooper (1987) found that significant learning time was needed in order for students to form connections and to demonstrate transfer. Given adequate time, students in the worked example group outperformed their peers whom were not provided with worked examples. In addition to allowing sufficient time on worked examples, other researchers have suggested that students be actively engaged in the knowledge-construction process of unpacking worked examples (Ding & Carlson, 2013), and yet other have argued for the need to space learning over time (Cepeda, Pashler, Vul, Wixted, & Rohrer, 2006; Ding, 2016; Pashler et al., 2007).

In other recent research on worked examples, Renkl and colleagues (2002) studied forty-eight student teachers' abilities to learn probability calculations. This study included one experimental group of student teachers who were given worked examples and one control group of student teachers who were left to learn the content on their own. Although there was no difference in performance between the experimental and control group for problems that were similar to the worked examples, the experimental group did significantly outperform the control group when presented with far-transfer problems in which the structure remained constant but the surface feature of the questions were

different from the worked examples. This suggests that the learned worked examples may be more of an influential factor when students form connections in order to activate the inference making process for situations that occur across, rather than within, mathematical contexts. A critique of this study is that subjects in the worked example experimental group were encouraged to partake in think-aloud activities, which could have had a significant effect on transfer. Further, empirical evidence surrounding worked examples and instructional tasks has primarily only focused on high school aged students and has not looked at their effect on long-term transfer.

The evidence that does exist for the use of worked examples suggests that introducing a formulated problem that includes steps leading to a final solution provides students with an expert's mental model that can be used to form connections and facilitate transfer (Chi & VanLehn, 2012; Sweller, & Cooper, 1985). In order to strengthen these mental models, "multiple examples with constant structure (i.e., underlying solution rationale) and varying surface characteristics" (Renkl et al., 1998, p. 94) should be used during instruction. When a limited amount of working memory is available worked examples help to reduce intrinsic cognitive load (Paas, Renkl & Sweller, 2003). The instructional purpose of worked examples is to develop schema that will increase the likelihood of transfer (Paas, Renkl & Sweller, 2003). This occurs when connections are formed between worked examples and as a result, learners can extract the underlying mathematical principle. In turn, enhanced mental models can be applied to problems with different surface characteristics (Renkl et al., 1998) and eventually to problems that arise within completely new relevant situations.

From the perspective of a mathematical situation model, the use of worked examples helps students develop a schema by facilitating connection-making between prior knowledge in order to increase the likelihood of transfer (Kirschner et al., 2006; Paas, Renkl & Sweller, 2003). Corresponding practice problems should therefore have connections to the worked examples in order to practice the learned knowledge. In addition, because worked examples should be built on student's prior understanding, review tasks used during instruction should also provide opportunities to review and form connections to relevant prior knowledge. Unfortunately, U.S. mathematics teachers spend little time unpacking worked examples (Ding & Carlson, 2013) and the amount of instructional time allocated to review is limited (Jones, 2012). This may in large part be due to the desire that U.S. teachers have for allowing enough classroom time for student practice (Stigler & Hiebert, 1999), which most likely results in learners' not having ample opportunities to develop connections to targeted mathematical concepts.

Paas and VanMerrienboer (1994) also suggested the use of highly variable examples (in contrast to using uniform examples) for increasing the likelihood of transfer. It should be noted however that although problem variability within instruction helps to support a learner's ability to form connections to fundamental underlying principles (Renkl et al., 2002), it has also been shown that as students develop greater expertise, decreased worked examples and increased problem solving opportunities appears to improve comprehension (Renkl, Atkinson & Grobe, 2004; Schwonke, et al., 2007). Moreover, considering the time constraint of a mathematics lesson, a classroom teacher may not be able to discuss too many worked examples within a short lesson (Ding & Carlson, 2013). Nonetheless, according to Kaluga and colleagues (2003),

worked examples are more favorable in earlier stages of learning. This is most likely because initial learning opportunities require students to use more working memory when forming connections to prior knowledge that are needed to acquire understanding (Renkl & Atkinson, 2007). It appears as if there may be many opportunities to enhance connection-making within the instructional tasks used during elementary school mathematics instruction.

Representations

Objects that students can physically grasp their hands around (e.g., blocks, rods, tiles) are often found within elementary school mathematics instruction (Clements, 1999). These concrete objects are designed to allow students to explore a hands-on experience with various mathematical concepts. By interacting with concrete manipulatives, the goal is that students will be able to form stronger connections to their internal mental models. The use of concrete representations during instruction have in fact been shown to activate student's internal mental representations (Baranes, Perry & Stigler, 1989; Kotovsky, Hayes, & Simon, 1985). Hauser (2009) mentioned, "students who use concrete materials develop more precise and more comprehensive mental representations" (p. 1). Martin and Schwartz (2005) support this notion and believe that by interacting with concrete manipulatives, students form stronger connections to their internal mental models, which helps to increase mathematical comprehension. The belief that concrete representations help to strengthen mathematical situation models, has been empirically supported by Harrison and Harrison (1986), who provided descriptions of successful learning activities that use concrete objects such as rulers and place value cards. This is similar to findings that suggest appropriately teaching students how to interact with concrete manipulatives

help them to better formulate connections to their own abstract ideas (Brown, McNeil & Glenberg, 2009).

Although literature suggests that concrete representations seem to best activate schemas during initial learning (Resnick & Omanson, 1987), they also often contain irrelevant information that can prevent students from making connections to underlying principles (Kaminiski, Sloutsky & Heckler, 2008). In fact, several studies have shown that using only concrete materials may hinder the creation of a situation model used for transfer to different unknown situations (e.g., Gentner, Ratterman, & Forbus, 1993; Goldstone & Sakamoto, 2003; Son, Smith & Goldstone, 2011). This includes a series of experiments by Goldstone and colleagues (Goldstone & Sakamoto, 2003; Goldstone & Son, 2005; Son & Goldstone, 2009; Son, Smith & Goldstone, 2001) who have investigated ideal ways to help students make connections across a variety of superficially dissimilar scientific topics. They have shown that although concrete representations may facilitate initial learning, focusing only on particular characteristics of situations makes transfer more difficult. For example, in one study (Goldstone, 2003) students were presented with the topic of the foraging behavior of ants, in which one group was presented with a concrete representation (pictures of ants and food) while the other group was taught with a more abstract representation (small dots and large blobs represented the ants and the food). Like several other studies, results of this study showed that the students in the concrete group found it more difficult to transfer their knowledge to a generalizable situation, namely the topic of complex adaptive systems. This study also revealed that among those students who had the most trouble with initial

comprehension, transfer was best increased through instruction that involved abstract representations.

Similar to these findings, Koedinger and Nathan (2004) found that high school students' comprehension of simple word problems increased when they were presented with grounded representations (e.g., verbal descriptions of concrete and familiar situations), rather than abstract representations (e.g., numeric expressions or equations). However, when Koedinger and colleagues (2008) extended their research to include more complex word problems, grounded representations were actually not as effective for increasing comprehension as were the use of abstract representations. Thus, it appears as if the benefit of different types of representations depends on the level of the learner and the type of task. This dependency is closely related to Realistic Mathematics Education (RME), a Dutch approach to mathematics education that is based on Freudenthal's (1977) belief that comprehension increases when mathematical concepts are initially connected to reality. As understanding increases, REM suggests that initial informal context-connected solutions help students to form schemas, which are then later used to make formal mathematical principles more general. In essence, connecting mathematical principles to real life experiences during initial learning, will later result in higher levels of inference-making. Van Den Heuvel-Panhuizen (2003) demonstrated the power of REM through describing how a bar model (i.e., a type of schematic diagram) was used in a U.S. middle school curriculum in order to support various levels of understanding. Specifically, the bar model was described as evolving "from a drawing that represents a context related to percentage, to a strip for estimation and reasoning, to an abstract tool that supports the use of percentage as an operator" (p. 9). Because of the aforementioned

research, it is commonly believed that concrete representations alone do not guarantee comprehension (McNeil & Jarvin, 2007), and thus should not be the only representations used to facilitate mathematical situation models.

In contrast to concrete representations, abstract representations in mathematics include the use of only numbers and symbols. For example, completing mathematics problems by paper and pencil, without the use of manipulatives or external drawings, is a common example of abstract representations in mathematics. Since abstract representations of quantitative situations are purely symbolic, reasoning abstractly occurs because of interactions with internal (abstract) situation models. From the perspective of a situation model, abstract representations therefore need to be an integral part of instruction because they are essential in the inference making process of many advanced mathematical tasks (Fyfe, McNeil & Borjas, 2015). Although reasoning abstractly is the goal of advanced mathematics, children often struggle to attain mathematical comprehension when only abstract representations are used during instruction (McNeil & Alibali, 2000; Rittle-Johnson & Alibali, 1999).

Carraher, Carraher and Schliemann (1985) found that the ability for Brazilian street vendor children to solve basic computational mathematics problems was dependent on the context and representations of the problems. Although the children could count aloud to determine the price of several products that a customer wished to purchase (e.g., "If I purchase four coconuts and each coconut costs 35 *centavos*, how much do I owe you?"), when asked to do the same computation with paper and pencil (e.g., $35 \times 4 =$ __), they could not arrive at the correct solution. One possible interpretation of these findings is that when given a concrete context, the children were able to work with their mental

representations of mathematical concepts because of "contexts which allow them to be sustained by human daily sense" (Carrraher et al., 1985, p. 28); however, when given an abstract situation they were unsuccessful at utilizing the procedures taught in school. On the other hand, one might interpret these findings to indicate that knowledge learned from concrete contexts alone, do not promote transfer. This is supported by more general research that indicates that the use of symbols can lead to inflexible application of learned procedures (McNeil & Alibali, 2005). Regardless of one's interpretation, these findings (Carraher, Carraher & Schliemann, 1985) at a minimum suggest that instruction should not rely solely on the use of concrete or abstract representations. Instead, instruction should be designed to facilitate connection-making between these concrete context and abstract representations.

Abstract representations of mathematical concepts are essential in the inference making process of mathematical tasks and they need to be an integral part of instruction (Fyfe, McNeil & Borjas, 2015). Pashler et al. (2007) suggested that by integrating concrete and abstract representations into instruction, students are better able to make connections to prior knowledge, which in turn improves the chances that they will be able to transfer new skills into different contexts. Instruction involving various representations has repetitively been shown to increase comprehension (Ainsworth, Bibby & Wood, 2002; Goldstone & Sakamoto 2003; Richland, Zur & Holyoak, 2007). Concreteness fading (Goldstone & Son, 2005), the act of using concrete representations for initial learning and over time replacing parts of these representations with abstract representations, appears to be an important feature in this growing body of research.

In coding distributive property representations found in a Chinese elementary mathematics textbook series, Ding and Li (2014) highlighted the importance of using semi-concrete representations (e.g., schematic diagrams such as number lines) to help with the transition from purely concrete to entirely abstract representations. They claimed that this intermittent type of representation helps to highlight the structure of a problem, which allows novice learners to form connections between representations that they otherwise may have missed. Overall, the idea of beginning with concrete examples and progressing to representations that are more abstract has been suggested by both cognitive theorists (e.g., Bruner, 1966; Piaget, 1952) and educational researchers (Fyfe, McNeil, Son & Goldstone, 2014; Gravemeijer, 2002; Lehrer & Schauble, 2002).

Empirical evidence has recently been found to support the notion that students' transfer ability increases when concreteness fading is used during instruction. For example, McNeil and Fyfe (2012) presented undergraduate students with one of three instructional conditions for learning modular arithmetic. These conditions included concrete (meaningful images), abstract (abstract symbols) and concreteness fading (meaningful images faded into abstract symbols). The transfer assessments completed by the subjects after instruction revealed that transfer was highest at all three time points (immediately, 1 week, and 3 week post-instruction) for those subjects who were in the concreteness fading condition. Although this provides evidence that concreteness fading increases transfer, it remains unknown what effects concreteness fading has on long-term transfer (more than 3 weeks post-instruction). Nonetheless, as noted by Barnett and Ceci (2002), if considerable time is spent teaching children important concepts, then transfer should hold up over months and even years between learning and assessment. According

to a situation model perspective of comprehension, connection-making opportunities that help students focus on the structure and relationships of these important concepts will best facilitate inference-making. It therefore becomes important to analyze the degree to which concrete and abstract representations are integrated into learning opportunities for the purpose of transitioning formed connections into inferences. Specifically, research is needed on how concreteness fading can be used to facilitate the development of elementary students' situation models for fundamental mathematical ideas such as inverse relations.

Deep Questions

According to the *Institute of Education Sciences* (IES) recommendations for improving student learning (Pashler et al., 2007), teachers need to "help students build explanations by asking and answering deep questions" (p. 29) in order to help students build connections to underlying principles. Defined as a question that elicits deep explanations, deep questions include questions that target "causal relationships" (p. 29) and structural connections to the underlying principles. Examples of deep questions that prompt deep explanations are "why, why-not, how, and what-if" (p. 29) type of questions. Specific to mathematics, these might include questions involving the logical progression of solving equations or proofs, or questions such as "what is the evidence for X" (p. 29) and "how does X compare to Y" (p. 29)? Comparison type questions might especially help students form effective connections between and within mathematical principles (Ding & Li, 2014), which may lead to increased comprehension.

More broadly speaking, classroom discourse, the use of language within social contexts (Gee, 2010), helps to facilitate the development of student conceptual

understanding (Chin, 2007; Mortimer & Scott 2003; Franke et al., 2009). Costa (2001) and Swartz (2008) provide empirical evidence that students attain higher comprehension when they are provided with opportunities to converse within instructional settings, which Greeno (1991) agrees may contribute positively to the development of mental models. These opportunities include verbal interactions with teachers, which often involves the act of asking and answering questions. Questioning student understanding during classroom instruction is a critical opportunity that shapes student learning (van den Oord & Van Rossem, 2002) through eliciting students' explanations of underlying principles (Craig, Sullins, Witherspoon, & Gholson, 2006). Different types of questions however, often have different effects on learning.

Creating a taxonomy for questioning techniques has been the focus of several recent studies surrounding learning opportunities (e.g., Chin, 2007; Ginsburg, 2009; Heritage & Heritage, 2013; Hopper, 2009; Smart & Marshall, 2013). Although types and levels of questions vary, all classification systems involve a spectrum beginning with lower-order recall questions and progressing toward higher-order inference questions. While the beginning of this spectrum represents procedural based questions, the end focuses on conceptual based questions that help students transfer knowledge to alternate contexts, the process of inference-making. This indicates that as questions approach the higher end of the spectrum, they help to better facilitate the development of situation models. Perhaps the best-known and most widely used system for classifying cognitive levels of learning is Bloom's (1956) taxonomy. Assuming that "the cognitive level of a question is determined by the response requested by the teacher" (Wimer, Ridenour, Thomas & Place, 2001, p. 85), Hopper (2009) used this taxonomy to classify cognitive

categories of question types. These categories classified lower-order questions as ones that elicit responses based on knowledge recall and procedural application; whereas, higher-order questions elicit analysis, synthesis and evaluation (i.e., the process of transforming connections into inferences). Higher-order/deep questions (used interchangeably from this point forward), are therefore often open-ended, challenging, and are of great importance for facilitating the use of situation models when students are presented with cognitively high demanding mathematical tasks.

Deep questions are the stimulus of classroom conversations and have been shown to directly influence student cognitive processes (Chapin & Anderson, 2003; Chin, 2006; Morge 2005). Sigel and Saunders (1979) suggested that deep questions are critical because they often force students to distance themselves from the present in order to think about past or future events. In doing so, students must mentally represent what has happened or what soon will happen in order to make valid inferential statements. This is in alignment with Nussbaum and Edwards (2011) suggestion that these type of questions can and should be used to help students think more critically about the plausibility of their own arguments as is related to connections between empirical evidence and scientific models. Furthermore, when questions are focused and deliberate, Rubin (2009) found that students were able to form stronger connections to prior knowledge. These connections led to the ability to transfer deeper levels of understanding across various contexts, the fundamental outcome of a well-defined situation model.

Student self-explanations of worked examples also seems to lead to an increase in knowledge transfer (Renkl et al., 1998). This agrees with Chi et al.'s (1994) findings that higher learning gains occur when learners are prompted with questions during the act of

reading. The elicitation of explanations is particularly beneficial for students who have low levels of prior knowledge (Renkl et al., 1998). In order to elicit students' self-explanations, Pashler et al. (2007) suggests that deep questions are essential because they promote students to form connections between explicit material and their "subjective explanations…that link the [explicit] material to personal knowledge and experiences" (p. 29). Moreover, in order to engage students in cognitively high demanding mathematical tasks, teachers must practice flexibility in adjusting questions based on student responses (Chin, 2007), and must provide adequate time for students to establish well thought out responses (Kazemi & Stipek, 2001; Weimer, 1993).

Several researchers have noted that an increase in question variety (i.e. lower- and higher-order questions) leads to an increase in comprehension (Ellis, 1993; Wilen, 1991). Creating a cognitive ladder to scaffold student understanding (Chin, 2006) that is based on the types of questions students are asked, may therefore support development of conceptual understanding and help to create learning environments that maximize students' connection-making opportunities (Boaler & Brodie, 2004; Kazemi & Stipek, 2001; Stein, Remillard & Smith 2007). Khan and Inamullah (2011) found however that in most U.S. secondary classrooms, very few deep questions are being asked. This is consistent with Wimer et al. (2001) who recorded that very few deep questions were posed during the instruction of sixteen third and fourth grade teacher's mathematics lessons. Instead, the teachers asked many lower-order questions which mainly focused on simple recall of information. While the researchers admit that lower-order questions do have a place in instruction, they argue that "higher level questioning leads to higher level learning" (p. 85). Especially for elementary school children who may have only limited

informal knowledge of a mathematical concept such as inverse relations, it seems as if lower-ordered questions that are based on formal memorized facts, will lead to inflexible and limited comprehension. The asking of deep questions on the other hand, should help novice learners develop connections to prior informal knowledge and in turn create a foundation for the development of children's situation models.

The Case: Domain of Inverse Relations

Inverse relations are a fundamental mathematical idea that often involve cognitively high demanding tasks. In elementary school, inverse relations mainly refers to the inverse relationship between addition and subtraction and between multiplication and division, which serve as a fundamental building block for many quantitative concepts (Baroody et al., 2009; Carpenter et al., 2003). Because inverse relations transcend across all levels and various contexts of mathematics, numerous connection-making opportunities exist when studying inverse relations. This critical mathematical topic therefore provides a promising domain for investigating how situation models facilitate mathematical comprehension.

Research reveals elementary school children generally lack a formal understanding of inverse relations (Baroody, Ginsburg & Waxman, 1983; De Smedt, Torbeyns, Stassens, Ghesquiere, & Verschaffel, 2010; Resnick, 1983). This presents a problem far beyond elementary classroom doors, since longitudinal empirical evidence (Baroody, 1987; Stern, 2005; Vergnaud, 1988) suggest that an elementary student's comprehension of inverse relations significantly predicts both algebraic and overall mathematical achievement in later years. The Common Core has recognized the importance of inverse relations by identifying them as a critical piece of mathematical

competency across all elementary grades levels (CCSSI, 2010). This supports the need to help elementary children facilitate the development of a well-connected situation model for inverse relations (i.e., # of groups × group size = total, total ÷ group size = # of groups).

Research on inverse relations has typically been conducted in both the mathematics education and the cognitive psychology fields (Nunes et al., 2007). While mathematics education research aims to improve teaching and learning of the content within inverse relations, cognitive psychologists have used the domain of inverse relations to investigate the dynamics of cognitive development (Nunes et al., 2009). Inverse relations provide a great domain in which to explore the longstanding developmental question of whether content should drive pedagogy or pedagogy should drive content. This is because "children's understanding and use of inverse relations provides an excellent vehicle for studying the interactions between different kinds of knowledge in the development of mathematical thinking" (Bisanz, Watchorn, Piatt & Sherman, 2009, p.11).

Comprehension of inverse relations was first linked to cognitive development by Piaget's (1952) assertion that very young children are not capable of reversible thought (Man, 2011). He claimed that reversibility, the ability to recognize that numbers or objects can be transformed and then returned to their original condition, is only achievable once children (aged 7-11) reach the concrete operational stage of his cognitive development theory. For example, during this stage, a child understands that a favorite ball that deflates is not gone but can be filled with air again and put back into play (Piaget, 1952). Interestingly, today's mathematics education literature on children's

comprehension of inverse relations suggests this very notion that students have a higher rate of success with inverse problem solving tasks that include the use of concrete objects (Nunes, Bryant & Watson, 2009). More recent research findings suggest that the comprehension of inverse relations is not just related to cognitive development; it is related to the effectives of classroom instruction (Man, 2011; Nunes, et al., 2007) and most likely also the curriculum (Ding, 2016).

Investigating children's use and understanding of inverse relations has led to improved instruction that is "designed to optimize conceptual and procedural competencies in mathematics" (Bisanz, et al., 2009, p.11). The literature however indicates that instructional practices primarily focus on procedural knowledge instead of conceptual understanding of inverse relations, which has largely resulted in past research omitting the process of developing comprehension. To illustrate this, the following sections provide a review of prior and current research on both additive and multiplicative inverse relations. Although the focus of the current study will only be multiplicative inverses, a review that includes additive inverses is necessary to situate the current study within the status quo of relevant mathematical learning opportunities within elementary school classrooms.

Additive Inverses

Research suggests that elementary aged students typically struggle with solving problems that involve additive inverses (Nunes et al., 2009; Stern, 1992). This has been shown to inhibit the level of comprehension that students have in regards to the overall operations of addition and subtraction (Bryant, Christie & Rendu, 1999). Specifically, research illustrates that many elementary age students do not use the inversion principle

when presented with three term arithmetic problems that have the form $a + b – b = a$ (Bisanz et al., 2009; Siegler & Stern, 1998). The mathematical inversion principle states that "inverse operations involving the same value results in no net change" (Prather & Alibali, 2009, p. 236) to an original quantity. Instead of using the more efficient inversion shortcut to cancel the *b*'s, most students tend to simplify these expressions by first adding the values of *a* and *b* before subtracting *b* from the resulting quantity (a left-to-right approach) (Bisanz et al., 2009). Perhaps this is due to students not wanting to deviate from their procedural understanding of the order of operations.

Students can often apply procedural knowledge to arrive at correct answers, but they frequently do so without a conceptual understanding of the inverse relation between addition and subtraction (Bisanz, et al., 2009). While the shortcut approach of cancelling the *b*'s might appear procedural in nature, Gilmore and Papadatou-Pastou (2009) argue that conceptual knowledge of the inversion principle directly underlies the use of the shortcut. Robinson and LeFevre (2012) agree with this notion that "fast and accurate solutions have been interpreted as evidence that solvers use their conceptual knowledge of the inverse property" (p. 410). Likewise, Crooks and Alibali (2014) believe that the use of the inversion shortcut demonstrates a deeply connected structural understanding of inverse relations. Because "most studies…do not distinguish between children's understanding of a concept and their ability to identify situations in which it might be relevant" (Gilmore & Bryant, 2008, p. 301), Gilmore and Bryant (2008) investigated this notion of connection knowledge. They compared children's use of the inversion shortcut in problems where inversion was transparent (i.e., 17 + 11 – 11) to problems where it was non-transparent (i.e., 17 + 11 – 5 – 6). As one might expect, students had more difficulty

recognizing and using the inversion shortcut for the non-transparent problems. Because transfer was limited in the face of a new unknown situation (non-transparent problems), this might suggest that students' situation models for the inversion principle were not yet well defined. Bisanz and colleagues (2009) also found that the way in which inverse relations are presented to students could influence comprehension.

The complement principle (if $a + b = c$, then $c - b = a$) is another way in which researchers study the teaching and learning of inverse relations (Baroody, 1983, 1987; Ding, 2016; Li, Hassler & Ding, 2016). The two term complement principle (if $a + b = c$, then $c - b = a$) is closely related to the three-term inversion principle ($a + b - b = a$, Baroody et al., 2009). In regards to the complement principle, Baroody et al. (1983) found that many elementary students (61% of those sampled) could not use addition as a method for solving subtraction problems. More recently, Li and colleagues (2016) noted that after effectively computing 9 + 3, only 41% of sampled grade 1 students could get the correct answer for 12 – 3 and 0% of those indicated that they did so based on using inverse thinking. Other studies (Baroody et al., 2009; De Smedt et al, 2010; Torbeyns et al., 2009; De Smedt et al, 2010) have also shown that even with instruction, elementary age students struggle with using addition to solve subtraction problems. Most commentaries on the complement principle (Resnick, 1983; Putnam, de Bettencourt, & Leinhardt, 1990) point out that *c* consist of two parts, *a* and *b*, and claim that the main reason for children's failure is due to the difficulty in making connections between the part-whole relationship (Li et al., 2016) or their part-whole schema might be loosely constructed. In other words, when attempting to use this principle in new situations,

students “struggle with retrieving inverse thinking strategies” (p. 13) which suggests that student situation models for inverse relations are not well-connected.

Another layer of research hypothesizes that comprehension of inverse relations consists of both a qualitative and quantitative aspect. Bryant, Christie and Rendu (1999) noted that elementary children had an easier time understanding the removal and replacement of physically identical bricks (qualitative) than they did with quantitative problems in which the same number of physically different bricks were added and then subtracted. Although their findings only involved additive inverses, if both a qualitative and quantitative aspect is needed for comprehension of inverse relations, Nunes et al. (2009) suggested that teaching children about the connections between these aspects would be an efficient way to improve comprehension. This further supports the need for teachers to form connections between concrete and abstract representations in order to help students enhance their situations models. However, little research has been conducted on the connections that teachers make when teaching inverse relations.

Multiplicative Inverses

The majority of prior research on inverse relations has only focused on additive inverses (Cowan & Renton, 1996; Squire, Davies & Bryant, 2004) and thus there exists a large gap in literature involving how connection-making facilitates the comprehension of multiplicative inverses. The limited research that is available (Robinson & Dubé, 2009b; Thompson, 1994; Vergnaud, 1988) suggests that similar to the well-documented problems children have with comprehending additive inverses (Nunes et al., 2009; Stern, 1992; Bryant, Christie, Rendu, 1999), multiplicative inverses are also a struggle for elementary aged students (Robinson & Dubé, 2009b; Thompson, 1994; Vergnaud, 1988).

Analogous to addition and subtraction, problems at the elementary level that use the inversion principle of multiplication and division take the form $a \times b \div b = a$. Like additive inverses, findings from recent studies (Dubé & Robinson, 2010; Robinson & Dubé, 2009a; 2009b) illustrate that students do not use an inverse shortcut to simplify these multiplicative inverse problems; rather, they routinely use the left-to-right solution approach instead of first simplifying $b \div b$ to obtain 1. This might again be due to students not wanting to deviate from their procedural understanding of the order of operations. Although Bisanz et al. (2009) only studied additive inverses it seems likely that students who can apply procedural knowledge to arrive at correct multiplicative inverse solutions might also be doing so without a conceptual understanding of inverse relations.

Elementary students' use of multiplicative inverses lags behind that of additive inverses. Baroody (2003) found that only 25% of children use the inversion principle as a shortcut in applicable multiplicative inverse problems as compared to 39% who do so in with additive inverses (Baroody et al., 1983). Not only is applying a shortcut thought to illustrate conceptual knowledge (Gilmore & Papadatou-Pastou, 2009), but McNeil (2007) found that U.S. students who did not apply the inversion shortcuts were more prone to procedural errors when simplifying inverse problems. In some European countries where connections to mental representations are emphasized and practiced, students apply shortcuts more frequently which leads to greater transfer across other mathematical operations (Verschaffel, Luwel, Torbeyns & Van Dooren, 2009). According to Hatano (2003), the most important issues in the psychology of (mathematics) education is how to facilitate the development of adaptive expertise, the ability to apply prior knowledge both "flexibly and creatively" (p. xi). Providing connection-making learning opportunities should

therefore help students strengthen their situation models and in turn lead to increased comprehension of inverse relations that extends far beyond routine expertise, the ability to "complete school mathematics exercises quickly and accurately without (much) understanding" (p. xi).

The complement principle (if $a \times b = c$, then $c \div b = a$) has also been researched in the domain of multiplicative inverses. Concerning this principle, both Grossi (1985) (as cited in Vergnaud, 1988) and Thompson (1994) found that elementary students were unable to recognize the appropriateness of using either equation when solving application problems. Recent research on student comprehension of the multiplicative complement principle seems limited; however Ding and Carlson (2013) did indicate that current instruction of inverse relations does not support conceptual connection making. As a result, students are more prone to make computational errors such as $7 \div 35 = 5$ or $5 \div 35 = 7$. Together these indicate that the teaching and learning of elementary inverse relations does not focus on facilitating well-connected situation models.

The Role of Curriculum and Instruction on Mathematical Comprehension

When learning a new mathematical concept, students must be afforded learning opportunities that activate existing mental representations that are used to facilitate connections between prior knowledge and the new targeted content (Sidney & Alibali, 2015; Zwaan & Madden, 2004). Although the amount and the ability to activate conceptually relevant prior knowledge has been shown to be a significant and reliable predictor of comprehension (Langer, 1984; McNamara et al., 1996; Pearson, Hansen & Gordon, 1979), novice learners often struggle to make connections to relevant prior knowledge (Novick, 1988). It is therefore of utmost importance to teach beginning learners how to make connections to relevant prior knowledge (Pearson et al., 1979),

which can be done by helping students construct situation models so as to activate the inference making process. Research indicates that curriculum and instruction should be as coherent and explicit as possible when attempting to facilitate understanding for learners with little prior knowledge (Kintsch, 1994; Reed, Dempster & Ettinger, 1985).

Textbooks

Textbooks are a critical part of a student learning environment in that they are a vital curriculum resource that provide opportunities to learn (Ding, 2016; NRC, 1999; Thompson, Kaur, Koyama & Bleiler, 2013). As reported by Malzahn (2013), 85% of grade K-5 mathematics classrooms in the U.S. use commercially published textbooks. It therefore is no surprise that teachers most often use textbooks as the primary source for content knowledge (Ding, 2016; Dossey, Halvorsen & McCrone, 2012). Most prior research that has been conducted on U.S. textbooks, analyzes teacher editions because they suggest "minimum opportunities that a teacher may use in the classroom" (Ding, 2016, p. 49) and often provide additional insight into instructional strategies that teachers may use with different types of learners. Thus, exploring the design of teacher edition textbooks in relation to analyzing the connection-making opportunities involving instructional tasks, representations and deep questions, should provide insight into how curriculum materials provide opportunities to facilitate student's situation models for multiplicative inverses.

As evident by the emphasis placed on numerical calculations, prior studies reveal that U.S. textbooks focus mainly on facilitating procedural understanding (Cai et al., 2005; Ding, 2016; Ding & Li, 2010). Underlying mathematical principles are also seldom made explicit in U.S. textbooks (Ding & Li, 2010). This is in contrast to international

textbooks that provide “more opportunities supporting meaningful and explicit learning” (Ding, 2016, p. 45). For example, in comparison to Chinese textbooks, Ding found that U.S. elementary mathematics textbooks do not always situate “the initial learning opportunities of a fundamental idea in concrete situations for sense making” (p. 48). This has also been found to be true for the fundamental mathematical ideas of the concepts of equivalence (Li, Ding, Capraro & Capraro, 2008), the distributive property (Ding & Li, 2010), the associative property (Ding, Li, Capraro, & Capraro, 2012), and most recently inverse relations (Ding, 2016).

Instructional tasks (i.e., worked examples and practice problems) for inverse relations found within two different representative U.S. textbooks mainly focus on procedural understanding (Ding, 2016). These tasks tend to emphasize multiple procedural-based solution strategies rather than attempting to form structural connections that could lead to increased conceptual understanding of inverse relations. Chinese textbooks on the other hand, tend to use instructional tasks to focus on sense making by stressing the structural relationships of inverses (Ding, 2016). This agrees with Zhou and Peverly’s (2005) observation that first grade Chinese textbooks used the composing-decomposing and part-whole methods to illustrate the underlying structural relationships of additive inverses. In general, Ding (2016) concluded that U.S. textbook worked examples primarily serve as “pretext for computation with a focus on procedures” (p. 64) instead of facilitating connection-making by activating “students’ personal experiences and informal understanding to aid in learning” (p. 64).

Another interesting finding from Ding’s (2016) analysis involves a cross-cultural difference in the proportion of worked examples that textbooks use for teaching inverse

relations. Whereas U.S. textbooks include a larger amount of worked examples for multiplicative inverses in comparison to additive inverses, Chinese textbooks exhibited the opposite frequency. A likely possibility for why this difference exists is because the structural connections established for additive inverses in Chinese textbooks helps students at an earlier age form well-connected situation models for inverse relations. If true, less explicit instruction on multiplicative inverses would be needed because Chinese students would be able to activate their inference-making process by using their previous inverse situation model to draw on connections from previous knowledge. This notion of strengthening a student's situation model seems to be in direct contradiction to U.S. classroom instruction that lacks connections to underlying principles (Crooks & Alibali, 2014; De Smedt et al., 2010). Likewise, the U.S. teacher belief that students will eventually master difficult concepts if given enough repetitive examples (Ding & Carlson, 2013), also does not indicate a U.S. instructional focus on facilitating the use of situation models.

Concerning the use of representations, Ding (2016) found that in both U.S. textbooks analyzed, less than 20% of coded additive inverse relation instances and less than 26% of coded multiplicative inverse relation instances were concrete in nature. Specific to worked examples, these concrete representation percentages dropped to 6% or below for both additive and multiplicative inverses. An even smaller percentage of worked examples were situated in real-world contexts (Ding, 2016). Instead, concrete representations used within instructional examples of U.S. textbooks, were "most often only physical or visual in nature (i.e., dominoes, cubes, diagrams) and therefore had very

little contextual support" (p. 64). Together with the fact that other textbook representations did not focus on the structural relationship of multiplicative inverses, Ding (2016) concluded that U.S. textbooks provide an incomplete use of representations. This incomplete use of representations within the U.S. textbooks appears to be a result of missed connection-making opportunities.

Missed opportunities to make connections between concrete and abstract representations most likely also occurs because U.S. textbooks generally are not designed to be "faded out from worked examples to practice problems with variations" (Ding, 2016, p.55). Instead, worked examples and practice problems in U.S. textbooks are mainly abstract in nature, creating minimal opportunities for students to form connections to concrete representations. In contrast to U.S. textbooks, Ding (2016) found that Chinese textbooks situate worked examples in concrete context, fade worked examples into abstract scenarios, include varied numerals across problems, and gradually minimize student instruction throughout a lesson. Together these attributes of Chinese textbooks promote connection-making opportunities that allow students to move beyond a surface-level understanding of worked examples. None of these concreteness fading (Goldstone & Son, 2005) tasks were found within U.S. textbooks, which may suggests that U.S. students are given limited opportunities to form connections to underlying mathematical concepts. This in turn may hinder the growth of their situation models for inverse relations.

Deep questions directly influence student cognition (Chapin & Anderson, 2003; Chin, 2006; Morge, 2005) because they elicit students' explanations to underlying mathematical concepts (Craig et al., 2006). The inclusion of deep questions within

textbooks thus seems to be an essential component for facilitating the use of situation models for cognitively high demanding mathematical tasks. However, U.S. elementary mathematics textbooks do not generally include many deep questions (Ding, 2006) and do not explicitly indicate a desire for teachers or students to provide deep explanations (Ding & Carlson, 2013).

Specific to the Common Core (CCSSI, 2010), Chingos and Whitehurst (2012) claim that new curriculum "standards will only have a chance of raising student achievement if they are implemented with high-quality materials" (p. 1). Thus, recent research has begun to explore methods for measuring the quality of Common Core curriculum materials. In one of the first studies of this nature, Polikoff (2015) reviewed four Common Core aligned fourth grade mathematics textbooks and concluded that there exists a "good deal of misalignment at the cognitive-demand level in textbooks—all of them systematically fail to cover the more conceptual skills called for by the standards" (p. 1203). There is a pressing need therefore "to investigate the extent to which textbook content may be associated with effectiveness" (p. 1207).

Some researchers have argued that textbooks that lack connections within and across topics might actually be desirable for the formation of situation models (Mannes & Kintch, 1987; McNamara et al., 1996). This argument is rooted in the belief that a less coherent text forces readers to actively search for their own ways to facilitate connection-making. On one hand, a minimally coherent text may be problematic for teaching elementary inverse relations because many elementary aged students are beginning learners who struggle to make connections to relevant prior knowledge (Novick, 1988; Pearson et al., 1979). On the other hand, a lack of coherence may be beneficial for

promoting connection-making opportunities when students possess prior knowledge and get comfortable with targeted content. An appropriate level of instructional coherence (i.e., when to use concreteness fading and when to ask deep questions to elicit deep understanding of targeted content) may therefore depend on the ability to assess the status of children's prior knowledge, an assessment that may require teacher expertise. Although U.S. elementary teachers are not typically mathematical content experts, expertise on knowing when to provide appropriate learning opportunities that facilitate connection-making appears to be an important component in the teaching of fundamental mathematical concepts. The root cause of students' limited comprehension of inverse relations may not be the curriculum itself, rather it may be due to elementary teachers not enacting curriculum in an adequate manner consistent with connection-making. While further research is arguably needed across other Common Core aligned textbooks, it nonetheless has become "essential to move from the textbook into the classroom to understand how curriculum materials influence teachers' instructional responses to the standards" (Polikoff, 2015, p. 1208).

Expert Teachers

It seems reasonable to begin and exploration of learning opportunities that facilitate mathematical situation models for multiplicative inverses with elementary aged students because of the relevance that prior knowledge has on connection-making. It also seems reasonable to begin the investigation by analyzing instruction that has the highest likelihood of creating a supportive learning environment that may lead to maximizing mathematical comprehension of inverse relations. Together these point to the analysis of expert elementary teachers' mathematics instruction. Not only has the study of expert

teachers been popular in recent mathematical comprehension research (Cai & Ding, 2015; Cai, Ding & Wang, 2013; Ding, Hassler, Li & Chen, 2016; Hassler, 2016; Hassler & Ding, 2016), but the study of experts' knowledge has also had a long tradition in cognitive psychology (Leinhardt & Smith, 1985). Chi (2011) describes an expert as "someone who is relatively more advanced, as measured in a number of ways, such as academic qualifications, years of experience on the job, consensus among peers, assessment based on some external intendent task, or assessment of domain-relevant content knowledge" (p. 18). While most expert U.S. elementary teachers are not mathematical content knowledge experts, general teaching knowledge along with any several of the aforementioned characteristics may classify them as experts. Bransford and colleagues (1999) suggest the use of experts in helping gain professional knowledge from classroom instruction because experts tend to notice meaningful patterns of information, have well organized and conditionalized knowledge and are able to flexibly retrieve and apply that knowledge to new situations. While the first two reasons for analyzing expert teachers seem to center around their ability to create connections to prior and current targeted knowledge, the last reason suggests that experts themselves understand when and how to activate their own situation models.

In an attempt to determine how practitioners view instructional coherence, Cai et al. (2014) interviewed 16 U.S. and 20 Chinese expert elementary mathematics teachers. The results showed that the majority of the U.S. teachers viewed coherence as the connectedness of teaching activities between and within instructional lessons. U.S. textbooks that include the same titled lessons across various grade levels, with only minimal surface level changes (e.g., vary numerals used in problems), may in part foster

this view (Ding, 2016). U.S. teachers also "referred to connections to prior knowledge as a result of rather than a condition of achieving instructional coherence" (Cai et al., 2014, p. 273) which suggests they might not view situation models as a tool useful for achieving comprehension. On the other hand, Chinese teachers "emphasized the interconnected nature of mathematical knowledge" (p. 267) and as confirmed by Cai and Ding (2015) view understanding as "a web of connections, which is a result of continuous connection making" (p. 17). Taken in combination with U.S. student's lower scores on international mathematical assessments (TIMSS, 2013; PISA, 2013), this cultural difference of expert teachers' views on connection-making suggest a need to examine how current U.S. learning environments (i.e., curriculum materials, classroom instruction) provide learners with opportunities to develop mathematical understanding.

Justification for Study and Research Questions

The reality of the Common Core's (CCSSI, 2010) call for strengthening connections between fundamental mathematical ideas, is that reformed curriculum materials are not meeting the necessary "advanced levels of cognitive demand" (Polikoff, 2015, p. 1188) and that U.S. children continue to struggle with comprehension. A review of literature has suggested that of utmost importance for improving comprehension is the need to teach beginning learners how to make connections to relevant prior knowledge (Pearson et al., 1979). I argue that students are best supported in this process when they are presented with connection-making opportunities useful for enhancing their situation models. To date however, little is known about how curriculum materials and classroom instruction in current learning environments afford elementary students these opportunities.

The goal of this research is to explore the opportunities that appear to be the most contributing factor in the creation of a situation model: (a) the presentation of instructional tasks, (b) the types of representations, and (c) the use of deep questions. Because text comprehension shares features with mathematical problem solving in that learners must form connections through an active inference making process, an integrated comprehension perspective in which mathematical understanding is explored by means of a situation model is justified for this study. Further, past research has primarily focused on *if* and *when* children show evidence of understanding inverse relations, not *how* and *why* this understanding occurs. The exploration of *how* connection-making opportunities enhance comprehension of inverse relations is thus also justified for study.

Due to the limited scope and the desire for an in-depth analysis, this study will focus solely on the complement principle of multiplicative inverses (if $a \times b = c$, then $c \div b = a$), a difficult principle to comprehend and an under researched domain as supported by the literature. Since curriculum and instruction should be as coherent and explicit as possible for learners with little prior knowledge (Kintsch, 1994; Reed, Dempster & Ettinger, 1985), learning opportunities for inverse relations should focus on facilitating the connection-making process in order to help elementary students develop situation models for multiplicative inverses that can be used for inference-making. Thus, in response to Linn's (2006) call for future empirical research to explicitly search for ways to facilitate children's connections to prior knowledge, the following three research questions have emerged:

(1) How do reformed elementary CCSS textbooks facilitate connection-making through the presentation of instructional tasks, representations, and deep

questions in order to promote students' comprehension of multiplicative inverses?

(2) How do expert elementary mathematics teachers facilitate connection-making through the use of instructional tasks, representations and deep questions in order to promote students' comprehension of multiplicative inverses?

(3) How do connection-making opportunities afforded by reformed CCSS textbooks and provided by expert teachers' classroom instruction relate to elementary students' comprehension of multiplicative inverse relations?

Specific to the first research question, and because past studies have revealed that U.S. curriculum materials generally lack connections within and across topics (Ding, 2016; Schmidt, Wang, & McKnight, 2005), I hypothesize that the materials on multiplicative inverses found within the two CCSS textbooks analyzed in this study will still share cultural-based features with missed opportunities for connection-making (Research Question #1). Past studies suggest that missed opportunities within instructional tasks may result from procedurally dominated worked examples and practice problems that do not focus on the structure of underling mathematical principles. I also believe that due to the literature suggesting that concrete representations have little contextual support in U.S. textbooks, missed opportunities to connect concrete representations to abstract number sentences will exist. I do not expect concreteness fading to occur, which similar to past findings will also limit the strength of connections. Several missed connection making opportunities due to a lack of deep questions are also expected. Although it might be expected that Common Core (CCSSI, 2010) linked textbooks integrate some of the aforementioned connection-making features, early studies

on these textbooks (e.g., Polikoff, 2015) suggest they fail to promote conceptual understanding.

Past research reveals that comprehension is strengthened when "teachers assist students in classifying and formalizing structures and explicitly linking them to existing knowledge" (Chapin & Anderson, 2003). I therefore hypothesize that the classroom instruction of the expert teachers in this study will overall tend to emphasize connection-making (Research Question #2). I further hypothesize that because of their expertise, teachers in this study will be able to fill in missed connection-making opportunities that may exit in their textbook (e.g., adding deep questions to instruction). Although Ding and Carlson (2013) found that non-defined expert teachers struggle to implement effect instructional techniques, the teachers in the current study are expert teachers and thus are expected to promote connection-making opportunities consistent with the literature review on instructional tasks, representations and deep questions. However, it should be noted that because literatures suggests that expert U.S. teachers view instructional coherence as the connectedness of teaching activities and not the interconnectedness of mathematical concepts, it would not be a surprise if some missed connections occur during expert teacher's instruction. After all, teaching has been identified as a cultural activity (Stigler & Hiebert, 1999). Connection opportunities that relate to the interconnectedness of mathematics (i.e., connections to prior or future mathematical content knowledge) might be the most vulnerable.

Finally, I hypothesize that both higher connection-making textbook scores and higher connection-making teachers scores will strengthen students' situation models for multiplicative inverses whereby resulting in increased comprehension (Research

Question #3). When connections within instructional tasks are established, concrete and abstract representations are integrated throughout a lesson, and deep questions are posed during instruction (Pashler et al., 2007), students are more likely to be able to transfer new skills into different contexts because of an increase in encoding and extraction of core underlying mathematical principles (Renkl et al., 2002). This also lends me to anticipate that students in classrooms with higher textbook and higher teacher connection-making scores will achieve greater learning gains from a pre- to a post-test assessment on understanding multiplicative inverse relations.

CHAPTER 3

METHODS

This chapter provides a detailed description of the methodology that I used to explore how textbooks and expert elementary mathematics teachers facilitate connection-making opportunities for enhancing students' comprehension of multiplicative inverses. The chapter begins with an explanation of the research design of this study and continues with a description of the participants. An explanation of the data sources and procedures follows. Finally, the last part of this chapter describes the data analysis approach that was taken in order to answer the three research questions.

Research Design

The purpose of this study was to explore how textbooks and expert elementary mathematics teachers provide connection-making opportunities for the learning of multiplicative inverse relations. Investigating the three research questions in this study involved a collective use of induction (discovering patterns within textbooks and classroom instruction), deduction (testing the theory of situation models based on connection-making) and abduction (uncovering the best set of connection-making opportunities that can lead to increases in comprehension), which classifies this study as mixed methods research (Johnson & Onwuegbuzie, 2004, p. 17). Originating in the late 1980's, mixed methods design is a relatively new methodology which Creswell (2014) defined as the class of research that "focuses on collecting, analyzing, and mixing both quantitative and qualitative data in a single study" (p. 6) and should be used when "quantitative and qualitative approaches in combination provides a better understanding of the research problems than either approach alone" (p. 6). As opposed to using only one

method, the analysis of each of the first two research questions in the current study involved the integration of qualitative and quantitative data. This provided a stronger understanding of how textbooks and expert elementary mathematics teachers facilitate connection-making. Further, although the third research question called for an exclusive quantitative analysis, it was designed to provide empirical evidence in support of the overall qualitative nature of connection-making opportunities. According to Creswell (2014), these specific features classify the current study as an embedded mixed methods design.

The embedded mixed methods research design involves using a secondary type of data to support a study's primary data set (Creswell & Plano Clark, 2011). Embedding a secondary type of data into a research design provides researchers with a methodology that can be used to investigate complex phenomena in which asking different types of questions within a single study may require different types of data. The premise of this advanced mixed methods design is that a single data type is not sufficient for a complete understanding of a study's research questions. Researchers therefore have traditionally used this type of mixed methods design to either nest a qualitative data analysis within a largely quantitative study or nest a quantitative data analysis within a largely qualitative study. In the current study, quantitative components are embedded within the overall qualitative content analysis (see Figure 3) in order to gain a more complete understanding of how learning opportunities may facilitate the creation of situation models that are useful for increasing student comprehension of multiplicative inverses.

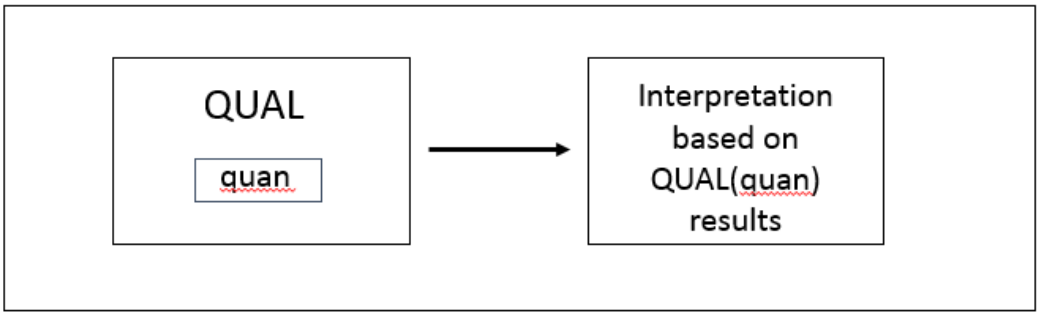


Figure 3. Embedded mixed methods design (Creswell & Plano Clark, 2011).

The qualitative approach to research involves investigating fundamental questions surrounding the how and why of decision making (Caelli, Ray & Mill, 2003; Merriam 1998). To gain a better understanding of current learning opportunities afforded to elementary students, a qualitative content analysis was conducted to identify the typical ways in which textbooks (Research Question #1) and expert teachers (Research Question #2) formed connections through the presentation of multiplicative inverse relations. Traditionally, content analysis has been used as a systematic method for compressing text data in order to examine trends and patterns (Cavanaugh, 1997; Stemler, 2001). However, Holsti's (1969) broader description that inferences can be made through analysis of most any type of conveyed message does not restrict content analysis to only textual examination. Instead, recent researchers have used content analysis to examine student drawings (Wheeelock, Haney & Bebell, 2000) and evaluate videotaped classroom instruction (Stigler et. al, 1999).

A directed content analysis (Hsieh & Shannon, 2005) was selected for the current study because this approach uses prior research to create an initial coding framework in which preconceived categories are operationally defined by existing theory. This is in contrast to a conventional content analysis in which coding categories evolve as data is collected and analyzed on a phenomenon that has rarely been researched. The direct

content analysis framework used in this study was derived from research corresponding to the *Institute of Education Sciences'* (IES) recommendations for improving student learning (Pashler et al., 2007), and from the existing situation model theory of comprehension (van Dijk & Kintsch, 1983). Specifically, the focus of this study was how textbooks and expert teachers facilitate connection-making through the use of instructional tasks, representations and deep questions. The goal of direct content analysis is to either confirm or find evidence against categories in the framework in order to improve and enrich existing theory (Hsieh & Shannon, 2005). It is expected that the content analyses in this study will lead to a rich and meaningful categorization of connection-making opportunities that ultimately may extend the situation model theory of comprehension.

In addition to this qualitative exploration, the categories within the content analysis framework were quantified for the purpose of providing a more complete answer to the first two research questions in this study. The reason behind quantifying the connection-making opportunities in this study was three-fold. First, because the quantitative approach to research views discovery of knowledge as a development that occurs because of examining a specific set of variables within a cause-and-effect mindset (Creswell, 2014), creating a quantitative connection-making scale allowed for easy comparison between the potential effect that each textbook and each teacher had on improving student comprehension. Second, by quantifying the variables in the framework, it became clearer how each variable (e.g., instructional tasks, representations and deep questions) contributed to the overall connection-making opportunities afforded by the textbooks and expert teachers. Lastly, the quantitative scale provided a means for

evaluating the extent to which different expert teachers enhanced textbook connection-making opportunities. It should be noted that the process of quantifying qualitative data has been conducted by numerous other researchers (Chi, 1997; Goldin-Meadow, Alibali, & Breckinridge Church, 1993; Jordan & Henderson, 1995). Overall, analyzing the first two research questions through the combination of both a qualitative and a quantitative lens provided a more complete understanding of how current learning environments (i.e., curriculum materials and classroom instruction) provide learners with opportunities to enhance mathematical comprehension.

The third research question in this study was designed to determine how connection-making opportunities relate to student comprehension of multiplicative inverses. Unlike the first two research questions, the investigation of this question was purely quantitative. Quantifiable data is often collected using an instrument that measures pre- and post-test performance and is analyzed through robust statistical procedures. Most often, the conclusion of a hypothesis test is used in quantitative research to make a decision that either supports or refutes an existing claim or theory (Creswell, 2014). In this study, student comprehension of inverse relations (Research Question # 3) was examined through a multivariate linear regression analysis consistent with the aforementioned quantitative approach. A more detailed explanation of both the qualitative and quantitative methods that were used to investigate the three research questions involving connection-making opportunities afforded by textbooks and expert instruction can be found in the data analysis section of this chapter.

Participants

Teachers

This study involved four elementary school expert teachers who are participants of a five-year National Science Foundation (NSF) funded project on early algebra in elementary schools. The data from these four teachers was gathered during the first year of the project. Although eight U.S. teachers participated in year one of the NSF project, only the four participants included in this study taught the targeted content for this study, multiplicative inverse relations. Amy, Esther, Jackson and Lily were the pseudonyms assigned for the four teacher participants. Amy and Esther were third grade teachers, and Jackson and Lily were fourth grade teachers at the time of this study. The teachers were selected from grades 3 and 4 because according to the Common Core State Standards (CCSSI, 2010) multiplicative inverses are taught across these two grade levels. In addition, all four teachers were employed by the same large high-needs urban school district in Pennsylvania; however, they each taught at different elementary schools within that district.

The following characteristics were considered when selecting these four expert teachers to be part of the overall NSF project (a) years of teaching experience, (b) having a good teaching reputation among peers (e.g., recommended by the principal or the school district), (c) having remarkable teaching recognition (e.g., earned teaching awards or certificates), (d) having a high score on a teacher survey instrument (Appendix A) and, (e) demonstrating high algebraic knowledge for teaching (AKT) on an open-ended multiplicative inverse teaching instrument (Appendix B). In particular, two of the teachers (Esther and Jackson) were National Board Certified Teachers (NBCT), and the

other two teachers (Amy and Lily) were highly recommended by their respective principals. All teachers except for Esther reported participating in 3+ professional development (PD) activities in the previous three years leading up to this study; Esther reported 1-2. Moreover, Amy reported that her average daily mathematics lesson (Time) lasted between 41-60 minutes, Lily reported 61-80 minutes, and Esther and Jackson both stated that they spent more than 81 minutes each day teaching mathematics. At the time of this study, all three females in the study held a master's degree in education (M.Ed) or an equivalence (M.Eqv), and Jackson had earned a doctorate degree in education (Ed.D). Table 1 provides a summary of the demographic information of the four expert teachers in this study.

Table 1: Teacher Participant Demographic Information

Name	Sex	Grade	Years	NBCT	PD	Time	Degree
Amy	F	3	0-6	No	3+	41-60	M.Ed.
Esther	F	3	21-25	Yes	1-2	81 +	M.Ed.
Jackson	M	4	6-10	Yes	3+	81+	Ed.D.
Lily	F	4	26 +	No	3+	61-80	M.Eqv

According to the teacher survey instrument (Appendix A), all four teachers believed that the inadequacy of a student's mathematics background can be overcome by good teaching. However, while Amy and Jackson agreed that a student's mathematics grade is likely to improve when a teacher employs a more effective instructional technique, Ester and Lily indicated that they were not sure if this association exists. Further, all four teachers alleged that they knew the steps necessary to teach mathematics concepts effectively. According to the open-ended multiplicative inverse teaching

instrument (Appendix B), all four teachers reported instructional techniques consistent with the reviewed literature on connection-making. Specifically, individual teachers noted a desire to use worked examples for building the structure of inverse relations, described the process of fading concrete manipulatives into abstract representations, and/or provided examples of deep questions that could be asked during instruction in order to solicit students' deep understanding.

Students

The students who participated in this study were from each of the four expert teachers' classes ($n_{Amy} = 24$, $n_{Esther} = 24$, $n_{Jackson} = 29$ and $n_{Lily} = 25$) and were participants in the overall NSF study. As indicated in Table 2, there were both similarities and differences in student demographics across the four classrooms. Concerning gender, the percent of female students in each of the four classrooms was around 50%. Although slight differences existed, the majority of students in all four classrooms were not disabled and were not classified as limited English proficiency (LEP) learners. In contrast, the race/ethnicity of students in these classrooms was not as consistent. For instance, while 79.2% of Esther's students were Caucasian, 72% of Jackson's students were African American. In addition, while approximately 30% of Amy, Esther and Lily's students were provided a free/reduced lunch, 68% of Jackson's students were granted this service. A more complete picture of the demographics of students in each of the four classrooms is provided in Table 2.

Table 2. *Student Demographic Information*

		Amy	Esther	Lily	Jackson
Gender	Female	50.0%	54.2%	44.8%	44.0%
	Male	50.0%	45.8%	55.2%	56.0%
Race/Ethnicity	Caucasian	29.2%	79.2%	44.8%	12.0%
	African American	16.7%	12.5%	20.7%	72.0%
	Hispanic	20.8%	4.2%	10.3%	4.0%
	Asian	20.8%	4.2%	13.8%	12.0%
	Multi-racial	12.5%	0.0%	10.3%	0.0%
Disability Status	No	95.8%	83.3%	86.2%	80.0%
	Yes	4.1%	16.7%	13.8%	20.0%
LEP Status	No	83.3%	100%	100%	96.0%
	Yes	16.7%	0.0%	0.0%	4.0%
Free/Reduced Lunch	No	70.8%	62.5%	69.0%	32.0%
	Yes	29.2%	37.5%	31.0%	68.0%

Data Sources and Procedures

Curriculum Sources

Investigations in Number, Data and Space (simply *Investigations*) was used by one third grade (Amy) and one fourth grade (Jackson) teacher in this study (Wittenberg et al., 2012). *Investigations* is a K-5 elementary mathematics reformed curriculum that is supported by the National Science Foundation (NSF) and was developed and maintained by the Technical Education Research Center, a nonprofit research and development organization whose mission is to improve mathematics, science, and technology teaching and learning. According to Barshay (2013), *Investigations* is one of the most widely used mathematics curriculums in U.S. elementary schools. This curriculum was developed

based on extensive classroom testing with a focus on allowing time for students to develop a strong mathematical conceptual skill set and has the full support of the National Science Foundation. *Investigations*, claims to "address the learning needs of real students in a wide range of classrooms and communities" in such a way that "invite(s) all students in mathematics—girls and boys; members of diverse cultural, ethnic, and language groups; and students with a wide variety of strengths, needs and interest" (Wittenberg et al., 2012). For each grade level, *Investigations* is broken down into various units that include individual textbooks for each unit. Implementation of 2 to 5 ½ weeks is suggested for each unit. At the time of this study, Amy and Jackson were using the second edition of *Investigations,* which was not yet aligned with the Common Core (CCSSI, 2010). However, both teachers had received common core supplemental materials from the publisher that they could use in conjunction with their existing textbooks.

The other third grade teacher (Esther) and the other fourth grade teacher (Lily) in this study used *GO Math,* another Common Core reformed curriculum (*GO Math!,* 2012). *GO Math* is a registered trademark of Houghton Mifflin Harcourt and claims to be the first K-8 mathematics program written to fully support the Common Core State Standards (CCSSI, 2010) through a focused, coherent and rigorous curriculum. *GO Math* claims to support critical thinking and application knowledge. Moreover, *GO Math* is a "21st-century educational technology with modern content, dynamic interactivities and a variety of instructional videos to engage today's digital natives" (*GO Math!*, 2012). To help facilitate a blended approach to instruction, *GO Math* claims to offer adaptive on-the-go instructional opportunities and comprehensive teacher support.

A full version of each curriculum (Grades K-5) was accessed for this study. From each textbook, four grade 3 and four grade 4 lessons were selected as part of the larger NSF project for examining classroom instruction. The lessons selected covered the major aspects of multiplicative inverse relations discussed in the literature and suggested by the Common Core (e.g., Baroody, 1999; Carpenter et al., 2003; CCSSI, 2010; Nunes et al., 2009; Torbeyns et al., 2009). In particular, among the four grade 3 lessons on inverse relations, two of them were related to fact families, one was about using multiplication to compute division, and one was a teacher choice on inverse relations. In grade 4, two lessons were related to multiplicative comparisons (find how many times, find the small or large quantity) which implicitly indicate inverse relations. In addition, there was one lesson on using multiplication to check division and one lesson on two-step word problems (operations were inverses). Table 3 lists the structure of the selected lessons topics.

Table 3. *The Overall Structure that Guided the Selection of Lessons*

	Multiplicative Inverse Topics		
G3	• Fact family (1) • Fact family (2) • Using multiplication to compute division • A topic suggested by teachers	G4	• Comparison word problem (1) - find how many times • Comparison word problem (2) - find the small or large quantity • Using multiplication to check for division • Two-step word problems

Across the two textbook series, common themed multiplicative inverse lessons were chosen in order to create the best scenario for comparison of connection-making opportunities across curriculums. In this study, both *Investigation*s and *GO Math* were used by one third and one fourth grade teacher participant, which allowed for comparison of textbooks across and within grade level. As suggested by Ding (2016), the teacher

textbook editions were used for the textbook coding and analysis. Although originally 16 textbook lessons were selected, only 14 were coded for analysis because both Jackson and Lily taught two lessons based on the same textbook lesson. This was not known to the researcher prior to data collection.

Videotaped Lessons

Each expert teacher agreed to be videotaped enacting his or her four corresponding textbook lessons. All 16 lessons were videotaped using two digital video cameras, one that followed the teacher throughout the lesson and one that was set up to capture student interactions. Members of the NSF project research team jointly collected and transcribed the footage from each lesson. These transcriptions were used to code the connection-making opportunities that the teachers provided during classroom instruction and to identify typical teaching episodes that promote the use of situation models for multiplicative inverses. The videotaped lessons had an average length of 43 minutes and were all collected during the 2014-2015 academic school year.

Teacher Interviews

A structured interview was immediately conducted with each teacher after each videotaped lesson was completed. These interviews were conducted for obtaining immediate self-reporting teacher feedback about the effectiveness of his or her instruction. The questions asked during these interviews included specific inquiries about the effectiveness of the instructional tasks, the uses of representations and the asking of deep questions that took place during instruction. For instance, the teachers were asked—What do you think about the representations you or students used during this lesson?—and—Did using the representations communicate mathematical ideas the way you

thought they would? The full questionnaire can be found in Appendix C. The transcribed teacher interviews were used for triangulation with the coded textbook and classroom enacted lessons and provided further evidence to help explain how and why teachers make instructional decisions during classroom teaching.

Student Assessments

To analyze students' comprehension of multiplicative inverses, a student pre-assessment was conducted at the beginning of the school year prior to any direct instruction on multiplicative inverses. The same test was administered as a post-assessment at the end of the school year after all four of the multiplicative inverse lessons were enacted by the teacher. Each item on this eight-question assessment (Appendix D) was adapted from the literature in order to assess comprehension of the complement principle of multiplicative inverses (Broody, 1987, 1999; Nunes et al. 2009, Resnick, 1989, Torbeyns et al. 2009). These items contain tasks on fact families, using inverse operations for computation or checking, and solving inverse unknown problems. In order to get a relatively complete picture of students' understanding of inverse relations, these items included both non-contextual and contextual items.

Data Analysis

The situation model theoretical framework generated from the review of literature suggests that mathematics instruction should focus on teaching students how to activate and facilitate the inference making process by making connections to relevant prior knowledge (Pearson et al., 1979). The instructional tasks, types of representations, and the use of deep questions during mathematics instruction have all been shown to affect a student's ability to form connections and draw inferences, the critical components of a

situation model. Based on the IES recommendations for establishing connections to underlying principles (Pashler et al., 2007), the author of this dissertation developed a coding framework (Appendix E) that was used for the content analysis of both textbook presentation and teacher instruction. This framework was based on Ding and Carlson's (2013) framework for coding teacher lesson plans, but was adapted for analyzing the existence and effectiveness of connection-making learning opportunities. It specifically was modified for coding both textbook and instructional opportunities that influence situation models. The initial version of the framework developed by the author was altered several times based on field tests and input from a second researcher working on the NSF project. These changes involved adding the words "targeted content" and "missed opportunities" for describing connection-making within several of the subcategories. For clarity, examples were also included in the highest connection-making category for concrete (i.e., story problems) and abstract (i.e., equations) representations. Finally, whereas the term "worked example" typically refers to instruction that involves providing students with already fully worked out problem solutions, it was decided this term would also refer to the sample tasks that teachers present during instruction. This decision was made because if the teachers in this study did not use "worked examples," their instructional examples would still be coded for connection-making opportunities.

The three main categories of this coding framework are based on the IES recommendations for improving student learning (Pashler et al., 2007) and were assembled from the reviewed literature (i.e., instructional tasks, Renkl et al., 2002; representations, Goldstone & Son, 2005; and deep questions, Chin, 2007). The connection-making framework included a 0-2 scale that was used to code the

effectiveness of forming connections within each of 9 subcategories. Concerning instructional tasks, the subcategories included review tasks, worked examples and practice problems. The subcategories for representations involved concrete, abstract and the sequence of representations. Finally, the framework included deep questions asked for establishing connections to prior, current and future knowledge.

To answer Research Question 1 – how reformed elementary CCSS textbooks facilitate connection-making—a qualitative content analysis involving the presentation of instructional tasks, representations and deep questions of each selected multiplicative inverse textbook lesson was conducted. The aforementioned connection-making framework based on the situation model theory of comprehension (Appendix E) was used for this analysis. In addition to identifying typical ways in which the two curriculums (*Investigations* and *GO Math*) facilitated connection-making, each textbook lesson was coded using a quantitative 0-2 scale which was applied to each subcategory in the framework. Table 4 provides an example of how the coding framework was implemented when coding Jackson's first textbook lesson involving multiplicative comparisons. The rationale of the textbook score for each category is provided (see Table 4) and the full textbook lesson can be found in Appendix F. More detailed explanations of the textbook codes are provided in the results section (Chapter 4) of this dissertation.

Table 4. *Example of Textbook Coding: Jackson's First Lesson*

Category	Subcategory	Score	Rational
Instructional Tasks	Review	0 - The task was a routine review of prior content but no connections to the targeted content was made.	The only task provided was "Ten-Minute Math" which reviewed basic arithmetic.
	Worked Examples	1 - Implicit connections to the targeted content were made, but not well established or discussed.	Activities 1 & 2 only discussed the multiplication structure. Discussion 3 states students might provide both a multiplication and a division

		Clear opportunities to make connections are missed.	equation; no suggestion on forming a connection to multiplicative inverses.
	Practice Problems	2 - Practice problems have an explicit connection to the targeted content.	Practice invoked multiplicative thinking (problems 1, 2, 4 & 6) and division thinking (problems 3 & 5).
Representations	Concrete	2 - Instructional tasks are situated in rich concrete contexts (i.e. story problems) and are used to form well developed connections to prior or targeted content within instructional tasks.	Every example and practice problem was situated within a real world story context. Concrete representations included pictures of apples and stick figures.
	Abstract	1 - Abstract representations are used to form connections to prior or targeted content within instructional tasks, but the connections are not well developed.	Multiplicative equations are provided in activities 1 and 2 but are not connected to division equations. Discussion 3 provides both but does not explicitly connect them to stress inverse relations.
	Sequence of Representations	1 - Connections between concrete and abstract representations are established during instructional tasks, but they do not always progress from concrete to abstract.	Activity 1 begins with pictures of apples & then provides a multiplicative equation. Activity 2 starts with multiplicative equation and then suggests drawing stick figures.
Questions	Prior	0 - No deep questions for the purpose of making connections to prior knowledge are posed.	No deep questions provided connections to prior knowledge.
	Current	1 - Some deep questions are posed for the purpose of making connections to targeted content, but connections remain at the surface level (ie. procedural)	Surface level questions: “where is the 2 in this problem?” -“what is unknown?”-“do you multiply or divide?” -“How did you solve this problem?”
	Future	0 - No deep questions are posed for the purpose of making connections to future content.	No deep questions provided connections to future knowledge.
	Total Score:	**8 / 18 Possible Points**	

All subcategory scores for each textbook lesson were summed resulting in a connection-making score for each textbook lesson (total possible score of 18). Averaging the four textbook lesson scores for each teacher yielded an overall textbook connection-

making score for each teacher. Further, an overall textbook connection-making score was calculated for each curriculum (see Appendix H). Along with a qualitative analysis that included typical ways in which these textbooks use instructional tasks, representations and deep questions to promote connection-making, these quantitative scores were used to examine and compare learning opportunities that contribute to the development of situation models for multiplicative inverses. To check the reliability of the coding process, a second researcher from the NSF project independently coded one textbook lesson from both curriculums in both grade levels (n = 4). Among these 36 codes (25% of all textbook codes), 2 codes were different, resulting in an initial reliability of 94%. In the case of the discrepancies, an in-depth dialogue led to a consensus and thus reliability reached 100%. After this reliability checking of the coding framework, all other textbook lessons were coded.

To answer Research Question 2 –how expert elementary mathematics teachers facilitate connection-making through the use of instructional tasks, representations and deep questions – each teacher's four enacted lessons were examined by means of the same content analysis framework (Appendix E) and through using similar procedures. Analogous to the coding of the textbook lessons, a 0-2 quantitative scale was used to code connection-making opportunities found within the subcategories of instructional tasks, representations, and deep questions within each teacher enacted lesson. Again, detailed explanations of these codes are provided in the results section (Chapter 4) of this dissertation.

Next, all subcategory scores were summed which resulted in a teacher connection-making score for each enacted lesson. The average of these four scores

yielded an overall teacher connection-making score for each teacher (see Appendix I). Along with qualitative analysis which identified typical teaching episodes that facilitated connection-making, these averages were used to determine the extent to which learning opportunities found within classroom instruction promote the development of situation models for multiplicative inverses. A second researcher coded the four videotaped enacted lessons corresponding to the textbook lessons that were used to check reliability of the textbook codes. Among these 36 codes (25% of all teacher codes) 3 were different, resulting in an initial 92% reliability. All 3 disagreements occurred in the asking of deep questions category, specifically between the scores of 0 (no deep questions) and 1 (some deep questions). For instance, while one researcher coded Jackson as asking no deep questions connected to prior knowledge because he did not review the meaning of multiplication (equal groups), the other researcher coded his review of procedural based questioning as indication that some deep questions had been asked for forming connections to multiplication. After negotiation and a discussion that involved this study's definition of deep questions (i.e., questions that elicits deep explanations), both researchers agreed that the "why" and "how" type questions that Jackson used to review the procedures involved in multiplication were indication that he had asked some deep questions (a score of 1) but important missing connections to prior knowledge remained. Thus, after reliability reached 100%, all other videotaped lessons were coded.

Based on the coding of the textbook lessons and enacted teaching, a quantitative comparison was conducted between teacher instruction and textbook presentation. The embedded mixed methods design used in this study allowed for further investigation of the second research question and was completed with an eye on determining if the expert

elementary teachers enhanced curriculum by increasing connection-making opportunities. When analysis suggested that expert teachers did enhance opportunities, calculated subcategory scores were analyzed to determine specific ways in which this occurred and qualitative analysis provided examples of those common enhancements. Although not generalizable due to the small sample size of teachers, the results of these comparisons were used to begin a discussion pertaining to how expert elementary teachers transition textbook connection-making opportunities into their enacted lessons. The teacher interviews conducted after the enacted lessons also helped to support this discussion because they provided a valuable qualitative component of practitioner insight surrounding the effectiveness of connection-making opportunities used during instruction.

To answer Research Question 3 –how textbook and instructional opportunities relate to student comprehension of multiplicative inverses– a quantitative multivariate linear regression analysis was conducted using SPSS version 22.0 software. Regression was selected because of the need to control for confounding variables when quantifying the effect that connection-making opportunities had on student comprehension. Further, regression models are considered robust tools for inference (Angrist & Pischke, 2010) if model assumptions are met. These assumptions include the existence of a linear relationship, multivariate normality, no or little multicollinearity between variables, residuals that are independent and fitted data points that are homoscedastic (equal variance) about the regression line (Montgomery, Peck & Vining, 2006). Moreover, the sample size of students (n = 102) in this study is sufficiently large enough to support statistical analysis as evident by the following power analysis. In a multivariate linear

regression analysis with $\alpha = .05$, k (the number of independent variables) = 9, an effect size of $R^2 = .20$ and a power = .90, a sample size of 90 is required (Cohen, Cohen & West, 2014).

The primary goal of the regression analysis in this study was to isolate the contribution that the textbook connection-making score and the teacher connection-making score had on student comprehension. As such, it was hypothesized that both textbook and teacher connection-making score would have a significant positive effect on comprehension. While correctness is certainly a huge contributing factor for comprehension, the focus of this study was on facilitating the creation of situation models for inverse thinking. Thus, instead of coding for correctness on the student assessment (Appendix D), the student pre- and post-test comprehension scores were determined based on a framework that specifically coded the existence of students' inverse understanding. In collaboration with the other NSF project researchers, the following rubric for inverse understanding was determined. Items 1-4 of the student assessment (Appendix D) were each coded as evidence of inverse understanding if a student generated all requested correct number sentences (all 3 parts of Q3 were required to be correct). Although providing all correct number sentences for these items may not guarantee a student possesses explicit understanding of inverse relations, this was considered the least evidence needed to show the possibility of inverse understanding. For items 5-8 on the assessment, inverse understanding was coded as existing if the student clearly used the complement principle to compute or check the solution. The full range of scores for inverse comprehension was therefore 0-8, with a score of 8 corresponding to the highest demonstration of a student's explicit understanding of

multiplicative inverses. Table 5 provides an example of how this inverse understanding coding framework was implemented for one of Amy's student's post-test (Appendix G).

Table 5. *Example of Inverse Understanding Post-Test Coding: A Student in Amy's Class*

Question	Score	Rationale
1	1	All 3 correct number sentences provided.
2	1	All 4 correct number sentences provided.
3	0	Part c was incorrect; Although the correct answer was provided the student did not provide the correct equation (27 ÷ 3 = 9) consistent with inverse understanding.
4	1	All 4 correct number sentences provided.
5	0	No indication of the inverse complement principle.
6	1	Belonging to "the same fact family" was considered use of the inverse complement principle.
7	0	The word "elimination" was not considered indication of the inverse complement principle.
8	0	The word "elimination" was not considered indication of the inverse complement principle.
Total Score:	**4/8**	

A second researcher on the NSF project team coded n = 24 students (384 total codes) for reliability and the number of inconsistent codes was 23. As such, the interrater reliability was 94%. After discussing the discrepancies, reliability reached 100% and the remaining student pre and post-test assessments were then coded in a manner consistent with the example in Table 5. During this discussion, two main decisions were made to address the unclear aspects of coding. First, it was agreed upon that repeated addition strategies represented "multiplicative thinking" and therefore would be coded as demonstration of inverse understanding. Figure 4(a) illustrates this case. Second, it was decided that an incorrect number sentence such as 2 ÷ 10 = 5 [Figure 4(b)] would be coded as no knowledge of inverse understanding. This decision was based on Ding and Carlson's (2013) discussion of computational errors. In addition, there were some instances in which a student arrived at a numerically correct solution but did not convey inverse understanding. Figure 4(c) provides an example of student work that was not

credited with demonstrating inverse understanding even though the numeric solution was correct.

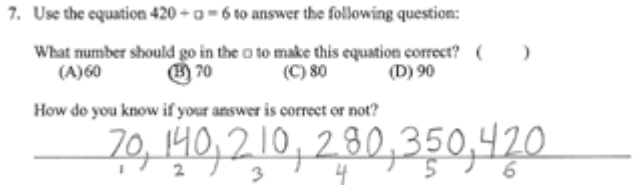
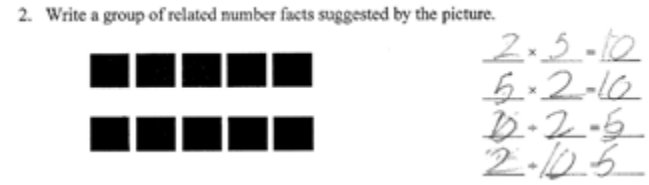
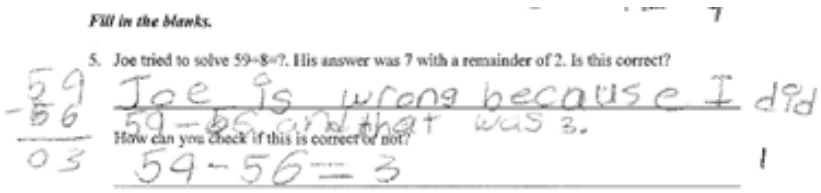
(a) The use of Skip Counting Coded as Inverse Understanding (1)	 7. Use the equation 420 ÷ □ = 6 to answer the following question: What number should go in the □ to make this equation correct? ( ) (A) 60 (B) 70 (C) 80 (D) 90 How do you know if your answer is correct or not? 70, 140, 210, 280, 350, 420
(b) Computational Errors Coded as No Inverse Understanding (0)	 2. Write a group of related number facts suggested by the picture. 2 × 5 = 10 5 × 2 = 10 10 ÷ 2 = 5 2 ÷ 10 5
(c) Correct but Coded as No Inverse Understanding (0)	 Fill in the blanks. 5. Joe tried to solve 59÷8=?. His answer was 7 with a remainder of 2. Is this correct? Joe is wrong because I did 59 − 56 and that was 3. How can you check if this is correct or not? 59 − 56 = 3

Figure 4. Samples of student solutions from the multiplicative inverse assessment.

The post-test inverse understanding score was the dependent variable in this multivariate regression analysis. The independent variables of interest included the textbook connection-making scores (Research Question #1) and the teacher connection-making scores (Research Question #2). Both of these connection-making variables could take on any value between 0 and 18 (see coding framework in Appendix E). In addition, the pre-test inverse understanding score (a value from 0 to 8) was entered as an independent variable to help control for variation among initial student understanding. This allowed conclusions to be made involving the role that prior knowledge had on comprehension. Further, the following student demographics were also considered as

independent variables during the analysis and served as further controls in the regression model: (1) disability status (indicator variable, 0 = not disabled and 1 = disabled), (2) gender (indicator variable, 0 = male and 1 = female), (3) race/ethnicity (indicator variable, 0 = Caucasian and 1 = not Caucasian), (4) LEP status (indicator variable, 0 = proficient English and 1 = limited English proficiency), (5) free/reduced lunch status (indicator variable, 0 = no and 1 = yes), (6) grade level.

A backwards elimination multivariate regression analysis was applied to these 9 independent variables. The backwards elimination method involves starting with the regression model that includes all independent variables and then testing the effect of deleting the least significant variables one at a time as indicated by the *p-values* of the standardized beta coefficients (Montgomery, Peck & Vining, 2006). Computing an *F* statistic between models determined if deletion of the variables improved the regression model. This process was repeated until no improvements in the model were possible and the assumptions of the regression were confirmed. The *adjusted-*R^2 value was then used to determine the percent of variation in comprehension that was accounted for by the remaining independent variables in the "*best*" multivariate regression model. When applicable, individual subcategory scores were entered as independent variables in an attempt to further determine the role that instructional tasks, representations, and deep questions have on facilitating situation models for multiplicative inverse understanding.

CHAPTER 4

RESULTS

In alignment with the three research questions, this study sought to explore how elementary reformed Common Core (CCSSI, 2010) textbooks and expert elementary mathematics teachers who use those textbooks facilitate connection-making in order to establish and enhance students' comprehension of multiplicative inverses. Viewing comprehension from a situation model perspective in which understanding is influenced by the nature of how learners form connections between current situations and prior knowledge, the findings below examine how the presentation of instructional tasks, uses of representations and the asking of deep questions afford students connection-making opportunities. In this chapter, I will first discuss the connection-making opportunities that exist in the teacher edition textbooks of both *Investigations* and *GO Math*. These opportunities were found as a result of performing a direct content analysis on multiplicative inverse lessons from each curriculum (Research Questions #1). Next, I will describe the opportunities for connection-making found during classroom instruction by revealing the findings from a content analysis on the corresponding lessons that were enacted by expert teachers (Research Question #2). Finally, I provide the results of a multivariate linear regression analysis that examined the effect that textbook and teacher connection-making opportunities have on facilitating situation models for multiplicative inverses (Research Question #3).

Connection-Making Opportunities Afforded by Textbooks

Averaging all of the textbook scores (n = 14) across curriculums revealed a total average textbook connection-making score of M = 12.13 (out of 18 possible points; SD =

3.45). This suggests that reformed Common Core curriculum materials provide at least some connection-making opportunities for learning the targeted content of multiplicative inverses. When broken down by curriculum however, there appears to be a sufficiently large difference between the *Investigations* (*M* = 9.25; *SD* = 1.91) and *Go Math* (*M* = 15.28; *SD* = 1.48) connection-making scores. Although differences exist among all sub-category scores, Figure 5 reveals that the greatest differences revolve around the use of instructional review tasks and worked examples, as well as the sequence of representations and the asking of deep questions (see also Appendix H). In the figure, the horizontal line at 18 represents the highest possible connection-making score.

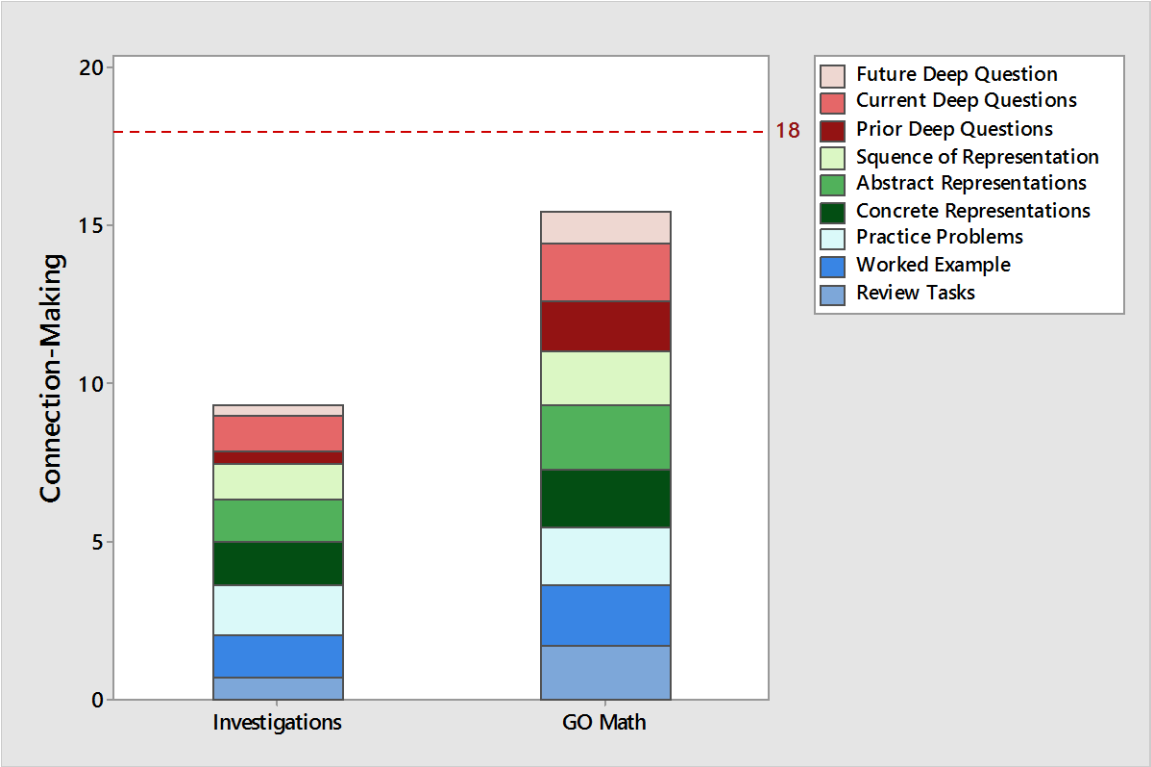


Figure 5. Average connection-making scores within each textbook.

Instructional Tasks

According to Figure 5, the instructional tasks provided by *GO Math* appear to offer well-developed explicit connections to multiplicative inverses. In contrast, the *Investigations* curriculum seems to form only implicit connections between and within instructional tasks. A closer inspection of how these instructional tasks are presented in both curriculums reveals three key findings. First, the level of cognitive demand found in review tasks is different between curriculums. Second, a difference exists in the use of worked examples for creating connections to the structural relationship of multiplicative inverses. Third, whereas *GO Math* tends to fade similarly structured worked examples into practice problems involving variations, *Investigations* does not. The following three sections explore these differences in detail.

Cognitive Demand of Review Tasks

The initial activation of conceptually relevant prior knowledge (schema) is the crucial first step in creating a mathematical situation model. Instructional review tasks that afford this initial activation opportunity are therefore an important part of learning. While it is promising that all textbook lessons in this study include some form of a review task, a close inspection of Figure 5 reveals a curriculum difference in how these tasks facilitate situation models for multiplicative inverses. In regards to the connection-making sub-category score for review tasks, *GO Math* scored an average of $M = 1.86$ (out of 2; $SD = 0.35$); whereas, *Investigations* scored an average of $M = 0.75$ ($SD = 0.45$). This suggests that the *Investigations* curriculum does not tend to explicitly connect review tasks to the targeted content. Instead, review tasks found in *Investigations* are categorized as “Ten-Minute Math,” and they often appear to have little or nothing to do

with multiplicative inverses. For example, the “Ten-Minute Math” for all of the third grade coded lessons focused on helping students tell time on an analog clock. On the surface, this task appears to have little to do with inverse relations; however, in two cases, multiplicative thinking (repeated addition) was invoked when students were told to count by 5’s. In another instance, students were asked to use five-minute intervals in order to identify the relative location of 11:18 on a clock. By dividing the clock into five-minute intervals, this task invoked division thinking. No discussion in the textbook however actually referred to using multiplication or division, and so at most, these review tasks offered only implicit connections to the targeted content. In the case of the fourth grade *Investigations* lessons, all “Ten-Minute Math” focused on practicing multiplication facts based on the method of repeated addition. This included having students write corresponding multiplication equations after listing out several multiples of a given number. Not having students count backwards in order to relate this process to division, revealed another missed connection-making opportunity during review. In general, the review tasks observed in the *Investigations* curriculum required low cognitive demand (i.e., stating facts, following procedures, solving routine problems; Van de Walle & Bay-Williams, 2012) and therefore provided only limited opportunities for activation of relevant prior knowledge, the first important component of developing a mathematical situation model.

On the other hand, the review tasks found within the *GO Math* curriculum afforded numerous opportunities to facilitate connection-making (i.e., high cognitive demand tasks; Smith & Stein, 1998). These opportunities were presented across three sections of the lesson: “Daily Routines,” “Response to Intervention Reteach Tier 1,” and

"Access Prior Knowledge." Typical "Daily Routines" in the lessons coded involved building fluency with a review of either multiplication facts or relevant vocabulary. While some of these tasks were procedural in nature, others explicitly stressed the conceptual relationships inherent in inverses. For example, Figure 6(a) illustrates part of a "Daily Routine" that required students to list explicit connections to the word quotient. The semantic nature of this task is likely to move a learner beyond the simple activation of prior knowledge and begins laying a structural foundation for the creation of a situation model for multiplicative inverses. "Daily Routines" also included a multiple choice "Problem of the Day" which often involved further explicit connections to targeted content. As indicated in Figure 6(b), these problems sometimes forced students to rely on reasoning involving counter-examples to refute or refine prior inference, a technique that can strengthen and create connections within one's situation model (Johnson-Laird, 1983). The "Reteach Tier 1" section of the "Response to Intervention" found within each *GO Math* lesson provided other review tasks that often explicitly connected multiplication and division. The "Reteach Tier 1" example in Figure 6(c) demonstrates one way in which the textbook reviewed the structural connection between a multiplication and a related division equation. Analyzing and comparing quantitative relationships that are explicitly connected to the targeted content appeared to also occur within the "Access Prior Knowledge" part of the *GO Math* lessons. Figure 6(d) includes a suggestion for reviewing the concept of additive comparison through use of the previously learned bar model (also known as strip or tape diagrams; Murata, 2008), which may contribute to students' new learning of the multiplicative comparison structure. Collectively, the connection-making opportunities afforded by the cognitively

high demanding review tasks in *GO Math* serve to activate prior knowledge in order to lay a foundation for increased encoding of core underlying mathematical principles, whereby initiating the process of forming a situation model.

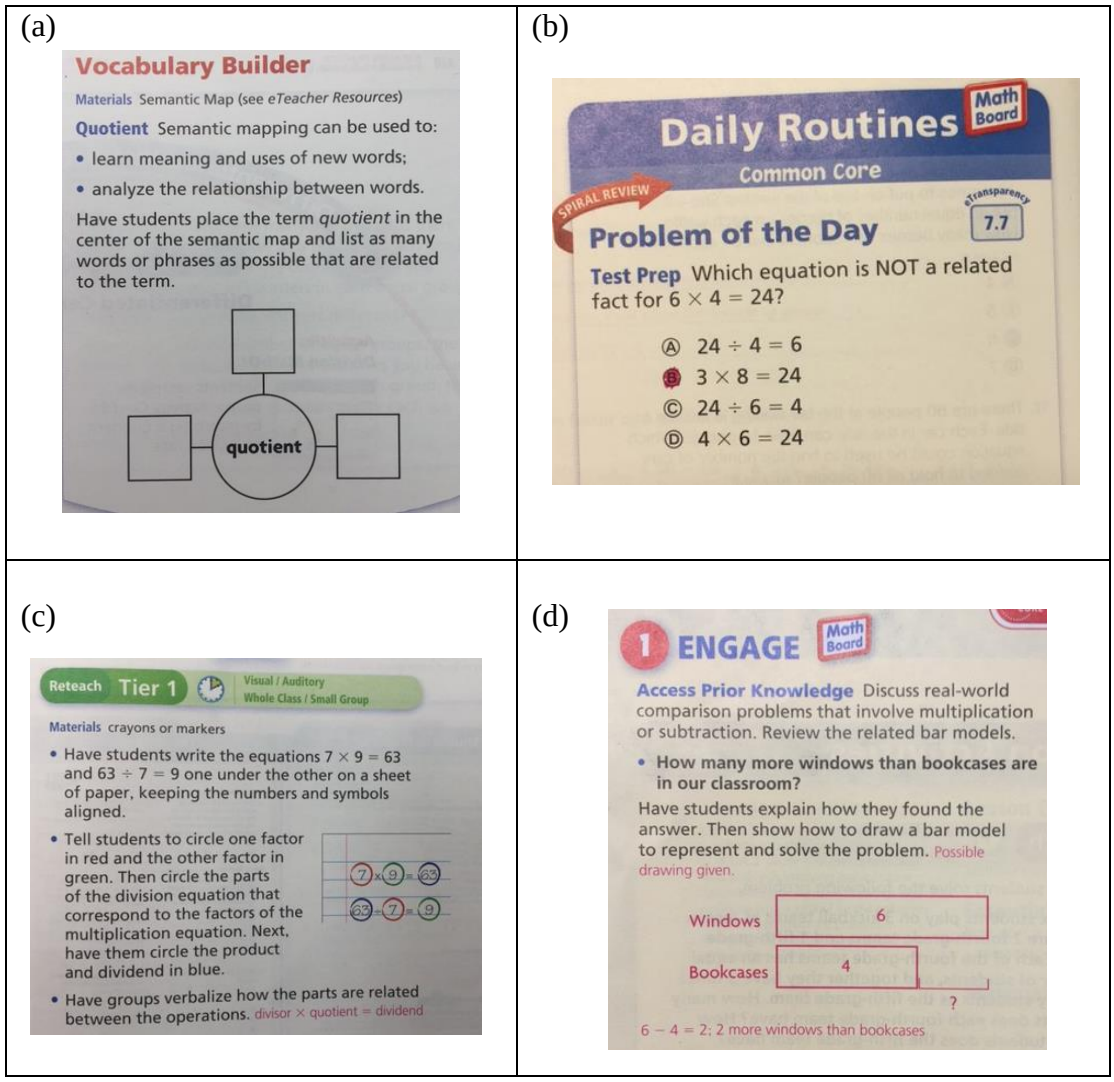


Figure 6. Explicit connections to multiplicative inverses found in *GO Math* review tasks.

The Use of Worked Examples to Create Constant Structure

In the situation model perspective of mathematical comprehension, worked examples provide students with the opportunity to observe how experts use activated prior knowledge during the inference-making process. Because this opportunity helps

novice learners create their own internal network of connections, worked examples should be explicitly connected to core underlying mathematical principles. Likewise, corresponding practice problems should also be connected to the targeted content so that learned knowledge can be reinforced for the purpose of strengthening connections within students' situation models. As indicated by Figure 7, while both *Investigations* and *GO Math* appear to provide practice problems that are explicitly connected to inverse relations ($M_I = 1.57$, $SD_I = 0.49$; $M_{GM} = 1.86$, $SD_{GM} = 0.35$), there seems to be a curriculum difference among worked examples ($M_I = 1.29$, $SD_I = 0.45$; $M_{GM} = 1.86$, $SD_{GM} = 0.35$). In the figure, the vertical line at 2 represents the highest possible connection-making score.

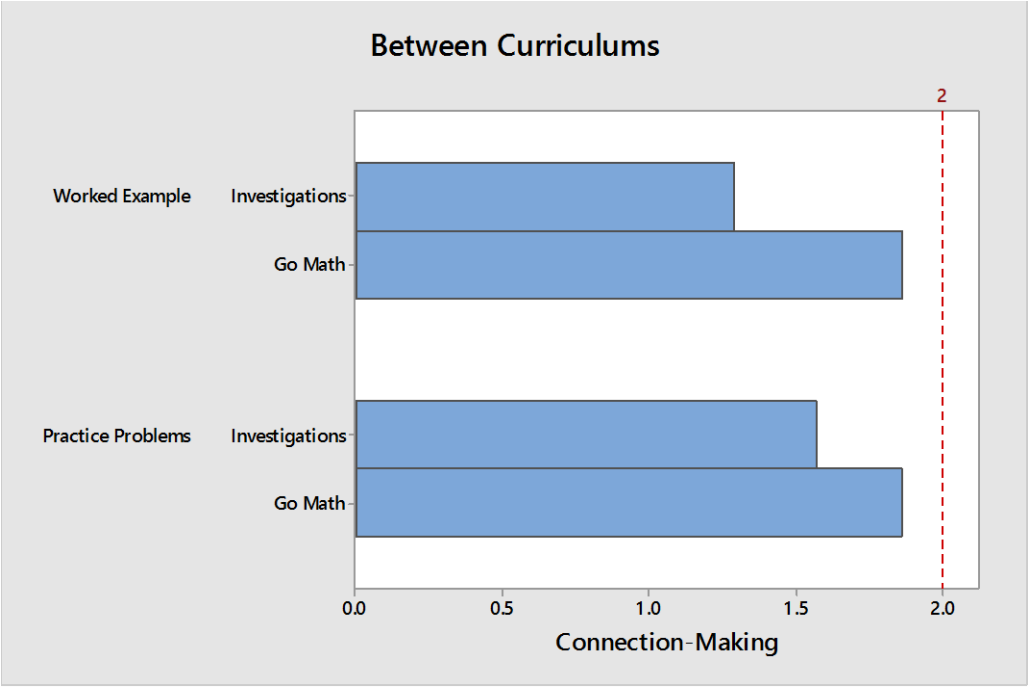


Figure 7. Between curriculum connection-making differences in worked examples and practice problems.

The worked examples in the coded textbook lessons of *Investigations* often lacked a clear connection to the structural relationship of multiplicative inverses. In six of the seven coded lessons in fact, only implicit connections between the worked examples and the targeted content were established. Although the worked examples in these lessons did involve both multiplication and division, they were often only computationally driven. In other words, *Investigations* tended to focus the use of worked examples around the procedures instead of the relationships within multiplication and division problems, which seems not to reinforce the development of a situation model for multiplicative inverses. For instance, in the third grade lesson entitled "Multiply or Divide," the worked example for multiplication and the worked example for division were presented in isolation. Even though the same numerals (4, 5 and 20) were used in both problems, the text did not explicitly suggest making the structural connection for how the division problem could be used to solve the related multiplication problem. Instead, at the very end of the lesson the textbook provided a chart (Figure 8) and instructed teachers to have students fill in the equation column with an equation that best represents each problem. It is promising that the first example in the chart provided both a multiplication and a division equation; however, the second does not, and the teacher edition of the textbook failed to emphasize the use of or connection between the two equations. Although a note to the teacher mentioned that this chart was designed to teach inverse relations, the potential learning opportunity for understanding inverse relations at a structural level (i.e., developing a situation model for multiplicative inverses) was never explicitly made.

Number of Groups	Number in Each Group	Product	Equation
?	4 muffins	20	20 ÷ 4 = ____ or ____ × 4 = 20
5 packs	4 yogurt cups	?	5 × 4 = ____

Figure 8. Multiply or Divide chart from *Investigations*.

Instead of focusing on the structure of multiplicative inverses, commentary across lessons within the teacher's edition of *Investigations* included side-bar "Math Notes" that often focused attention on notation and placed an emphasis on the desire for students to develop various solution strategies (i.e., tallies, skip-counting, and multiplication facts). As seen through these "Math Notes," the main goal of worked examples in *Investigations* appeared to be for teaching procedural computations as opposed to helping students gain a structural foundation for inverse relations. This goal was reinforced by another coded *Investigations* lesson in which the worked examples involved writing story problems for the expressions 6 × 3 and 18 ÷ 3. This was another potential opportunity to teach inverse relations in a meaningful way. Unfortunately, the worked examples used different contexts when writing these two stories, and as a result, the relationships between the parts of the two expressions remained implicit. The practice problems for this particular lesson also did not request the use of the same context for similar expressions, thus confirming another missed opportunity to form explicit connections to the structural relationships necessary for the development of a situation model for multiplicative inverses.

In contrast to *Investigations*, all but one of the *GO Math* lessons used worked examples to make explicit connections to multiplicative inverses. Further, these connections were almost entirely focused on the structural relationship of multiplicative inverses and were often presented in the format of side-by-side solutions related to multiplication and division equations. Figure 9 shows two instances which the *GO Math* textbook suggested teachers use to form a connection to the structural relationship of multiplicative inverses.

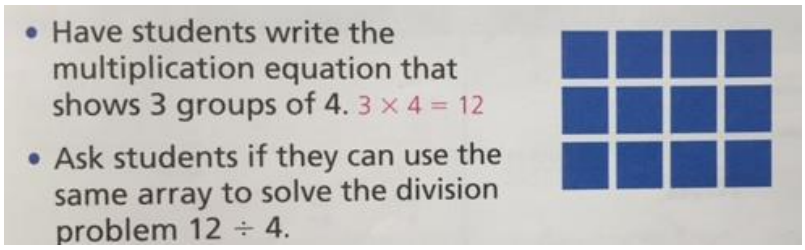
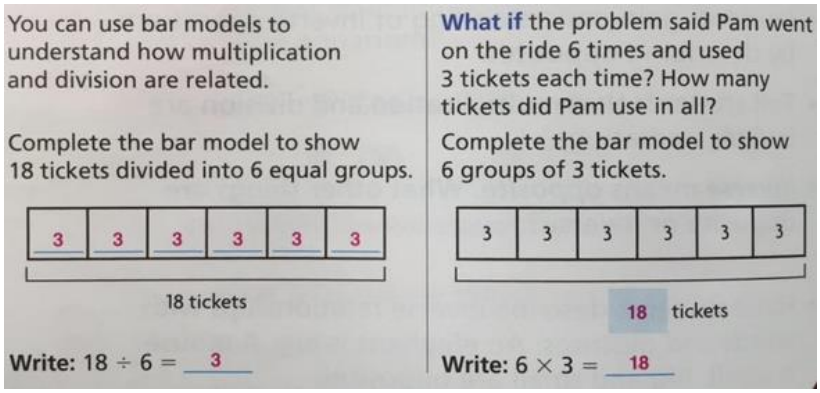
(a) Array	
(b) Bar Model	

Figure 9. Side-by-side examples in which the text suggested teachers use to form a connection to the structural relationship of multiplicative inverses in *GO Math.*

The first visual depiction [Figure 9(a)] of the structural relationship of multiplicative inverses was accompanied in the *GO Math* teacher textbook with a note that read, “the same array can represent a multiplication fact and its related division fact. The array is used in different ways to find a product or quotient…depending on the situation” (Grade

3, p. 235b). The second example [Figure 9(b)] included instructions for the teacher to use "Math Talk to have students focus on how multiplication and division are related by comparing parts of the two models" (Grade 3, p. 235), the part-whole relationship (i.e., the situation model) for multiplicative inverses. This directive went on to state that "it is important for students to notice the relationships between factors, products, divisors and quotient" (Grade 3, p. 235) and provided exemplary deep questions for the teacher to use when making these explicit connections. The "Reteach Tier 1" section of Figure 6(c) provides yet another alternative method which *GO Math* suggested that teachers use when forming connections within the structure of inverse relations.

What is also particularly powerful about the teacher's edition of the *GO Math* textbook is the "Professional Development" sections that are included within each lesson in order to provide commentary to teachers "About the Math." These excerpts serve to help teachers form connections between targeted content and prior or future knowledge. In the reviewed lessons, the "About the Math" included examples of inverse thinking relative to the operations of addition/subtraction (prior), multiplication/division (current) and algebraic equations (future). These explicit connections extended beyond what was simply learned yesterday or what will be learned tomorrow to include the reasons why students need a solid understanding of the structural relationship of multiplicative inverses. One of these "Professional Development" passages in *GO Math* that is titled "Look for and make use of structure" is provided below:

> When students are able to identify important mathematical relationships by noticing a pattern, they can use those relationships to solve problems. Students can use the structure of a division problem to solve a related multiplication problem in order to find a quotient, dividend, or divisor. They should recognize that a quotient and divisor of a division problem

> are related to the factors of a multiplication problem, and that the dividend is related to the product. Using this structure will help students solve more complex multiplication and division problems in future lessons and algebraic equations in future grades. (Grade 3, p. 237)

Comparing this level of support with the *Investigations* curriculum which simply tells teachers to have students "think about what is the same or different" (Grade 3, p. 117) but provides no instructional support for how to connect possible student responses to the targeted content of multiplicative inverses, perhaps best summarizes the curriculum differences in facilitating connection-making through the use of worked examples. The focused attention that *GO Math* places on the underlying structure of multiplicative inverses helps to illustrate the inter-connectedness of mathematics which is crucial for the strengthening of a student's situation model.

Fading Worked Examples into Practice

From the situation model perspective, worked examples should all have an underlying constant structure that students can use to develop and apply their own schema when presented with new situations such as practice problems that have varied surface characteristics. The *GO Math* curriculum was found to agree with this perspective in that worked examples (M_{GM} = 1.86, SD_{GM} = 0.35) established a constant structure that was maintained in practice problems (M_{GM} = 1.86, SD_{GM} = 0.35) that had varied surface characteristics (see Figure 10). *Investigations* on the other hand tended to vary the structure within practice problems. Although the *Investigations* lessons included practice problems (M_I = 1.57, SD_I = 0.49) that were more likely to be explicitly connected to multiplicative inverses, only implicit connections to targeted content seemed to be made through the presentation of the worked examples (M_I = 1.29, SD_I = 0.45). In the figure, the vertical line at 2 represents the highest possible connection-making score.

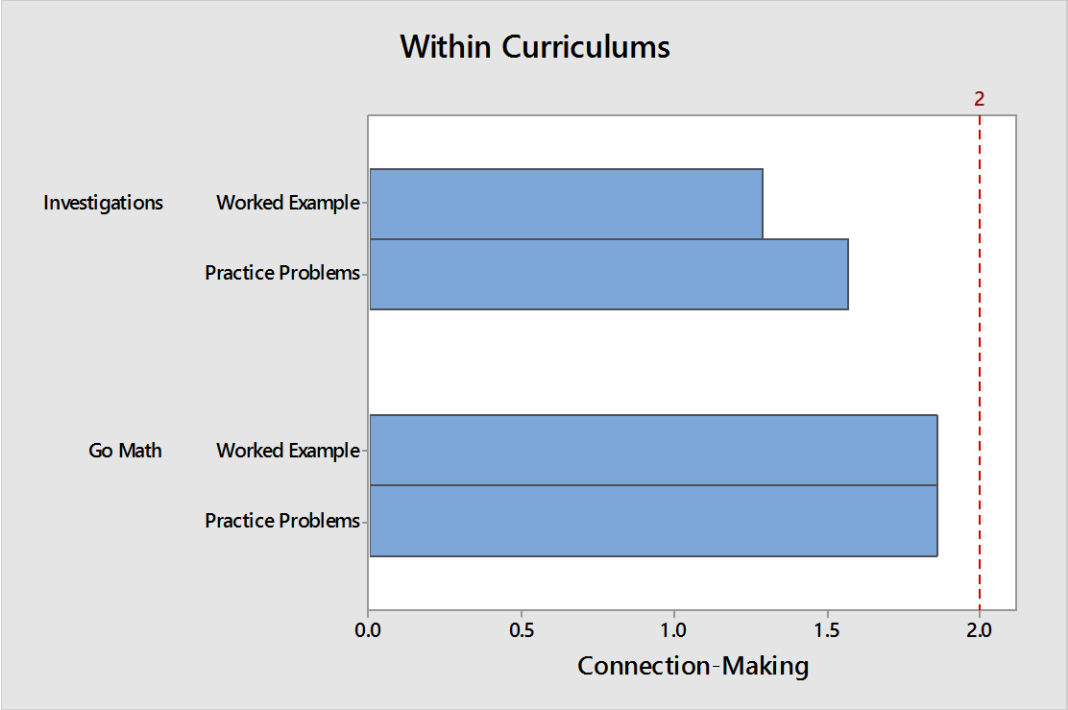


Figure 10. Within curriculum connection-making differences in worked examples and practice problems.

In other words, the worked examples in *Investigations* did not tend to discuss inverse relations, yet students were required to use inverses for self-guided practice. A close inspection of the individual textbook lesson scores revealed that worked examples had lower connection-making scores than practice problems in half (n = 4) of the coded *Investigations* lessons. For example, the two worked examples in one of the fourth grade lessons involving multiplicative comparisons were as follows:

1. *Darlene picked 7 apples. Juan picked 4 times as many apples. How many apples did he pick?*

2. *Franco's daughter is 2 feet tall. Franco is 3 times as tall as his daughter. How tall is he?*

Clearly, these examples were repetitive in nature and only stressed multiplicative thinking. When completing independent practice however, students were prompted with the following questions that invoked division thinking:

3. *A tree in Helena's yard is 25 feet tall. Helena is 5 feet tall. The tree is how many times as tall as Helena?*

4. *Amelia has 24 marbles. She has 6 times as many marbles as Steve. How many marbles does Steve have?*

These problems illustrate how *Investigations* did not tend to fade worked examples into practice problems that consist of varied surface characteristics. Instead, an incomplete structure seemed to be presented in the worked examples, and practice problems tended to vary at a structural level. Only after students had the opportunity to possibly form a structural connection to inverse relations themselves during practice did the text suggest the teacher make this structural connection explicit.

This finding was consistent with the third grade *Investigations* lesson on solving division problems which presented students with only one worked example but expected them to be able to visualize each situation (i.e., grouping and sharing) when completing practice problems. Further, only at the very end of these practice problems did the textbook suggest the need to "explain to students that the story problems they solved were all division problems" (Grade 3, p. 119). This further supports the notion that *Investigations* overall tended to not form explicit connections during worked examples. Interestingly, in a different lesson that did use worked examples to form explicit connections to multiplicative inverses, the practice problems only involved division. In this case, expecting students to use multiplication to find the answer to a division problem was emphasized but only illustrated the use of inverse relations for the purpose

of computation, not for creating a structural connection to the targeted content. Taken together, the presentation of content in *Investigations* did not tend to support the fading of worked examples into varied practice problems. Rather, students are often expected to form their own connections to the underlying principle of inverse relations during their own independent practice. Overall, the presentation of instructional tasks in the *Investigations* curriculum did not tend to contribute to the development of a situation model for multiplicative inverses.

The *GO Math* curriculum on the other hand tended to use worked examples that were connected to the underlying structure of inverse relations and reiterated the structure with practice problems that had varied surface characteristics. Instead of having students "investigate" connections on their own, *GO Math* "Unlocks the Problem" during the first worked example in each lesson. This "Unlocking" in the initial example serves to establish a connection to the core underlying mathematical concept being presented in the lesson, the important first step in the mathematical situation model perspective of comprehension. This connection is made explicit with step-by-step interactive procedures in the teacher's edition of the textbook and is facilitated by examples of deep questions that teachers can use in order to illicit deep conceptual understanding. The worked examples and practice problems that follow are grounded by this foundational structure and are gradually varied in order to further comprehension. Figure 10 provides an example of how this fading of worked examples into varied practice problems occurred in a typical *GO Math* lesson involving multiplicative inverses.

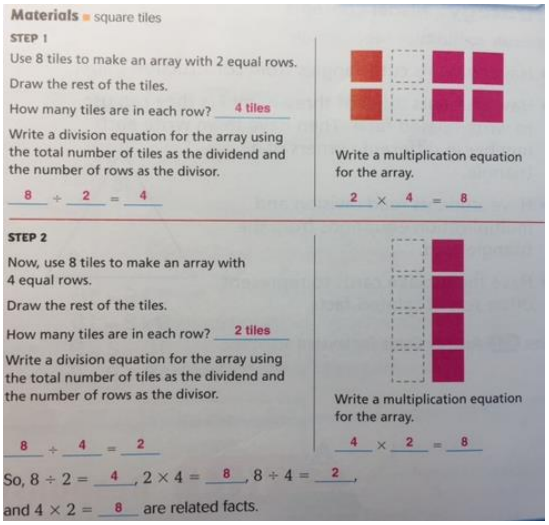
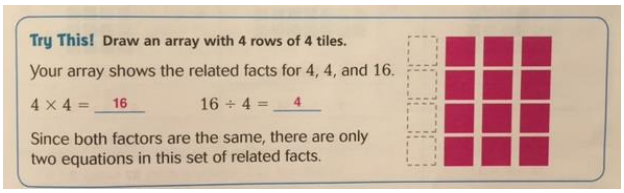
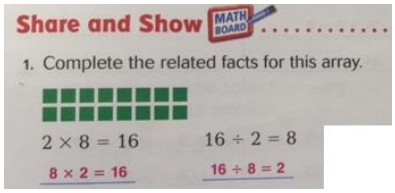
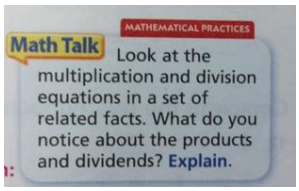
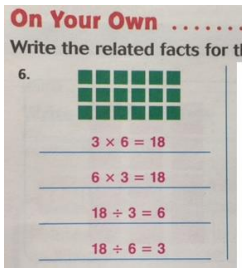
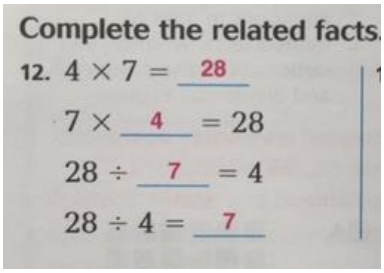
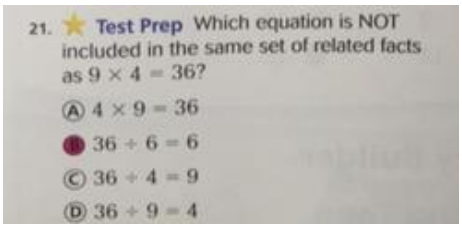
(a) Unlock the Problem	 Materials square tiles STEP 1 Use 8 tiles to make an array with 2 equal rows. Draw the rest of the tiles. How many tiles are in each row? 4 tiles Write a division equation for the array using the total number of tiles as the dividend and the number of rows as the divisor. 8 ÷ 2 = 4 Write a multiplication equation for the array. 2 × 4 = 8 STEP 2 Now, use 8 tiles to make an array with 4 equal rows. Draw the rest of the tiles. How many tiles are in each row? 2 tiles Write a division equation for the array using the total number of tiles as the dividend and the number of rows as the divisor. 8 ÷ 4 = 2 Write a multiplication equation for the array. 4 × 2 = 8 So, 8 ÷ 2 = 4, 2 × 4 = 8, 8 ÷ 4 = 2, and 4 × 2 = 8 are related facts.
(b) Try This!	 Try This! Draw an array with 4 rows of 4 tiles. Your array shows the related facts for 4, 4, and 16. 4 × 4 = 16 16 ÷ 4 = 4 Since both factors are the same, there are only two equations in this set of related facts.
(c) Share and Show	 Share and Show MATH BOARD 1. Complete the related facts for this array. 2 × 8 = 16 16 ÷ 2 = 8 8 × 2 = 16 16 ÷ 8 = 2  MATHEMATICAL PRACTICES Math Talk Look at the multiplication and division equations in a set of related facts. What do you notice about the products and dividends? Explain.
(d) Independent Practice	 On Your Own Write the related facts for the array. 6. 3 × 6 = 18 6 × 3 = 18 18 ÷ 3 = 6 18 ÷ 6 = 3  Complete the related facts. 12. 4 × 7 = 28 7 × 4 = 28 28 ÷ 7 = 4 28 ÷ 4 = 7
(e) Test Prep	 21. Test Prep Which equation is NOT included in the same set of related facts as 9 × 4 = 36? A 4 × 9 = 36 B 36 ÷ 6 = 6 C 36 ÷ 4 = 9 D 36 ÷ 9 = 4

Figure 11. Fading of worked examples into varied practice problems in *GO Math*.

In this lesson on related facts, the “Unlock” problem [Figure 11(a)] used an array model to make the explicit connection to the structure of inverse relations by presenting all four related multiplication and division equations for the fact family of 2, 4, and 8. Although this is a specific numeric example and therefore may not promote transfer, the teacher is told to make explicit connections between the rows and columns in both images which lays the structural foundation of multiplicative inverses. It should be noted however that the highlighting of the boxes (white dashed-lines) does not seem to agree with the structure of the multiplication problems under the arrays. For instance, in the first array where a group of 2 boxes is highlighted to represent 4 groups of 2, it would have been better to connect this representation to $4 \times 2 = 8$, the multiplication statement that directly represents 4 groups of 2. The second worked example “Try This” [Figure 11(b)] used the same multiplicative inverse structure but varied a surface characteristic of the problem which required students to think about how the underlying concept could be applied in a slightly new situation. Specifically, a fact family that involved two of the same factors (a square number) was presented which resulted in only two related equations. Next, the textbook lesson presented students with “Share and Show” [Figure 11(c)] which bridges the gap between worked examples and completely independent practice. The first problem in that section contained one of the same factors (2) as the first worked example, and students were provided with the same concrete representation (array with 2 rows) to support the constant underlying structure. Students also only had to provide two related equations since the other two (one multiplication and one division) were already provided. To emphasize the connection to multiplicative inverses, students were encouraged by the “Math Talk” to “look at the multiplication and division equations

in the set of related facts" (Grade 3, p. 240) in order to determine that the "product of a multiplication is the dividend of a related division" (Grade 3, p. 240). Further independent practice problems [Figure 11(d)] included a chance for students to practice writing related equations both with (write the related facts for the array) and without (write the related facts for the set of numbers) the support of the array model. Next, instead of writing out all related equations, students were asked to complete a set of related facts (complete the related facts). While this task maintained the structural connection to inverse relations, the process of filling in missing unknown factors challenged students to move beyond the surface characteristics of the worked examples and in turn began to facilitate algebraic thinking. In addition, this task required students to find missing factors, which was different from the worked examples that used the product or the quotient as the unknown components. Varying the missing unknown components represents a change in the surface characteristic of the worked examples. This may also help students strengthen their own schema surrounding the underlying constant structure of multiplicative inverses and thus help in the development of their situation models. Finally, the last practice problem entitled "Test Prep" presented a drastic surface characteristic change from all previous examples and problems. Instead of focusing on which equations were related, this problem involved asking students to determine which equation was not part of a related fact family. Reasoning with counter-examples, as presented by *GO Math* in this question, helps students to refute or refine prior inferences and in turn should create and strengthen the numerous connections within their situation models for multiplicative inverses.

Representations

All 14 coded textbook lessons included both concrete and abstract representations. Many of the instructional tasks found in both curriculums involved story problems that were situated in concrete contexts (e.g., frogs in a pond, desks in a classroom, and tickets per ride). Situating early learning of inverse relations in completely real-world contexts likely creates an opportunity to initiate the beginning stages of creating a situation model, because connection-making begins as a result of activating students' informal prior knowledge. As a result, concrete representations are frequently used to model these real-world situations. In *Investigations,* these representations most commonly involved drawings and tally marks. In *GO Math*, they included arrays and bar models. Both curriculums also suggested that classroom instruction include the use of concrete manipulatives such as cubes or counters. In addition, abstract representations used in both curriculums included the standard multiplication notation of $a \times b$ and both the $a \div b$ and $b\overline{)a}$ representations of division. Upon inspection of the connection-making scores involving representations, there seems however to be curriculum differences in the purpose behind the use of representations and in the sequence of how representations appeared throughout a lesson.

Purpose of Representations across Grades and Curriculums

The purpose behind using concrete representations for teaching inverse relations appeared to be different for both the third grade and fourth grade lessons. The third grade lessons coded in *Investigations* tended to use concrete representations as a means for calculation. The third grade lessons in *GO Math* on the other hand were more inclined to use concrete representations to form connections to the structural relationship of

multiplicative inverses. In the fourth grade lessons, *GO Math* tended to further develop these connections; whereas, *Investigations* tended to initiate these connections. In other words, *GO Math* used concrete representations to develop connections sooner and to a better degree than *Investigations*, which most likely leads to the increased development of a student's situation model. Figure 12 (the horizontal line at 2 represents the highest possible connection-making score) illustrates that the third grade connection-making scores for both concrete and abstract representations are collectively lower than the fourth grade scores, possibly indicating that the connections established between representations and the target content of multiplicative inverses seemed to be better developed in both fourth grade curriculums. This suggests that the textbooks might have intended to develop implicit understanding in third grade and explicit understanding in fourth grade.

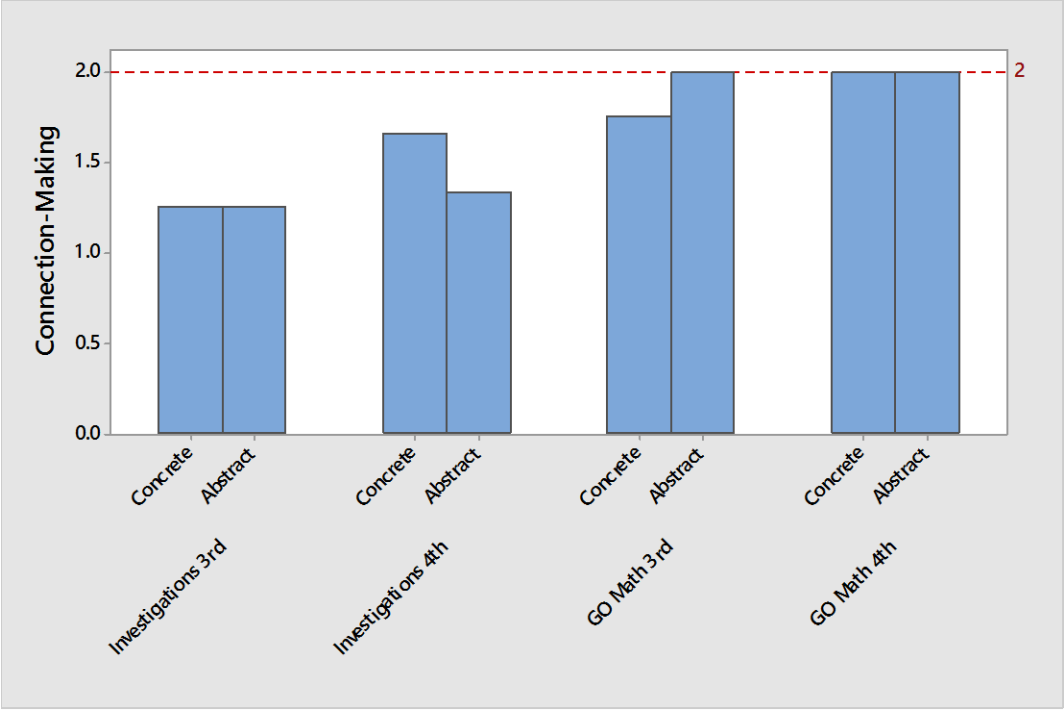


Figure 12. The use of representations to form connections in textbooks.

The third grade *Investigations* lessons tended to place a large emphasis on arriving at a correct solution. As a result, multiple solution strategies were endorsed. Developing efficient strategies however was not emphasized, and thus representations tended to focus on numerical calculations. Analyzing the representations in the third grade *Investigations* lessons revealed that tallies and stick figures were mainly used for the purpose of calculating the product or quotient within worked examples. For instance, in one example that involved dividing 28 classroom desks into groups, the teacher's textbook suggested using an illustration of 28 tally marks for computing the quotient [Figure 13(a)]. Although the math focus point for that lesson was "using the inverse relationship between multiplication and division to solve problems," only the last of a list of six different possible solution strategies mentioned the use of known multiplication facts. Further, the abstract equations that were used in the third grade *Investigations* lessons were often not connected to the concrete representations and rarely were they used to form connections to multiplicative inverses. Figures 13(b)(c) are two examples of sample student work included in the third grade *Investigations* curriculum that do not connect to abstract representations, which confirms the computational focus of concrete representations.

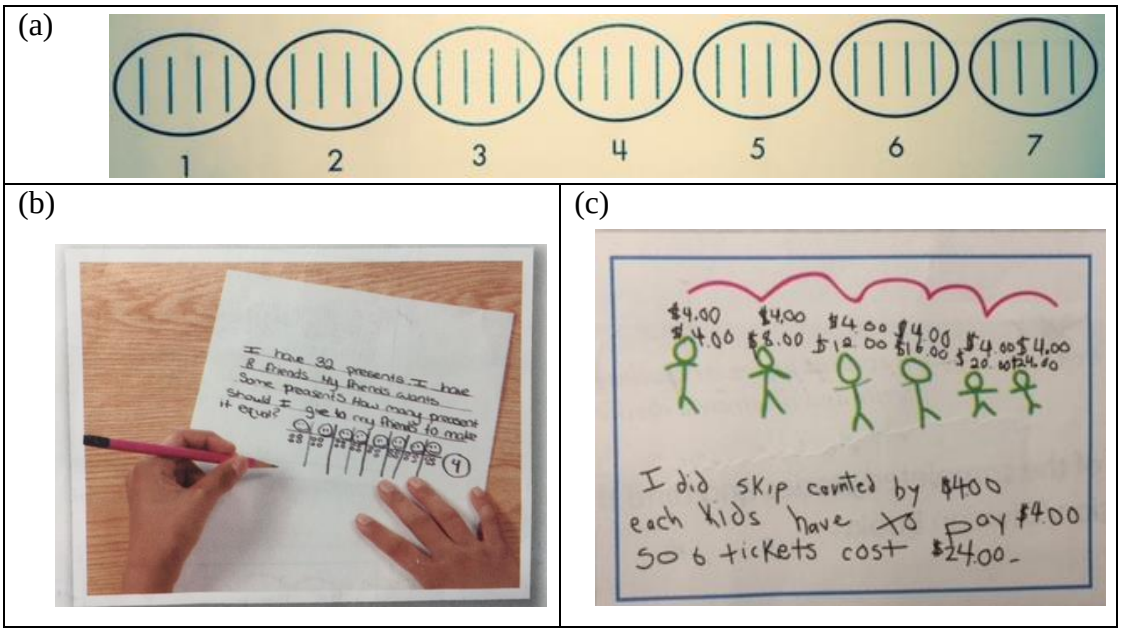


Figure 13. Computational purpose of concrete representations in third grade *Investigations.*

Although concrete representations were also used for computation in the *GO Math* third grade curriculum, unlike *Investigations*, these representations were supplemented by structural support. For example, when determining how many apples were used to make 28 pies, *GO Math* presented two different concrete representations. First, students were instructed to create an array model by drawing 1 tile in each of 7 rows until all 28 tiles were drawn. Second, students were instructed to draw 7 circles and place 1 counter at a time into each circle until all counters were assigned a circle. The first representation invoked multiplicative-thinking, the second division-thinking. By including both of these concrete representations in the worked example, the computational aspect of each individual representation was overshadowed by the structural connection to multiplicative inverses that was established as a result of including both types of thinking.

As previously shown in Figure 9(b), *GO Math* also introduced the concrete bar model representation during third grade instruction for the purpose of forming units in

order to build connections to the part-whole structural relationship (i.e., the situation model) of multiplicative inverses. Figure 14 provides an illustration of how *GO Math* used the bar model in conjunction with the concrete representations explained in the apple pie example in order to further emphasize the structural nature of multiplicative inverses. This example involved first sharing 20 dog treats evenly among 5 dogs (using multiplication thinking to determine how many in each group), and then second, distributing 20 treats five at a time in order to determine how many dogs (using division thinking to determine how many groups). Although a connection to the abstract equation 20 ÷ 5 = 4 was made by the textbook, 5 × 4 = 20 was never mentioned. Although rare in the *GO Math* curriculum, this was a missed opportunity to use abstract representations to form connections to the targeted content of multiplicative inverses.

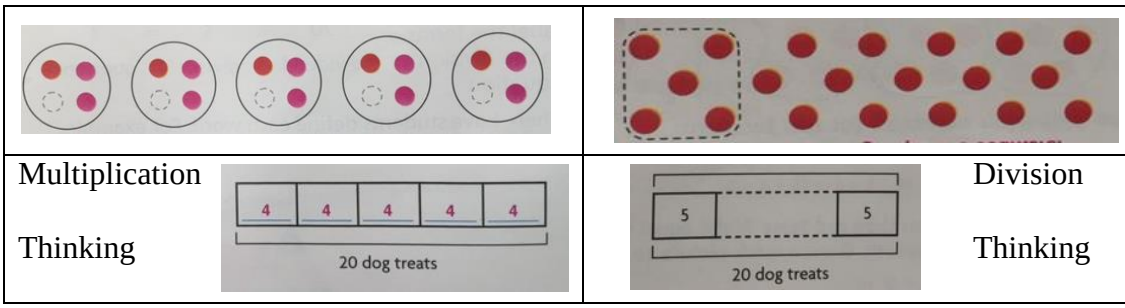
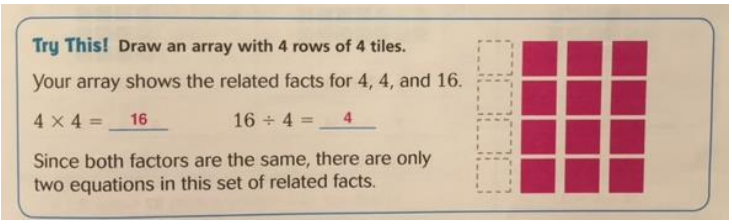


Figure 14. Concrete representations connected to multiplicative inverses in *GO Math.*

Abstract equations were commonly provided alongside concrete representations in the *GO Math* third grade lessons, and they often were used to form explicit connections to multiplicative inverses. This was different from the third grade *Investigations* curriculum that tended not to include side-by-side concrete and abstract representations. However, in one instance in which *Investigations* did attempt to form connections between concrete and abstract, the connections were not as explicit as the connections found in *Go Math*. Figure 15 illustrates this difference in how *GO Math* and

Investigations used the semi-concrete array model to represent the inverse relationship between multiplication and division. As seen in Figure 15(a), *GO Math* explicitly provided both the abstract multiplication and division equations that represent the pictured array. This was different from *Investigations*, which only depicted the abstract multiplication expressions 4 × 6 and 6 × 4 on a similar array representation [See Figure 15(b)]. Although a connection to multiplicative inverses exists based on the structure of the array in Figure 15(b) (i.e., the rows and columns), this connection at first only appeared implicit since no division expressions were provided alongside the array. Instead, *Investigations* only made this connection explicit by telling teachers to ask students for the division expressions if students did not first suggest them. This was consistent with the finding that no other worked examples or practice problem that involved arrays in the *Investigations* curriculum began with an abstract division expression. Showing side-by-side multiplication and division expressions and asking students to provide all fact family expressions during independent practice most likely would have helped students form better connections between concrete and abstract representations, whereby strengthening their situation models for multiplicative inverses. Both of these instructional techniques were used by the third grade *GO Math* curriculum.

(a) *GO Math*



(b) *Investigations*

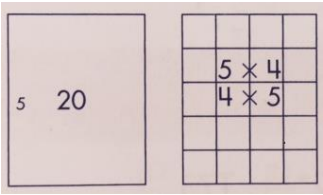
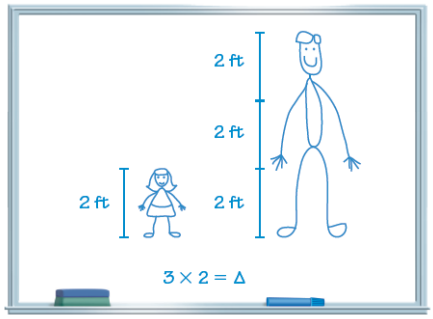


Figure 15 . The use of arrays to teach multiplicative inverses in third grade lessons.

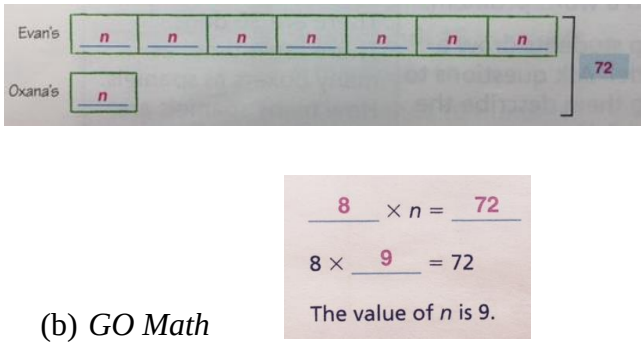
Even though differences in the purpose of representations existed in the third grade curriculums, multiplicative inverse connections to representations were strengthened from the third to the fourth grade lessons in both curriculums. This is consistent with the Common Core's (CCSSI, 2010) emphasis of forming fundamental mathematical connections across grade levels. In *GO Math*, this occurred as a result of strengthening the foundational connection already created by the previous use of the bar model representation. The third grade *GO Math* lessons used both concrete arrays and counters to supplement the semi-concrete bar model representation. By fourth grade, these purely concrete representations were no longer used, and the bar models had become the focal point of instruction for multiplicative inverses. In fact, in most of the coded fourth grade worked examples, students were explicitly asked to draw and explain how a bar model could be used to represent both multiplication and division situations. Independent practice problems in the fourth grade *GO Math* lessons also required students to create bar models which was an extension to the third grade practice problems that only expected students to be able to use the bar models. Using the constant structure of the bar model but providing a reduced amount of support (i.e., having students self-create bar models), illustrates how *GO Math* strengthened the structural connections for inverse relations between grade levels that in turn most likely enhanced students' situation models for multiplicative inverses.

In the fourth grade *Investigations* lessons, multiplicative inverse connections to representations were strengthened as a result of the creation of a foundational connection that was missing in the third grade curriculum. Although not as well-developed as the third grade *GO Math* representational connections to inverse relations (see Figure 12), the

purpose of using representations in the fourth grade *Investigations* lessons appeared to be less computational and more structural in nature. For example, in a lesson on multiplicative comparisons where students were presented with the following question—*Franco's daughter is 2 feet tall. Franco is 3 times as tall as his daughter. How tall is he?*—a side-by-side picture of the two individuals was drawn in order to compare heights [Figure 16(a)]. The structure of this representation was similar to how *GO Math* extended the use of the bar model for fourth grade multiplicative comparison problems [Figure 16(b)]; however, there was an inconsistent use of the abstract representations in this *Investigations* problem. Although the textbook instructed the teacher to write $2 \times 3 = \Delta$ on the board, the image showed $3 \times 2 = \Delta$. Clearly, the structure of the problem is 3 groups of 2, but this was not made explicit to students through the use of abstract equations. This inconsistency was similar to the aforementioned discrepancies found by the highlighting of boxes in the *Go Math* array representations [Figure 11(b)]. Further, even though teachers were told that students could use either a multiplication or a division equation for these comparison problems, neither *Investigations* nor *GO Math* provided abstract division equations when presenting the multiplicative comparisons examples illustrated in Figure 16. This illustrates that missed opportunities to use representations to form connections to the targeted content of multiplicative inverses were found in both fourth grade curriculums. To various degrees, the concrete and semi-concrete representations used by both curriculums were sometimes found to form inconsistent and incomplete connections to the abstract, which most likely would hinder the development of a student's situation model.



(a) *Investigations*



(b) *GO Math*

Figure 16. Fourth grade multiplicative comparison representations.

In general, *Investigations* did use representation to establish connections to the structure of multiplicative inverses; however, those connections were not as well developed and did not occur as early as those in the *GO Math* curriculum. Nonetheless, the use of concrete and abstract representations in both curriculums, provided students with initial connections they could use to develop their own situation models that could be applied in future inverse situations.

Sequence of Representations

Situating initial learning opportunities in real-world concrete settings has been shown to help activate students' informal knowledge of inverse relations. The goal of mathematics instruction is to help transform informal concrete knowledge into formal abstract understanding that can be applied across various mathematical contexts. The use of a situation model can therefore facilitate the transformation that involves converting connections into inferences. Using a sequence of representations that starts with concrete connections and progresses towards abstract thought is thus in alignment with the situation model perspective of comprehension. Connection-making scores for the sequence of representations within the textbook lessons coded for this study indicated a

curriculum difference in the suggested placement of representations throughout instruction.

The sequence of representation connection-making score for *GO Math* (M_{GM} = 1.71, SD_{GM} = 0.45) indicates that connections were established between concrete and abstract representations, and these connections tended to be presented as a linear progression from concrete to abstract (concreteness-fading, Goldstone & Son, 2005). The third grade dog treat problem represents a typical example of how *GO Math* facilitated this progression (see Figure 17).

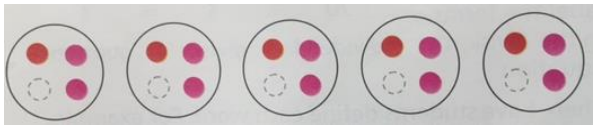
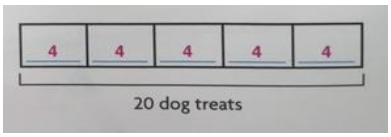
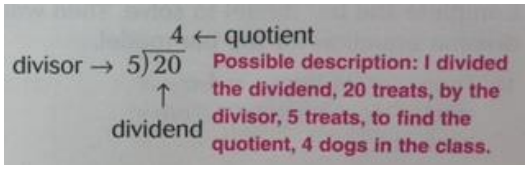
(a) Concrete representation (story situation & pictures)	*A dog trainer has 20 dog treats for 5 dogs in his class. If each dog gets the same number of treats, how many treats will each dog get?* 
(b) Semi-Concrete representations (dots & bar diagrams)	 
(c) Abstract representation (number sentence)	

Figure 17. Sequence of representations used within worked examples in *GO Math.*

By accompanying the problem in Figure 17(a) with the vivid image of the five dogs (concrete representation), the worked-example was situated in a real-world context which may better invoke students' informal knowledge. Based on the concrete representation (image of the dogs), the teacher was told to first elicit a classroom discussion that focused on the task at hand (e.g., the number of treats each dog would get). Once this was established, a semi-concrete representation of five circles [Figure 17(b)] was drawn to represent the five dogs, and students were instructed to place one dot at a time into each circle until all 20 dots were used. The solution was then elicited from this semi-concrete representation, and the teacher's textbook provided a reference to multiplicative thinking (i.e., 5 groups of 4 = 20). This example however did not cease with computation. Rather, the bar model in Figure 17(b), a more abstract semi-concrete representation, was introduced for the purpose of focusing student thinking on the part-whole relationship of multiplicative inverses, an essential component of comprehension. Whereas the discrete dots diagram is more closely related to the vivid image of the dogs, the bar-diagram is more abstract, and its depiction of the part-whole relationship makes it a more efficient semi-concrete representation. In fact, because of the more abstract nature of the bar model, the teacher's textbook suggested that students should draw a line from each dog to one of the boxes on the bar model. This was an explicit attempt to connect a concrete with a semi-concrete representation, which may ease students' transition to abstract understanding. Students were also directed to compare how the bar model and the dots were alike and different, further strengthening the connections between representations. Finally, Figure 17(c) connected the abstract division number sentence to the other representations, which completed the overall progression from concrete to abstract. It

should also be noted that the typical set of practice problems presented to students in the *GO Math* lessons similarly progressed from concrete representations to tasks that involved only abstract expressions.

The *Investigations* sequence of representation connection-making score ($M_I = 1.13$, $SD_I = 0.64$) indicates that connections between concrete and abstract representations were established but did not fade from concrete to abstract. Although most worked examples were situated in rich concrete contexts, on several occasions these contexts were introduced after students were presented with abstract representations and thus did not necessarily serve to initiate students' prior knowledge. For example, in one fourth grade *Investigations* lesson on relating multiplication and division problems, the introductory activity involved presenting students with the expressions 15 × 6 =___ and 90 ÷ 6 = ___. The teacher was told to allow several minutes for students to solve the problems before soliciting various solution strategies. Commentary from the teacher textbook also suggested that "some students may notice that the two problems are related and use the work they did in the first problem to help answer the second. Others may not notice this relationship or may notice it only after both problems have been solved" (Grade 4, p. 85). This statement clearly connects the abstract representations to the targeted content of multiplicative inverses; however, the next step in the worked example was to allow students to use cubes or grid paper to construct a concrete representation for the expression 15 × 6. The sequence of representations in this example revealed "abstractness-fading," a reverse of the concreteness-fading approach which treats concrete representations as tools for finding answers to abstract number sentences. The progression from abstract to concrete was found to also exist in multiple third grade

Investigations lessons. Moreover, the *Investigations* chart provided in Figure 8 included both concrete contexts and abstract equations; however, it appeared to be used as an organization tool as opposed to forming connections between concrete and abstract representations. The sequence of representations in the *Investigations* curriculum therefore did not always support ideal connection-making opportunities needed for the development of situation models for multiplicative inverses.

Deep Questions

According to Figure 5 (see also Appendix H), deep questions appears to be the category in which there exists the greatest opportunities for improving connection-making within both coded textbook curriculums. As might be expected based on discussions involving underlying content of worked examples, the teacher's edition of both *Investigations* and *GO Math* provided a substantial amount of deep questions aimed at eliciting students to form connections within the current targeted content of multiplicative inverses (see Figure 18; the horizontal line at 2 represents the highest possible connection-making score). Few deep questions that targeted forming connections to prior or future knowledge were found in either curriculum. Overall, questions that were found in the *GO Math* curriculum ($M_{GM} = 1.47$, $SD_{GM} = 0.46$) were more likely to facilitate connection-making than were the questions found in *Investigations* ($M_{I} = 0.62$, $SD_{I} = 0.37$), indicating a curriculum difference in questioning. Upon closer inspection of the questions presented in all 14 coded textbook lessons, the difference appears to be a due to how questions were presented and for what purpose they were asked.

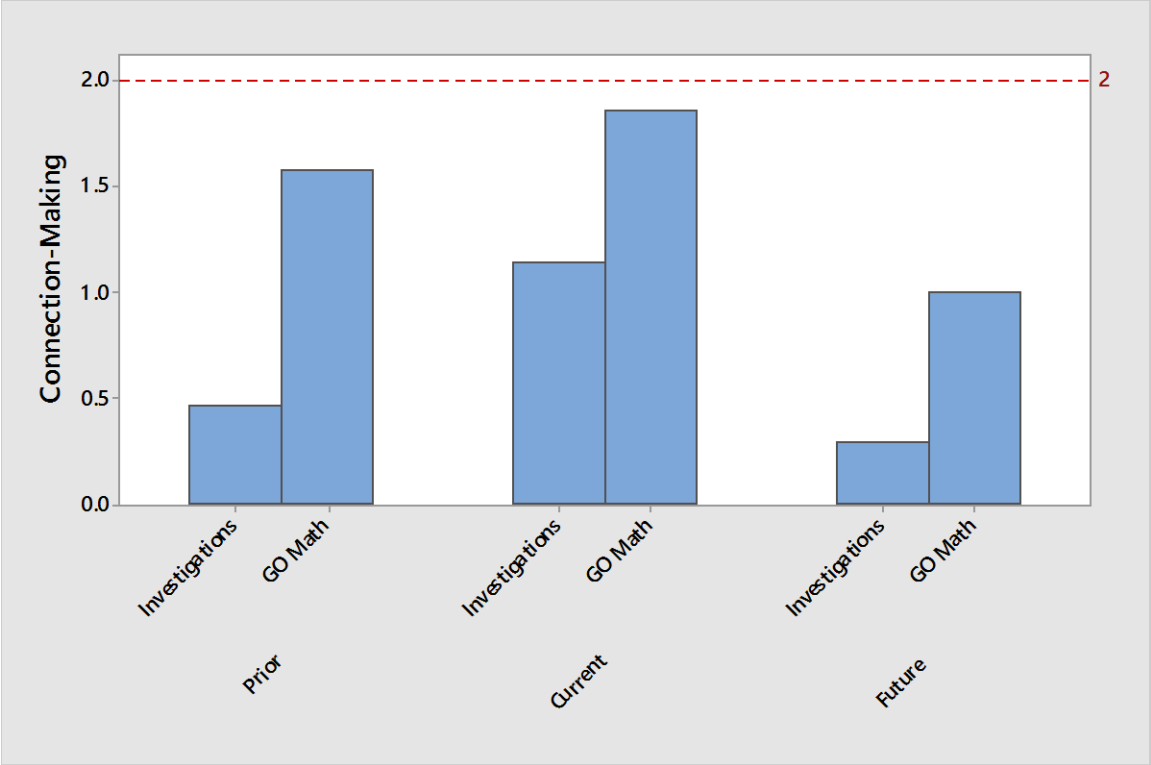


Figure 18. The use of deep questions to promote connection-making in *Investigations* versus *GO Math.*

Presentation of Questions

The physical manner in which questions were presented in the teacher's textbook of *Investigations* and *GO Math* differed. In the *Investigations* curriculum, questions were imbedded into the main body of the text and were integrated within the worked examples. In the *GO Math* curriculum, questions were provided within side-bars that provided commentary to teachers about how to present the worked examples found in the main body of the text. In addition, although the layout of the textbooks was different, both curriculums provided example questions that teachers could use when differentiating instruction (i.e., English Language Learners and intervention). Another curriculum difference observed was based on whom the questions targeted. Whereas all of the

questions in *GO Math* involved the teacher asking questions to students, *Investigations* actually asked questions to the teacher. Common questions asked to the teacher included, "can students answer the questions in each story problem" (Grade 3, p. 122), and "are students able to write multiplication equations and division equations to represent the problem accurately" (Grade 4, p. 80)? These questions were found under the "Ongoing Assessment: Observing Students at Work" section of the teacher's *Investigations* textbook, and in general, they elicited simple yes/no responses without providing teachers with support on how to evaluate student understanding. *GO Math* on the other hand, provided teachers with example questions that could be asked of students in order to assess comprehension, which seems to be an important component in helping students use and enhance their situation models. These questions often took the form of "how do you know your answer is correct" (Grade 3, p. 221), and "how do you know when to divide and when to multiply to solve a word problem" (Grade 3, p. 235)? These examples also suggested that there exists a fundamental difference in the purpose behind why each textbook included questions.

Existence and Purpose of Deep Questions

An analysis across curriculums revealed that the largest number of deep questions were posed within worked examples for the purpose of forming connections to the current targeted content (see Figure 18). Between curriculums however, the questions posed by *GO Math* appeared to provide deeper connection-making opportunities. This was in large part because most of the questions provided in the *Investigations* curriculum focused on procedures and computations; whereas, *GO Math* predominantly asked questions that were comparative in nature and elicited conceptual understanding. For

example, *Investigations* often suggested that teachers ask questions such as “How many groups of 4 can you make with 28 cubes” (Grade 3, p. 118) or “What did she have to do next to find out how many apples will be in each row” (Grade 4, p. 62)? This does not stimulate connection-making for inverse relations. When *Investigations* did ask deep questions, the anticipated connections from students often remained at a surface level. For instance, when comparing a multiplication problem to a division problem, one third grade lesson suggested that the teacher ask students, “What is the same about these problems? What is different” (Grade 3, p. 123)? Follow-up teacher commentary suggested that students might recognize that both problems involved the same numbers, a surface level similarity according to Ding and Carlson (2013). This same surface level connection was also made in a fourth grade *Investigations* lesson that asked students “What do you notice about the numbers in these two problems” (Grade 4, p. 85)?

In contrast, the purpose of questions provided in the *GO Math* curriculum appeared to be for developing deep connections to the targeted content of multiplicative inverses. This purpose was achieved by posing questions about the structural relationship of inverses and by forming deep connections within and between representations. The deep structural relationship was emphasized by two different third grade lessons that asked students to explain “which number in the multiplication fact is the quotient in the division fact” (Grade 3, p. 280) and to discuss “how the products relate to the dividends in a set of related facts” (Grade 3, p. 241). One question even asked students to re-write the multiplication equation factor × factor = product into a division equation, reinforcing their understanding and use of their situation model for multiplicative inverses. Interestingly, these questions were listed under a section in the teacher’s textbook entitled

"Go Deeper." *GO Math* also asked deep questions for the purpose of forming connections within and between representations. For example, a connection to multiplicative inverses within representations was made when students were asked, "How can an array show both division and multiplication" (Grade 3, p. 239)? Further, after an array model and a counter representation for the same division equation were created, the teacher's textbook suggested asking the question "How is making equal groups like making an array to solve the problem" (Grade 3, p. 2.80)? This deep question formed a connection between representations for the purpose of encouraging students to reason more abstractly. Similar attempts to use questions to make deep connections between representations throughout the *Investigations* lessons likely only resulted in surface-level understanding. In one third grade *Investigations* lesson for instance, students were asked to use cubes to model the abstract expression 3×4, but were never prompted with questions that connected these representations to multiplicative inverses.

The biggest difference in deep questions that formed connections to prior knowledge revolved around comprehension of knowing when to use multiplication or division for different situations. Whereas *Investigations* tended to ask students to recall strategies that could be used for solving these problems, *GO Math* asked students why and how they knew which operation was appropriate. Further, the only deep questions provided in the textbooks for making connections to future knowledge involved having students think about the case when only two equations could be written for a set of related facts (i.e., $4 \times 4 = 16$ and $16 \div 4 = 4$). These questions created connections for the future study of square numbers. Overall, very few deep questions in either curriculum targeted prior knowledge, and even fewer questions attempted to form connections to

future content (see Figure 18). From the situation model perspective of comprehension, deep questions therefore do not appear to be a tool that textbooks suggest using for the purpose of activating students already existing schema (prior deep questions) or for helping students recognize how to use their multiplicative inverse situation models to make inferences in unknown situations (future deep questions).

Summary of Curriculum Findings

In summary, although both curriculums provided connection-making opportunities for learning multiplicative inverses, the reviewed *GO Math* textbook lessons created more explicit connections than did *Investigations* and thus was more in alignment with the situation model perspective of mathematical comprehension. Concerning instructional tasks, the review exercises found in *Investigations* required low cognitive demand (i.e., stating facts, following procedures, solving routine problems; Van de Walle & Bay-Williams, 2010), and therefore did not provide optimal opportunities for activation of relevant prior knowledge. In contrast, *GO Math* tended to use cognitively high demanding review tasks that often explicitly stressed conceptual connection-making. *GO Math* also tended to use worked examples for creating connections to the structural relationship of multiplicative inverses, whereas, *Investigations* often only used procedural and computationally focused examples. As a result of varying surface characteristics, the practice problems found in *GO Math* reiterated the underlying structure of inverse relations. This was different from *Investigations*, which did not fade worked examples into practice, but rather expected students to investigate structure on their own during independent practice. With regards to representations, *GO Math* used concrete representations to form connections to the structural relationship of multiplicative

inverses sooner and to a better degree than did *Investigations*. In addition, connections established between concrete and abstract representations tended to progress linearly (concreteness-fading, Goldstone & Son, 2005) in the *GO Math* curriculum, but not in the *Investigations* curriculum. Finally, deep questions posed by the *GO Math* curriculum were more likely to facilitate connection-making within the targeted content of multiplicative inverses, because they emphasized structural relationships within and between different representations. However, very few questions in either curriculum were asked for forming connections to prior or future content.

Connection-Making Opportunities Afforded by Teachers

Averaging all of the classroom instructional lesson scores (n = 16) across teachers revealed a total average teacher connection-making score of M = 13.13 (out of 18 possible points; SD = 3.77). This score was slightly higher than the overall textbook-connection making score of M = 12.13 (SD = 3.45), which suggests that on average these teachers provided more connection-making opportunities than were found in the elementary school curriculum materials. While one might expect this to be true based on the reviewed literature of expert practitioners (e.g., Bransford et al., 1999; Cai et al., 2014), with regards to facilitating connection-making opportunities during instruction, there were notable differences between the teachers in this study. Whereas Amy appeared to provide connection-making opportunities across all aspects of her instruction (highest connection-making score of M = 17.25, SD = 0.43), Lily missed many connection-making opportunities (lowest connection-making score of M = 7.50, SD = 1.50). Esther's connection-making score of M = 14.25 (SD = 1.09) and Jackson's score of M = 13.50 (SD = 1.80) fell in between these two endpoints. Figure 19 and Appendix I provide detail

about the differences in teacher connection-making scores relative to instructional tasks, representations, and the use of deep questions. In the figure, the horizontal line at 18 represents the highest possible connection-making score.

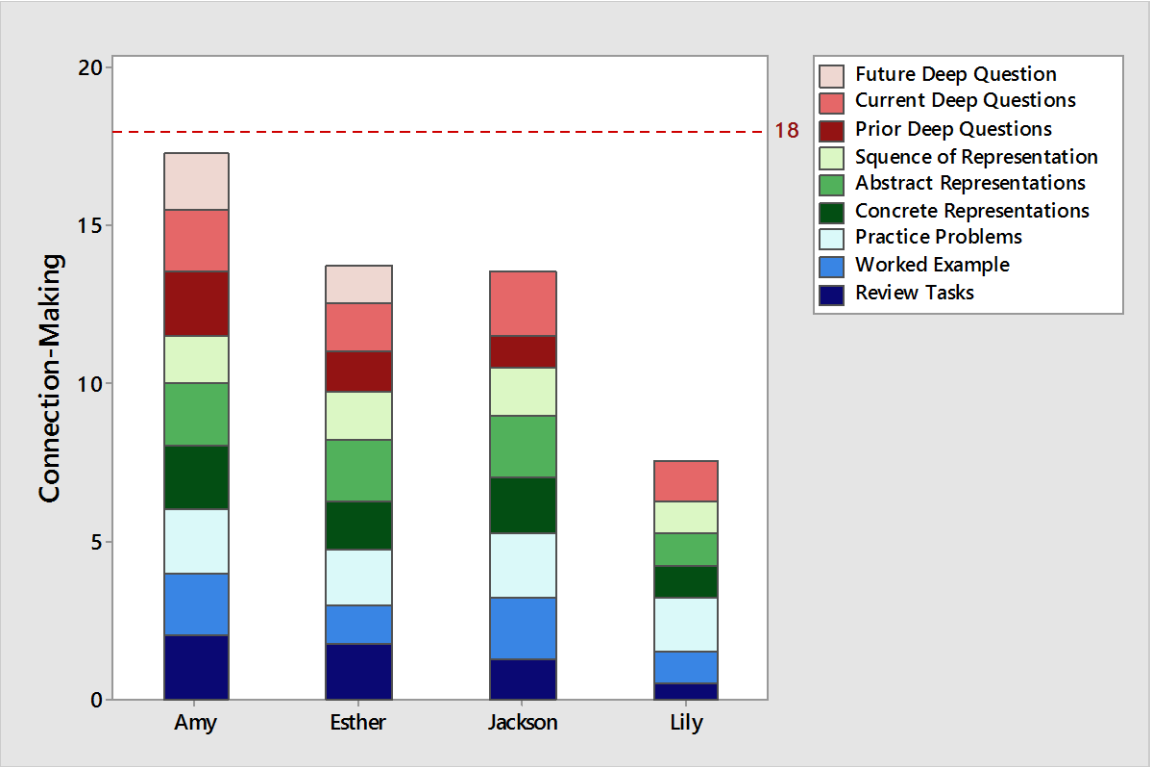


Figure 19: Average connection-making scores across teacher lessons.

According to Figure 19, even though not all teachers in this study explicitly connected their instructional worked examples to multiplicative inverses, the practice problems used by all teachers seemed to reinforce targeted content. In addition, Amy, Esther and Jackson appear to have effectively used multiple concrete and abstract representations during instruction; however, Lily's use of representations to facilitate connection-making was not as well developed. All four teachers also do not fully embrace the idea of concreteness-fading, as indicated by the sequence of representations subcategory scores. Figure 19 also indicates that while all of the teachers posed deep

questions aimed at forming connections within the current targeted content, this instructional technique was not often used to form connections to future content. Finally, significant differences involving the use of instructional review tasks and the posing of deep questions that target prior knowledge were found among the teachers' instruction.

When pairing individual teacher connection-making scores against their corresponding textbook connection-making scores, differences between curricula also became apparent (see Figure 20; the horizontal line at 18 represents the highest possible connection-making score). The two teachers who used the *Investigations* curriculum appeared to enhance the connection-making opportunities that were found in their textbook. On the other hand, the two teachers who used the *GO Math* curriculum appeared to fall short in implementing their textbook's connection-making opportunities. The following sections provide detailed findings involving both the classroom instruction and the differences between textbook and classroom instruction in relation to the connection-making opportunities afforded by the use of instructional tasks, representations, and deep questions.

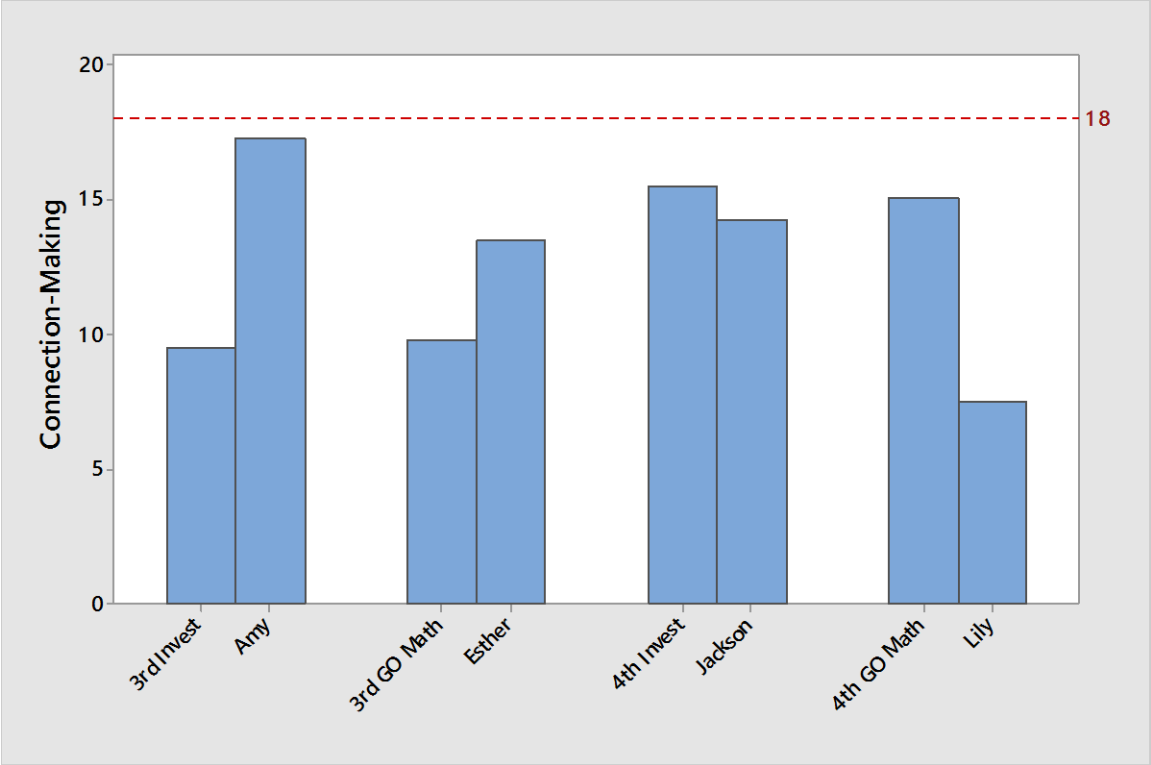


Figure 20: Overall textbook and teacher connection-making scores within each classroom.

Instructional Tasks

All four teachers in this study included some form of review, worked examples and independent student practice in each of their four enacted lessons. A close inspection of how these instructional tasks were used revealed two main differences. First, there exists a difference in the degree to which teachers used review tasks to form explicit connections to targeted content. Second, although teachers tended to use the instructional tasks provided in their respective textbooks, a difference exists in the degree to which individual teachers enhanced those tasks during instruction. Specifically, some teachers enhanced textbook worked examples by forming more explicit connections to the structural relationship of multiplicative inverses. No substantial teacher difference was

found in the use of practice problems, as all teachers used practice problems that were aligned with their worked examples.

Purpose of Review

A failure to activate relevant prior knowledge may cause difficulties with comprehension that relies heavily on an interconnected web of fundamental concepts that are needed for inference-making. From the situation model perspective of comprehension, review tasks therefore appear to be an important component in classroom instruction of mathematics. All teachers in this study appeared to recognize the need for review; however, these review tasks were used for three different purposes during instruction. These purposes corresponded to the three levels of connection-making found in the connection-making framework: (0) routine review with no connections to multiplicative inverses; (1) implicit connections to multiplicative inverses; and (2) explicit connections to multiplicative inverses.

In all four of her lessons, Amy used review tasks to make explicit connections to the targeted content of multiplicative inverses. This was evident by her sub-category review task score of $M = 2.00$ ($SD = 0$). Her review tasks were always connected to the content from the prior day's lesson and focused on formerly learned problem solving strategies involving these conceptually relevant concepts. For example, in one lesson, Amy reviewed the structure of multiplication (prior lesson) as multiple sets of equal groups. This involved using a student's previously created word problem involving placing shoes into shoeboxes to review the concept of how many groups (boxes), how many were in each group (shoes), and how many there were altogether. Although she never used abstract vocabulary (i.e., factor and product), she made an explicit connection

between addition and multiplication (skip-counting) when talking about the solution strategies for this problem. Later in the lesson, Amy used this connection to set-up the structure (i.e. the situation model) of the inverse relationship between multiplication and division, the targeted content. In a different lesson, Amy linked geometry knowledge (area of a rectangle) to multiplicative inverses. This facilitated a deep connection to content that had been learned in the more distant past. Through the process of bisecting the rectangle into rows and columns, she helped students understand why the area of a rectangle was calculated by the expression, length × width. Interestingly, past studies (e.g., Simon & Blume, 1994) have found that most elementary teachers only understand the procedural formula for the area of a rectangle, and therefore, they do not possess the conceptual understanding needed to form connections between the multiplicative relationships inherent in area problems. Along with other facets of her instruction, this example illustrated Amy's use of interconnected mathematical knowledge, which the reviewed literature (e.g., Cai & Ding, 2015; Ding, 2016) suggests is generally not part of elementary teacher instruction. In summary, Amy's use of review tasks not only involved activating prior knowledge about the structure of multiplication; rather, it included forming deep connections to this knowledge by applying it to other familiar content (area of rectangles) in an unfamiliar way (inverse relationship) whereby creating new connections. In terms of a situation model, Amy afforded her class the opportunity to both activate and manipulate their already existing schema surrounding inverse relations.

In comparison to Amy, Esther formed slightly less-developed connections to multiplicative inverses during review tasks. This was evident by her review task connection-making score (M = 1.75; SD = 0.43). For instance, Esther helped her students

recall the explicit connection between addition and subtraction, but only used this connection for having students recall the word inverses and to review prior computation strategies that involved addition and subtraction fact families (surface-level connections). Although she did use this opportunity to make a connection between addition and multiplication (repeated-addition), no explicit connection to the structure of multiplicative inverses occurred during the time spent reviewing additive inverses. Further, instead of using concrete representations to review informal concepts (i.e., groups and total) like Amy did with the shoeboxes, Esther had students "shed some light" on relevant vocabulary words such as multiplication, division and relationship. This review activity involved students using a flashlight (to highlight) or a yardstick (to point out) to identify and define key words that they recognized in the daily learning objectives. In general, Esther used review tasks to remind students about non-contextual abstract ideas (i.e., vocabulary and fact families), and therefore she did not always create explicit connections between prior knowledge and the targeted content.

Among all four teachers, Jackson spent the most amount of time on reviewing students' prior knowledge. The review tasks that he used during instruction however were primarily procedural in nature and were often only implicitly connected to multiplicative inverses. Jackson's review task connection-making score (M = 1.25; SD = 0.43) reflects this finding. Taken from the *Investigations* curriculum, Jackson used the "Broken Calculator" activity as a review task in three of his lessons. This activity involved having students discover various methods for arriving at certain quantities without using "broken buttons" on a calculator. For instance, during one review students had to produce the number 40 without using a 0, 1, 2 or the operations of addition (+) and

multiplication (×). After several correct solutions were provided by students, Jackson noted to his class that no one had used division. This created an opportunity for Jackson to discuss the use of division as the inverse for the broken multiplication button; however, he did not make this explicit connection to the targeted content. Thus, the "Broken Calculator" review task provided at most an implicit connection to multiplicative inverses.

On the other hand, Jackson's use of skip-counting ("Ten Minute Math" in the *Investigations* lesson) during one lesson to review multiples of various numbers facilitated the use of situation models and created deep connections to multiplicative inverses. Excerpt 1 (involving skip-counting by 11's) reveals Jackson's intention to use this review task for promoting inference-making.

Excerpt 1:

Matthew: Counting by 11's is easy before you get to the 100's because when you do 11 and 22 that's adding the numbers.

Jackson: So adding 1 to both the 10's place and the ones place. So that's pretty up to here. Matthew. Why was it easier counting by 11's than counting by 9's?

Matthew: …because people know the pattern

Jackson: So people know the patterns…how it works. Is there still a pattern after you get to 100?

Students: Yes (collectively)

Jackson: Lisa

Lisa: So after you get to 110, so you are in the 100's but in the 10's and 1's, the 10's just goes up 1 and the 1's is the number before the 10's number.

Jackson: But that is also going up by 1 isn't it? So, if we had 110 now we have 121, so the 10's place went up by 1 and the 1's place went up by 1. But you said the 1's place is behind. So, let's see. We have 132 so it goes from 2 to 3 and 1 to 2. 3 to 4. 2 to 3. 4 to 5. 3 to 4. So we kind of have this pattern that could come up pretty quickly and easily. So if I were to put down the 1's place for the next 5 numbers, what would it be? If we ended with 242, what would the next 1's place be?

Asking students to generate and analyze patterns is an important aspect of promoting fourth graders' algebraic thinking (CCSSI, 2010); however, the pattern that Jackson emphasized in this excerpt was recursive (i.e., the next term depends on the previous term) which is not very powerful in developing connections among fundamental mathematical concepts. Although not related to the targeted content of multiplicative inverses, perhaps Jackson could have formed a connection to the pattern that involved the distributive property. For example, he could have used $16(10 + 1)$ to show that $16 \times 10 = 160$, and then when one more 16 is added, the 10's place becomes $6 + 1 = 7$ and the 1's place becomes $0 + 6 = 6$. Nonetheless, by helping his students first realize a pattern that exists with the multiples of 11 and then asking them to make conclusions based off that pattern, Jackson used his expert knowledge (Bransford et al., 1999) to help students develop the reasoning skills inherent in using situation models to transform

connections into inferences. During this review task, Jackson also stressed the structure of multiplication when he had students explain to him that in the expression 3 × 11, the three represented the third student (analogous to the number of groups) who provided him with a multiple of 11 (analogous to the number in each group). He took this review task one step further and created an explicit connection to multiplicative inverses when he asked students to provide him with the division equation that could be used to determine which student provided the multiple of 165 (i.e., 165 ÷ 11= 15th student). This represents an instance in which Jackson enhanced a connection-making opportunity afforded by his textbook, because the *Investigations* curriculum only suggested reviewing multiplication during this task.

In contrast to the other three teachers, Lily's review tasks (M = 0.50; SD = 0.50) were routine in nature and did not create well-developed connections to the targeted content of multiplicative inverses. This review often included drilling students on abstract vocabulary (e.g., identifying in an expression which number represented the divisor, dividend and quotient) and memorized multiplication facts that could be used for checking long division computations. Lily did begin one lesson by explicitly asking students "What does the word inverse mean?" However, by accepting the first student's response of "opposite," Lily illustrated that perhaps she herself had a conceptual misunderstanding about inverse relations. In fact, throughout the course of Lily's four lessons, she emphasized to students that the words opposite, reciprocal, reversal, and backwards were all synonyms for the word inverses. In mathematical language however, these words do not always translate into the same meaning and do not seem to be consistent with the complement principle of multiplicative inverses adopted for this

study. For instance, the word opposite is used to describe two numbers that have a sum of zero (e.g., 8 + -8 = 0), and the word reciprocal is used to describe two numbers that have a product of 1 (e.g., 8 and 0.125 are reciprocals). These vocabulary words are therefore more in line with the additive ($a + b - b$) and multiplicative ($d \times e \div e$) inversion principles. Although Lily was most likely trying to activate informal knowledge of inverses by using language that students might have encountered in other instances, her students' minimal explanations suggest that her inconsistent use of vocabulary appeared to hinder the understanding and development of a situation model involving the complement principle of multiplicative inverses.

Interestingly, while Lily tended to use review tasks for the purpose of simply triggering students' vocabulary memory, she was the only teacher who explicitly used the word "connections" during review. To introduce the use of variables in one lesson, she asked students how the ABC's connected to mathematics. After student responses that included using letters in geometry (e.g., let x represent an unknown angle or finding the lines of symmetry in the letter X), she wrote the expression $3 \times R = 18$ on the board and told students, "I am making connections for you…instead of saying blank…you can put in a letter…you will do this more and more with Algebra as you get older." All four of Lily's interviews recorded immediately after her enacted lessons also illustrated her desire to help her students facilitate connection-making. In fact, after her first lesson, she broadly admitted that she needed to find ways to make better connections during instruction. It appears as if she was referring to forming connections to informal knowledge and to the interconnectedness of mathematics. For instance, when discussing her attempt to form connections between multiplicative inverses and literacy she stated,

"I was trying to make that connection with multiplication…compare and contrast in literacy…I feel like I try to simplify, and sometimes I don't…I try my best to put higher level thought processes into teaching" (Lily Lesson 3 Interview). Even though Lilly claimed to "know some of the higher level skills and…throw out the higher level terms" (Lily Lesson 2 Interview), she also appeared to have a clear desire to improve her ability to facilitate connection-making. This became evident when she described her own missed opportunities to include fractions and percents in order to develop "more of a connection to other math" (Lily Lesson 2 Interview). Lily's connection-making score was the lowest of the four teachers in this study; however, she appeared to understand the importance that connection-making has in "helping them [students] retain and apply" (Lily Lesson 3 Interview) targeted content and was eager to seek out ways in which she could better facilitate these connections. This understanding would seem to be an important characteristic of a teacher who embraces a situation model perspective of mathematical comprehension.

Enhancement of Worked Examples

Worked examples were used during instruction by each teacher in this study. Almost all of these examples were drawn from the teacher's respective textbook lessons. As seen in Figure 21 (the horizontal line at 2 represents the highest possible connection-making score), the teachers in this study differed in the delivery of these textbook examples. For instance, Amy and Jackson both scored $M = 2.00$ ($SD = 0$) on the teacher worked example connection-making score, which was higher than both of their respective textbook worked example connection-making scores of $M = 1.25$ ($SD = 0.43$) and $M = 1.33$ ($SD = 0.47$) respectively. On the other hand, Esther ($M = 1.25$; $SD = 0.43$) and Lily ($M = 1.00$; $SD = 0$) both scored lower than their individual textbook scores ($M = 1.75$, SD

= 0.43; *M* = 2.00, *SD* = 0) respectively. This difference corresponded to the two different curriculums. For the inquiry-based curriculum of *Investigations*, Amy and Jackson were able to enhance the non-explicit connections to multiplicative inverses found in the textbook. For the *GO Math* curriculum, which provided more explicit textbook connections within worked examples, Esther and Lily were unable to fully transfer those connections to their classroom instruction. This was similar to the pattern found for the instructional review tasks.

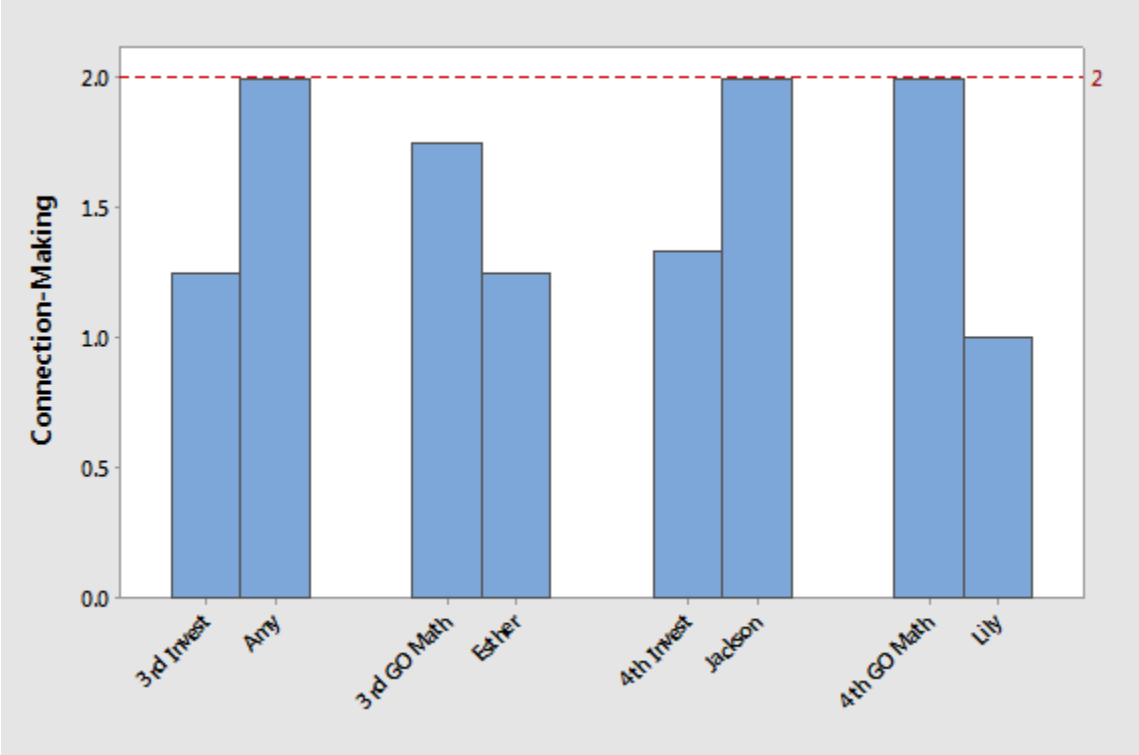


Figure 21: Differences in connection-making within worked-examples between textbook and teacher.

Across lessons, Amy showed explicit awareness for creating deep connections to inverse relations during classroom instruction of worked examples. She primarily facilitated connection-making by comparing side-by-side examples (see Figure 22) involving the same context in order to illustrate the structure of multiplicative inverses. In

contrast, her curriculum (*Investigations*) often presented worked examples in isolation (i.e., provided a multiplication example with no explicit connection to a division example). One case that illustrated her tendency to enhance the worked examples found in her textbook, involved the multiplication story problem– *A robot has 4 hands. Each had 6 fingers. How many fingers does the robot have altogether?* After finding the solution through a process that formed deep structural connections to multiplication, Amy facilitated a classroom discussion about the process of rewriting this problem to illustrate a division situation. Using the ideas of "how many in each" and "how many in total," the class determined the division problem to be –*There are 24 fingers from a robot. This robot has 4 hands. How many fingers in each hand?* Although *Investigations* did emphasize the need for the teacher to compare and contrast a multiplication and a division problem, the problems suggested by the textbook involved different contexts.

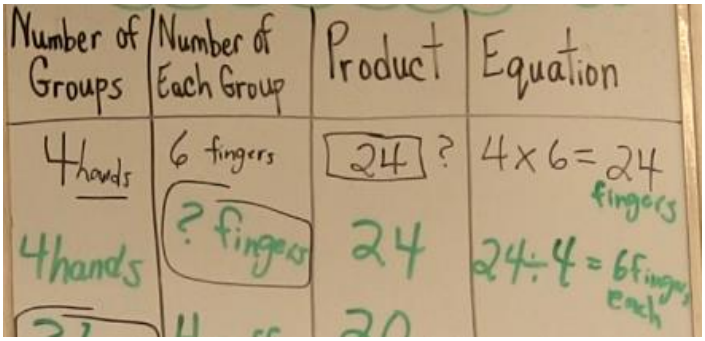


Figure 22: Amy's use of side-by-side multiplicative inverses involving robots.

Amy's ability to effectively unpack this one example provided her students two key connection-making opportunities that were not afforded by the textbook. First, using the same context for the multiplication and the division problem allowed for a meaningful connection to the structural relationship (i.e., the situation model) of

multiplicative inverses. For example, Amy made the connection that both problems involved a total (number of fingers), the number of groups (hands) and the number in each group (fingers). She also pointed out that the only difference was which component the problem asked the students to determine. Focusing on the structure and the relationships in this example, most likely helped Amy's students extract and make connections to the underlying mathematical concepts needed for strengthening their own situation models for multiplicative inverses. If Amy would have used the examples suggested by the curriculum, the context of the groups would have changed to muffins in one example and yogurt cups in the other, whereby eliminating the deep structural connection afforded by using the same context. Secondly, by making her students rewrite the multiplication problem into a division example, she engaged her students in a higher-level cognitive task that moved beyond the computational features of the textbook to include the process of transforming connections into inferences, the main function of using a situation model.

Similar to Amy, Jackson was able to enhance the worked examples in the *Investigations* curriculum in order to change missed textbook opportunities into explicit connections to multiplicative inverses. He also did so as a result of providing paralleled multiplication and division problems involving the same story context. In one of his enacted lessons, Jackson included the textbook's missing factor example which simply involved using an array card to determine the solution to $56 \div 4$. Like Amy, he created a story problem (chocolate candies) when using this example and required his students to think about multiplicative inverses. Excerpt 2 below indicates this request:

Excerpt 2:

Jackson: So what I want you to think about today is…I'll have you use these tiles, and you'll have a chance to organize them into your rows with whatever the question is asking you. But, I also want you thinking about how you use multiplication to solve the problem as well. Okay, so here is how I use these arrays, and Eric told us that he got 40 first, added 16 more to get 56 pieces of chocolate. Use your piles first and then see if you can come up with equations to help you solve a division problem. Because I want to tell you, sometimes solving a multiplication problem is a lot easier than solving a division problem.

Mallory: No…

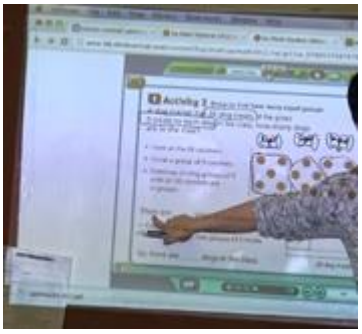
Student(s): Yes, yes it is!

This small excerpt shows an explicit connection to the structure of multiplicative inverses and illustrates Jackson's desire to create efficient methods of problem solving. When having his students use array models to represent worked examples similar to the above problem, Jackson made the connection that the rows and columns of the array correspond to the number of groups and the number in each group. In addition, during the lesson on multiplicative comparisons, the textbook presented two worked examples that had a repetitive multiplicative structure (how many times more). Instead of using both of these examples, Jackson used one of the textbook's practice problems that involved a division structure (how many times as large) for his second worked example. Although he did not use the same context to compare these two examples, he did make explicit connections to the structure of multiplicative inverses by requesting that his students search through a

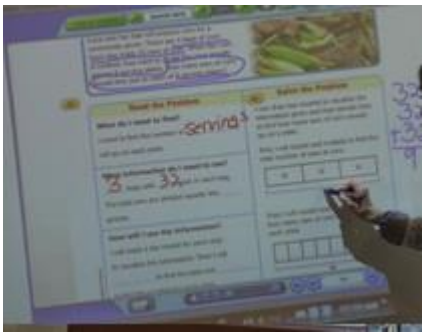
sample of problems to identify and explain how the structure of the problems were similar and different to the worked examples. This distinction most likely helped his students develop stronger situation models for multiplicative inverses.

According to Figure 21, both Esther and Lily were unable to maximize the explicit connections that were provided by *GO Math* in the curriculum's worked examples. During Esther's first lesson, students made a 3×4 array card and were asked to write a representative multiplication and division statement. Although most students were able to write both $3 \times 4 = 12$ and $12 \div 4 = 3$, Esther did not make an explicit connection to how the array connected to the structure of the two multiplicative inverse number sentences. Furthermore, when presenting students with the notion of reversing a story problem from multiplication to division, she showed an already reversed worked example from the textbook. This did not allow her students to have the same connection-making opportunities afforded by Amy and Jackson's instructional approach that involved the unpacking of worked examples by engaging students' thinking. Similarly, on one occasion, Lily requested that the boys in her class create a multiplication representation for a given example and that the girls create a division representation. Even though the students then shared-out their individual examples, the connection to multiplicative inverses was only implicit since each student only worked with one operation. This was a missed opportunity to help students create connections within their own situation models for multiplicative inverses. If Lily would have created and worked with side-by-side examples such as Amy did, most likely her students would have been able to better extract the underlying mathematical structure of inverse relations, thereby increasing the likelihood of transfer (Paas, Renkl & Sweller, 2003).

Relative to Amy and Jackson, both Esther and Lily did not spend as much time unpacking worked examples for drawing deep explicit connections to the targeted content. This may partially be due to a difference in classroom technology use that was noted among the four teachers in this study. Although all four teachers used a digital smart-board in almost every lesson, Esther and Lily mainly used it for accessing the online digital *GO Math* textbook. This was in contrast to Amy and Jackson who used the smart-board for their own teacher-created guided notes that seemed to provide them with more flexibility during instruction. During all four of Esther's lessons and one of Lily's lessons, the projected textbook was used as a systematic framework for presenting the curriculum's worked examples. Students in both classes were even exposed to the audio capabilities of the digital text when the teachers had the textbook read aloud instructions that requested students underline certain words or fill in the blanks (see Figure 23). The underlining of words during these lessons was not focused on creating connections to the structural relationship of inverses, but rather was used to have students concentrate on using key words for distinguishing the problem type. In one instance, Lily even explained to her students, "here, look at the wording, look at the wording. What operation might you use? The wording should help you figure it out." Instructing children to look for key words during mathematical problem solving has often been criticized by educational researchers (Drake & Barlow, 2007; Stigler & Hiebert, 1999) who believe that this process inhibits the ability for students to reason and make sense of why certain operations are used in certain situations. Unfortunately, the practice of searching for key words still appears to be part of some teachers' classroom instruction, and does not seem to be of any benefit in the development of a student's situation model.



(a) Esther



(b) Lily

Figure 23: Esther and Lily using the digital GO Math textbook to deliver classroom instruction.

Although the examples in the *GO Math* textbook did tend to provide many explicit connection-making opportunities, this was most often due to the fact that the side-notes (i.e. “About the Math” and “Go Deeper”) in the teacher’s textbook provided guidance and support for unpacking the worked examples. Using the student textbook as a template and not referring to the teacher’s textbook during instruction resulted in Esther and Lily missing several opportunities to fully transfer the curriculum’s deep connections into their classroom instruction. Insufficiently unpacking worked examples made each teacher’s instruction seem rushed. Specifically, during the occasions in which Esther (for one lesson) and Lily (for three lessons) did not use the digital textbook to present worked examples, their instruction did not contain as much depth and lacked variability. For example, worked examples in Lily’s one lesson on using multiplication to check division were repetitive and involved only procedural computations. Overall, Esther’s and Lily’s inability to fully transfer the explicit *GO Math* connections into their classroom instruction appears to be a result of not using the suggestions in the teacher’s textbook for unpacking the worked examples. As admitted by Esther during the interview following

her third lesson, her inability to fully unpack worked examples appeared to be at least partially due to the U.S. cultural belief that enough classroom time needs to be allocated for student practice (Stigler & Hiebert, 1999).

Representations

All four teachers in this study used multiple representations throughout their enacted lessons. This included the use of both purely concrete and abstract representations, as well as semi-concrete representations (Ding & Li, 2014) which were used to bridge the gap between concrete and abstract. The common concrete manipulatives that were used during instruction included children's fingers, base-ten blocks, number-cubes, paper tiles and jolly ranchers. The common semi-concrete representations used included pictures, tallies, arrays, bar models and fact triangles. In each lesson observed, abstract expressions or equations involving some connection to multiplicative inverses were also used. Figure 24(a)(b) show common examples of multiple representations that were used by Amy and Esther during instruction, and Figure 24(c)(d) show multiple representations that were produced by Jackson and Lily's students.

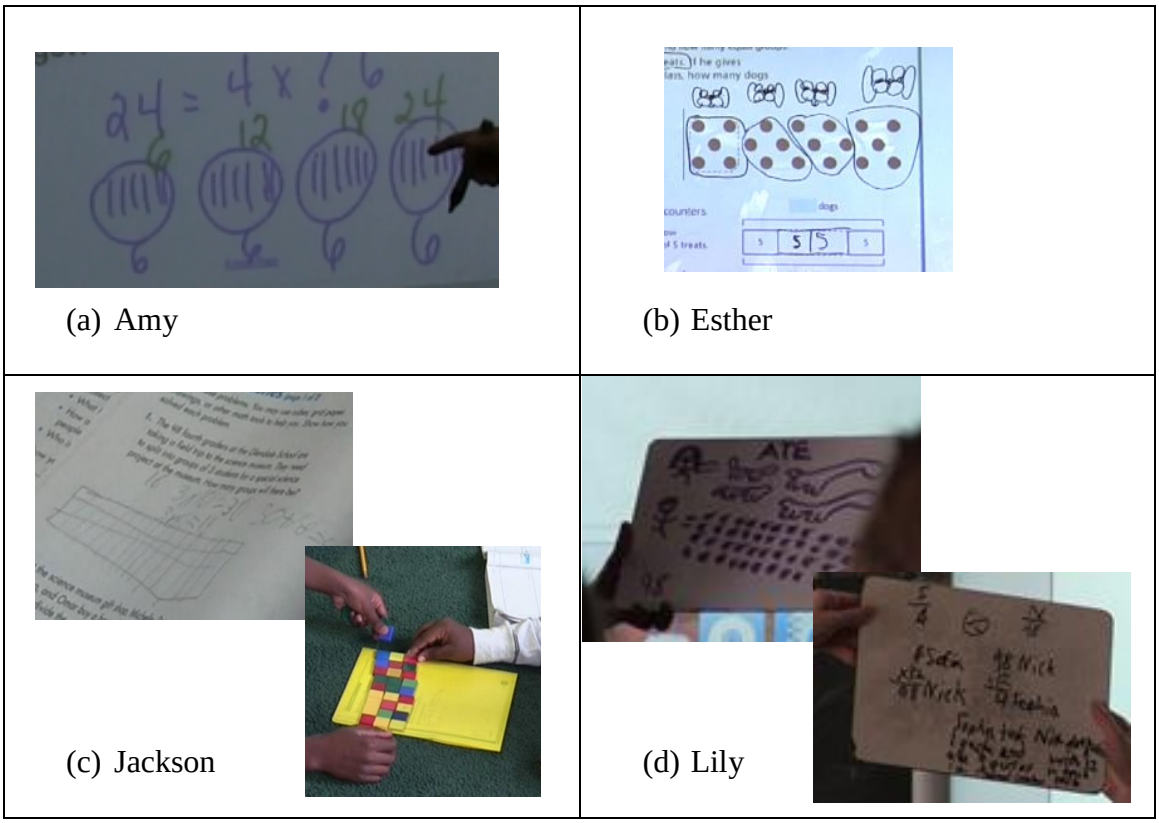


Figure 24: Multiple representations found in teacher's classrooms.

Given that all teachers used both concrete and abstract representations during instruction, differences in connection-making scores seem to be due to the differences in using representations for the following three purposes: (1) situating instructional tasks in personal concrete contexts, (2) teaching the structure of multiplicative inverses, and (3) promoting efficiency. From the situation model perspective of comprehension, these correspond to: (1) using children's informal or prior knowledge to activate schema, (2) facilitate connection-making for developing a situation model for multiplicative inverses, and (3) using the multiplicative inverse situation model to transform connections into inferences.

Using Representations to Promote Activation

Almost every worked example and practice problem used by the teachers in this study were situated in rich concrete contexts. Although these contexts were typically extracted from the curriculum because of the teacher's tendencies to use the textbook's instructional tasks, on several occasions the teachers attempted to make these contexts personal to their own students. According to the literature (Van Den Heuvel-Panhuizen, 2003), making connections to realistic informal contexts helps students form schemas, an important part in the development of a situation model. For instance, Amy referred to her own classroom when discussing a practice problem about the grouping of desks and used the names of her own children when creating a worked example involving the sharing of balloons. Amy also often made use of her students' prior work (i.e., a storybook problem) involving additive inverses as a reference for the new targeted content. This seemed to incite personal connections as the students recalled the specifics of their own written problems. Likewise, in one textbook example that involved comparing Franco and his daughter's height, Jackson changed the story context to include the name of a student (Mibsam) and a teacher (Chris) who each of the students knew [Figure 16(a)]. Having this personal context to individuals in their everyday life appeared to activate students' informal knowledge as they joked about the idea of Mibsam only being two feet tall, but nonetheless reasoned that it would take three Mibsam's to make up a teacher Chris (6 feet tall). In addition, although the dog treat worked example (Figure 17) used in Esther's instruction was provided by *GO Math*, on several occasions she enhanced the example by creating contexts that were more personal for students in her class. This included asking students if they themselves had dogs. Also, when stressing that division represented equal

parts and therefore each dog should be given an equal number of treats, Esther said to her class, “I want to be fair…we don’t want a dog biting you if it doesn’t get enough treats.” The notion of a dog biting most likely activated a vivid image for students and reiterated the structural importance of creating equal groups.

In addition to sometimes changing the context of the curriculum’s story problems, teachers in this study also tended to create story problems for the textbook’s concrete examples that did not have realistic contexts. For example, Jackson included the context of chocolate candies for the problem 56 ÷ 4 that was only represented as an array in the textbook. He also used tiles as physical representations of the candy pieces in order to have students form a more personal connection as they created an array model for the given problem. Unlike Jackson who did not actually use real candies, Lily introduced multiplicative comparisons based on having two students compare the weights of two envelopes containing different amounts of Jolly Ranchers. According to Lily, she used Jolly Ranchers because it was a context that “children can relate to” and in which they can make “self-connections” (Lily Lesson 3 Interview). Even for the students who did not feel the weights of the envelopes, using these candies most likely activated informal knowledge based on a previous experience the children had with holding a Jolly Rancher in their hands. This personal context appeared to help students’ reasoning skills associated with the targeted content of multiplicative comparisons. Moreover, the student practice worksheet that Lily made for this lesson was also personal, as it involved family situations about the children in her class (i.e., Jen’s older brother is 3 times her height). All four teachers in this study appeared to agree with the need to situate initial learning in

a concrete personal context, whereby activating children's informal knowledge for creating a schema that lays the foundation for a mathematical situation model.

Using Representations to Promote Structure

Three of the four teachers in this study seemed to effectively use both concrete and abstract representations to develop explicit connections to multiplicative inverses. Figure 25 indicates that Lily was the only teacher who on average created only implicit connections or missed clear opportunities to use representations for connection-making. In the figure, the horizontal line at 2 represents the highest possible connection-making score.

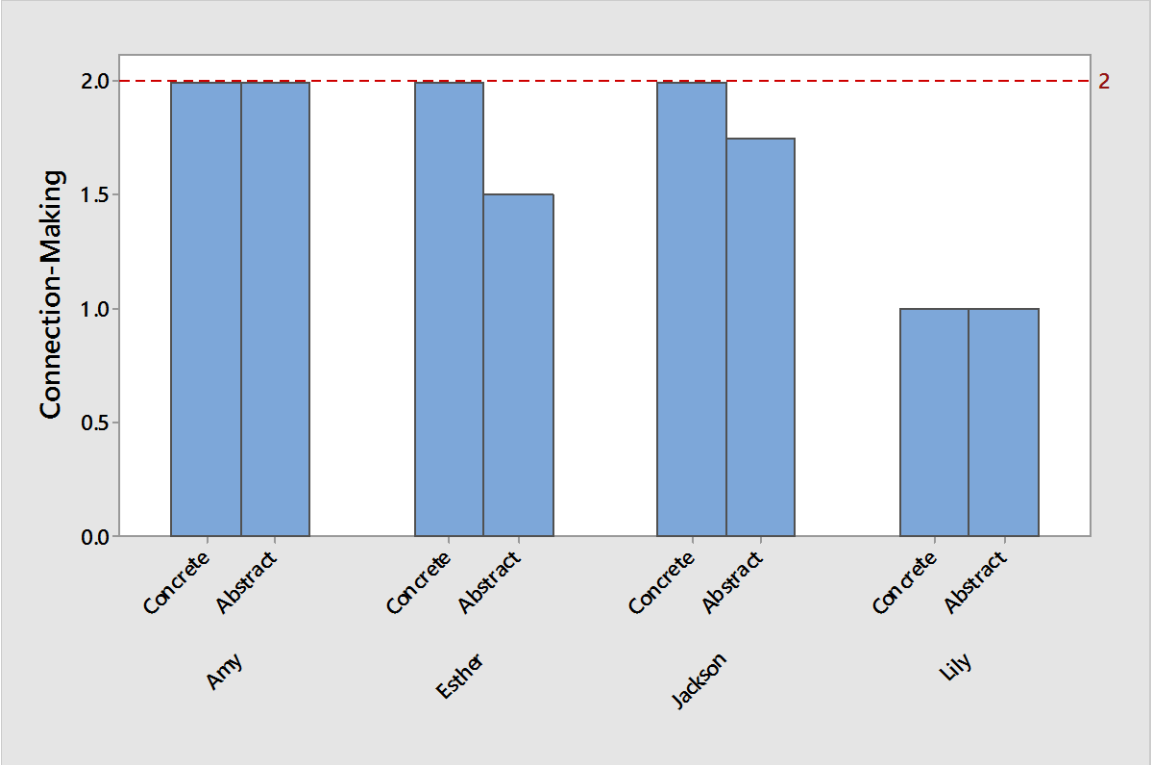
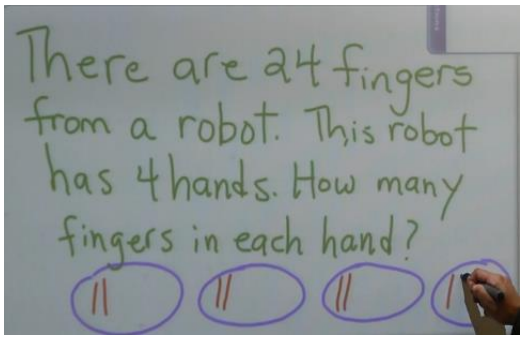


Figure 25: Connection-making within concrete and abstract representations.

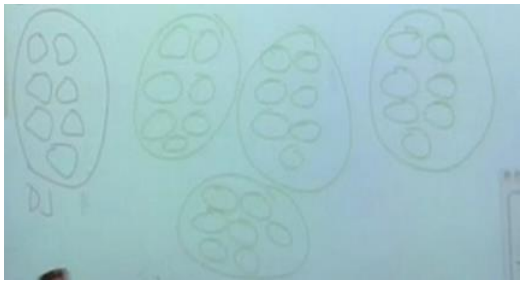
All four teachers in this study first used concrete representations in the surface-level format, which entailed using representations as a tool for helping children find the

solutions to a multiplication or a division problem. Figure 26 reveals that this was most often done by using counters and circles that were drawn to group the counters together. In Amy's and Esther's representations [Figure 26(a)(b)], the total number of groups was drawn out first, and then counters were placed one at a time into each group until the total number of counters were given out. The answer was then found by counting how many counters were in each group. Jackson's representation [Figure 26(c)] illustrates how he used counters and circles to model a multiplicative comparison problem in which the first group contained 7 and the goal was to determine the solution to 5 times as many. By drawing out and counting all of the counters, the answer of 35 was revealed. Similarly, Figure 26(d) illustrates a representation created by a student in Lily's class for the division problem 48 ÷ 4. In this picture, Lily's hand can be seen holding the student's slate as she circled the first group of four and explained to students how they would create equal groups in order to find the solution. The use of representations to find answers appeared to be common across all teachers.

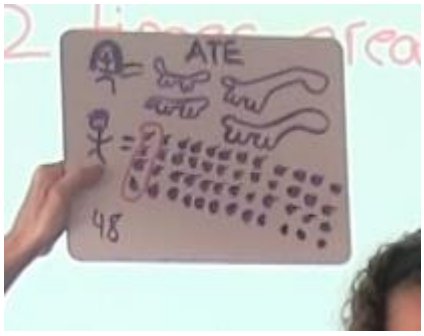


(a) Amy

(b) Esther



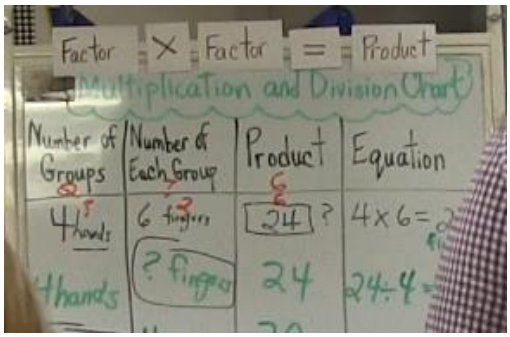
(c) Jackson



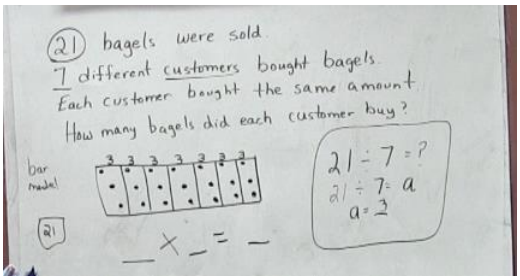
(d) Lily

Figure 26: The use of concrete representations to find answers.

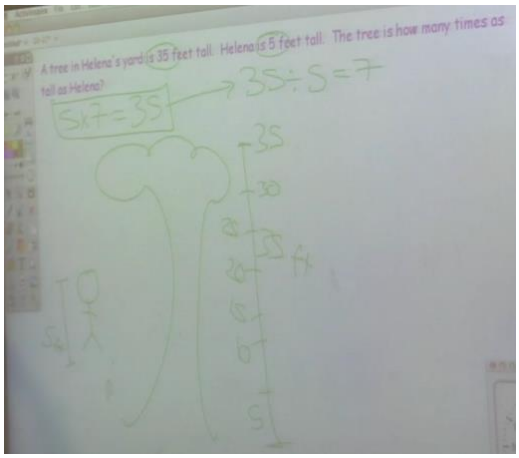
However, using representation to find an answer was not the main goal in Amy, Esther or Jackson's classrooms. This became obvious in Jackson's classroom when he said to his students, "Now that we have the answer out of the way, now we can move into the more important question of how do you know?" Specifically, Jackson requested that his students explain the "how" question by "using pictures, using diagrams, using models, to help [them] show the relationship." As illustrated by Figure 27, all teachers except for Lily proceeded to deepen their use of representations in order to facilitate connection-making involving the structural relationship inherent in multiplicative inverses.



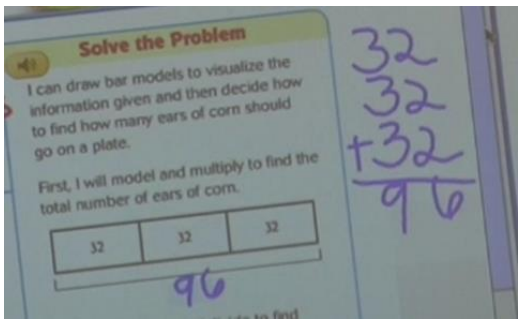
(a) Amy



(b) Esther



(c) Jackson



(d) Lily

Figure 27: The use of representations to promote structure.

In Amy's classroom, when using concrete representations to create groups of equal quantities, students were repetitively forced to think about how the parts of a multiplication and a division problem were related. As seen in Figure 27(a), Amy made an explicit connection to the structural relationship of multiplicative inverses through her use of the multiplication and division chart. This chart shows three important characteristics of Amy's use of representations to promote structure (i.e., the development of a situation model for multiplicative inverses). First, the examples filled in

on the first two lines of the chart involved the same context, namely the aforementioned robot problem. By using the chart to organize the components of both the multiplication problem and the student-formed reversed division problem, students could see how the structure of the two problems align. Second, Amy used this structural relationship to facilitate a classroom discussion that led to the creation of both abstract equations listed in the last column of the chart. Lastly, and of most importance, at the top of the chart Amy derived the general structure (*Factor* × *Factor* = *Product*) of a multiplication equation. This illustrates that her intention of the chart was to form connections between a particular concrete example and the abstract structural relationship of inverse relations. In other words, she used the chart to derive the constant underlying structure needed for students' situation models of multiplicative inverses. Amy's concrete (M = 2.00; SD = 0) and abstract (M = 2.00; SD = 0) connection-making scores (Figure 25) are a reflection of her ability to fully develop the previously discussed incomplete structural relationship that was missing in the third grade *Investigations* curriculum.

Figure 27(b)(c) illustrates how Esther and Jackson also deepened their use of representations for facilitating a connection to the structure of multiplicative inverses. Although Esther's representation is of a bar model and Jackson's is of a number line, both are forms of schematic diagrams that clearly illustrate structural relationships. Specifically, Esther used the bar model to illustrate the part-whole relationship for multiplication and division. This included breaking the initial whole bar into 7 equal parts (customers) in order to evenly distribute 21 bagels. She then had students determine the division equations 21 ÷ 7 based off using the part-whole structure of the bar model representation. At this point however, she missed the opportunity to connect this structure

to multiplicative inverses because she asked students to write the inverse multiplication problem based off of the abstract expression, not based off of the structure that she had just facilitated by creating the bar model. Jackson's use of a number line in Figure 27(c) resulted in the same missed connection-making opportunity. Although he had his students determine the multiplication problem based off the part-whole structure of the number line, as indicated by the arrow going from the multiplication to the division equation, he did not complete the representation's connection to multiplicative inverses. This lack of completeness involving their respective use of schematic diagrams is reflected in both Esther's (M = 1.50, SD = 0.43) and Jackson's (M =1.75; SD = 0.34) abstract connection-making scores (Figure 25). Further, Esther's abstract connection-making score was slightly lower than Jackson's because of her inconsistent use of abstract notation. In one example, she used 3 × 4 to represent a story problem involving 4 groups of 3, but later used 7 × 4 to represent 7 groups of 4. This may have been partially due to the inconsistent use of the abstract notation found in the *GO Math* curriculum [Figure 11(a)]. Nonetheless, this inconsistency appears to be particularly troublesome for the creation of situation models that rely on building connections and extracting underlying mathematical principles from constant structured problems.

In contrast to the other three teachers, Lily did not use representations to form connections to the underlying structure of multiplicative inverses. Her concrete (M = 1.00; SD = 0) and abstract (M = 1.00, SD = 0) representations instead involved only surface-level (implicit) connections dealing with finding and checking answers. This is illustrated in Figure 27(d) by the emphasis she placed on calculating 96 and by the absence of any abstract expression that would have connected this representation to the

targeted content. This missed connection-making opportunity was not a result of an incoherent text, as *GO Math* explicitly directed teachers to use the bar model to create the structural relationship of multiplicative inverses. Instead, Lily's inability to use representations to facilitate deep connections appears to be a result of her own confusion surrounding the use of bar models. This is confirmed in the following excerpt taken from the interview following her third lesson.

Excerpt 3:

Interviewer: Since this was a follow-up lesson… do you remember them struggling more?

Lily: I do remember, they were confused about it. The way it was presented in *GO Math* was a bit confusing. It took me two solid lessons for them to get… to understand even how to draw the picture by themselves. I don't like the way GO Math presented it. I will present it differently next time.

Interviewer: Can you talk more about the confusion?

Lily: It was tricky because they said to draw the pictures…then they were asking for the total…I would have to show you…but I do remember it was very confusing and it took me awhile.

Sequencing Representations to Promote Efficiency

After using representations to both activate schema (initiate a situation model) and to create structure (develop the situation model), the final component of the situation model perspective of comprehension is to apply the established structure (use the

situation model) to promote an efficient method for converting connections into inferences. Mathematical efficiency is most often associated with abstract understanding, and therefore students should be afforded learning opportunities that stress the importance of abstract reasoning. In addition to forming clear connections between concrete and abstract representations in order to promote structure, a sequence of representations that progressively becomes more abstract will likely result in students more efficient use of this structure. A common theme found among the teachers in this study was the use of multiple solution strategies (i.e., using pictures, skip-counting, repeated addition and equations). Although connections were often made between strategies, these connections only sometimes progressed from concrete to abstract (concreteness-fading) and thus did not always promote the development of efficient strategies. This inconsistent sequence of representations is reflected in the corresponding teacher connection-making scores ($M_{Amy} = M_{Esther} = M_{Jackson} = 1.50$, $SD_{Amy} = SD_{Esther} = SD_{Jackson} = 0.50$; $M_{Lily} = 1.00$, $SD_{Lily} = 0.71$). Further, although all of the teachers in this study mentioned a desire for students to develop more efficient solution strategies, there appeared to be slight differences in how they conveyed that message inside their classrooms.

During initial learning of multiplicative inverses, Amy emphasized the importance of multiple solution strategies and portrayed no preference for any one strategy. For example, acceptable solution methods for an instructional task involving a product of 24 and a factor of 4 [Figure 24(a)] included the strategies of repeated addition and skip counting, as well as the concrete tally and the abstract equation representations. How Amy presented these methods is indicated by the following excerpt:

Excerpt 4:

Amy: We can think of it as 24 pencils divided by 4 people equals how many pencils will they get each? How did you solve that? Damon?

Damon: I did um...I started to figure out what numbers again and again would equal 24. I came up with 6 + 6 + 6 + 6 = 24.

Amy: Ok, so you were using multiplication, you knew the product 24, equals 4, here is 4 times something gave you 24 [teacher writes 24 = 4 × __] …so you were using multiplication to figure out division and that is a strategy that we said works.

Amy: Anybody else? Did anyone draw a picture with the lines we have been learning? Celeste.

Celeste: I did tallies.

Amy: …I know there are already 4 groups and I know the total is 24 so I'm going to share out equally 24 [teacher draw 4 circles on board and proceeds to place tallies into circles]. I can probably share 2 at a time…. 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24…I notice I got to 24 so how many are in each group?

Students: 6

Amy: 6+6+6+6 [teacher writes 6's under each circle] so I get 24, this is repeated addition. Did anyone use skip counted? Melvin?

Melvin: I skip counted by 6 four times. 6, 12, 18, 24 [teacher writes these numbers above the circles].

Amy: So again, another strategy…skip counting to get to the answer.

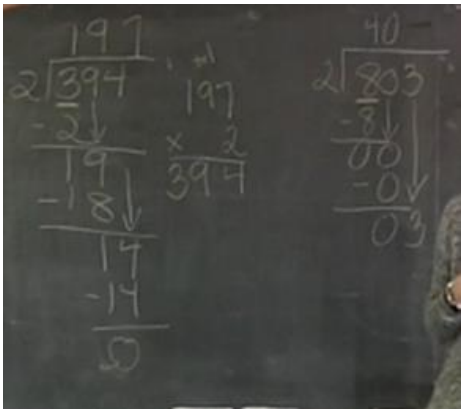
Because the presentation of these methods did not progress from concrete to abstract (i.e., concreteness-fading), Amy's sequence of representations connection-making score for the corresponding lesson was only coded as a 1. Beginning a worked example with an abstract solution (in this case the multiplication equation 24 = 4 × __) and then soliciting other more concrete strategies (i.e., tallies and skip counting) was also found during Esther's and Jackson's instruction. Even though individual students often manifest different levels of understanding, a teacher should purposefully guide students to make connections between concrete and abstract representation (e.g., concreteness-fading) so as to facilitate structural understanding.

In this study, even if a student first provided a solution strategy based on abstract thought, the teachers often encouraged the use of other strategies that included using concrete tools for computation. By stressing the use of concrete representations as tools for finding answers, the importance of reasoning abstractly was downplayed. In turn, this possibly decreased the effect that concrete tools have on aiding students' structural understanding. Not emphasizing efficiency during the initial stage of learning was the main reason for lowering a teacher's sequence of representation connection-making score. It should be noted here that the *Investigations* curriculum (used by Amy and Jackson) also tended to suggest this "abstractness-fading" sequence during textbook examples.

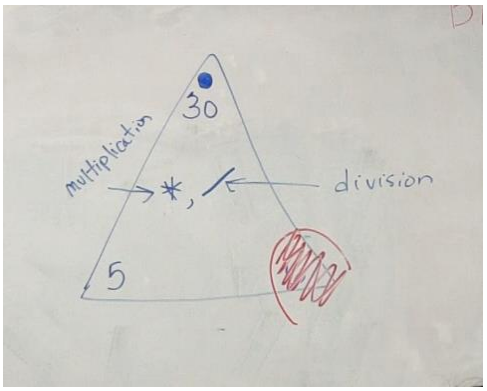
Conversely, in several instances teachers in this study did explicitly adhere to the idea of concreteness fading. For example, in contrast to Amy's sequence of strategies found in Excerpt 4, she used the multiplication and division chart [Figure 27(a)] to clearly facilitate the task of fading a concrete story problem into an abstract equation. Further, she promoted an abstract progression by using array cards [Figure 15(b)] to play a game called "the missing factor." Her students had previously used the array cards to determine the product of two numbers based on the semi-concrete representation of rows and columns illustrated on the multiplication side of the cards. With the missing factor game however, the division side of the cards contained no drawn arrays. Amy facilitated a discussion about the similarities and differences between finding the product versus finding the missing factor and stressed the importance of using semi-concrete representations (arrays) to assist with abstract reasoning. Moreover, several of Esther's and Jackson's worked examples also faded from concrete to abstract. In the previously discussed chocolate candy example, Jackson faded from the real-life image of chocolate (concrete, contextual) into manipulative tiles (concrete, no context). The tiles (concrete, no context) were then used to create an array picture (semi-concrete), which was later used to set up multiplicative inverses equations (abstract).

Although Amy, Esther and Jackson's sequence of representation scores were often lowered because they began instruction with abstract representations, Lily's score was most often lowered because of missed connections between concrete and abstract representations. This was evident in one of the worked examples that she used during her lesson on multiplicative comparisons. In the example, students were told to use their individual white boards to illustrate a representation for the problem –*Sophia ate 4*

animal crackers. Nathan ate 12 times as many. How many did Nathan eat? Students in Lily's class created a variety of representations that included various levels of abstractness; however, Lily never had students draw comparisons between representations. The side-by-side student examples in Figure 24(d) could have been used to facilitate a discussion about efficiency, but instead she simply asked the class if both representations yielded correct solutions. In contrast to the other teachers in this study who often requested that students create both a concrete and an abstract representation for the same story problem, Lily's lack of doing so resulted in missed connection-making opportunities. Further, Lily's instruction during a lesson on checking division with multiplication involved only computational long division examples that were not connected to concrete representations [Figure 28(a)]. Similarly, Esther's use of the fact triangle [Figure 28(b)] also did not connect to a concrete context, which was different from Amy's use of a fact triangle to fade a concrete story situation into abstract equations. Using abstract representations that have not been built on structure appears to promote only number manipulation and root memorization, not deep connections that could strengthen students' situation models for the targeted content of multiplicative inverses.



(a) Lily



(b) Esther

Figure 28: Teacher abstract representations that are not connected to concrete.

Several indications do suggest that the teachers in this study had at least some desire to implement concreteness fading for increasing efficiency; however, it just seemed as if this desire was not always made explicit during instruction. The use of concreteness fading by teachers appeared to be related to the teacher's knowledge and desire to create an appropriate level of instructional coherence. For instance, in Amy's interview after the lesson from which Excerpt 4 was drawn, she explained,

> I like the fact that we learn that every strategy is important, and we share out, and as long as you get to the answer, it's okay to use that. Now of course I want those students who are drawing it out to be more efficient and get to where the other students are, and they will, and those students will just need extra practice.

It is worth noting that the only time Amy explicitly talked about efficiency with her class was when students had incorrectly attempted to use their own hands as concrete representations for robots that had six fingers. After a quick discussion of why this

representation would not work, she stated that “we want more efficient…quicker ways” and then discounted the next student’s suggestion of using repeated addition. She proceeded to use the abstract equation. Jackson was the only other teacher in this study who brought up efficiency. During his last lesson he said to one student,

> If you are still doing repeated addition like this, you need to get some assistance from me or from your family at home to figure out how you can use multiplication equations, because this is no longer efficient. This takes too long and can be too prone to errors. We want to be moving toward ways that are efficient and make more sense to us.

In general, these four teachers appeared to understand how forming connections between representations, and fading representations from concrete to abstract, could help promote efficiency. Unfortunately, they rarely articulated this understanding during classroom instruction.

Deep Questions

The biggest difference in connection-making scores across teacher instruction appeared to involve the use of deep questions, questions that target relationships and structural connections to underlying principles. As indicated in Figure 29, whereas all four teachers asked deep questions to elicit students to form connections within the current to-be-learned content knowledge, only three teachers used deep questions to form connections to prior knowledge, and only two teachers posed questions aimed at facilitating connection to future mathematical topics. Upon further analysis, it appears as if the deep questions that were posed during instruction either (a) targeted sense-making or (b) supported the interconnectedness of mathematics.

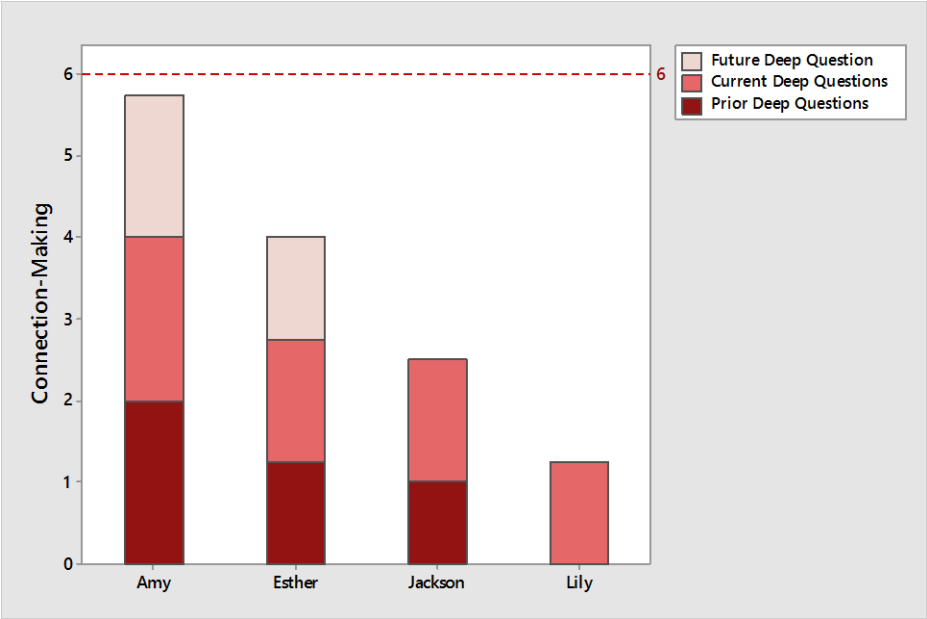
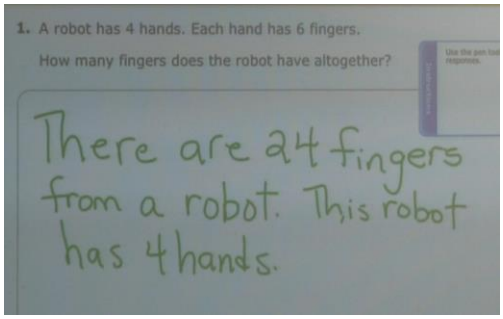


Figure 29: Connection-making scores for teacher's deep questions.

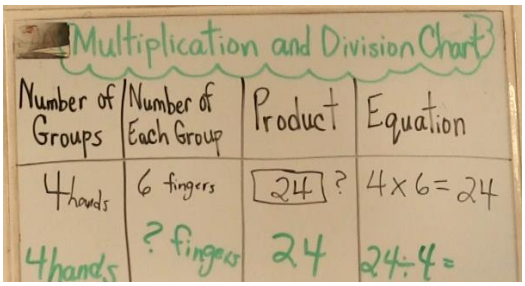
Questions that Target Sense-Making

According to Figure 29 (the horizontal line at 6 represents the highest possible connection-making score), all four teachers asked at least some deep questions for forming connections within multiplicative inverses (current deep questions). Of the deep questions that related to this targeted content, most were based around eliciting students sense-making. This included asking deep questions for having students examine why they were applying certain operations to new situations and exactly what those operations meant in context of that situation. In other words, teachers asked deep questions to help students not just form connections, but also to create situational awareness involving the understanding of those connections within their situation models. Thus, according to a situation model perspective of comprehension, these deep questions provided connection-making opportunities that focused student reasoning on both the structure and relationships inherent within multiplicative inverses in order to facilitate inference-making.

Amy created a learning environment that was quite inference oriented and that was based on continuous conversations related to children's mathematical reasoning. As a result, she used deep questions throughout her entire lessons as a means to guide instruction and force students to build connections by making sense of their own reasoning. On several occasions, Amy asked deep questions that were focused on having her students determine the relationships between multiplication and division scenarios. For instance, after discussing a solution to the aforementioned multiplicative robot problem, Amy asked her students to "come up with the reverse" story problem that would show division [Figure 30(a)]. This level of deep questioning (i.e., asking her students to create an inverse story) illustrates why her connection-making score for deep questions about multiplicative inverses was the highest possible score (M = 2.00; SD = 0). During the classroom discussion that followed students' attempts to write the reversed robot problem, Amy asked questions that forced her students to reason about the known and unknown parts within the current situation. In addition to deep questions surrounding the structure of multiplicative inverses, her guidance included contextual support that helped her students (and even herself) make sense of the problem.



(a)



(b)

Figure 30: The robot problem used during Amy's instruction.

Amy initially began the solution to the division problem by writing out 24 tallies, the number of total fingers given in the problem. Quickly however, she realized that the problem did not provide her with the number of fingers in each group, and thus she did not know how many tallies to circle at one time. This prompted a discussion about the importance of determining which pieces of information given in the problem were related to the structural components of multiplicative inverses. Deep questions asked during this discussion included, "Is 24 fingers the number of groups? Would 24 be the number of fingers in each group? Is 24 the total number of fingers? The product?" She also helped students make further sense out of inverse relations by asking questions to compare and contrast the similarities and differences seen in the multiplication and the division chart [Figure 30(b)]. Excerpt 5 below further illustrates her use of deep questions to orient a student's sense-making.

Excerpt 5:

Amy: What's the problem?

Jayden: I have something to say about the last problem. You could do it the first way.

Amy: What's that?

Jayden: You could just do 24…[student draws 24 tallies on paper] You could circle all of the 4's, and you can see how many circles there are, and that's the fingers.

Amy: But did we know how many were in each group yet?

Jayden: Oh….

Amy: Do you see what I mean? Are there 4 fingers in each group? Was there 6 groups or 4 groups?

Jayden: Well I counted and got 6…

Amy: It is a strategy, but as a visual way, I want you to think…was there really 6 hands?

Jayden: No

Amy: There were 4 hands, and that is why we switched it. Does that make sense?

Jayden: Oh…yes.

By using specific contextual questions to unpack a broader question, this excerpt reveals that Amy enhanced sense-making by providing students opportunities to correct their own reasoning. Other typical deep questions found within Amy's classroom included "Can you describe?—What makes you think it is multiplication?—Can you compare these two strategies?"—all of which elicited deep conversations surrounding the conceptual understanding of multiplicative inverses.

In a similar manner, Jackson often posed questions to his students to direct their reasoning to inverse operations and to encourage sense-making (current deep question connection making score $M = 2.00$; $SD = 0$). For example, after students had used graph paper to form an array representing $15 \times 6 = 90$, he asked students to explain how the number 90 was represented by the array. Next, he solicited a division equation that could represent the array and asked deep questions involving the connection between the semi-concrete array representation and the abstract equations. One question involved asking students, "Where can you see those numbers on this array?" This led to a discussion

about how the number of rows and the number of blocks in each row connected (structurally) to both the multiplication and the division equation. At several points throughout Jackson's instruction, the structure of multiplicative inverses (i.e. the use of a situation model) was reiterated with the question, "How are the two related?" Like Amy, he also stressed sense-making through asking comparison type questions such as, "What made this different than the other problems we were just doing?"

Another aspect of Jackson's questioning involved stressing the meaning of multiplication and division. Although some of his questions did tend to target word recognition (i.e., using the word "times" to infer a multiplication problem), on several other occasions he explicitly asked students to make sense of mathematical operations by posing the questions, "What does multiplication mean?"—"What does division mean?" These questions ultimately appeared to help his students make sense of why certain operations were being applied in different situations. Excerpt 6 includes one of these sense-making occasions drawn from Jackson's lesson multiplicative comparisons. The problem being discussed in the excerpt is:

DJ picks 7 apples. Teacher Kelly picked 4 times as many apples. How many apples did Teacher Kelly pick?

Excerpt 6:

Jackson: What does times mean? Sarah.

Sarah: Uh, it means that when she has 7 more but 4 times.

Jackson: What do you mean, she has 7 more but 4 times?

Sarah: Like, she has 7 more and then she has another 7 more, it its like…

Jackson: Call on someone to help you out to clarify your thinking.

Mark: Can I give an example?

Jackson: Please.

Mark: Do you see how you have 7 apples?

Jackson: I do see I have 7 apples. That's my favorite number.

Mark: You just add on 4 more like. They saying like you're adding on 4 more bags of 7 apples.

Jackson: Well, what does it mean that I have to add on 4 more bags of 7 apples?

Mark: Cause it says 4 times.

Jackson: Okay, because it says 4 times, but why do the bags have to have 7?

Mark: Because of the number that you already have, that's like the..

Jackson: Ah, because of the number I have already. I picked 7, and then we said that teacher Kelly picked 4 times as many apples as I did. You can't just pick a number out of the sky and say that I'm going to do 4 times 7. Okay, because it's 4 times as many apples as I already picked. So she has to have 4 groups with the same number inside of it. So now, who can tell us an equation that can represent how many apples teacher Kelly picked?

This conversation illustrates Jackson's expertise involving the act of turning incomplete student reasoning into deep questions, which ultimately helped guide his students to develop deeper situational connections. In making sense of the current situation, students

were able to turn “times 4” into four additional equal groups of 7 apples. Further, because reasoning that occurs as a result of the creation of mental models is based on the structure of the external situation that the models are formed to represent (Johnson-Laird & Byrne, 1991), Jackson’s deep structural connection that multiplication involves “groups, with the same number inside” most likely enhanced student’s situation models for multiplication.

In contrast to Amy and Jackson, Esther’s (M = 1.50; SD = 0.50) and Lily’s (M = 1.25; SD = 0.43) ability to ask deep questions for the purpose of forming connections to the current targeted content, appears to have room for improvement. For Esther, although she often asked deep questions, she also often provided students with her own deep explanations. The following excerpt illustrates this finding.

Excerpt 7:

Esther: Alright, let’s look over here. What if the problem said Pam went on the ride six times and used three tickets each time, how many tickets did she use in all? So look at this bar model. How is it the same and how is it different than the other one we just looked at? Turn and talk to your partner, I want to hear one way it’s the same and one way that it’s different. [Teacher monitors partners]. Alright, let’s take a look. Who wants to share with the whole class what you and your partner came up with? Joey.

Joey: Um it’s the same because, um, it’s the same types of numbers and it’s different because you use a different kind of strategy.

Esther: Yes. So for this one they told us the total tickets didn't they? And we had to split them into different groups—that's division. But in the bar model one, they gave us how many tickets the ride costs, three, and how many times she went on, six, but what did we need to figure out? We needed to figure out how many tickets she used altogether the six times she went on the ride. And how many was that?

Students: Eighteen.

Esther: Eighteen, very good. So they used multiplication in this one: six times three equals eighteen. So that is just a way to show you how multiplication and division are related and how you can use them to find the answer. So if you knew, if you had to say to yourself, "What times six would equal eighteen," you could do this division problem. Okay, because you knew your fact family, or you knew the division was opposite. So Pam used blank tickets each time she went on the ride. What would you fill in here? Felicia.

Felicia: Three.

Esther: Three, okay. Can you read this part out loud for us? And boys and girls can you track this with your eyes while she is reading? This is important.

Felicia: "Multiplication and division are opposite operations. They are inverse operations."

Although Esther posed the initial deep question involving a side-by-side comparison of multiplication and division problems, student responses included only surface-level comparisons (i.e., same numbers and different strategies). Instead of probing her students with more deep questions in order to guide their reasoning (as Jackson did in the previous excerpt), she simply explained the differences that existed between the two problems. This process involved having the students perform several procedural calculations as they concurrently confirmed her own answer to the original deep question. During this discussion, students were not given the opportunity to make sense of the current situation and were therefore deprived connection-making opportunities that could have enhanced their situation models for multiplicative inverses. While Esther did make a connection to multiplicative inverses, this connection focused only on the procedure of using multiplication to check division. Many of Lily's questions (e.g., Can you prove it to me?) also only focused student reasoning on the computational checking procedure of inverse relations and thus did not support sense-making. Unlike Esther, Lily herself provided very few deep explanations, and so the students in her class were presented with even fewer connection-making opportunities.

The differences found among the teacher's deep (current) questioning abilities appears to be in large part due to differences in classroom discourse. Whereas Amy's and Jackson's students were continuously involved in investigative tasks throughout classroom instruction, Esther's and Lily's students were exposed to a more show-and-tell environment. Not surprisingly, this aligns with the presentation of content found within the two different curriculums used by teachers in this study (i.e., Amy and Jackson used *Investigations* and Esther and Lily used *GO Math*). The teachers who used the textbook

which appeared to be less coherent (*Investigations*) actually provided more deep questions surrounding the targeted content, than did the teachers who used the textbook that was more coherent (*GO Math*). Moreover, the teachers who used *Investigations* seemed to add deep questions into instruction that were not found in their curriculum, whereas the teachers who used *GO Math* actually missed asking several of the deep questions provided by that curriculum.

Questions that Support the Interconnectedness of Mathematics

Based on the connection-making scores for asking deep questions involving prior or future content, the teachers in this study appeared to have different views on instructional coherence. Although recent literature (e.g., Cai, Ding & Wang, 2014) suggests that U.S. teachers view instructional coherence as the connectedness of teaching activities rather than the interconnectedness of mathematical concepts, Amy's and Esther's use of deep questions which connected multiplicative inverses to prior and future knowledge, at least somewhat tend to contradict this claim. Amy provided further evidence that she viewed instructional coherence as a continuous process of connection-making when she stated in a post-lesson interview that, "Obviously, whatever we have done previously I want them to relate that to division. Especially in this unit because it is so much related" (Amy Lesson 1 Interview). On the other hand, Jackson and Lily provided little indication that their view of instructional coherence was any deeper than forming connections between daily teaching activities.

Specific to facilitating connections to prior knowledge, Amy and Esther both used deep questions to form connections between multiplicative inverses and students' previous knowledge about additive inverses. In one case, when referencing students'

prior work with addition and subtraction word problems, Amy asked her students, “What is different about this problem compared to problems that we had been working with before?” Later, she was more explicit when asking, “Do you think multiplication and division are related like addition and subtraction?” Illustrating the connection-making that occurred as a result of asking these prior knowledge deep questions, one student in Amy’s class deduced, “Multiplication is like addition...You are adding them all up, and division is separating them.” Likewise, after being asked a similar question by Esther (i.e., Do we know two other operations that are opposites?), a student formed the connection that, “Inverses…when you do division, you are subtracting groups. When you do multiplication you are adding groups.” Both of these student responses provided strong indication that well-connected situation models for inverse relations were being formed because of the teacher’s deep questions involving prior knowledge. In addition, Amy, Esther and Jackson all facilitated connections to prior knowledge by asking deep questions about concepts and previously learned quantities that were needed to set up the current targeted content of multiplicative inverses (e.g., groups, factors, times, and product).

Deep questions were also used by Amy and Esther for forming connections between multiplicative inverses and future content. For Amy, these questions primarily focused on connections between division and the future study of fractions. One of these deep questions—“How come you didn’t say 2 divided by 6?”—was asked after students instructed Amy to write 6 ÷ 2 as the representation for the situation involving splitting six into two equal parts. Student responses included, “because you couldn’t do it,” and “it wouldn’t make sense,” a clear indication that her deep question prompted students to use

their current knowledge in order to make inferences about what would happen in a future unknown situation (i.e., when the dividend < divisor). Amy also asked several deep questions for forming connections to the future topic of improper fractions (i.e., when the dividend > divisor). For example, when discussing a practice problem that involved sharing 18 cards evenly among four friends, she asked, “What will happen if it was not even?—19—Who gets the last card?” Responses that involved the words “extra” and “remainder” indicated that her students were making inferences given these new unknown circumstances, the very outcome of using a situation model for mathematical comprehension.

While Amy’s series of deep questions involving the future study of fractions was not suggested by her curriculum (*Investigations*), forming connections between multiplicative inverses and the future content of squared numbers was found in Esther’s corresponding textbook (*GO Math*). The following excerpt and Figure 32(a), together illustrate Esther’s enactment of this connection to future knowledge.

Excerpt 8:

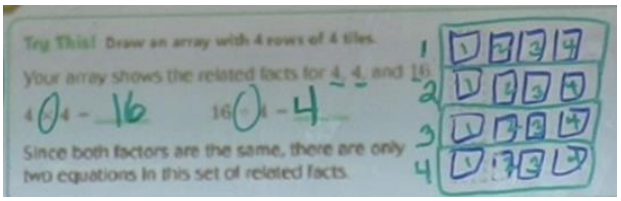
Esther: All right, draw an array with four rows of four. Four rows of four tiles in each row. Who thinks they can come add tiles to that? They already started it and you have to do it in your book. They have one tile but you need four tiles in each row. So you have 1, 2, 3, 4, but you have to keep adding so that all of your rows have four tiles.

Patrick: (Draws tiles on the board).

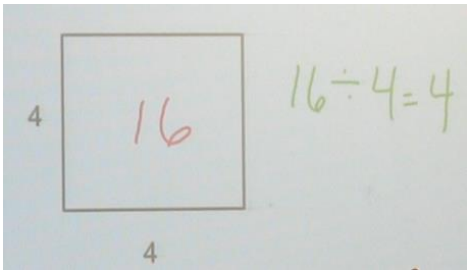
Esther: All right, let me see. I need four rows, four tiles in each row. Very good. If you’re not sure, look what Patrick did on the board. Here’s our four rows, first row, second row, third row, fourth row, here they are. There’s 1, 2, 3, 4 blocks in each row. He did a good job.

	Okay, now the array shows related facts, 4, 4, and 16. 4, 4, and 16. Fill in four times four is the multiplication sentence that matches that. Who knows the answer for that?
Justin:	Sixteen.
Esther:	Mhm. If I counted all those blocks, it would be sixteen. Sixteen divided by four rows equals how many in each row? Charmaine what is 16 divided by four?
Charmaine:	Four.
Esther:	It equals four. Can I write another division and another multiplication fact like we did for two, four, and eight?
Students:	No
Esther:	Why not? The last time we had three numbers we wrote two multiplication and two division equations, right? Can I write two more?
Students:	No.
Esther:	Zachary, what do you think?
Zachary:	Yeah.
Esther:	Give me the other multiplication fact I could write for this.
Zachary:	Wait, I don't think you can because…oh yeah, you couldn't because there's only four times four. Like if it was two times four you could, but you're not able to do that because it's two fours.
Esther:	Can you read this aloud?
Zachary:	Since both factors are the same, there are only two related factors.
Esther:	Yeah, so since the factors four and four are the same, then we can only write one division and one multiplication. Here's what I want you to try, and you can work with the person next to you. I want you to try and think of another set of related facts that only has one set of related facts for multiplication and division, just like four times four equals sixteen, and sixteen divided by four equals four. Give me another example. Work with your partner, or talk to your partner.

Esther never mentioned the formal topic of a perfect square, but by asking a series of probing questions her students were able to determine that the fact family for a square number only contains two related inverses facts. Interestingly, as seen in Figure 31(b), Amy also brought up the idea of a fact family that contained two of the same factors. However, unlike Esther (and the *GO Math* textbook), Amy missed the opportunity to connect a squared number to multiplicative inverses because she only explicitly provided the abstract division equation. Further, paired together with soliciting her students to provide other examples of squared numbers, Esther's discussion of equal rows and equal columns [based off her square image in Figure 31(a)] created a deeper connection to the future content of squared numbers. Nonetheless, both teachers used deep questions to lay a foundation for the creation of situation models involving a future mathematical topic.



(a) Esther



(b) Amy

Figure 31. The use of squared numbers in multiplicative fact families.

Summary of Teacher's Implementation of Curriculum

According to the level to which instructional tasks, representations and deep questions were used for connection-making, *GO Math* was identified to be a more coherent text (*M* = 15.28*; SD* = 1.48) than *Investigations* (*M* = 9.25*; SD* = 1.91). This was also true for each category and subcategory (Appendix H). As might be expected due to the expertise of the teachers, connection-making scores for teachers who used the less coherent *Investigations*, was higher than the textbook scores (see Figure 32). In other words, they enhanced the curriculum and provided instruction that was more coherent. Interestingly, both teachers who used the more coherent *GO Math* curriculum, scored lower than their textbook scores. That is, they provided less coherent instruction than the textbook (See Figure 32). However, it should be noted that Lily, the fourth grade teacher who used *GO Math*, only explicitly used her textbook during one enacted lesson. Interestingly, this was the lesson that corresponded to her highest connection-making score of 10. Her other three lessons only have an average connection-making score of *M*= 6.67 *(SD* = 0.47*)*. It thus appears as if the more coherent textbook was beneficial for Lily's instruction.

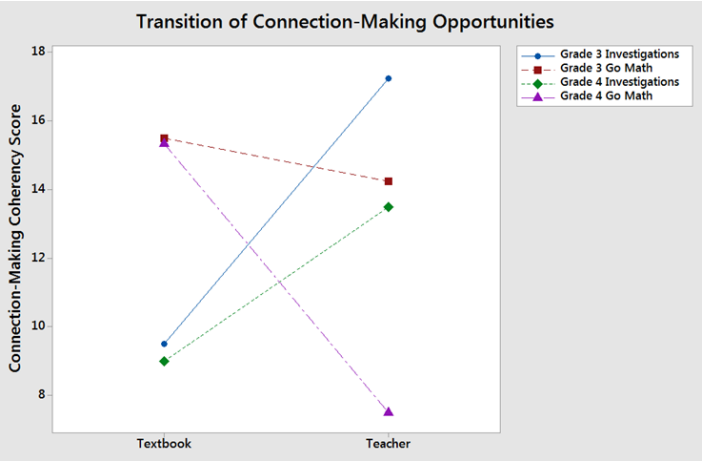


Figure 32: Transition of connection-making opportunities across curriculums.

Figure 32 reveals that the teachers who used the less coherent textbook (*Investigations*) actually provided stronger connections both within and across instructional tasks, within and between representations, and asked more deep questions than the teachers who used the more coherent *GO Math* textbook. In general, the high level of coherency found in the *GO Math* textbook, did not fully transfer into classroom instruction. Based on the situation model perspective of comprehension, this finding is similar to research that suggests that less coherency may actually support learning for high-learners. Perhaps the less coherent textbook (*Investigations*) was actually more appropriate for the teachers in this study (e.g., better supported teaching for expert-teachers). In other words, the inference-making nature of *Investigations* might force teachers to actively search for their own ways to facilitate connection-making.

Regardless of the coherency, when teaching multiplicative inverses, it appears as if elementary mathematics teachers best facilitate connection-making as a result of enhancing the connection-making opportunities found within their curriculums. Figure 32 below provides a side-by-side comparison of each teachers' instructional connection-making scores compared to his/her respective curriculum scores for each of the nine subcategories in this studies coding framework.

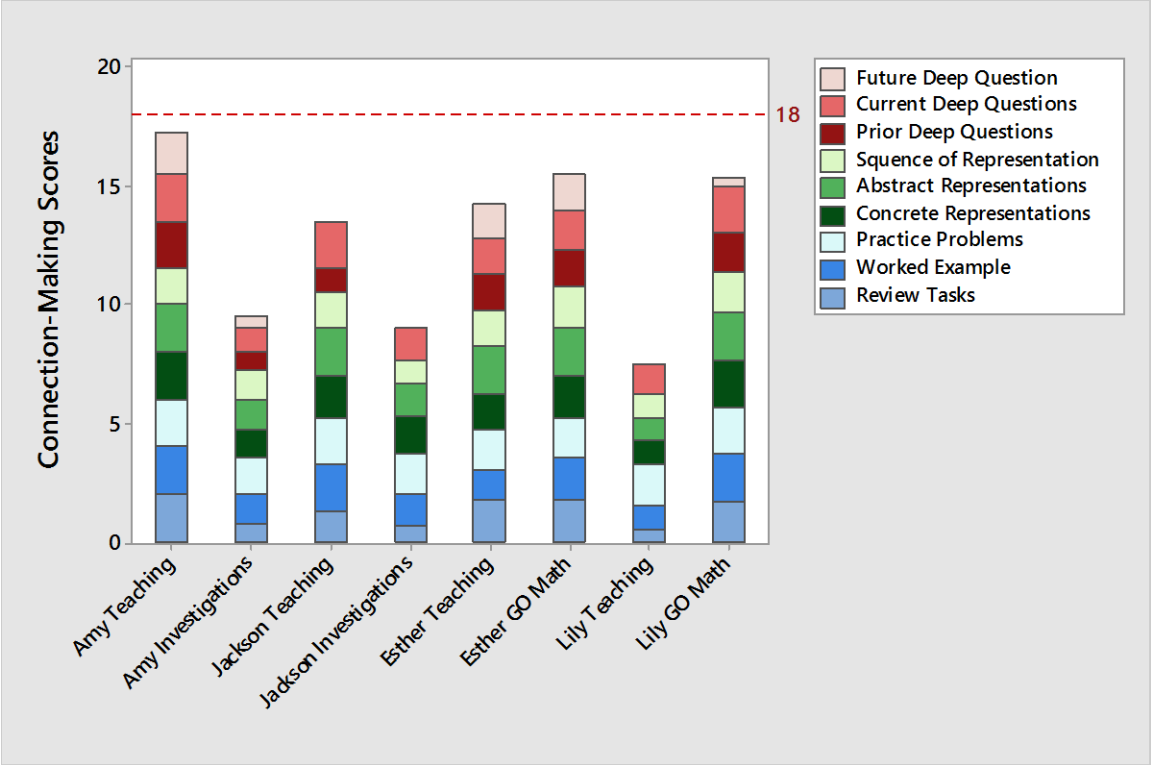


Figure 33: Instructional vs. curriculum connection-making scores for each teacher.

Concerning instructional tasks, although review tasks and worked examples were used by all four teachers, differences existed in the degree to which teachers used these tasks to form explicit connections to the targeted content. Those who formed the strongest connections tended to focus their instruction on deeply unpacking side-by-side comparison type examples. With regards to representations, the teachers who best facilitated connection-making situated instructional tasks in personal concrete contexts, used semi-concrete representations to illustrate the structural relationship of multiplicative inverses, and promoted efficiency through the sequence of their presented representations. Finally, the teachers who asked deep questions appeared to do so in support of using instructional tasks and representations to promote connection-making. More specifically, the teachers who provided students with the highest level of

connection-making opportunities posed deep questions for the purpose of supporting students' sense-making while emphasizing the interconnectedness of mathematics.

Student Comprehension Relative to Connection-Making Opportunities

In order to determine the effect that connection-making opportunities afforded by textbooks and expert elementary teachers had on student comprehension of multiplicative inverses (Research Question #3), a multivariate linear regression analysis was conducted on the inverse understanding scores (dependent variable) calculated from the post-instructional student assessment. Table 6 below provides summary statistics for the pre and post-test student understanding scores for each teacher in this study.

Table 6. *Descriptive Statistics for Pre and Post-test Understanding Scores.*

Teacher	Sample Size	Pre-Mean	Pre-SD	Post-Mean	Post-SD
Amy	24	0.95	1.91	4.67	2.68
Esther	24	0.64	1.49	4.05	2.29
Jackson	29	2.29	2.34	4.04	2.19
Lily	25	4.09	1.84	3.78	2.17

Independent variables considered for regression included the textbook connection-making score, the teacher connection-making score, the pre-test inverse understanding score (included to control the effect that prior knowledge had on comprehension), along with the student demographics of disability status, gender, race/ethnicity, LEP (limited English proficiency) status, free/reduced lunch status and grade level. A backwards elimination analysis produced the multivariate regression model that included teacher connection-making scores, pre-test inverse understanding scores, disability status and ethnicity as significant predictors of students' post-test inverse comprehension. The descriptive statistics for the variables entered into the final

analysis are listed in Table 7 (Note: * is a percentage as the variable is an indicator with only two options, present or not present).

Table 7. *Descriptive Statistics for Key Variables in the Regression Model.*

	Minimum	Maximum	Mean	Standard Deviation
Post-test (DV)	0.00	8.00	4.19	2.35
Pre-test (IV)	0.00	7.00	1.99	2.37
Teacher Connection (IV)	7.50	17.25	12.83	3.59
Disability Status (IV)	0.00	1.00	0.14*	0.35
Ethnicity (Caucasian) (IV)	0.00	1.00	0.38*	0.49

Table 8 provides the bivariate correlations between all of the variables included in the final regression model.

Table 8. *Correlations Between Variable in the Regression Model.*

	Post-test	Pre-test	Teacher Connection	Disability Status	Ethnicity
Post-test (DV)	1	.330	.092	-.300	.206
Pre-test (IV)		1	-.536	-.129	-.115
Teacher Connection (IV)			1	-.064	-.049
Disability Status (IV)				1	.072
Ethnicity (Caucasian) (IV)					1

Table 9 provides the overall summary for the regression model, which includes an *adjusted-*R^2 value of .289. This can be interpreted to mean that 28.9% of the variation in the post-test inverse comprehension scores has been accounted for by the four independent variables that were included in the regression model.

Table 9. *Model summary of the "best" regression model using backwards elimination.*

Model	R	R Square	Adjusted R Square	Standard Error of the Estimate
1	0.538	0.289	0.253	2.061

Although the *adjusted-*R^2 value provides an estimate of the strength of the relationship between the independent and dependent variables in a multivariate regression model, it

does not provide a formal hypotheses test for this relationship. Table 10 provides the outcome of an *F*-test that determined that the relationship explained by the regression model is statistically significant (*p-value* < 0.001).

Table 10. *ANOVA Test for the Overall Significance of the Regression Model*

Model	Sum of Squared	df	Mean Square	F	Sig.
Regression	134.82	4	33.70	7.93	< 0.001
Residual	331.35	78	4.35		
Total	466.17	82			

Further, the *adjusted-*R^2 value has few implications when the goal of research is exploratory in nature. For this exploratory study, determining how changes in connection-making opportunities (independent variables) affected student comprehension (prediction of a response variable) was of most importance. Table 11 provides the unstandardized regression coefficients (*B*), test statistics (*t*) and their significance levels (*Sig.*) and the collinearity statistics for each independent variable included in the final model. Unstandardized regression coefficients are reported for this study because standardizing coefficients removes the units from the variables and thus would only indicate the strength of each variable relative to the other variables in the model. Leaving units on the variables (unstandardized regression coefficients) allows for discussion about the implications that a one-point change in a variable would have on a student's overall comprehension (see discussion below).

Table 11. *Regression Model Coefficients*

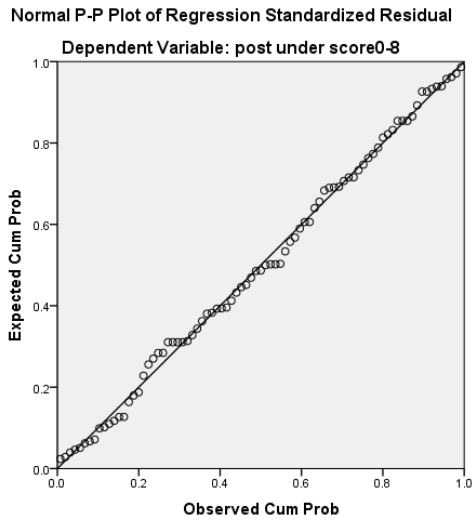
Model	Unstandardized Coefficients				Collinearity Statistics	
	B	Standard Error	*t*	Sig.	Tolerance	*VIF*
(Constant)	-.469	1.211	-.387	.700		
Teacher Connection	.263	.076	3.437	.001	.716	1.398
Pre-Test	.521	.114	4.564	<.001	.690	1.450
Disability	-1.449	.708	-2.047	.044	.964	1.037
Ethnicity	.900	.466	1.930	.057	.982	1.018

Although it was hypothesized that both textbook and teacher connection-making scores would have a significant positive effect on comprehension, only the teacher score was statistically significant (*p-value* = .001) and therefore included in the final model. The regression coefficient for the teacher connection-making score indicates that for every one point higher a teacher scored on the facilitating connection framework, a student's post-test comprehension score was predicted to increase by *B* =.263 points, if in fact the other predictor variables in the model were held constant. It should be noted that although this effect might appear small, this effect is only for a one point increase in connection-making scores. In other words, the difference between the highest teacher connection-making score (M_{Anna}= 17.25; SD_{Anna}= 0.43) and the lowest score (M_{Lily}= 7.5; SD_{Lily}= 1.50) would be $M_{Difference}$= 9.75 points, which when multiplied by the regression coefficient (*B* = .263) results in an average post-test difference of 9.75 × .263 = 2.564 points. As would be expected, a student's pre-test score was statistically significant (*p-value*< 0.001) and was the most closely related predictor (*Beta* = .525) to the post-test score. The regression coefficient for the pre-test score was *B* = .521, suggesting that prior knowledge influences comprehension. In addition, the presence of a disability had a significant negative effect on the post-test score (*p-value* = 0.044 and *B* =

-1.449) and a student's ethnicity (i.e., being Caucasian) was found to be marginally significant (*p-value* = 0.057 and *B* = .900). It should be noted that effects of these variables are additive. For instance, if a teacher scored 10 on the connection-making scale but a student was disabled, there would be a predicted (10 × .263) + (1 × -1.449) = 1.181 point increase on the post-test inverse understanding score.

All of the multiple linear regression assumptions were met for this analysis. Those assumptions included multivariate normality, residuals that were independent and homoscedastic about the regression line, and minimal multicollinearity. The multivariate normality assumption was verified through the linear trend illustrated on the normal probability plot [see Figure 34(a)]. A check of the standardized residual plot [Figure 34(b)] revealed random scatter and equal variance among residuals. Because no clear pattern or unequal vertical distribution of residuals was found, the independent and homoscedastic assumptions were satisfied. Finally, the *Tolerance* and *VIF* statistics reported in Table 10, indicate that multicollinearity is not of concern in this model. The *Tolerance* statistic indicates the percent of variance in the independent variable that cannot be accounted for by the other independent variables, and thus small *Tolerance* values (typically less than 0.10) indicate redundancy in the predictors. The variance inflation factor (*VIF*) is defined as the inverse of the *Tolerance* statistic, and thus as a rule of thumb, *VIF* values greater than 10 are of concern. Because all *Tolerance* statistics are larger than 0.10 and all *VIF* statistics are smaller than 10 according to Table 10, there is at most only minimal multicollinearity in this regression model.

(a)



(b)

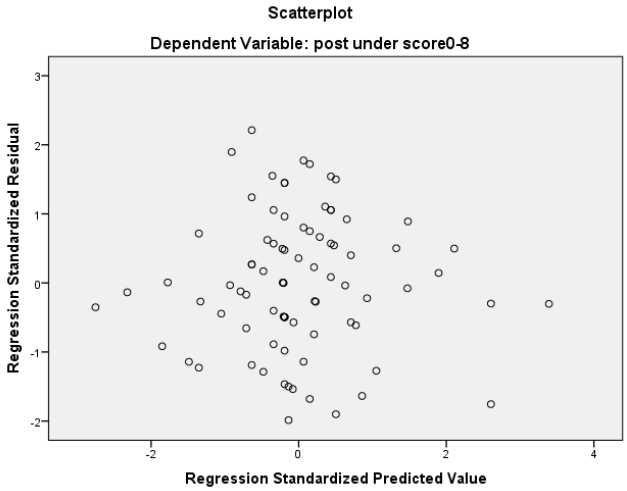


Figure 34. Graphs used to assess multiple linear regression model assumptions.

After verifying the assumptions of the initial backwards elimination regression model, the teacher connection-making score was broken into the three individual category scores (i.e., instructional tasks, representations, and deep questions) that were previously summed to determine the overall teacher connection-making score. Using the same dependent variable (post-test inverse understanding scores), a second backwards elimination regression model was run on this new set of independent variables (i.e., teacher instructional task scores, teacher representation scores, teacher deep questions scores, pre-test scores, disability and ethnicity). The resulting statistically significant

(F = 9.404; *p-value*< 0.001) regression model included only the teacher deep question connection-making score, the pre-test inverse understanding score, and the disability indicator as significant predictors of the post-test scores. As a result of the teacher instructional task connection-making score and the teacher representation connection-making scores being highly correlated with the teacher deep questions connection-making score (r = .878; r = .866 respectively), both were removed from the model. This is also reflective of the qualitative finding that deep questions were used to facilitate connection-making among instructional tasks and with representations. Interestingly, this simplified model (only two predictor variables) yielded an *adjusted-*R^2 of .225, only slightly less than the initial regression model. An ANOVA test conducted on the two models also determined no significant difference between the models (*p-value* = .060). Table 12 provides the regression coefficients (B) of this simplified model. All model assumptions were verified. Of particular interest with this second model, is the fact that the disability and ethnicity variables are no longer significant predictors of the post-test inverse understanding score.

Table 12. Coefficients for Simplified Regression Model

Model	Unstandardized Coefficients				Collinearity Statistics	
	B	Standard Error	*t*	Sig.	Tolerance	*VIF*
(Constant)	.849	.754	1.127	.263		
Pre-Test	.561	.112	5.028	<.001	.699	1.431
Teacher Deep Questions	1.837	.476	3.864	<.001	.699	1.431

The regression analysis has revealed that connection-making opportunities that expert teachers provide during classroom instruction have a significant impact on student comprehension of multiplicative inverse relations. Specifically, the asking of deep

questions appears to have the greatest effect on the situation model perspective of comprehension.

CHAPTER 5

CONCLUSIONS

This study sought to examine the extent to which current learning environments expose students to connection-making opportunities that may help facilitate mathematical understanding of elementary multiplicative inverses. As part of this embedded mixed-methods study, curriculum materials, classroom instruction, and student assessments were analyzed from a situation model perspective of comprehension. The aim was to determine how instructional tasks, representations, and deep questions are used for connection-making, the foundation of a situation model that can be used for inference-making. Although results of this study are only directly related to four teacher's classrooms, they provide important instructional and curriculum suggestions surrounding the Common Core State Standards' (CCSSI, 2010) call for students to make connections between fundamental mathematical concepts. This chapter provides a summary of major findings, a list of limitations, and implications for practice and future research involving mathematical comprehension.

Discussion of Findings

The findings from this study support the situation model perspective of comprehension, which suggests that understanding is influenced by the nature of how connections between current situations and prior knowledge are formed (Zwaan & Radvansky, 1998). As indicated by the regression analysis, comprehension of multiplicative inverses was found to depend more on connection-making opportunities afforded by classroom teachers, rather than on learning opportunities found solely within a curriculum. Interestingly, the two curriculums used by teachers in this study provided

very different connection-making opportunities. The connection-making opportunities in the *GO Math* curriculum were more explicit than in the *Investigations* curriculum. In fact, according to Appendix H, this was true for all categories and all subcategories of the connection-making framework. An increase in curriculum connection-making opportunities however, did not exactly translate into a higher level of instructional opportunities, or vice-versa. Instead, the elementary mathematics teachers who best facilitated connection-making, did so as a result of integrating the curriculum into their students' already existing knowledge systems. In other words, the teachers who demonstrated a more interconnectedness view of mathematics offered instructional opportunities for students to move beyond a simple *textbase* (Kintsch, 1986) understanding of the targeted content. This facilitated the creation and use of the most critical component for increasing comprehension, a *situation model* (Kintsch, 1986), a catalyst that can be used to convert connections into inferences. The following three sections summarize the connection-making opportunities that were afforded by the curriculums and teachers in this study for helping students initiate, develop, and apply situation models for increasing their comprehension of multiplicative inverses.

Initiation of a Situation Model:

Initiating conceptually relevant prior knowledge is the crucial first step in creating a mathematical situation model. This involves activating a learner's already existing schema involving a past action of a stereotypical situation (Zwaan & Radvansky, 1998). For the connection-making framework developed in this study, initiation of a situation model encompassed the use of instructional review tasks, concrete representations, and the asking of deep questions for making connections to prior knowledge. The review

tasks that appeared to best initiate the creation of a multiplicative inverse situation model connected content from prior lessons in a way that did not just focus on previously learned procedural facts or computational strategies. Instead, they tended to also incorporate the cognitively high demanding tasks of forming conceptual relevant connections. These review tasks that pursued both procedural skills and conceptual understanding, coincide with the Common Core's (CCSSI, 2010) call for more rigor. Specifically, these optimal review tasks were focused on activating both formal and informal prior knowledge surrounding the structure of multiplication. The formal knowledge involved an interconnectedness view of mathematics, which applied familiar content in an unfamiliar way. For instance, Amy reviewed the formula for calculating the area of a rectangle, by emphasizing the multiplicative inverse relationship between length, width and area. As suggested by past research (i.e., Cai & Ding, 2015; Cai, Ding, & Wang, 2014), forming connections to prior knowledge as a condition instead of a result of achieving instructional coherence appears to agree with the situation model perspective of comprehension.

The teachers in this study activated students' informal prior knowledge most effectively by situating instructional tasks in concrete contexts. Although literature (Resnick & Omanson, 1987) reveals that concrete manipulatives help activate existing schema, providing students with contextual support of those manipulatives appears to be most beneficial. In fact, the importance of contextual support in the initial stage of learning has been previously noted by research involving both the targeted content of inverse relations (Ding, 2016), as well as other fundamental mathematical concepts (Ding & Li, 2010; Ding et al., 2012). In this study, whereas almost every instructional task in

both curriculums was situated in rich concrete contexts, the teachers who activated the deepest connections to prior knowledge enriched those contexts by making them personal for their students. For each of the four teachers, this included changing the names of characters in their curriculum's instructional problems to reflect the names of students in their own classes. In addition, some of the teachers modified their curriculum's story contexts in order to better resemble situations that their students were sure to encounter in every-day life (e.g., when Esther changed a dog trainer problem into a problem which involved her students feeding their own dogs). For an instructional task in which the curriculum did not provide a concrete context, Lily even activated students' informal knowledge by using candy pieces, a physical concrete representation that was much more personal than a non-contextual concrete manipulative such as blocks or tiles. Situating initial learning in concrete personal contexts may activate students' personal experiences and their existing schema (Resnick & Omanson, 1987), whereby laying a foundation for the creation of a situation model for multiplicative inverses.

Deep questions do not appear to be a tool that the textbooks in this study used to activate a learner's already existing schema. In fact, very few deep questions in either curriculum targeted forming connections to prior knowledge. During instruction however, three of the teachers did ask deep questions that helped students form connections between multiplicative inverses and their prior knowledge of additive inverses. For example, both Amy and Esther asked there students to make comparisons between current multiplicative inverse problems and previously learned additive inverse problems. Deep questions about previously learned quantities that were needed to set up the targeted content of multiplicative inverses (e.g., groups, factors, times, and product),

were also asked by Amy, Esther and Jackson. Overall, student responses to these questions provided a strong indication that the situation model for multiplicative inverses was being initiated because of the teachers' deep prior knowledge questions.

Development of a Situation Model

After the initial activation of a student's prior knowledge, the construction of a situation model is facilitated by an ongoing development of schema (Rumelhart et al., 1986). This development involves providing students with adequate connection-making opportunities so that they can draw on their prior knowledge to create a more complete mental representation of the to-be-learned content. According to Zwaan and Radvansky (1998), this stage of development includes transforming schema (e.g., mental models of stereotypical situations) into situation models (e.g., mental representation of a specific real life experience) that can be used in future inference-making opportunities. For the connection-making framework in this study, development of a situation model for multiplicative inverses encompassed the use of worked examples, the sequence of representations, and the asking of deep questions for making connections to the targeted content. As noted in the literature (Paas, Renkl & Sweller, 2003), worked examples are used during instruction to develop schema so that learners can more easily extract underlying mathematical principles. In this study, the worked examples that best facilitated the development of a situation model for multiplicative inverses were deeply unpacked through the use of side-by-side comparison problems that illustrated the inverse relationship between multiplication and division. These types of problems were primarily found in the *GO Math* curriculum and in Amy, Esther, and Jackson's classroom instruction. As noted by past research (i.e., Gentner, Lowenstein & Thompson, 2003;

Rittle-Johnson & Star, 2011; Star et al., 2015), comparison is an especially powerful tool in mathematics instruction, especially for helping novice learners develop a more general schema (i.e., a situation model) that "primarily captures the common structure of the cases rather than the surface elements" (Genter et al., 2003, pg. 394). Although some comparison problems were found in both curriculums, the teachers in this study often enhanced textbook examples by situating them in the same concrete context (e.g., Amy's use of the robot example), which allowed a greater focus to be placed on the underlying structure of multiplicative inverses.

Focusing mathematics instruction on creating structural knowledge is perhaps the most important method for developing a students' situation model for multiplicative inverses. This is because in order to use a situation model, students need to be able to understand and recognize deep structural connections between problems (Chi & CanLehn, 2012). According to Gick and Holyoak (1983), deep structure is achieved when learners form a "convergence schema" (i.e., a situation model). The most explicit opportunity in this study to create deep structure occurred through the use of semi-concrete (Ding & Li, 2014) representations (i.e., schematic diagrams), an intermittent type of representation that helps novice learners bridge the gap between purely concrete and abstract knowledge. Further, several international studies (e.g., Cai et al., 2005; Ding & Li, 2014; Murata, 2008) have shown that schematic diagrams are powerful for creating deep structural relationships. As suggested by the Common Core (CCSSI, 2010), bar-models (i.e., tape diagram; Murata, 2008) and number line models are two schematic diagrams that were used by the curriculums and teachers in this study. Unfortunately, students had little exposure creating these models because they were primarily used for

computation instead of for illustrating the part-whole structural relationship that is critical in understanding the connection between multiplication and division. This may partially be due to the fact that schematic diagrams have only recently drawn attention within the U.S. mathematics education field. It should also be noted that in both curriculums and in some teacher instruction, inconsistent and incomplete connections were sometimes formed between representations. Furthermore, connections established between concrete and abstract representations did not always progress linearly, as suggested by the research on concreteness-fading (Goldstone & Son, 2005). This perhaps inhibited the ability of learners to "strip away extraneous concrete properties and distill the generic, generalizable properties" (Fyfe et al., 2014, p. 9) that are foundational to the development of a situation model.

To support the use of side-by-side comparison problems and schematic diagrams, the teachers in this study often posed deep questions to learners which were comparative in nature and which provoked conceptual understanding of the structural relationship inherent in multiplicative inverses. As suggested by the literature (Ding & Li, 2014; Pashler et al., 2007), the comparison type questions that were asked in this study—"What makes you think it is multiplication and not division?"—"How are these two related?"—"How can this model illustrate both multiplication and division?"—seemed to be effective in helping students form connections between and within mathematical principles. This is supported by the regression analysis where revealed that the deep questions connection-making category was most predictive of student achievement. Moreover, deep questions that stressed meaning—"What does multiplication mean?"—"What does division mean?"—appeared to be especially beneficial for supporting

students' sense-making. Facilitating discussions that promote reasoning and which target sense-making has been shown to be an integral component for increasing students' mathematical comprehension (Ball & Bass, 2003; Cengiz, 2013). The most effective teachers in this study actually provided opportunities for students to correct their own reasoning through asking specific contextual questions, which ultimately helped guide students to develop deeper situational connections. The previously discussed Excerpt 5 reveals how Amy provided these opportunities for students. By asking students questions such as—"Are there 4 fingers in each group?"—and—"Was there 6 groups or 4 groups?"—Amy helped students make sense of their reasoning. However, there were also occasions in which students in this study were simply provided with deep explanations instead of being afforded the opportunity to develop connections on their own.

Application of a Situation Model

The final component of the situation model perspective of comprehension is to use the developed structure (i.e., the situation model) in order to efficiently draw conclusions in unfamiliar situations. In other words, once a situation model has been developed, it may act as a catalyst for converting connections into inferences. In comprehension research, this is most commonly referred to as transfer, the ability to apply knowledge beyond initial learning (Lobato, 2006). For the connection-making framework developed in this study, using a situation model for multiplicative inverses occurred through practice problems, abstract representations, and the asking of deep questions for making connections to future situations or future content. Worked examples that faded into practice problems created the best opportunities to use a multiplicative inverse situation model in this study. This occurred when corresponding practice

problems were connected to the underlying structure developed in worked examples, but contained varied surface characteristics that most likely helped students reinforce learned knowledge while at the same time strengthening and creating new connections (Renkl et al., 1998). Although the practice problems in the *Investigations* curriculum at times varied at the structural level, all four teachers in this study seemed to embrace establishing a constant structure that was maintained throughout practice. This appeared to have promoted the use of a student's situation model for multiplicative inverses.

Because a situation model consists of an internal network of mental connections, they in essence are a learner's abstract representation of knowledge. In mathematics education, efficiency is associated with abstract understanding (Cai, 2001; CCSSI, 2010); therefore, the goal of advanced mathematics is to reason abstractly (i.e., develop and use situation models). Learning opportunities that use mathematical situation models should thus stress the importance of abstract reasoning. In this study, this was not always the case. For instance, although all of the teachers and both curriculums mentioned a desire for students to develop efficient solution strategies, students were often exposed to "abstractness-fading." This meant that reasoning abstractly was sometimes downplayed in place of using concrete tools for the sole purpose of computation. In turn, this created less than ideal opportunities for students to use multiplicative inverse situation models for converting connections into inferences. This finding was similar to Ding and Li's (2010) critique of textbook presentation that used concrete context as a pretext for computation. In addition, students in this study were sometimes presented with abstract representations that either had not been built on structure (i.e., number sentences in isolation) or which were inconsistent with the structure they were built upon (i.e., 3×4 represents 4 groups

of 3, but 7 × 4 represents 7 groups of 4). Together, these instances created an atmosphere that promoted number manipulation and root memorization, not deep connections that enhanced students' use of their abstract situation models of multiplicative inverses.

Similar to the findings surrounding prior knowledge, deep questions do not appear to be an instructional tool that the textbooks in this study endorsed for helping students to recognize how to use their multiplicative inverse situation models in the inference-making process. This seems problematic since the end goal of any comprehension perspective is to transfer learned knowledge into novice situations (Bransford et. al., 1999). If teachers in this study viewed mathematics as an interconnected web of fundamental concepts, one would think that they might pose more deep questions aimed at facilitating connections to a much broader situation model, a model that could be used for future content involving the complement principle of the inversion. This did not happen. In fact, only two teachers asked questions that invoked forming connections to future knowledge, neither of which involved inverse relations.

Limitations

Due to a few limitations in this study, one should be cautious when making generalizations based on the results described above. Because of a desire for an in-depth investigation of curriculum, instruction, and student assessment concerning connection-making, only four elementary classrooms were chosen to be part of this study. First, this small sample of classrooms resulted in a curriculum limitation, since only two different textbook series (*GO Math* and *Investigations*) were used by participants. Even though these two curriculums have rather different levels of coherency, an analysis of other elementary mathematics curriculums might have yielded an overall different picture

(qualitative) of the connection-making opportunities that are afforded by textbooks. Other curriculums might have also had a greater impact on the regression analysis (quantitative) for student comprehension. In addition, although the *GO Math* curriculum used by two teachers in this study was a fully adapted Common Core edition, *Investigations* had not yet been aligned with the Common Core (CCSSI, 2010). Instead, the publisher of *Investigations* had provided supplemental Common Core materials that could be used in conjunction with the primary textbook. Perhaps the three key shifts called for by the Common Core (i.e., rigor, focus and coherence) were therefore not as integrated into the *Investigations* curriculum as they were the *GO Math* curriculum. This potentially could have contributed to the difference in curriculum connection-making scores. Moreover, although the teachers in this study were aware of which textbook lessons they were supposed to enact, no guidance was given about the extent to which they should use the curriculum. In fact, the teachers were not even aware that this study involved analyzing both curriculum and instruction. This lack of clarity resulted in one teacher solely using the textbook for one lesson, and another teacher spreading one curriculum lesson across two different classroom lessons. Direct comparison between similar instructional lessons was therefore sometimes difficult. To avoid this curriculum limitation, future research should analyze other Common Core aligned textbooks and future studies should be designed explicitly evaluate how different curriculums may support teacher learning.

Second, as a result of only studying expert teachers, an instructional limitation exists in this study. Although experts were chosen because of an interest to examine instruction that had the highest potential for increasing students' mathematical comprehension, characteristics specific to the type of teacher who participated in this

study may prohibit more generalized findings. Expert teachers are practitioners that are more experienced and who have deeper content knowledge than their peers. Specifically, research shows that experts tend to notice meaningful patterns of information, have well organized and conditionalized knowledge and can flexibly retrieve and activate that knowledge (Bransford et al., 1999). While the first two characteristics suggest that experts tend to see and create structure by forming connections between prior and current targeted content, the third reveals that they might better understand how to use their own situation models. Therefore, perhaps expert teachers may use textbooks with less coherency (e.g., *Investigations*) better than textbooks that are more coherent (e.g., *GO Math*). This might not be true for non-expert teachers. If this study was reproduced with novice U.S. elementary teachers, perhaps several of the explicit and deep connection-making opportunities afforded during instruction (e.g., use of schematic diagrams, creating structure, emphasizing sense-making) might not be as prevalent. Further, if this study was reproduced with Chinese expert teachers who more commonly embrace an interconnected view of mathematics (Cai et al., 2014; Cai & Ding, 2015; Ding, 2016), perhaps classroom instruction would include more explicit and deeper connection-making opportunities. This might have also occurred if the expert teachers in this study held advanced degrees in mathematics. To avoid this instructional limitation, future mathematics comprehension studies should consider involving more varied teacher participants.

The third major limitation of this study was that connection-making opportunities were only assessed on student comprehension of one core underlying mathematical principle, multiplicative inverse relations. Given that, multiplicative inverses are

essentially an examination of the relationship between two quantities (i.e., multiplication and division), the very nature of the content itself may have led to more numerous connection-making opportunities. An increased emphasis on connection-making within this study's targeted content, might also be due to the third grade Common Core (CCSSI, 2010) standard that explicitly mentions the need to help students "understand the relationship between multiplication and division." Perhaps other important mathematical topics that are not as overtly emphasized or in which relational quantities are not as obvious may result in different connection-making opportunities. To avoid this assessment limitation, future mathematics comprehension studies should investigate how curriculum and instruction support connection-making for a wider range of fundamental mathematical concepts.

Finally, the characteristics of the learners in this study might also have influenced the comprehension analysis. Although participants were from four different elementary schools, they were all students of the same large urban school district. One should therefore be cautious when generalizing the student comprehension results to populations of students who do not have similar demographics as those in this study. In addition, because this study was conducted in an elementary school setting, the students in this study can be considered novice learners. Novice learners benefit from curriculum and instruction that is coherent (i.e., explicit connection-making opportunities) (Kintsch, 1994; Reed, Dempster & Ettinger, 1985); however, as students develop greater expertise, less coherency has actually been shown to improve comprehension (Renkl, Atkinson & Grobe, 2004; Schwonke, et al., 2007). Some of the explicit connection-making opportunities found in this study may therefore not be as beneficial for improving a more

experienced learner's mathematical comprehension. Lastly, because student comprehension was not assessed immediately following each lesson, perhaps their final post-test understanding scores were influenced by other instructional lessons or other connection-making opportunities that were not revealed in this study. Future studies should therefore involve more varied student participants and should more precisely assess the effects that connection-making opportunities have on student comprehension.

Implications for Practice and Future Research

Since mathematical understanding generally relates back to one's ability to form connections (Hiebert & Carpenter, 1992), exploring how learning environments afford opportunities to facilitate connection-making should be a critical component of future research that aims to improve U.S. students' mathematical comprehension. This includes research on both curriculum design and classroom instruction. Although the findings in this study suggest that the coherency level of a curriculum does not necessarily restrict or benefit a teachers' instructional coherency, more research on this relationship is warranted. This should include a wider variety of textbooks and analysis of instruction from a larger group of teachers with a broader range of expertise. After all, this was one of the first comprehensive studies that examined how reformed curriculum materials influence teachers' responses to the Common Core's (CCSSI, 2010) call for more focused, rigorous, and coherent mathematics instruction.

Specific to the theoretical framework in this study, educational researchers should continue to look for ways to help students improve the initiation, development, and application of mathematical situation models. Because of the emphasis that the Common Core (CCSSI, 2010) has placed on forming better connections in order to make better

inferences, expanding upon the connection-making framework used in this study seems promising for future research on mathematical comprehension. Of particular interest, from the findings in this study, would be research that further examines the role that comparison problems and schematic diagrams have in creating mathematical structure for aiding a student's sense-making. Moreover, because the regression analysis in this study reveals that deep questions appear to have the biggest influence on comprehension, future studies should explore how to help teachers master the art of questioning. Further, exploring the relationship between teachers' deep questions and the nature of student explanations to these questions may be of particular interest to future research on mathematical comprehension. Analyzing detailed components of classroom discourse may therefore provide additional guidance for helping teachers create connection-making opportunities during their use of instructional tasks and representations.

Future research on mathematical comprehension should also more closely involve practitioners, as implementing an appropriate level of instructional coherence (i.e., knowing when to use concreteness fading and how to ask deep questions to elicit deep understanding of targeted content) seems to depend on a teacher's view of connection-making. The short post-instructional teacher interviews conducted in this study attempted to begin this exploration; however, much still remains unknown. The teachers in this study all seemed to understand the importance of connection-making, thus design-based research that involves collaboration between researchers and practitioners might be helpful in bridging the gap between curriculum and instruction. Although U.S. elementary teachers are not typically mathematical content experts, it comes as no surprise that their instruction is more intra-connected (i.e., forming connections within

day-to-day teaching activities) rather than inter-connected (i.e., forming connections between fundamental mathematical concepts; Cai, Ding & Wang, 2014). Giving elementary teachers professional development that focuses on the interconnectedness of mathematics would thus be beneficial for improving students' mathematical comprehension. This might first involve training teachers how to assess the status of children's prior knowledge; but also, should later focus on knowing how and when to provide appropriate learning opportunities that facilitate the development and use of mathematical situation models. Future research should thus surround the development of a taxonomy of connection-making opportunities.

Conclusions

This study examined how current learning environments expose elementary students to connection-making opportunities for the learning of multiplicative inverse relations. It further explored the effect that those opportunities had on students' mathematical comprehension. By uncovering effective connection-making strategies found in both curriculum materials and classroom instruction, this study is one of the first comprehensive analyses (i.e., curriculum, instruction and student assessment) to analyze the effects that the Common Core (CCSSI, 2010) has had on a specific fundamental mathematical concept. While the findings in this study have provided important implications for practice and future research of multiplicative inverses, of most importance is the justification for adopting a situation model perspective of mathematical comprehension. Although previous research has pointed out the importance of worked examples, representations, and deep questions, the situation model perspective adopted for this study has provided a deeper understanding of the whys behind these key aspects

involved in mathematical comprehension. Ultimately, this study has provided a foundation for helping students facilitate their transfer of prior knowledge into novel mathematical situations.

REFERENCES CITED

Ainsworth, S., Bibby, P., & Wood, D. (2002). Examining the effects of different multiple representational systems in learning primary mathematics. *The Journal of the Learning Sciences, 11*, 25-61.

Angrist, J. & Pischke, J. (2010). The credibility revolution in empirical economics: How better research design is taking the con out of econometrics. *The Journal of Economic Perspectives, 24*(2), 3-30.

Anthony, G. & Walshaw, M. (2009). Characteristics of effective teaching of mathematics: A view from the west. *Journal of Mathematics Education, 2*(2), 147-164.

Ausubel, D. P. (1960). The use of advance organizers in the learning and retention of meaningful verbal learning. *Journal of Educational Psychology, 51*, 267-272.

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W.G. Martin, & D. Schifter (Eds), *A research companion to principles and standards for school mathematics*, (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Baranes, R., Perry, M., & Stigler, J. W. (1989). Activation of real-world knowledge in the solution of word problems. *Cognition and Instruction, 6*, 287-318.

Barmby, P., Harries, T, Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics, 70*, 217-241.

Barnett, S. M., & Ceci, S. J. (2002). When and where do we apply what we learn? A taxonomy for far transfer. *Psychological Bulletin, 128*, 612-637.

Baroody, A. J. (1987). *Children's mathematical thinking: A developmental framework for preschool, primary, and special education teachers.* New York: NY: Teacher College Press.

Baroody, A. J. (1992). The development of preschoolers' counting skills and principles. In J. Bideaud, C. Meljac, & J. P. Fischer (Eds.), *Pathways to number* (pp. 99-126). Hillsdale, NJ: Erlbaum.

Baroody, A. J. (1999). Children's relational knowledge of addition and subtraction. *Cognition and Instruction, 17,* 137-175.

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 1-33). Mahwah, NJ: Erlbaum.

Baroody, A. J., Ginsburg, H., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education, 14,* 156-168.

Baroody, A.J., Torbeyns, J., & Verschaffel, L. (2009). Young children's understanding and application of subtraction-related principles: Introduction. *Mathematics Thinking and Learning*, *11,* 2-9.

Bartlett, F. (1932). *Remembering: A study in Experimental and Social Psychology*. Cambridge: Cambridge University Press.

Barshay, J. (2013). Data analysis discredits widely used TERC math curriculum. Education by the numbers [web log]. Retrieved from http://educationbythe numbers.org/content/data-analysis-discredits-widely-used-terc-math-curriculum_535/

Bisanz, J., Watchorn, R., Piatt, C., & Sherman, J. (2009). On “understanding” children’s developing use of inversion. *Mathematical Thinking and Learning, 11*(1-2), 10-24.

Bloom, B. S. (1956). *Taxonomy of educational objectives, handbook I: The cognitive domain.* New York: David McKay Company Inc.

Bloom, B. S., Krathwohl, D. R., & Massia, B. B. (1984). *Taxonomy of educational objectives: the classification of educational goals*. New York: Longman.

Blum, W., Galbraith, P. L., Henn, H. W. & Niss, M. (Eds.) (2007). *Modelling and applications in mathematics education: The 14th ICMI-study.* New York: Springer.

Boaler, J., & Brodie, K. (2004). The importance of depth and breadth in the analyses of teaching: A framework for analyzing teacher questions. *Proceedings of the 26th meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Toronto, Ontario, Canada.

Booth, J. L., Lange, K. E., Koedinger, K. R., & Newton, K. J. (2013). Using example problems to improve student learning in algebra: Differentiating between correct and incorrect examples. *Learning & Instruction, 25*, 24-34.

Bransford, J. D., Barclay, J. R., & Franks, J. J. (1972). Sentence memory: A constructive versus interpretive approach. *Cognitive Psychology, 3*, 193-209.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (1999). *How people learn.* Washington, DC: National Academy Press.

Brown, M. C., McNeil, N. M., & Glenberg, A. M. (2009). Using concreteness in education: real problems, potential solutions. *Child Development Perspectives, 3(3)*, 160-164.

Brownell, W. A. & Chazal, C. B. (1935). The effects of premature drill in third-grade arithmetic. *The Journal of Educational Research, 29*(1), 17-28.

Bruner, J. S. (1960). *The process of education*. Cambridge, MA: Harvard University Press.

Bruner, J. S. (1966). *Toward a theory of instruction*. Cambridge, MA: Belknap Press of Harvard University Press.

Bruner, J. S. (1990). *Acts of meaning.* Cambridge, MA: Harvard University Press.

Bryant, P., Christie, C., & Rendu, A. (1999). Children's understanding of the relation between addition and subtraction: Inversion, identity and decomposition. *Journal of Experimental Child Psychology, 74*, 194-212.

Businskas, A. M. (2008). *Conversations about connections: How secondary mathematics teachers conceptualize and contend with mathematical connections*. Unpublished PhD thesis, Simon Fraser University, Barnaby, Canada.

Byrne, D. (2005). *Social Exclusion.* Maidenhead: Open University Press.

Caelli, K., Ray, L., & Mill, J. (2003). 'Clear as Mud': Toward greater clarity in generic qualitative research. *International Journal of Qualitative Methods, 2*(2), 1-24.

Cai, J., & Ding, M. (2015). On mathematical understanding: perspective of experienced Chinese mathematics teachers. *Journal of Math Teacher Education.*

Cai, J., Ding, M., & Wang, T. (2014). How do exemplary Chinese and U.S. mathematics teachers view instructional coherence? *Educational Studies in Mathematics, 85*, 265-280.

Cai, J., Lew, H. C., Morris, A., Moyer, J. C., Ng, S. F., & Schmittau, J. (2005). The development of students' algebraic thinking in earlier grades: A cross-cultural comparative perspective. *ZDM: The International Journal of Mathematics Education, 37*(1), 5-15.

Carpenter, T. P., Franke, L. P., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic & algebra in elementary school.* Portsmouth, NH: Heinemann.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology, 3*, 21-29.

Cavanagh, S. (1997). Content analysis: concepts, methods and applications. *Nurse Researcher, 4(3)*, 5-16.

Cengiz, N. (2013). Facilitating productive discussions. *Teaching Children Mathematics, 19*(7), 450-456.

Cepeda, N. J., Pashler, H., Vul, E., Wixted, J. T., & Rohrer, D. (2006). Distributed practice in verbal recall tasks: A review and quantitative synthesis. *Psychological Bulletin, 132*, 354-380.

Chapin, S. H., & Anderson, N. C. (2003). Crossing the bridge to formal proportional reasoning. *Mathematics Teaching in the Middle School, 8*(8), 4–20.

Chi, M., DeLeeuw, N., Chiu, M., & LaVancher, C. (1994). Eliciting self-explanations improves understanding. *Cognitive Science, 18*, 439-477.

Chi, M. T. H. (1997). Quantifying qualitative analyses of verbal data: A practical guide. *The Journal of the Learning Sciences, 6*(3), 271-315.

Chi, M. T. H. (2011). Theoretical perspectives, methodological approaches, and trends in the study of expertise. In Y. Li, & G. Kaiser (Eds.), *Expertise in mathematics instruction: An international perspective* (pp. 17-39). New York: Springer.

Chi, M. T. H. & VanLehn, K. A. (2012). Seeing deep structure from the interactions of surface features. *Educational Psychologist, 47*(3), 177-188.

Chin, C. (2006). Classroom interaction in science: Teacher questioning and feedback to students' responses. *International Journal of Science Education, 28*(11), 13–15.

Chin, C. (2007). Teacher questioning in science classrooms: Approaches that stimulate productive thinking. *Journal of Research in Science Teaching, 44(6),* 8–15.

Chingos, M. M., & Whitehurst, G. J. (2012). *Choosing blindly: Instructional materials, teacher effectiveness, and the Common Core.* Washington, DC: Brookings Institution.

Clark, R. C., Nguyen, F., & Sweller, J. (2006). *Efficiency in learning: evidence-based guidelines to manage cognitive load.* San Francisco: Pfeiffer.

Clements, D. H. (1999). Concrete manipulatives, concrete ideas. *Contemporary Issues in Early Childhood, 1*(1), 45-60.

Common Core State Standards (2010). *Common core state standards for mathematics.* Retrieved from http://www.corestandards.org/the-standards.

Costa, A. L. (2001). Teacher behaviors that enable student thinking. In A. L. Costa (Ed.), *Developing minds* (pp. 359–369). Alexandria, VA: Association for Supervision and Curriculum Development.

Cowan, R., & Renton, M. (1996). Do they know what they are doing? Children's use of economical addition strategies and knowledge of commutativity. *Educational Psychology, 16*, 407-420.

Craig, S. D., Sullins, J., Witherspoon, A., & Gholson, B. (2006). Deep-level reasoning questions effect: The role of dialog and deep-level reasoning questions during vicarious learning. *Cognition and Instruction, 24*(4), 565-591.

Craik, F.M. & Lockhart R.S. (1972). Levels of processing: A framework for memory research. *Journal of Verbal Learning & Verbal Behavior, 11*(6), 671–84.

Craik, K. (1943). *The Nature of Explanation.* Cambridge: Cambridge University Press.

Creswell, J. W. (2014). *Research design: Qualitative and quantitative, and mixed approaches*. Thousand Oaks, CA: Sage.

Creswell, J. W., & Plano Clark, V. L. (2011). *Designing and conducting mixed methods research*. Thousand Oaks, CA: Sage.

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review, 34*(4), 344-377.

Cummins, D. D., Kintsch, W., Reusser, K., & Weimer, R. (1988). The role of understanding in solving word problems. *Cognitive Psychology, 20*, 405-438.

De Smedt, B., Torbeyns, J., Stassens, N., Ghesquiere, P., & Verschaffel, L. (2010). Frequency, efficiency and flexibility of indirect addition in two learning environments. *Learning and Instruction, 20,* 205-215.

Ding, M. (2016). Opportunities to learn: Inverse operations in U.S. and Chinese textbooks. *Mathematical Thinking and Learning, 18,* 45-68.

Ding, M., & Carlson, M. A. (2013).Elementary teachers' learning to construct high quality mathematics lesson plans: A use of IES recommendations. *The Elementary School Journal, 113*, 359–385.

Ding, M., Hassler, R., Li, X. & Chen, W. (April 2016). *Algebraic knowledge for teaching: An analysis of US experts' lessons on inverse relations.* Paper presentation at the 2016 Annual Meeting of the American Educational Research Association, Washington, DC.

Ding, M., & Li, X. (2010). A comparative analysis of the distributive property in U.S. and Chinese elementary mathematics textbooks. *Cognition and Instruction, 28*, 146-180.

Ding, M., & Li, X. (2014). Transition from concrete to abstract representations: The distributive property in a Chinese textbook series. *Educational Studies in Mathematics, 87*, 103-121.

Ding, M., Li, X., Capraro, M. M., & Capraro, R. M. (2012). Supporting meaningful initial learning of the associative property: Cross-cultural differences in textbook presentations. *International Journal for Studies in Mathematics Education, 5*(1), 114-130.

Dole, J. A. & Sinatra, G. M. (1998). Reconceptualizing change in the cognitive construction of knowledge. *Educational Psychologist, 33*(2/3), 109-128.

Dossey, J. A., Halvorsen, K. T., & McCrone, S. S. (2012). *Mathematics education in the United States 2012: A capsule summary fact book.* Reston, VA: National Council of Teachers of Mathematics.

Drake, J. M., & Barlow, A. T. (2007). Assessing Students' Levels of Understanding Multiplication through Problem Writing. *Teaching Children Mathematics, 14*(5), 272-277.

Dubé, A. K., & Robinson, K. M. (2010). Accounting for individual variability in inversion shortcut use. *Learning and Individual Differences, 20*, 687-693.

Duncan, A. (2013). *The threat of educational stagnation and complacency: Remarks of U.S. Secretary of Education Arne Duncan at the release of the 2012 Program for International Student Assessment (PISA).* Washington, DC: U.S. Department of Education.

Ellis, K. (1993). *Teacher questioning behavior and student learning: What research says to teachers.* Albuquerque, NM: Paper presented at the Annual Meeting of the Western States Communication Association.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*, 7-26.

Franke, M., Webb, N., Chan, A., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education, 60*, 380-392.

Frase, L. T. (1968). Effect of question location, pacing, and mode upon retention of prose materials. *Journal of Educational Psychology, 60*, 49-55.

Freudenthal, H. (1977). Wiskunde Onderwijs anno 2000 – Afscheidsrede IOWO. *Euclides, 52*, 290-295.

Fyfe, E., McNeil, N., & Borjas, S. (2015). Benefits of "concreteness fading" for children's mathematics understanding. *Learning and Instruction, 35*, 105-120.

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: a systematic review. *Educational Psychology Review, 26*(1), 9-25.

Gagatsis, A., & Elia, I. (2004). The effects of different modes of representations on mathematical problem solving. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 447-454). Bergen, Norway: PME.

Gee, J. P. (2010). A situated sociocultural approach to literacy and technology. In E. Baker (Ed.), The new literacies: Multiple perspectives on research and practice (pp. 165–193). New York: Guilford.

Gentner, D., Loewenstein, J., & Thompson, L. (2003). Learning and transfer: A general role for analogical encoding. *Journal of Educational Psychology, 95*(2), 393-408.

Gentner, D., Ratterman, M., & Forbus, K. (1993). The roles of similarity in transfer: Separating retrievability from inferential soundness. *Cognitive Psychology, 25*, 524-575.

Gick, M., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology*, *15*, 1-38.

Gilmore, C. K., & Bryant, P. (2008). Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill. *British Journal of Developmental Psychology*, *26*, 310-316.

Gilmore, C. K., & Papadatou-Pastou, M. (2009). Patterns of individual differences in conceptual understanding and arithmetical skill: A meta-analysis. *Mathematical Thinking and Learning, 11* (1-2), 25-40.

Ginsburg, A., Leinwand, S., Angstrom, T., & Pollok, E. (2005). *What the United States can learn from Singapore's world-class mathematics (and what Singapore can learn from the United States): An exploratory study.* Washington, DC: American Institutes for Research.

Ginsburg, H. P. (2009). Early mathematics education and how to do it. In Oscar A. Barbarin and Barbara H. Wasik (eds.), *Handbook of Child Development and Early Education* (pp. 403-428). New York: The Guilford Press.

Glenberg, A. M., Kruley, P., & Langston, W. E. (1994). Analogical processes in comprehension: Simulation of a mental model. In M. A. Gernsbacker (Ed.), *Handbook of psycholinguistics* (pp. 609-640). San Diego: Academic Press.

Goldin-Meadow, S., Alibali, M. W., & Breckinridge Church, R. (1993). Transitions in concept acquisition: Using the hand to read the mind. *Psychological Review, 100*, 279-298.

Goldstone, R. L., & Sakamoto, Y. (2003). The transfer of abstract principles governing complex adaptive systems. *Cognitive Psychology, 46*, 414-466.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *The Journal of the Learning Sciences, 14*, 69-110.

GO Math! (Common Core edition). (2012). Boston, MA: Houghton Mifflin Harcourt.

Graesser, A. C., Millis, K. K., & Zwaan, R. A. (1997). Discourse comprehension. Annual *Review of Psychology, 48*, 163-189.

Gravemeijer, K. (2002). Preamble: from models to modeling. In K. Gravemeijer, R. Lehrer, B. Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 7-22). Dordrecht: Kluwer.

Greeno, J. G. (1978). Understanding and procedural knowledge in mathematics instruction. *Educational Psychologist, 12*(3), 262-283.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*(3), 170-218.

Greeno, J. G. & Riley, M. S. (1987). Processes and development of understanding. In F. E. Weinert & R. H. Kluwe (Eds.), *Metacognition, Motivation, and Understanding* (pp. 289-313). Hillsdale, NJ: Lawrence Erlbaum Associates.

Grouws, D. A. & Cebulla, K. J. (2000). *Improving Student Achievement in Mathematics*. Geneva, Switzerland: International Academy of Education.

Harrison, M., & Harrison, B. (1986). Developing numeration concepts and skills. *Arithmetic Teacher, 33,* 18-21.

Hassler, R. (April 2016). *From elementary textbooks to classroom teaching: A situation model perspective on US mathematics lessons.* Paper presentation at the 2016 Annual Meeting of the American Educational Research Association, Washington, DC.

Hassler, R. & Ding, M. (April 2016). *Situation model perspective on mathematics classroom teaching: A case study on multiplicative inverse relations.* Paper presentation at the 2016 Annual NCTM Research Conference, San Francisco, CA.

Hatano, G. (2003). Forward. In A. J. Baroody & A. Dowker (Eds), *The development of arithmetic concepts and skills* (pp. xi-xiii). Mahwah, NJ: Lawrence Erlbaum Associates.

Hauser, J. (2009). *Concrete-representational-abstract instructional approach.* Retrieved from http://www.k8accesscenter.org/training_resources/CRA_Instructional_Approach.asp

Heritage, M., & Heritage, J. (2013). Teacher questioning: The epicenter of instruction and assessment. *Applied Measurement in Education, 26*, 176-190.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Mcmillian.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding:* Portsmouth, NH: Heinemann.

Holsti, O. R. (1969). *Content analysis for the social sciences and humanities*. Reading, MA: Addison-Wesley.

Hopper, C. H. (2009). *Practicing College Learing Strategies.* (5th Ed). Boston, MA: Cenage Learning Inc.

Hsieh, H., & Shannon, S. E. (2005). Three approaches to qualitative content analysis. *Qualitative Health Research, 15*(9), 1277-1288.

Wittenberg, L., Economopoulos, K., Bastable, V., Bloomfield, K. H., Cochran, K., Earnest, D., …Sillman, K. (2012). *Investigations in Number, Data, and Space.* Glenview, IL: Pearson Education.

Johnson-Laird, P. N. (1983). *Mental models.* Cambridge, MA: Harvard University Press.

Johnson-Laird, P. N. & Byrnes, R. M. J. (1991). *Deduction.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Johnson-Laird, P. N. & Miller, G. A. (1976). *Language and Perception*. Belknap Press.

Johnson-Laird, P. N., & Byrne, R. M. J. (1991). *Deduction*. Hope, UK: Lawrence Erlbaum Associates.

Johnson, R. B., & Onwuegbuzie, A. J. (2004). Mixed methods research: A research paradigm whose time has come. *Educational Researcher, 33*(7), 14-26.

Jonassen, D. (2009). Externally modeling mental models. In *Learning and Instructional Technologies for the 21st Century* (pp. 49-74). University of Missouri, St. Louis.

Jones, A. M. (2012). Mathematics teacher time allocation. (Unpublished master's thesis). Brigham Young University, Utah.

Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *The Journal of the Learning Sciences, 4*, 39-103.

Kalyuga, S., Ayres, P., Chandler, P., & Sweller, J. (2003). The expertise reversal effect. *Educational Psychologist, 38*(1), 23-31.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science, 320*, 454-455.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal, 102*, 59-80.

Khan, W.B. & Inamullah, H. M. (2011). A study of lower-order and higher-order questions at secondary level. *Asian Social Science, 7*(9), 149-157.

Kintsch, W. (1974). *The representation of meaning in memory*. Hillsdale, NJ: Lawrence Erlbaum Associates Inc.

Kintsch, W. (1986). Learning from Text. *Cognition and Instruction, 3*(2), 87-108.

Kintsch, W. (1988). The use of knowledge in discourse processing: A construction - integration model. *Psychological Review, 95,* 163-182.

Kintsch, W. (1994). Text comprehension, memory and learning. *American Psychologist, 49,* 294-303.

Kintsch, W. (1998). *Comprehension: A paradigm for cognition.* Cambridge: Cambridge University Press.

Kintsch, W. & Greeno, J. G. (1985). Understanding and solving word arithmetic problems. *Psychological Review, 92*(1), 109-129.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologists, 41*(2), 75–86.

Klein, D. (2003). A brief history of American K-12 mathematics education in the 20th century. In J. M. Royer (Ed.), *Mathematical cognition* (pp. 175-225). Greenwich, CT: Information Age Publishing.

Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2008). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science, 32,* 336-397.

Koedinger, K.R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representation on quantitative reasoning. *Journal of the Learning Sciences, 13*, 129-164.

Kotovsky, K., Hayes, J. R., & Simon, H. A. (1985). Why are some problems hard? Evidence from the tower of Hanoi. *Cognitive Psychology, 17*, 248-294.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics, 33*, 159-174.

Langer, J. A. (1984). Examining background knowledge and text comprehension. *Reading Research Quarterly, 19*(4), 468-481.

Lehrer, R., & Schauble, L. (2002). Symbolic communication in mathematics and science: co-constituting inscription and thought. In E. D. Amsel, & J. P. Byrnes (Eds.), *Language, literacy, and cognitive development* (pp. 167-192). Mahwah, NJ: Lawrence Erlbaum.

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*, 247-271.

Leiss, D., Schukajlow, S., Blum, W., Messner, R., & Pekrun, R. (2010). The role of the situation model in mathematical modelling—Task analyses, student competencies, and teacher interventions. *Journal für Mathematik-Didaktik,31*(1), 119-141.

Li, X., Ding, M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and the United States. *Cognition and Instruction, 26*, 195-217.

Li, X., Hassler, R., & Ding, M. (April 2016). *Elementary students' understanding of inverse relations in the United States and China.* Paper presentation at the 2016 Annual Meeting of the American Educational Research Association, Washington, DC.

Linn, M. C. (2006). The knowledge integration perspective on learning and instruction. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 243-264). New York: Cambridge University Press.

Lobato, J. (2006). Alternative perspectives on the transfer of learning: History, issues, and challenges for future research. *The Journal of the Leaning Sciences, 15*, 431-449.

Lombardi, D., Sinatra, G. M., & Nussbaum, E. M. (2013). Plausibility reappraisals and shifts in middle school students' climate change conceptions. *Learning and Instruction, 27,* 50-62.

Long, D. L., Seely, M.R., Oppy, B.J., & Golding, J. M. (1996). The role of inferential processing in reading ability. In *Models of Understanding Text* (pp. 189-214). Mahwah, NJ: Lawrence Erlbaum Associates.

Lorch, R. F. & van den Broek, P. (1997). Understanding reading comprehension: Current and future contributions of cognitive science. *Contemporary Educational Psychology, 22,* 213-246.

Malzahn, K. A. (2013). *2012 national survey of science and mathematics education: Status of elementary school mathematics.* Chapel Hill, NC: Horizon Research.

Man, Y. S. (2011). Students' understanding of inverse relation between addition and subtraction at primary levels. (Unpublished master's thesis). The University of Hong Kong, China.

Mannes, S. M., & Kintsch, W. (1987). Knowledge organization and text organization. *Cognition and Instruction, 4*, 91-115.

Martin, T., & Schwartz, D. L. (2005). Physically distributed learning: adapting and reinterpreting physical environments in the development of fraction concepts. *Cognitive Science, 29*, 587-625.

McKenna, M. C. & Robinson, R. D. (1990). Content literacy: A definition and implications. *Journal of Reading, 34*(3), 184-186.

McNamara, D. S., Kintsch, E., Songer, N. B., & Kintsch, W. (1996). Are good texts always better? Interactions of text coherence, background knowledge, and levels of understanding in learning from text. *Cognition and Instruction, 14*(1), 1-43.

McNeil, N. M. (2007). U-shaped development in math: 7-year-olds outperform 9-year-olds on equivalence problems. *Developmental Psychology, 43*, 687-695.

McNeil, N. M., & Alibali, M. W. (2000). Learning mathematics from procedural instruction: externally imposed goals influence what is learned. *Journal of Educational Psychology, 92*, 734-744.

McNeil, N. M., & Alibali, M. W. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development, 76*, 883-899.

McNeil, N. M., & Fyfe, E. R. (2012). "Concreteness fading" promotes transfer of mathematical knowledge. *Learning and Instruction, 22*, 440–448.

McNeil, N. M., & Jarvin, L. (2007). When theories don't add up: disentangling the manipulative debate. *Theory into Practice, 46*(4), 309-316.

Merriam, S. B. (1998). Qualitative Research and Case Study Applications in Education. San Francisco: Jossey-Bass.

Montgomery, D. C., Peck, E. A., & Vining, G. G. (2006). *Introduction to linear regression analysis.* Hoboken, New Jersey: John Wiley & Sons, Inc.

Morge, L. (2005). Teacher-pupil interaction: A study of hidden beliefs in conclusion phases. Research report. *International Journal of Science Education, 27*(8), 9–35.

Mortimer, E. F., & Scott, P. H. (2003). *Meaning making in science classrooms.* Milton Keynes: Open University Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for Ccherence*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics Commission on Standards for School Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: The Council.

National Research Council. (1999). *Global perspectives for local action: Using TIMMS to improve U.S. mathematics and science education: Professional development guide.* Washington, DC: National Academy Press.

Nickerson, R. S. (1985). Understanding understanding. *American Journal of Education, 92*(2), 201-239.

Novick, L. R. (1988). Analogical transfer, problem similarity, and expertise. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 14,* 510-520.

Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. (2007). The contribution of logical reasoning to the learning of mathematics in primary school. *British Journal of Developmental Psychology, 25*, 147-166.

Nunes, T., Bryant, P., Hallet, D., Bell, D., & Evans, D. (2009). Teaching children about the inverse relation between addition and subtraction. *Mathematical Thinking and Learning, 11(1-2),* 61-78.

Nunes, T., Bryant, P., & Watson, A. (2009). *Key understandings in mathematics learning: A report to the Nuffield Foundation.* Nuffield Foundation.

Nussbaum, E. M., & Edwards, O. V. (2011). Critical questions and argument stratagems: A framework for enhancing and analyzing students' reasoning practices. *Journal of the Learning Sciences*, 20(3), 443-488.

Osterholm, M. (2006). Characterizing reading comprehension of mathematical texts. *Educational Studies in Mathematics, 63,* 325-346.

Paas, F., Renkl, A., & Sweller, J. (2003). Cognitive load theory and instructional design: Recent developments. *Educational Psychologist, 38*(1), 1-4.

Paas, F. & VanMerrienboer, J. (1994). Variability of worked examples and transfer of geometric problem-solving skills: A cognitive-load approach. *Journal of Educational Psychology, 86*, 122-133.

Pashler, H., Bain, P. M., Bottge, B. A., Graesser, A., Koedinger, K., McGaniel, M. et al., (2007). *Organizing instruction and study to improve student learning* (NCER 2007-2014). Washington, DC: National Center for Education Research.

Pearson, P. D., Hansen, J., & Gordon, C. (1979). The effect of background knowledge on young children's comprehension of explicit and implicit information. *Journal of Literacy Research, 11*(3), 201-209.

Perfetti, C. A. (1989). There are generalized abilities and one of them is reading. In L. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Piaget, J. (1928). The Child's Conception of the World. London: Routledge and Kegan Paul.

Piaget, J. (1952). *The child's conception of number*. New York, NY: Humanities.

Pinker, S. & Bloom, P. (1990). Natural language and natural selection. *Behavioral and Brain Sciences, 13*(4), 707-784.

Program for International Student Assessment (2009). *Highlights from PISA 2009: Performance of U.S. 15-year-old students in mathematics, science, and reading literacy in an international context*. Retrieved from http://nces.ed.gov/pubs2014/2014024rev.pdf.

Program for International Student Assessment (2012). *Performance of U.S. 15-year-old students in mathematics, science, and reading literacy in an international context: First look at PISA 2012*. Retrieved from http://nces.ed.gov/pubs2014/2014024rev.pdf.

Polikoff, M.S. (2015). How well aligned are textbooks to the Common Core standards in mathematics? *American Education Research Journal, 52*(6), 185-1211.

Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011). Common Core standards: The new U.S. intended curriculum. *Educational Researcher, 40*, 1103-116.

Prather, R. W., & Alibali, M. W. (2009). The development of arithmetic principle knowledge: how do we know what learners know? *Developmental Review, 29*, 221-248.

Putnam, R. T., deBettencourt, L. U., & Leinhardt, G. (1990). Understanding of derived-fact strategies in addition and subtraction. *Cognition and Instruction, 7*(3), 245-285.

Reed, S. K., Dempster, A., & Ettinger, M. (1985). Usefulness of analogous solutions for solving algebra word problems. *Journal of Experimental Psychology, 11*, 106-125.

Renkl, A. (1997). Learning from worked-out examples: A study on individual differences. *Cognitive Science, 21*(1), 1-29.

Renkl, A., & Atkinson, R. K. (2007). Interactive example-based learning environments: Using interactive elements to encourage effective processing of worked examples. *Educational Psychology Review, 19*(3), 375-386.

Renkl, A., Atkinson, R. K., & Grobe, C. S. (2004). How fading worked solution steps works—a cognitive load perspective. *Instructional Science, 32*, 59-82.

Renkl, A., Atkinson, R. K., Maier, U., & Staley, R. (2002). From example study to problem solving: Smooth transitions help learning. *Journal of Experiential Education, 70*, 293-315.

Renkl, A., Stark, R., Gruber, H., & Mandl, H. (1998). Learning from worked-out examples: The effects of example variability and elicited self-explanations. *Contemporary Educational Psychology, 23*, 90-108.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109-151). New York, NW: Academic Press.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (Vol. 19, pp. 275–323). Hillsdale, NJ: Lawrence Erlbaum.

Resnick, L. B., & Ford, W. W. (1981). *The psychology of mathematics for instruction.* Hillsdale, NJ: Lawrence Earlbaum Associates.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 3, pp. 41-95). Hillsdale, NJ: Erlbaum.

Richland, L. E., Zur, O., & Holyoak, K. J. (2007). Cognitive supports for analogy in the mathematics classroom. *Science, 316*, 1128-1129.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*, 175-189.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346-362.

Rittle-Johnson, B. & Star, J.R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561-574.

Rittle-Johnson, B. & Star, J. R. (2011). The power of comparison in learning and instruction: Learning outcomes supported by different types of comparisons. *Psychology of Learning and Motivation, 55*, 199-225.

Robinson, K. M., & Dubé, A. K. (2009a). A microgenetic study of the multiplication and division inversion concept. *Canadian Journal of Experimental Psychology, 63*, 193–200.

Robinson, K. M., & Dubé, A. K. (2009b). Children's understanding of the inverse relation between multiplication and division. *Cognitive Development, 24*, 310-321.

Robinson, K. M., & LeFevre, J. (2012). The inverse relation between multiplication and division: Concepts, procedures, and a cognitive framework. *Educational Studies in Mathematics, 79*, 409-428.

Rothkopf, E. Z. (1966). Learning from written materials: An exploration of the control of inspection behavior by test-like events. *American Educational Research Journal, 3*, 241-249.

Rubin, C. (2009). The curious classroom: Answers about questions. *ASCD Express, 4*(18).

Rumelhart, D.E., Hinton, G. E., and Williams, R. J. (1986). Learning internal representations by error propaganda. In *Parallel distributed processing*, D. E. Rumelhart, J.L. McClelland, and the PDP Research Group, eds., Vols. I and II, Bradford Books and MIT Press, Cambridge, MA.

Schmalhofer, F., & Glavanov, D. (1986). Three components of understanding a programmer's manual: Verbatim, propositional, and situational representations. *Journal of Memory and Language, 25*, 279-294.

Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H., Wiley, D. E., Cogan, L. S., & Wolfe, R. G. (2001). *Why schools matter: A cross-national comparison of curriculum and learning.* San Francisco: Jossey-Bass

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of mathematics and science content standards from an international perspective. *Journal of Curriculum Studies, 37*, 525-559.

Schwonke, R., Wittwer, J., Aleven, V., Salden, R. J. C. M., Krieg, C., & Renkl, A. (2007). *Can tutored problem solving benefit from faded worked-out examples?* Paper presented at The European Cognitive Science Conference, Delphi, Greece.

Seel, N. (2006). Mental models in learning situations. In *Mental models and the mind* (pp. 85-107). Amsterdam: Elsevier B.V.

Shepherd, M. D., Selden, A. & Selden, J. (2012). University students' reading of their first-year mathematics textbooks. *Mathematical Thinking and Learning, 14*(3), 226-256.

Sidney, P.G. & Alibali, M.W. (2015). Making connections in math: Activating a prior knowledge analogue matters for learning. *Journal of Cognition and Development, 16*(1), 160-185.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: a microgenetic analysis. *Journal of Experimental Psychology: General, 127*, 377-397.

Sigel, I. E., & Saunders, R. (1979). An inquiry into inquiring. Question asking as an instructional model. In L. Katz (Ed.), *Current topics in early childhood education* (Vol. 2, pp. 169-193). Norwood, NJ: Ablex.

Silver, E. A., Mesa, V. M., Morris, K. A., Star, J. R., & Benken, B.M. (2009). Teaching mathematics for understanding: An analysis of lessons submitted by teachers seeking NBPTS certification. *American Educational Research Journal, 46*(2), 501-531.

Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*(5), 472-494.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*(December), 20-26.

Skemp, R. R. (1978). Goals of learning and qualities of understanding. *Mathematics Teaching, 88*, 44-49.

Smart, J. B., & Marshall, J. C. (2013). Interactions between classroom discourse, teacher questioning, and student cognitive engagement in middle school science. *Journal of Science Teacher Education, 24*(2), 249-267.

Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks. *Mathematics Teaching in the Middle School, 3*, 344-350.

Son, J. Y., & Goldstone, R. L. (2009). Fostering general transfer with specific simulations. *Pragmatics and Cognition, 17*, 1-42.

Son, J. Y., Smith, L. B., & Goldstone, R. L. (2011). Connecting instances to promote children's relational reasoning. *Journal of Experimental Child Psychology, 108*(2), 260-277.

Squire, S., Davies, C., & Bryant, P. (2004). Does the cue help? Children's understanding of multiplicative concepts in different problem contexts. *British Journal of Educational Psychology, 74*(4), 515-532.

Star, J., Pollack, C., Durkin, K., Rittle-Johnson, B., Lynch, K., Newton, K., & Gogolen, C. (2015). Learning from comparison in algebra. *Contemporary Educational Psychology, 40*, 41-54.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. in F. J. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte, NC: information Age.

Stein, M. K., Smith, M. S., Henningsen, M., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development.* New York: Teachers College Press.

Stemler, S. (2001). An overview of content analysis. *Practical Assessment, Research & Evaluation, 7*(17). Retrieved from http://PAREonline.net/getvn.asp?v=7&n=17

Stern, E. (1992). Spontaneous use of conceptual mathematical knowledge in elementary school children. *Contemporary Educational Psychology, 17*, 266-277.

Stern, E. (2005). Pedagogy- Learning for teaching. *British Journal of Educational Psychology, Monograph Series, II, 3*, 155-170.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S. & Serrano, A. (1999). *The TIMSS Videotape Classroom Study: Methods and Findings from an Exploratory Research Project on Eight –Grade Mathematics Instruction in Germany, Japan, and the United States.* U.S. Department of Education National Center for Educational Statistics: NCES 99-074. Washington, D.C.: Government Printing Office.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the worlds' teachers for improving education in the classroom.* New York: Free Press.

Stylianides, A. J.**,** & Stylianides, G. J. (2007). Learning mathematics with understanding: A critical consideration of the Learning Principle in the Principles and Standards for School Mathematics. *The Mathematics Enthusiast, 4*(1), 103-114.

Swartz, R. (2008). Energizing learning. *Educational Leadership, 65*(15), 26-31.

Sweller, J., & Cooper, G. A. (1985). The use of worked examples as a substitute for problem solving in learning algebra. *Cognition and Instruction, 2*, 59-89.

Thompson, D. R., Kaur, B., Koyama, M., & Bleiler, S. K. (2013). A longitudinal view of mathematics achievement of primary students: Case studies from Japan, Singapore, and the United States. *ZDM Mathematics Education, 24*, 73-89.

Thompson, P. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.). *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: State University of New York.

Torbeyns, J., De Smedt, B., Stassens, N., Ghesquiere, P., & Verschaffel, L. (2009). Solving subtraction problems by means of indirect addition. *Mathematical Thinking and Learning, 11,* 79-91.

Trafton, J. G., & Reiser, B. J. (1993). The contributions of studying examples and solving problems to skill acquisition. In M. Polson (Ed.), *Proceedings of the 15th Annual Conference of the Cognitive Science Society* (pp. 1017-1022). Hillsdale, NJ: Erlbaum.

Trends in International Mathematics and Science Study. (2003). *Highlights from the trends in international mathematics and science study (TIM22) 2003*. Retrieved from http://nces.ed.gov/pubs2005/2005005.pdf.

Trends in International Mathematics and Science Study. (2007). *Highlights from TIMSS 2007: Mathematics and science achievement of U.S. fourth and eighth-grade students in an international context*. Retrieved from http://nces.gov/pubs2009/ 2009001.pdf

Trends in International Mathematics and Science Study. (2011). *Highlights from TIMSS 20011: Mathematics and science achievement of U.S. fourth and eighth-grade students in an international context*. Retrieved from http://nces.ed.gov/pubs 2013/2013009_1.pdf

Van De Walle, J., Karp, K., & Bay-Williams, J. (2012). *Elementary and middle school mathematics: Teaching developmentally*. New York*:* Pearson

Van Den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematical education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54,* 9-35.

van den Oord, E.J. & Van Rossem, R. (2002). Differences in first graders' school adjustment: the role of classroom characteristics and social structure of the group. *Journal of School Psychology, 40*(5), 369-394.

van Dijk, T. A., & Kintsch, W. (1983). *Strategies of discourse comprehension*. San Diego, CA: Academic Press.

Van Merriënboer, J. (1997). *Training complex cognitive skills: A four-component instructional design model for technical training.* Englewood Cliffs, NJ: Educational Technology Publications.

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24*, 335-359.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert., & M. Behr (Eds.) *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: National Council of Teacher of Mathematics.

Walmsley, A. L.E. (2007). *A history of mathematics education during the twentieth century.* Lanham, MD: University Press of America.

Weaver, C. A., III, Bryant, D. S., & Burns, K. D. (1995). Comprehension monitoring: Extensions of the Kintsch and van Dijk model. In Weaver, S. Mannes, and C. R. Fletcher (Eds.), Discourse comprehension: *Essays in honor of Walter Kintsch*, pp. 177-192. Hillsdale, NJ: Erlbaum.

Weaver, C. A., & Kintsch, W. (1992). Enhancing students' comprehension of the conceptual structure of algebra word problems. *Journal of Educational Psychology, 84*(4), 419-428.

Weimer, M. (1993). *Improving your classroom teaching: Volume 1*. Newbury Park, CA: Sage Publications.

Wheelock, A., Haney, W., & Bebell, D. (2000). What can student drawings tell us about high-stakes testing in Massachusetts? *Teachers College Record*.

Wilen, W. (1991). *Questioning skills for teachers: What research says to the teacher*. Washington DC: National Education Association.

Wimer, J. W., Ridenour, C. S., Thomas, K., & Place, A. W. (2001). Higher order teacher questioning of boys and girls in elementary mathematics classrooms. *Journal of Educational Research, 95*(2), 84-92

Wu, H. (1999). Basic skills versus conceptual understanding. *American Educator/American Federation of Teachers, 23*(3), 14-22.

Zhou, Z., & Peverly, S. (2005). The teaching addition and subtraction to first graders: A Chinese perspective. *Psychology in the Schools, 42*, 259-272.

Zwaan, R. A., & Madden, C. J. (2004). Updating Situation Models. *Journal of Experiential Psychology: Learning, Memory, and Cognition, 30*(1), 283-288.

Zwaan, R. A., Magliano, J. P., & Graesser, A. C. (1995). Dimensions of situation model construction in narrative comprehension. *Journal of Experimental Psychology: Learning, Memory, & Cognition, 21*, 386-397.

Zwaan, R. A., & Radvansky, G. A. (1998). Situation models in language and memory. *Psychological Bulletin, 123*, 162-185.

APPENDICES

APPENDIX A

TEACHER SURVEY INSTRUMENT

This survey takes about 20 minutes. We ask for your name so that we can match your responses now with your responses at the end of the program. Your name will not be included with your responses when data is reviewed, analyzed and reported in aggregate form to understand the effects of the program.

1. **Your full name (last, first).** _____

2. **Your school name** _____

3. **Representative teaching honors or awards that you have received**

   _____

4. **How many years have you taught? Please check one box.**

   ☐ 6-10 ☐ 11-15 ☐ 16-20 ☐ 21-25 ☐ 26 and above

5. **What grade level are you teaching at your current school? Please check one box.**

   ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

<u>Please tell us a bit about your own past experiences learning mathematics:</u>

6. Please indicate what kinds of mathematics you took during your post-secondary studies (e.g., college and your certification process). Also please indicate if it was required, if you liked it, and if you did well in it. (Circle one response in each applicable box.)

Did you take one course or more in the following subject matter? (Circle yes or no for each subject area.)			**If yes**, you did take at least one course…						
			Why did you take the course? Was it required, did it fulfill credit hours, or was it an elective? If you have taken more than one course in the subject, please circle ALL answers that apply.			Did you like the subject matter?		Did you consider that you did well in it?	
	Yes	No				Yes	No	Yes	No
Calculus	☐	☐	Required	Credit Hours	Elective	☐	☐	☐	☐
Linear Algebra	☐	☐	Required	Credit Hours	Elective	☐	☐	☐	☐
Modern Algebra	☐	☐	Required	Credit Hours	Elective	☐	☐	☐	☐
Probability and Statistics	☐	☐	Required	Credit Hours	Elective	☐	☐	☐	☐
Differential Equations	☐	☐	Required	Credit Hours	Elective	☐	☐	☐	☐

Numerical Analysis	☐	☐	Required	Credit Hours	Elective	☐	☐	☐ ☐
Non-Euclidean geometry	☐	☐	Required	Credit Hours	Elective	☐	☐	☐ ☐

7. **How many professional development sessions in mathematics have you attended during the past three years? Please check ONE box.**
   ☐ *None* ☐ *1-2* ☐ *More than three*

8. Please indicate the areas where you would like to receive more professional development support in mathematics, ranking them (1 – 4) in order of importance to you, with 1 being the most important

   ☐ Learn more content (subject-matter) knowledge.
   ☐ Learn more inquiry/investigation oriented strategies for the classroom.
   ☐ Learn more about understanding student thinking with regard to MATHEMATICS learning.
   ☐ Learn more about assessing student learning in mathematics.

9. Please indicate how well prepared you feel to do each of the following. Please check ONE box per line.

		Not Adequately Prepared	**Somewhat Adequately Prepared**	**Fairly Well Prepared**	**Very Well Prepared**
a.	Lead a class of students using investigative strategies.	☐	☐	☐	☐
b.	Manage a class of students engaged in hands-on/project-based work.	☐	☐	☐	☐
c.	Help students take responsibility for their own learning.	☐	☐	☐	☐
d.	Recognize and respond to student diversity.	☐	☐	☐	☐
e.	Encourage students' interest in mathematics.	☐	☐	☐	☐
f.	Use strategies that specifically encourage participation of females and minorities in mathematics.	☐	☐	☐	☐
g.	Involve parents in the mathematics education of their students.	☐	☐	☐	☐

10. **How many lessons per week do you typically teach mathematics in your class? Please check ONE box.**

    ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 or more

11. Approximately how many minutes is a typical mathematics lesson? Please check ONE box.

    ☐ 20 or fewer ☐ 21-40 ☐ 41-60 ☐ 61-80 ☐ 81 or more

12. How many mathematics units has your class (or a typical class if you have more than one) worked on so far this academic year? (We are defining a "unit" as a series of related activities, often on a single topic such as addition or subtraction) Please check ONE box.

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10

13. **How many weeks do your mathematics units typically last? (Circle one response.)**

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 or more weeks

We just have a few more questions about your view on mathematics teaching. Your responses are <u>very important</u> for our program evaluation, and we appreciate your time and thought.

14. **Please tell us how much you disagree or agree with the following statements about mathematics teaching and learning. Please check ONE box per line.**

		Strongly Disagree	**Disagree**	**Not Sure**	**Agree**	**Strongly Agree**
a.	When a student does better than usual in mathematics, it is often because the teacher exerted a little extra effort.	☐	☐	☐	☐	☐
b.	I am continually finding better ways to teach mathematics.	☐	☐	☐	☐	☐
c.	Even when I try very hard, I don't teach mathematics as well as I do most subjects.	☐	☐	☐	☐	☐
d.	When the mathematics grades of students improve, it is most often due to their teacher having found a more effective teaching approach.	☐	☐	☐	☐	☐
e.	I know the steps necessary to teach mathematics concepts effectively.	☐	☐	☐	☐	☐
f.	I am not very effective in monitoring mathematics experiments.	☐	☐	☐	☐	☐

15. **About how often do the students in your class (or typical class) take part in each of the following types of activities as part of their mathematics instruction? Please check ONE box per line.**

		Never	**Rarely (e.g., a few times a year)**	**Sometimes (e.g., once or twice a month)**	**Often (e.g., once or twice a week)**	**All or almost all math lessons**
a.	Work on solving a real-world problem.	☐	☐	☐	☐	☐
b.	Share ideas or solve problems with each other in small groups.	☐	☐	☐	☐	☐
c.	Engage in hands-on mathematics activities.	☐	☐	☐	☐	☐

d.	Interact with a professional scientist, engineer, or mathematician, either at school or on a field trip.	☐	☐	☐	☐	☐

16. **Please tell us how much you disagree or agree with the following statements about mathematics teaching and learning. Please check ONE box per line.**

		Strongly Disagree	**Disagree**	**Not Sure**	**Agree**	**Strongly Agree**
a.	If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching.	☐	☐	☐	☐	☐
b.	I generally teach mathematics ineffectively.	☐	☐	☐	☐	☐
c.	The inadequacy of a student's mathematics background can be overcome by good teaching.	☐	☐	☐	☐	☐
d.	The low mathematics achievement of some students cannot generally be blamed on their teachers.	☐	☐	☐	☐	☐
e.	When a low achieving child progresses in mathematics, it is usually due to extra attention given by the teacher.	☐	☐	☐	☐	☐
f.	I understand mathematics concepts well enough to be effective in teaching elementary mathematics.	☐	☐	☐	☐	☐
g.	Increased effort in mathematics teaching produces little change in some students' mathematics achievement.	☐	☐	☐	☐	☐
h.	The teacher is generally responsible for the achievement of students in mathematics.	☐	☐	☐	☐	☐
i.	Students' achievement in mathematics is directly related to their teacher's effectiveness in mathematics teaching.	☐	☐	☐	☐	☐
j.	If parents comment that their child is showing more interest in mathematics at school, it is probably due to the performance of the child's teacher.	☐	☐	☐	☐	☐
k.	I find it difficult to explain to students why mathematics procedures work.	☐	☐	☐	☐	☐
l.	I am typically able to answer students' mathematics questions.	☐	☐	☐	☐	☐
m.	I wonder if I have the necessary skills to teach mathematics.	☐	☐	☐	☐	☐
n.	Effectiveness in mathematics teaching has little influence on the achievement of students with low motivation.	☐	☐	☐	☐	☐
o.	Given a choice, I would not invite the principal to evaluate my mathematics teaching.	☐	☐	☐	☐	☐
p.	When a student has difficulty understanding a mathematics concept, I am usually at a loss as to how to help the student understand it better.	☐	☐	☐	☐	☐
q.	When teaching mathematics, I usually welcome student questions.	☐	☐	☐	☐	☐

r.	Even teachers with good mathematics teaching abilities cannot help some kids learn mathematics.	☐	☐	☐	☐	☐

THANK YOU VERY MUCH FOR COMPLETING THIS SURVEY!

APPENDIX B

MULTIPLICATIVE INVERSE TEACHING INSTRUMENT

Teacher Name _____ School Name _____
Grade_____
Date _____Time used_____(minutes)

1. Imagine that your students have never formally learned the relationship between addition and subtraction *(e.g.,* $a \times b = c$; $b \times a = c$; $c \div a = b$; $c \div b = a$*).*

   - What example problem may you design to teach this relationship?
   - What kinds of representations will you use when you teach this example?
   - What kinds of questions will you ask when you teach this example?

2. Imagine that your students have never formally learned why multiplication can be used to check for division (*e.g., to check if "*$a \div b = c$*" is correct,* one can compute "$b \times c$" and see if it is equal to *"a"*)

   - What example problem may you design to help students make sense of this procedure?
   - What kinds of representations will you use when you teach this example?
   - What kinds of questions will you ask when you teach this example?

APPENDIX C

INTERVIEW PROTOCAL

Post Instruction Teacher Interview Questions

1. Do you think you accomplished your teaching goals in today's lesson? Why do you think so?

2. Were there any unexpected things that happened during your teaching of this lesson? How did you deal with it?

3.Would you teach the lesson again this way? Why or why not?

4. How do you think your sample problems worked out? Did you use them as you planned? Did you accomplish what you wanted to mathematically by using them? Please explain.

5. What do you think about the representations you or students used during this lesson? Please explain. Did using the representations communicate mathematical ideas the way you thought they would? Did you use them as you had planned? Explain.

6. What do you think about the questions you asked in today's class? Were they helpful for eliciting students' deep understanding of mathematics? Explain. Are there other questions you wished you had asked students?

7. How satisfied were you with children's reasoning during math class? How satisfied were you with any of the discussions that occurred during math class? Explain.

8. At this point what are you planning to teach next?

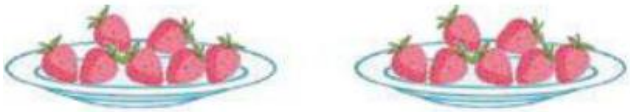
9. Is there anything else I should know about today's math lesson, your teaching, or your students?

APPENDIX D

STUDENT ASSESSMENT

Name _____ Grade _____ School _____ Teacher_____

1. Write a family of related number facts suggested by the picture.

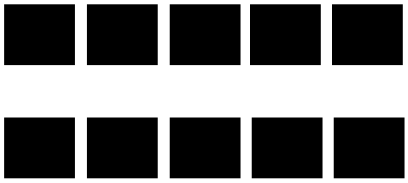


_____ × _____ = _____

_____ ÷ _____ = _____

_____ ÷ _____ = _____

2. Write a family of related number facts suggested by the picture.



_____ × _____ = _____

_____ × _____ = _____

_____ ÷ _____ = _____

_____ ÷ _____ = _____

3. (a) Hillary spent $9 on Christmas gifts for her family. Geoff spent 3 times as much money as Hillary. How much did Geoff spend? Show how you found your answer.

(b) Hillary spent $9 on Christmas gifts for her family. Geoff spent $27. How many times as much did Geoff spend as Hillary? Show how you found your answer.

(c) Hillary spent some money on Christmas gifts for her family. Geoff spent 3 times as much as Hillary. If Geoff spent $27, how much money did Hillary spend? Show how you found your answer.

4. Please write a family of related number facts using 63, 9, and 7.

Fill in the blanks.

5. Joe tried to solve 59÷8=?. His answer was 7 with remainder 2. Is this correct?

   _____

   How can you check if this is correct or not?

   _____

6. 3 × 7 = ( )
   21 ÷ 7 = ( )

   How did you get the answer for 21÷7 = ( )?

   _____

7. Use the equation 420 ÷ □ = 6 to answer the following question:

   What number should go in the □ to make this equation correct? ( )
   (A) 60 (B) 70 (C) 80 (D) 90

   How you know if your answer is correct or not?

   _____

8. There are 3 tables. Each table has 2 plates. If 48 apples are evenly put on these plates, how many apples does each plate have? Show how you found your answer.

APPENDIX E

CODING FRAMEWORK

Coding Framework for Connection-Making: Facilitating a Situation Model

Category	Subcategory	0	1	2
Instructional Tasks	Review	The task was a routine review of prior content but no connections to the targeted content was made.	An implicit connection to the targeted content was made, but not well developed.	An explicit connection to the targeted content was established and well developed.
	Worked Examples	No connections to prior or targeted content were made.	Implicit connections to the targeted content were made, but not well established or discussed. Clear opportunities to make connections are missed.	Explicit connections to the targeted content were made. No clear opportunities to make connections are missed.
	Practice Problems	Practice problems have no connection to the targeted content.	Practice problems have an implicit connection to the targeted content.	Practice problems have an explicit connection to the targeted content.
Representations	Concrete	No concrete representations (ie. manipulatives, pictures, or story situations) are used to form connections to prior or targeted content within instructional tasks.	Concrete representations are used to form connections to prior or targeted content within instructional tasks, but the connections are not well developed.	Instructional tasks are situated in rich concrete contexts (i.e. story problems) and are used to form well developed connections to prior or targeted content within instructional tasks.
	Abstract	No abstract representations (ie. numbers, symbols, or equations) are used to form connections to prior or targeted content	Abstract representations are used to form connections to prior or targeted content within instructional tasks, but the	Abstract representations (i.e. equations) are used to form well developed connections to prior or targeted content

		within instructional tasks.	connections are not well developed.	within instructional tasks.
	Sequence of Representations	No connections between concrete and abstract representations are established during the instructional tasks.	Connections between concrete and abstract representations are established during instructional tasks, but they do not always progress from concrete to abstract.	Connections between concrete and abstract representations are established during instructional tasks and they progress from concrete to abstract.
Questions	Prior	No deep questions for the purpose of making connections to prior knowledge are posed.	Some deep questions for the purpose of making connections to prior knowledge are posed, but important missed connections remain.	Deep questions for the purpose of making connections to prior knowledge are posed and no important missed connections remain.
	Current	No deep questions are posed for the purpose of making connections to targeted content (ie. between and within worked examples)	Some deep questions are posed for the purpose of making connections to targeted content, but connections remain at the surface level (ie. procedural)	Deep questions are posed for the purpose of making connections to targeted content and the connections go beyond the surface level (ie. conceptual)
	Future	No deep questions are posed for the purpose of making connections to future content.	Some deep questions are posed for the purpose of making connections to future content, but these connections are implicit.	Deep questions are posed for the purpose of making connections to future content and these connections are explicit.

APPENDIX F

JACKSON'S FIRST TEXTBOOK LESSON (*INVESTIGATIONS*)

SESSION 1.6A

Multiplicative Comparison

Math Focus Points

- Solving word problems that involve multiplicative comparison

Today's Plan		Materials
1 ACTIVITY **Introducing Multiplicative Comparison Problems**	15 MIN, CLASS	
2 ACTIVITY **Multiplicative Comparison Problems**	30 MIN, INDIVIDUALS	• *Student Activity Book*, p. 15A or **C2, Multiplicative Comparison Problems** Make copies. (as needed)
3 DISCUSSION **Comparison Problems**	15 MIN, CLASS	• Students' completed copies of *Student Activity Book*, p. 15A or C2 (from Activity 2)
4 SESSION FOLLOW-UP **Daily Practice**		• *Student Activity Book*, p. 15B or **C3, More Multiplicative Comparison Problems** Make copies. (as needed) • *Student Math Handbook*, pp. 29–34

Ten-Minute Math

Today's Number Students write expressions that equal 348. They must use multiples of 10 and only subtraction in each expression. For example: 400 – 50 – 2 = 348 and 348 = 1,000 – 500 – 100 – 40 – 12. Collect a few expressions to write on the board.

- How do you know this expression equals 348?
- How did you use multiples of 10?

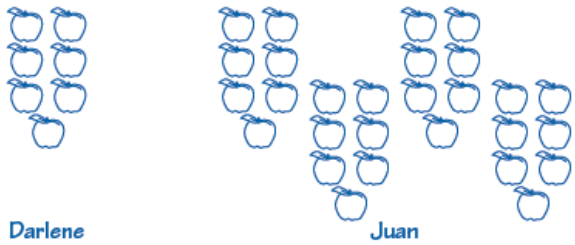
ACTIVITY

15 MIN CLASS

1 Introducing Multiplicative Comparison Problems

You have worked on multiplication problems about equal groups of things. You have also worked on multiplication problems using arrays. Here is another type of multiplication problem: *Darlene picked 7 apples. Juan picked 4 times as many apples. How many apples did he pick?*

Ask a student to draw a picture that shows what is happening in the problem.



What equation can we write that represents what is happening in this problem?

Students might say:

"Since Juan has 4 times as many apples, he has 28 apples. The equation is $7 \times 4 = 28$."

Some students may suggest drawing a picture that shows 7 apples and 4 more apples or might suggest $7 + 4$ as an equation. If this is the case ask the students to relate their picture or equation back to the problem and ask, "Did Juan pick just 4 more apples?"

Here is another problem: *Franco's daughter is 2 feet tall. Franco is 3 times as tall as his daughter. How tall is he?* In this problem, Franco's height is compared to his daughter's height. What equation can we write that represents what is happening in this problem?

Write $2 \times 3 =$ __ on the board.

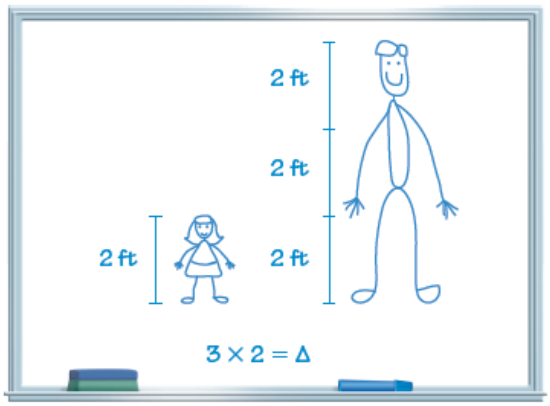
Where is the 2 in this problem? Where is the × 3 in this problem?… What is unknown?… We could use an underline to represent what is unknown, but we could also represent the unknown in other ways.

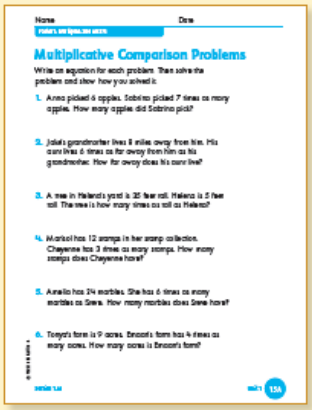
Write 2 × 3 = ? and 2 × 3 = Δ on the board.

What represents what is unknown in each of these equations?

[Jake] said the unknown is how tall Franco is. Work with a partner to draw a picture that shows this problem.

Ask one student to draw a picture that shows the problem on the board. Ask students where they see each part of the equation in the picture. Record the answer to the problem.





Multiplicative Comparison Problems

Write an equation for each problem. Then solve the problem and show how you solved it.

1. Anna picked 6 apples. Sabrina picked 7 times as many apples. How many apples did Sabrina pick?
2. Jake's grandmother lives 8 miles away from him. His cousin lives 6 times as far away from him as his grandmother. How far away does his cousin live?
3. A tree in Helena's yard is 35 feet tall. Helena is 5 feet tall. The tree is how many times as tall as Helena?
4. Marisol has 12 stamps in her stamp collection. Cheyenne has 3 times as many stamps. How many stamps does Cheyenne have?
5. Amelia has 24 marbles. She has 6 times as many marbles as Sara. How many marbles does Sara have?
6. Tonya's farm is 9 acres. Brooke's farm has 4 times as many acres. How many acres is Brooke's farm?

▲ Student Activity Book, Unit 1, p. 15A; Resource Masters, C2

2 ACTIVITY

Multiplicative Comparison Problems

30 MIN INDIVIDUALS

Students solve the problems on *Student Activity Book* page 15A or C2. For each problem, they write an equation that represents the problem, solve it, and show how they solved it.

For some problems, you might want to draw a picture to help you make sense of what is happening in the problem or to help you solve it.

ONGOING ASSESSMENT: Observing Students at Work

Students solve multiplicative comparison problems. For Problems 3 and 5, students may write either a multiplication or a division equation.

- **Do students see these as another type of multiplication problem?** Do they know they need to multiply or divide, not add or subtract, to get the correct answer?
- **How do students solve the problems?** Do they draw pictures? Do they write an equation with a symbol for the unknown quantity? Do they just know the answers? Do they use known facts?

DIFFERENTIATION: Supporting the Range of Learners

Intervention Some students may be unsure whether "4 times as many" means "a group plus 4 more groups" or "4 times the number of groups." It may help these students to think about how 2 times, or twice, as many of something would look.

ELL Students may be unfamiliar with the phrase "times as many." Write down a few simple problems with this phrase and together draw pictures for the problems. Then rearticulate that each picture shows "__ times as many."

Extension For students who easily solve these problems, give them some multiplication equations and ask them to write multiplicative comparison problems for the equations.

DISCUSSION

3 Comparison Problems

15 MIN CLASS

Math Focus Points for Discussion

- Solving multiplicative comparison problems

Write Problem 3 from *Student Activity Book* page 15A or C2 on the board.

This problem was different from some of the other problems on this sheet. Can someone draw a picture of what is happening in this problem? … What do we know in this problem? What are we trying to find out?

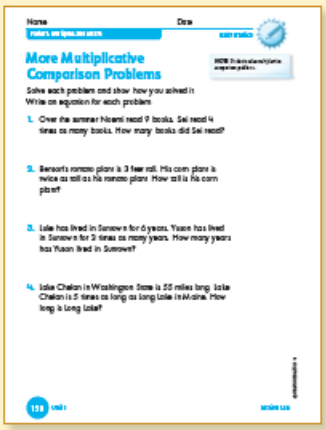
Students should understand that in this problem, the heights of Helena and the tree are both known and they are trying to find out how many times as tall the tree is.

What equation did you write for this problem? How did you solve this problem?

Students might say:

"My equation is 35 ÷ 5 = ? I know 35 divided by 5 is 7. So the tree is 7 times as tall as Helena."

"I thought: 'what times 5 is 35?' So I wrote ? × 5 = 35, and I knew it was 7."



Name Date

More Multiplicative Comparison Problems

Solve each problem and show how you solved it. Write an equation for each problem.

1. Over the summer Naomi read 9 books. Sai read 4 times as many books. How many books did Sai read?
2. Benson's ranchero plant is 3 feet tall. His corn plant is twice as tall as his ranchero plant. How tall is his corn plant?
3. Isiah has lived in Sunowe for 6 years. Yuson has lived in Sunowe for 3 times as many years. How many years has Yuson lived in Sunowe?
4. Isiah Chelan in Washington State is 55 miles long. Isiah Chelan is 5 times as long as long Isiah in Maine. How long is Long Isiah?

15B Unit 1

▲ Student Activity Book, Unit 1, p. 15B; Resource Masters, C3

4 SESSION FOLLOW-UP
Daily Practice

Daily Practice: For reinforcement of this unit's content, have students complete *Student Activity Book* page 15B or C3.

Student Math Handbook: Students and families may use *Student Math Handbook* pages 29–34 for reference and review. See pages 134–139 in the back of Unit 1.

Name Date

Factors, Multiples, and Arrays

Multiplicative Comparison Problems

Write an equation for each problem. Then solve the problem and show how you solved it.

1. Anna picked 6 apples. Sabrina picked 7 times as many apples. How many apples did Sabrina pick?

2. Jake's grandmother lives 8 miles away from him. His aunt lives 6 times as far away from him as his grandmother. How far away does his aunt live?

3. A tree in Helena's yard is 35 feet tall. Helena is 5 feet tall. The tree is how many times as tall as Helena?

4. Marisol has 12 stamps in her stamp collection. Cheyenne has 3 times as many stamps. How many stamps does Cheyenne have?

5. Amelia has 24 marbles. She has 6 times as many marbles as Steve. How many marbles does Steve have?

6. Tonya's farm is 9 acres. Emaan's farm has 4 times as many acres. How many acres is Emaan's farm?

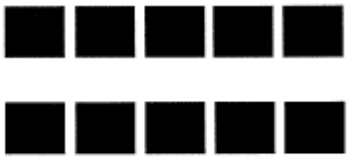
APPENDIX G

A STUDENT POST-TEST FROM AMY'S CLASSROOM

1. Write a group of related number facts suggested by the picture.

2 × 7 = 14
14 ÷ 7 = 2
14 ÷ 2 = 7

2. Write a group of related number facts suggested by the picture.



2 × 5 = 10
5 × 2 = 10
10 ÷ 5 = 2
10 ÷ 2 = 5

3. (a) Hillary spent $9 on Christmas gifts for her family. Geoff spent 3 times as much money as Hillary. How much did Geoff spend? Show how you found your answer.

3 × 9 = 27

(b) Hillary spent $9 on Christmas gifts for her family. Geoff spent $27. How many times as much did Geoff spend as Hillary? Show how you found your answer.

27 ÷ 9 = 3

(c) Hillary spent some money on Christmas gifts for her family. Geoff spent 3 times as much as Hillary. If Geoff spent $27, how much money did Hillary spend? Show how you found your answer.

3 × 9 = 27

4. Please write a group of related number facts using 63, 9, and 7.

63÷9=7 63÷7=9
9x7=63 7x9=63

Fill in the blanks.

5. Joe tried to solve 59÷8=?. His answer was 7 with a remainder of 2. Is this correct?

59÷8=7 with a remainder of

How can you check if this is correct or not?

by doing the problem

6. 3 × 7 = (21)
21 ÷ 7 = (3)

How did you get the answer for 21÷7 = ( )?

They are in the same fact family

7. Use the equation 420 ÷ □ = 6 to answer the following question:

What number should go in the □ to make this equation correct? ( )
(A) 60 (B) 70 (C) 80 (D) 90

How do you know if your answer is correct or not?

process of elimahation

8. There are 3 tables. Each table has 2 plates. If 48 apples are split equally among the plates, how many apples does each plate have? Show how you found your answer.

48÷ 8 =6

proccess of elimanation

APPENDIX H

TEXTBOOK CONNECTION-MAKING SCORES

Categories	*Investigations*	*Go Math*	Subcategories	*Investigations*	*Go Math*
			Review	0.71	1.71
Instructional Tasks	1.21	1.86	Worked Examples	1.29	1.86
			Practice Problems	1.57	1.86
			Concrete	1.43	1.71
Representations	1.34	1.81	Abstract	1.29	2.00
			Seq. of Rep.	1.14	1.71
			Prior	0.43	1.57
Deep Questions	0.67	1.43	Current	1.14	1.86
			Future	0.29	1.00
Total Average				**9.25**	**15.28**

APPENDIX I

TEACHER CONNECTION-MAKING SCORES

Subcategories	*Amy*	*Esther*	*Jackson*	*Lily*
Review	2.00	1.75	1.25	0.50
Worked Examples	2.00	1.25	2.00	1.00
Practice Problems	2.00	1.75	2.00	1.75
Concrete	2.00	1.50	2.00	1.00
Abstract	2.00	2.00	1.75	1.00
Seq. of Rep.	1.50	1.50	1.50	1.00
Prior	2.00	1.50	1.00	0
Current	2.00	1.50	2.00	1.25
Future	1.75	1.50	0.00	0
Total Average	**17.25**	**14.25**	**13.50**	**7.50**