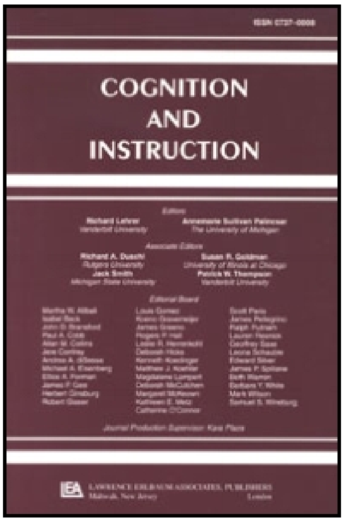

Cognition and Instruction
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t775648096

A Comparative Analysis of the Distributive Property in U.S. and Chinese Elementary Mathematics Textbooks

Meixia Ding [a];Xiaobao Li [b]
[a] University of Nebraska-Lincoln, [b] University of Houston,

Online publication date: 14 April 2010

To cite this Article Ding, Meixia andLi, Xiaobao(2010) 'A Comparative Analysis of the Distributive Property in U.S. and Chinese Elementary Mathematics Textbooks', Cognition and Instruction, 28: 2, 146 — 180
To link to this Article: DOI: 10.1080/07370001003638553
URL: http://dx.doi.org/10.1080/07370001003638553

COGNITION AND INSTRUCTION, *28*(2), 146–180, 2010

ISSN: 0737-0008 print / 1532-690X online
DOI: 10.1080/07370001003638553

Routledge
Taylor & Francis Group

A Comparative Analysis of the Distributive Property in U.S. and Chinese Elementary Mathematics Textbooks

Meixia Ding
University of Nebraska-Lincoln

Xiaobao Li
University of Houston

This study examines presentations of the distributive property (DP) in two widely used U.S. elementary text series and one main Chinese text series along three dimensions: problem contexts, typical problem types within each problem context, and variability in using the DP. In general, the two U.S. texts were found to resemble each other but to differ considerably from the Chinese text series. Both U.S. texts are computation-dominated, presenting various strategies centering on "breaking apart a factor to perform multiplication." These strategies limit the use of the DP mainly with whole numbers and in a regular direction. The underlying principle of these strategies is seldom made explicit. In contrast, the Chinese text approaches focus on the underlying principle and are well aligned with cognitive research suggestions. Multiple-step word problems with particular structures are used in a systematic and hierarchic manner across grades to help students learn and transfer the DP. The Chinese texts also tend to ask students to "compute in convenient ways" involving various numbers (e.g., whole numbers, decimals, fractions, and percents) and using the DP in both regular and opposite directions. The introduction of repeated variables is a timely application of the DP, which provides an entry to algebra (e.g., expressions and equations with repeated variables). The Chinese approaches (e.g., contextual interferences, spaced practice, and encoding variability) suggest alternative insights into developing U.S. students' understanding of the DP and readying them for algebra.

The distributive property (DP), along with the commutative and associative properties, has been recognized as a critical foundation for school mathematics (National Mathematics Advisory Panel, 2008). The combination of these properties allows powerful flexibility for students doing arithmetic (National Research Council [NRC], 2001). These properties are also viewed as three of the fundamentals necessary for working with equations (Bruner, 1960/1977). Recently, other researchers (e.g., Carpenter, Franke, & Levi, 2003; Schifter, Monk, Russell, & Bastable, 2008) have emphasized that these properties are at the heart of early algebra because they provide a solid foundation for exploration of generalizations about number and operation, and for justification of such generalizations. With regard to the DP, the *Curriculum Focal Points* (National Council of

Correspondence should be addressed to Dr. Meixia Ding, 214A Henzlik Hall, UNL, Lincoln, NE, 68588. E-mail: mding2@unl.edu

Teachers of Mathematics [NCTM], 2006) points out that it needs to be taught in the third grade and particularly emphasized in fourth grade. It also needs to be used by students, along with other properties, to support the learning of division in fifth grade and solving equations in sixth grade.

Although the significance of these fundamental ideas has been recognized for a long time, researchers have just begun to discuss ways to teach these ideas in the elementary grades. It remains largely unknown how these properties can be systematically and effectively introduced and developed across the elementary grades, and how knowledge of the principles can be developed from arithmetic contexts thus preparing students for later algebraic learning. In this study, for the purpose of in-depth research, we will only focus on the DP. Our study aims to contribute to the current literature, through a cross-national curriculum examination, by presenting alternative approaches and contexts for developing the concept of the DP at the elementary level.

CURRENT RESEARCH PROGRESS REGARDING THE USE OF THE DP IN ELEMENTARY GRADES

The NCTM (2000) Standards suggest students in grades 3–5 should be able to "identify such properties as commutativity, associativity, and distributivity, and use them to compute with whole numbers" (p. 158). Schifter et al. (2008) provided detailed classroom episodes illustrating how elementary students could be engaged with these basic properties. With regard to the DP, Schifter et al. presented two episodes from third grade classrooms. When calculating 12×6, students broke either 6 or 12 in different ways, resulting in various calculation approaches such as $12 \times 6 = 12 \times (3 + 3) = (12 \times 3) + (12 \times 3)$ and $12 \times 6 = (10 + 2) \times 6 = (10 \times 6) + (2 \times 6)$. Based on these classroom examples, Schifter et al. concluded, "We might define 'understanding distributivity' for a third grader to mean breaking apart one factor to perform multiplication and successfully explaining why it works" (p. 439).

The aforementioned cases of "breaking apart one factor to perform multiplication" are applying the distributive property, $a(b + c) = ab + ac$. Two steps are involved: (1) breaking apart a factor into $b + c$, and (2) expanding $a(b + c)$ into $ab + ac$. In fact, this approach has been consistently advocated by existing research (e.g., Carpenter et al., 2003; Mason, 2008; NCTM, 2000; Schifter, 1999) and has become a dominant way to teach and use the DP. Carpenter et al. (2003) pointed out that students could learn multiplication facts by relating the difficult or complicated ones to the facts they already know by using the DP (e.g., $3 \times 8 = 2 \times 8 + 8$, and $9 \times 7 = 10 \times 7 - 10$). In the cases of multiplication with larger numbers, it was suggested that the DP embedded in the algorithms be brought to the forefront. Examples given by researchers were $8 \times 14 = 8 \times 10 + 8 \times 4$ (NCTM, 2000), $18 \times 12 = 18 \times 10 + 18 \times 2$ (Schifter, 1999), and $645 \times 123 = (600 \times 123) + (40 \times 123) + (5 \times 123)$ (Ball, 1988). The area model (breaking up an array) was suggested for modeling the distributive property (NCTM, 2000) (see Figure 1).

When breaking apart both factors to perform multiplication, the DP could be used multiple times to produce the partial products. For example, the DP could be used three times for 48×37 to generate four partial products $(40 \times 30) + (40 \times 7) + (8 \times 30) + (8 \times 7)$ (Carpenter et al., 2003, p. 115). Without understanding the DP embedded in the above multiplication, students may produce $(40 + 8) \times (30 + 7) = (40 \times 30) + (8 \times 7)$, which may cause further mistakes in binomials such as $(a + 7) \times (a + 4) = a^2 + 28$ (Carpenter et al., 2003, NCTM, 2000, Schifter,

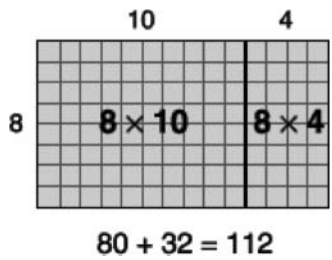

FIGURE 1 An array model showing the distributive property of multiplication in NCTM (2000).

1999). Instead, if students understand the above process, "they will be able to use that knowledge to solve algebra problems like $(a + b) \times (c + d) = ac + ad + bc + bd$" (Baek, 2008, p. 152).

Existing research supports the NCTM (2000) recommendation and has contributed to our understanding of how to lay the foundations for students' future algebraic learning. However, this research is limited to a single context: multiplication with whole numbers. In this context, the most common use of the DP, regardless of how many digits the numbers contain and how many times this property is used, is "breaking apart one factor to perform multiplication," thus limiting the use of the DP only to the regular direction $a(b + c) = ab + ac$. Using the DP in the opposite direction, $ab + ac = a(b + c)$ is, however, rarely explored.

A few researchers (e.g., Carpenter et al., 2003) are cognizant of the importance of students' ability to use the DP in an opposite direction, $ab + ac = a(b + c)$ [e.g., $5 \times 10 + 3 \times 10 = (5 + 3) \times 10$]. According to Carpenter et al., this use of the DP is essential to later algebraic learning with repeated variables such as $9a + 3a = (9 + 3)a = 12a$. Lacking a good sense of the DP, many students tend to make mistakes such as $7d + 5e = 12de$ (Carpenter et al., 2003). The significance of the use of the DP in an opposite direction, however, seems not fully acknowledged by existing research.

An opportunity for students to explore the DP more deeply is briefly encouraged by the NRC (2001) in the perimeter problem:

> It would be helpful, for example, if the curriculum included perimeter problems in which students were asked to calculate the perimeter of a 7-by-4 rectangle in three ways that yield equivalent expressions: $2(7 + 4)$, $(2 \times 7) + (2 \times 4)$, and $7 + 7 + 4 + 4$. Such situations are ideal for initiating discussions of the equivalence of arithmetic expressions and of the properties underlying that equivalence. (p. 271)

Solving the perimeter problem in multiple ways to initiate a discussion of the DP is different from the common approach of "breaking apart a factor to multiply" because the perimeter problem does not limit students' thinking to one direction [e.g., $2(7 + 4) = (2 \times 7) + (2 \times 4)$ or $(2 \times 7) + (2 \times 4) = 2(7 + 4)$]. In addition, the perimeter problem provides concrete contextual support for students to make sense of the DP because the expressions on both sides of the equal sign are computing the perimeters for the same rectangle. However, as the NRC (2001) points out, such opportunities are quite rare in current elementary school curricula.

Understanding fundamental principles like the DP is critical for any domain (Goldstone & Son, 2005). However, it is notoriously difficult for students to learn these principles, in part because students' interpretation of the principles is usually context-dependent (Colhoun, Gentner, &

Loewenstein, 2005), and, as has been indicated, the widely acknowledged context for using the DP is limited to whole number multiplication with a regular direction. Significant alternatives are rarely discussed. In order to explore more effective contexts and tasks for learning the DP, we first seek general guidance from cognitive findings that could inform the teaching and learning of abstract principles in classroom settings.

COGNITIVE FINDINGS REGARDING THE LEARNING OF PRINCIPLES

Our study draws on the research of Pashler et al. (2007), which provides evidence-based instructional recommendations to help children learn about principles and transfer that learning. The relevant guidance for our study includes recommendations to space learning over time, use worked examples (solutions provided), connect concrete and abstract representations, and ask deep questions to build self-explanations.

Space Learning Over Time

Numerous laboratory experiments have demonstrated that students can markedly retain information when they are re-exposed to concepts over time (Cepeda, Pashler, Vul, Wixted, & Rohrer, 2006). Based on this finding, important curriculum content should be reviewed at least several weeks or months after it was initially taught. Such delayed reviews could provide spacing for learning. According to Pashler et al. (2007), one of the limitations of current research on this issue is that the majority of studies are assessments of students' acquisition of isolated bits of information such as facts and definitions because these are easily measured. Building structural understanding of principles like the DP is rarely studied.

Worked Examples

Worked examples, as cases of principles, can facilitate students' understanding because the comparison between principles and cases can effectively constrain the ways that students interpret those abstract principles (Colhoun et al., 2005). Additionally, comparisons between examples may lead to better encoding of principles by making the implicit structures underlying these examples explicit (Gentner, Lowenstain, & Thompson, 2003, 2004). Some researchers (e.g., Rittle-Johnson & Star, 2007) have claimed that comparing different solutions within one worked example could facilitate student learning. However, as Colhoun et al. (2005, p. 1663) warn, not all examples are equally effective. "A clear example can enable an abstract principle to be better understood (but a poor example can degrade learning from a clear principle)". Worked examples were found to be more effective when they were interleaved with practice problems (Renkl, 2002; Renkl, Atkinson, & Große, 2004; Zhu & Simon, 1987). Fading examples into problems by providing early steps in a problem but requiring more and more later steps from students was found beneficial (e.g., Renkl et al., 2004). It has also been suggested that examples and problems with greater variability could better enable students to encode the key principles (Renkl, Atkinson, Maier, & Staley, 2002).

Concrete and Abstract Representations

Both concrete and abstract representations should be used and connected to help students understand principles (Pashler et al., 2007). For example, graphics can be added to text presentations to enhance student learning (Mayer, 2001; Moreno & Mayer, 1999; Sweller, 1999). Word problems, as concrete events, can provide sources of meanings for understanding abstract formalisms (e.g., Resnick, Cauzinille-Marmeche, & Mathieu, 1987; Uttal, Liu, & Deloache, 1999). However, concrete representations might support initial learning but not the transfer of that knowledge to novel contexts (Resnick & Omanson, 1987). This is because perceptually rich information or concrete word problem situations may focus students' attention to non-relevant or surface information thus distracting students from the underlying ideas (Goldstone & Sakamoto, 2003; Kaminiski, Sloutsky, & Heckler, 2006). Therefore, "concreteness fading" is suggested for promoting both learning and transfer (Goldstone & Sakamoto, 2003; Goldstone & Son, 2005). Sometimes, a more abstract or schematic picture (e.g., the number line diagram widely used in many high-achieving countries, see Cai & Moyer, 2008; Murata, 2008) may be more effective to illustrate deep structures (Pashier et al., 2007). Regarding word problem solving, a recent study (Koedinger, Alibali, & Nathan, 2008) furthered our understanding of the effects of concrete and abstract representations by introducing a new factor: level of problem complexity. This study found that abstract representations were more powerful for solving complex problems such as double-reference algebraic problems while concrete representations were more helpful for solving simple problems with less computational load.

Self-Explanations and Deep Questions

Students can effectively learn new concepts through self-explanations (Chi, 2000; Chi, de Leeuw, Chiu, & LaVancher, 1994) that may help them clarify the conditions of a conclusion (VanLehn, Jones, & Chi, 1992), map the new knowledge onto existing knowledge, and integrate procedural and declarative knowledge (Chi et al., 1994). NCTM (2000) strongly advocated students to explain and justify their mathematical thinking. However, students themselves usually have little motivation or ability to generate high-quality explanations without external help. Deep questions, as recommended by Pashler et al. (2007), may prompt students to look for underlying principles and causal relationships (Craig, Sullins, Witherspoon, & Gholson, 2006).

With these cognitive findings in mind, we are interested in knowing the types of effective experiences that students should be offered to better understand the DP. For example, what are effective ways for spacing the learning of the DP with delayed reviews over time? What types of concrete and abstract representations and contexts should be provided? How can word problems be well designed for learning and transferring this property? What variations in problem contexts are useful? And what kinds of deep questions can be asked to support students' understanding of the DP?

EXPLORATIONS OF ALTERNATIVES THROUGH CROSS-NATIONAL CURRICULUM COMPARISON

One of the important ways to explore alternative learning contexts is through a cross-national curriculum comparison, examining how the curriculum resources in high-achieving countries provide students opportunities to learn these fundamental concepts. There are two reasons for doing so. First, there is little disagreement that curricula have a large impact on learning (Ball & Cohen 1996; Porter, 1989) principally through textbook presentation, including problem contexts, types of problems, and sequences of presenting the concepts (Hamann, & Ashcraft, 1986; Li, Ding, Capraro & Capraro, 2008; McNeil et al., 2006; Stigler, Fuson, Ham, & Kim 1986).

Second, existing international comparative studies have already identified important differences in textbook presentation between the United States and high-achieving countries, potentially promoting more sophisticated mathematical understanding in the elementary grades. For example, with regard to the notorious fear of algebra by U.S. students, Cai, Lew, Morris, Moyer, Ng & Schmittau (2005) compared curriculum resources from five countries and found that U.S. textbooks focus more on students' intuitive algebraic sense, and postpone formal opportunities to learn. These researchers also gleaned ideas from Chinese and Singaporean curricula (e.g., using pictorial equations, using both arithmetic and algebraic approaches to solve problems), which encourage algebraic thinking in early grades (Cai et al., 2005; Cai & Moyer, 2008). Murata (2008) investigated the presentation practices and curriculum approaches regarding addition and subtraction between U.S. and Japanese elementary texts. This researcher found that U.S. texts are more likely to use non-contextual (numerical) problems while the Japanese series mainly uses contextual (story) problems with a particular emphasis on using tape diagrams (similar to Cai & Moyer's "pictorial equations"). Similarly, Stigler et al. (1986) analyzed addition and subtraction word problems in U.S. and Soviet elementary texts (grades 1–3). Striking differences in problem distribution, frequency of occurrence, and level of difficulty were revealed. It is impressive that the Soviet texts presented many more two-step word problems, which are considered difficult in America, while the U.S. textbooks mainly provided one-step problems that are more easily solved. Moreover, the Soviet texts presented two-step (and one-step) equations much earlier in the curriculum. These variations in textbook presentation are viewed by Stigler et al. as important factors affecting children's capability in solving word problems.

These cross-national textbook comparisons provide alternatives for teaching and learning key mathematical ideas. Given the importance of the DP and a lack of research on this property, it is our expectation that the cross-cultural curriculum comparison will bring insights into developing students' understanding of the DP, thus better preparing students for algebraic learning.

PURPOSE

In this study, we explore the differences in textbook presentation of the distributive property between U.S. and Chinese elementary textbook series. We selected Chinese textbooks because China is one of the high-achieving countries in mathematics (Stevenson & Stigler, 1992) demonstrated by international assessments where Chinese students have consistently outperformed their U.S. counterparts (Trends in International Mathematics and Science Study [TIMSS], 1999, 2003, 2007). Empirical studies that directly compare early algebraic learning also demonstrate that Chinese elementary students have a better understanding of the concept of equivalence (Li et al., 2008) and tend to solve word problems in abstract and algebraic ways (Cai, 2004). Chinese elementary textbooks have many features that may aid student learning of early algebra (e.g., Cai & Moyer, 2008; Li et al., 2008). In fact, for decades Chinese text presentation has been drawing on successful classroom learning and teaching experiences (personal communications with the Chinese textbook editors). In addition, Chinese elementary teachers who demonstrated *profound understanding of fundamental mathematics* including the use of the DP in Ma's (1999) study also admitted that one of the important sources of their professional knowledge was the Chinese textbooks.

For this study, we particularly examine the text presentation regarding the DP along three dimensions: (a) problem contexts, (b) typical types of problems within each problem context, and (c) variability in using the DP. With regard to problem contexts, we want to know whether a textbook mainly uses computation or word problems, or some other problem contexts. The second dimension, "typical types of problems," is a closer examination into each problem context. For example, concerning the word problem context, does it mainly include one-step or two-step problems? How are these word problem structures different from each other? It is worthwhile to explore this aspect because we may expect that not all problems are equally effective in supporting students' understanding of key principles and concepts (Colhoun et al., 2005; McNeil & Alibali, 2005). The third dimension, "variability in using the DP," includes two sub-questions: What are the directions (e.g., regular, opposite) and what are the types of numbers (e.g., whole numbers, fractions) that are used in the problems involving the DP? As previously noted, examples/problems with greater variability can enable students to better encode knowledge of principles (Renkl et al., 2002).

Since "it may take time for students to develop this understanding [of the distributive property], but it can pay big dividends in the long run by making learning meaningful" (Carpenter et al., 2003, p. 39), we investigate the aforementioned three aspects across grade levels with regard to how the related curriculum tasks are presented before, during, and after the DP is formally introduced. This cross-grade level examination of text presentation allows us to explore possible effective ways to space the learning of the DP over time (Carpenter, Pashler, & Cepeda, & Alvarez, 2007). In addition, the cross-grade level examination provides information regarding how the understanding of the DP from arithmetic may possibly contribute to later algebraic learning, thus paying its "big dividends" in the upper elementary grades and beyond.

METHOD

Selection of Textbooks

The Chinese textbook series from *Jiang Su Educational Press* (JSEP) (Su & Wang, 2005) and the U.S. textbook series from *Houghton Mifflin* (HM) (Greenes et al., 2005) and from *Scott Foresman–Addison Wesley* (SF-AW) (Charles et al., 2004) were analyzed for comparison of the manner in which the distributive property is developed. The Chinese textbook JSEP is one of the three main Chinese curricula, with little qualitative difference from the other two textbook series (Li et al., 2008). Both U.S. textbook series (HM and SF-AW) are also widely sold and widely used (Horizon Research, 2002; Murata, 2008; Stigler et al., 1986). Each textbook page in grades 1–6 for the Chinese curricula and grades K–6 for the U.S. curricula were first examined for the coding of instances of the DP.

The teachers' instructional manuals that accompanied the textbooks were also examined. For the U.S. textbook series, we examined the teacher guides; for the Chinese textbook series, we examined the *Curriculum Analysis* written by the original textbook authors (e.g., Sheng, 2008). While the U.S. teacher guides present brief information on the learning objectives and teaching steps, the Chinese manuals provide detailed information on the design/purposes/rationales of each unit/lesson and suggest teaching approaches to the content, including the related practice problems. As will be exemplified below, these manuals were valuable resources to resolve possible discrepancies in the process of coding of instances and data analysis.

Coding of the Instances

In general, a content analysis method was used to code and analyze the curriculum materials (NRC, 2004). We coded all of the problems in the student textbooks, including the worked examples (solutions provided) and practice problems (no solutions provided). A worked example would be considered as an instance if its solution either (a) explicitly uses the DP, or (b) implicitly involves the DP with a clear possibility to develop student intuitive sense of the DP. For example, regarding the worked example, "If 17 farms each had 25 goats, how many goats would all the farms have together?" (US_SF-AW, grade 4, p. 332), the textbook presents an array model along with three steps to clearly show the partial products for 17×25 (step 1: $7 \times 25 = 175$; step 2: $10 \times 25 = 250$; and step 3: $175 + 250 = 425$). This process is relatively consistent with the NCTM (2000) recommendation about the teaching of the DP. As a result, this worked example was coded as an instance. A practice problem would be coded as an instance if the text explicitly asks students to use the DP to solve problems. When no clear requirement for using the DP was visible, our decisions on coding practice problems were then referred back to the text presentation of corresponding worked examples. In addition, there were two special cases regarding coding instances worthy of attention (mainly with Chinese textbooks): (a) A problem may include two or more sub-questions, solving each of which involves the DP. For such a case, we considered each sub-question as an instance; and (b) Two problems (e.g., "$2 \times 5 + 5$" and "3×5") are listed as a pair vertically, and taking them together could illustrate the DP. For such a case, we considered the pair as one instance.

The Process of Developing and Refining Instance Coding. The first author coded all of the instances from the three text series using a coding framework agreed on by both of the researchers. The recorded information included page numbers, chapter titles, lesson titles, example problems, the frequencies of the same type of problem occurring on the same page, and the coders' comments. Most instances could be easily identified and categorized. However, difficulties did occur when determining whether some worked examples and practice problems should be coded as instances or not. These challenges were resolved through discussions with the second author. For example, the HM third-grade textbook introduces two strategies to calculate 5×6. The first strategy is called "uses doubles," that is, $5 \times 6 = 5 \times 3 + 5 \times 3$. The second strategy is termed "uses the distributive property," that is, $5 \times 6 = 5 \times (4 + 2) = 5 \times 4 + 5 \times 2$. The textbook clearly states that the doubles strategy is different from the use of the DP. The teacher guide displays a similar view. This raises a challenge for determining whether the doubles strategy problems should be coded or not. After discussion, the doubles strategy problems were coded because both authors agreed that since the doubles strategy is indeed a special case of the DP, it could at least provide students with an intuitive sense of this property.

The above difficulty with the doubles strategy occurred in coding the HM text, but not the other two text series.

When coding the SF-AW series, the challenges were associated with those multiplication problems where the standard computation algorithm was a focus. For example, a lesson entitled "Multiplying two-digit and one-digit numbers" presents the standard algorithm with an example of 3×26 (Charles et al., 2004, p. 270). Two steps are demonstrated: (a) $3 \times 6 = 18$ ones. Regroup 18 ones as 1 ten and 8 ones, write down 8; and (b) 3×2 tens $= 6$ tens, 6 tens $+ 1$ ten $= 7$ tens, write down 7. This standard algorithm does involve the DP. However, in comparison with previous lesson where "partial products" is a focus (e.g., 5×15 is solved through three steps: $5 \times 10 = 50$, $5 \times 5 = 25$, $50 + 25 = 75$, p. 264), the DP in this standard algorithm is much more hidden and much less useful to develop students' intuitive sense of the DP. After discussion, we decided to count as instances only those multiplication problems that show partial products and not the ones that teach standard algorithms. Since the SF-AW text series heavily emphasizes the partial products, we obtained a high frequency of instances. When such a "rule" of coding instances was made, the first author went back to re-code the HM textbook. As a result, fourteen missing instances were added.

The difficulty in determining instances in the Chinese textbooks was mainly related to the word problems in the sixth grade. Since the DP had been formally introduced in previous grades, it is no longer a focus at this time. However, the application of this property appears throughout the sixth-grade texts. Two types of problems can be clearly coded as instances: (a) the problem requires two different solutions the connection between which illustrates the DP, and (b) the problem is expected to be solved by using equations where repeated variables are involved (e.g., $x + 1.4x = 48$). Challenges for coding occurred when the word problems do not explicitly require two solutions or solutions with equations. After discussion, we agreed to code a problem as an instance if either of the following conditions exists: (a) the problem is presented in a context that asks for an equation response involving repeated variables; or (b) the teacher manual encourages the teacher to provide a second solution, or similar problems in previous contexts request two solutions. The three worked examples presented in Table 1 provide a sense of how we resolved our difficulty in coding the Chinese text.

There was no difficulty in coding both Examples 1 and 2 (see Table 1). In Example 1, the textbook first proposes two ways of thinking: $2/5 \times 18 + 3/5 \times 18$ and $(2/5 + 3/5) \times 18$.[1] It then reveals the DP through a comparison of the two expressions, which both contain fractions. In Example 2, the textbook first provides one way of thinking and then asks: "What is another way to solve it?" As a result, both solutions [$45 - 45 \times 4/9$ and $45 \times (1 - 4/9)$] can be expected. However, with regard to Example 3, the textbook only presents one way of thinking, $24 + 24 \times 1/4$. The second solution $24 \times (1 + 1/4)$ is not suggested. At first, we were uncertain whether Example 3 should be coded as an instance. We solved this difficulty by consulting two sources: the previous contexts and the teacher manual. Since both worked examples (Examples 1 and 2) in previous lessons in this chapter consistently present two solutions involving the DP, it is reasonable to expect such a way of thinking can be transferred to Example 3. In addition, the *Curriculum Analysis* points out that the second method had been introduced in Example 3 in the

[1]The meaning of multiplication is represented differently in Chinese texts. Thus, the total length of 18 knots with each 3/5 meter long is represented as $3/5 \times 18$ rather than $18 \times 3/5$ (as is common in the United States). This type of thinking is consistent across word problems in Chinese texts.

TABLE 1
An Example of Solving the Difficulty in Coding Instances With the Chinese Texts

Mixed Operations of Fractions (Chinese sixth grade text, vol. 11)		
Example 1 (p. 80)	*Example 2 (p. 83)*	*Example 3 (p. 84)*
It takes 2/5 meters of colorful rope to make one type of Chinese knot, while 3/5 meters for the other type. How many meters of colorful rope do we need if we make 18 Chinese knots of each type?	45 six graders in Lingnan Elementary School participated in the school sports meeting. Among these students, 5/9 are boys, how many of the students are girls?	Linyang Elementary School had 24 classes last year. There are 1/4 more classes this year than last year. How many classes does this school have this year?

traditional texts but not the current one because of the consideration of students' cognitive load. However, it suggests teachers should encourage students to solve this problem with the second method. With the above evidence, Example 3 was coded as an instance.

Reliability Checking for Instance Coding. All of the textbook pages were re-coded by the first author three months later. A few missed instances were added. After this, the second author, who had a shared understanding of the coding categories, examined each page of Chinese and U.S. second-, fourth-, and sixth-grade curricula using the same coding framework. We compared our coding results grade by grade, and marked and counted the problems that were coded by one coder but not the other. We then added the frequencies of common and non-common instances as our total. The reliability was calculated using "common instances/total instances." The reliabilities for coding the Chinese texts for each grade were 92.3%, 95.2%, and 93.4%, respectively; for coding the three U.S. HM texts the reliabilities were 100%, 95.1%, and 94.6%; and for coding the U.S. SF-AW texts, they were 100%, 90.7%, and 91.2%. The discrepancies between the two authors (non-common instances) were discussed and resolved before doing data analysis.

Data Analysis

After all of the instances were identified, we first classified them into one of three contexts: word problem, computation, or definition. The word problem and computation contexts are two categories frequently suggested and used by previous studies (e.g., Murata, 2008; Schifter, 1999). The definition problems in this study are those that focus on "what the DP is." It is not appropriate to classify these problems into either word problem or computation context. For example, some problems ask students to recall the vocabulary "Distributive Property" [e.g., "An example of the _____ is $4 \times (5 + 6) = (4 \times 5) + (4 \times 6)$"] and others demand an application of the DP without actual computation [e.g., $(49 + 25) \times 8 = \square \text{ O } \square \text{ O } \square\text{O}\square$]. After the instances were classified, we indentified the period of their appearance as *before, during,* or *after* the formal introduction of the DP.

Next, we analyzed typical problem types within each of the aforementioned three problem contexts. After a comparison and combination of the problem types across countries, we came up with 3 types of word problems (5 sub-types), 6 types of computation problems (13 sub-types), and

3 types of definition problems. Since the word problem context distinguishes U.S. and Chinese texts, we particularly reexamined all of these problems, type by type, with a focus on how a problem structure is systematically used and evolved across grades to develop/apply the idea of the distributive property.

Finally, we examined the "variability" in terms of the direction of using the DP and the types of numbers used (simply called "numbers used") for each recorded instance. The analysis of "numbers used" enables us to see clearly how the DP is developed over varied mathematical topics and the ways it integrates arithmetic and algebra. We examined the numbers used in all three contexts and classified types of numbers used into arithmetic categories (whole number, decimal, fraction, percent) and algebraic ones (expression, equation involving whole numbers, decimals, fractions, and percents, respectively). In addition, we examined the directions of using the DP in the computation and definition contexts. We did not include the word problem contexts because of the uncertainty of the solutions that might be used. We classified the directions into three categories: the regular direction ($\Rightarrow$), the opposite direction ($\Leftarrow$), and the dual direction ($\Leftrightarrow$).

Reliability Checking for Data Analysis. Each of the aforementioned classifications was first completed by the first author. After the initial analysis was done, the second author, who had shared in producing the categories along with the typical examples, randomly selected 20% of the instances from each text to do reliability checking. The procedure of calculating reliability for data analysis was similar to the one used in checking reliability for coding instances. The agreement for each classification was over 90%. A few disagreements (mainly about the directions) were resolved through discussion. For example, with regard to "*using known facts to find unknown facts*" in the SF-AW series (e.g., How can you use $2 \times 9 = 18$ to find 3×9?), the student text often teaches "breaking apart a fact" (e.g., breaking 3 into 2+1) which is the regular direction. The teacher guide suggests "$(2 \times 9) + (1 \times 9)$ is 3×9," which indicates an opposite direction. The first author coded it as a dual direction while the second author coded it as a regular direction. Through discussion, both agreed to classify the direction in such problems as "dual direction." After all the disagreements were resolved, we calculated the frequencies and percentages for the final results.

RESULTS

We identified the instances from the Chinese texts for grades 2–6 and both U.S. texts for grades 3–6. Each Chinese grade has two volumes of texts, each of which includes about 110 pages (dimensions 4×6 in.). In contrast, each U.S. grade has one textbook with about 750 pages (dimensions 8×11 in.). The total instances in each of the three text series vary considerably. The SF-AW has the highest frequency while HM has the lowest one ($n_{\text{China}} = 319$, $n_{\text{US\_HM}} = 115$, $n_{\text{US\_SF-AW}} = 552$). Despite the frequency differences, both U.S. text series resemble each other while the Chinese text series is qualitatively different from the U.S. texts in all areas including problem contexts, types of problems used in each context, and the variability of using the DP.

Problem Contexts Involving the DP Across Grades

All three text series include the three problem contexts: word problem, computation, and definition. A detailed tabulation of the frequency of instances for problem context, broken down by

TABLE 2
Problem Contexts Regarding the Distributive Property in the U.S. and Chinese Texts

			Word Problem		*Computation*		*Definition*	
Textbook	*Period*	*Grade*	*Freq*	*Percent*	*Freq*	*Percent*	*Freq*	*Percent*
China_ JSEP ($n = 319$)	Before	2nd	0	0	12	3.8	0	0
		3rd	21	6.6	8	2.5	0	0
		4th	2	0.6	5	1.6	0	0
	During	4th	12	3.8	43	13.5	12	3.8
	After	4th	18	5.6	12	3.8	1	0.3
		5th	0	0	9	2.8	0	0
		6th	123	38.6	40	12.5	1	0.3
	Total		**176**	**55.2**	**129**	**40.4**	**14**	**4.4**
US_HM ($n = 115$)	Before	3rd	2	1.7	6	5.2	0	0
	During	3rd	0	0	8	7.0	0	0
	After	4th	6	5.2	23	20	11	9.6
		5th	3	2.6	33	28.7	2	1.7
		6th	1	0.9	7	6.1	13	11.3
	Total		**12**	**10.4**	**77**	**67.0**	**26**	**22.6**
US_ SF-AW ($n = 552$)	Before	3rd	9	1.6	150	27.2	0	0
	During	4th	1	0.2	36	6.5	0	0
	After	4th	22	4	211	38.2	0	0
		5th	2	0.4	54	9.8	4	0.7
		6th	6	1	40	7.2	17	3.1
	Total		**40**	**7.2**	**491**	**88.9**	**21**	**3.8**

grade level and text series, is presented in Table 2. We calculated the corresponding percentage within each text (e.g., we used $n = 319$, or $n = 115$, or $n = 552$ as a total to calculate the percentage for the Chinese, the HM, or the SF-AW text, respectively). An examination of the table reveals several interesting findings.

First, both U.S. texts are computation-dominated with a rare use of word problems (HM: 67.0% vs. 10.4%; SF-AW: 88.9% vs. 7.2%). In contrast, the Chinese text utilizes both computation and word problem contexts (40.4% vs. 55.2%). In fact, the heavy utilization of word problems is a distinguishable factor that separates the Chinese text from the U.S. texts ($n_{\text{China}} = 176$, $n_{\text{HM}} = 12$, $n_{\text{SF-AW}} = 40$).

Second, the Chinese text begins to develop students' intuitive sense of the DP much earlier than U.S. texts, and with more varied contexts. Because the Chinese texts introduce multiplication at the beginning of second grade, the second-grade text starts to offer relevant problems involving the DP, while both of the U.S. texts start to do so at the third grade. In addition, Chinese texts provide a certain number of problems, including both word problems ($n = 23$) and computation problems ($n = 25$) to develop students' intuitive sense of the DP. In contrast, HM only provides 8 informal instances before the DP is introduced. Although SF-AW provides many more opportunities, most of these problems are computation only ($n_{\text{word problems}} = 9$, $n_{\text{computation}} = 150$).

Third, the Chinese text demonstrates a clear focus on the DP when it is first formally introduced, while neither U.S. text has such a focus. The Chinese fourth-grade text (vol. 8) has a specific chapter titled *Operational Properties* (only about the DP) including three lessons incorporating 67 problems ($n_{\text{word problems}} = 12$, $n_{\text{computation}} = 43$, $n_{\text{definition}} = 12$, accounting for 21.1% of the total instances). In contrast, HM first formally introduces the DP in the third grade, with only

9 computation problems (7% of the total). The SF-AW first introduces the DP in the fourth grade. Although there are about 37 problems (mainly computation) involving this property, these problems only account for 6.7% of the total instances in this text.

Finally, after the DP is formally introduced, the fourth-grade Chinese texts immediately ask students to apply the DP with repeated variables and problem solving ($n_{\text{word problems}} = 18$, $n_{\text{computation}} = 12$). A heavy use of the DP appears in the sixth grade, including solving equations and using equations to solve word problems ($n_{\text{word problems}} = 123$, $n_{\text{computation}} = 40$). In contrast, both U.S. texts mainly applied the DP to computations. Both U.S. texts also tend to use the definition context (or revisit the definition) more frequently than the Chinese texts during this period.

Problem Types Involving the DP Within the Word Problem Context

Within the word problem context, we identified three types of word problems based on the cross-national analysis: "one-step multiplication," "equal-groups involved," and "multiplicative-comparison involved." The latter two types of problems involve multiple steps (mainly two steps). One of the steps involves a common multiplication structure, either "equal-groups" or "multiplicative comparison" (Carpenter, Fennema, Franke, Levi, & Emspon, 1999; Greer, 1992). The multiplication steps in such a problem share either the same number of groups, or the same group size, or the same reference number, allowing these problems to be solved in two ways [either "$ab \pm ac$" or "$a(b \pm c)$"], which illustrates the use of the DP. With regard to the "equal-groups involved" problems, the Chinese textbook has two distinguishable types of problems across grades, "perimeter/area" and "distance." Therefore, we specified these two subtypes and referred to the rest as "common factor" problems. Table 3 provides examples of each of the above problem types. Table 4 summarizes the frequency of each type of word problem involving the DP in each text, broken down by period (before, during, or after the formal introduction of the DP).

The Chinese texts mainly use multiple steps problems including 120 "multiplicative-comparison involved" and 54 "equal-groups involved" problems. The one-step problems are rarely used for teaching the DP ($n = 2$). The same problem structures are used across periods. In contrast, both U.S. texts use one-step problems more often ($n = 10$ for each text) than the Chinese texts. While the one-step problem is the main problem type for the HM text, the SF-AW has more multiple-step problems (mainly "common factor") than the HM ($n_{\text{SF-AW}} = 30$; $n_{\text{HM}} = 2$).

Textbook Approaches and Goals of Using the Word Problems. Even within the same type of word problems, the textbook approaches and the goals appear to be different across countries. In general, both U.S. texts ignore the contextual support of word problems in teaching the DP, while the Chinese texts utilize the word problem situation to help students make sense of the DP. For example, with regard to "one-step multiplication," both U.S. texts use the word problems essentially as computation problems but under the pretext of word problems. With regard to the given examples of "one-step multiplication" in Table 3 *[(a weekly salary of* \$986 *with 52 weeks in a year (HM); 17 farms each had 25 goats (SF-AW)],* the HM text directly

TABLE 3
Examples of Problems Types Within Word Problem Context

Word Problem Type	*Example*
One-step multiplication	*(US_HM, grade 5, p. 76)* One train engineer makes a weekly salary of $986. How much does that train engineer earn in a year? There are 52 weeks in a year.
	(US_SF-AW, grade 4, p. 332) If 17 farms each had 25 goats, how many goats would all the farms have together?
	(China, grade 4, vol. 8, p. 56) If we buy 102 shirts and every shirt is 32¥, how much should we pay?
Equal-groups involved Common factor	*(US_HM, grade 4, p. 180)* At a science store, a leaf fossil costs $5.95, a geode costs $8.50, and a space pen costs $6.75. How much less would it cost Tim to buy 6 fossils than 6 pens as gifts?
	(US_SF-AW, grade 4, p. 156) Trisha and her brother, Kyle, collect and sell baseball cards. Kyle has 6 cards to sell. Trisha has 3 cards to sell. If they sell the cards for 8¢ each, how much money will they get all together?
	(China, grade 4, vol. 8, p. 54) A T-shirt is 32¥, a pant is 45¥, and a jacket is 65¥ (shown by pictures). Someone buys 5 jackets and 5 pants. How much do they pay altogether?
Perimeter/Area	*(China, grade 4, vol. 8, p. 55)*. (A picture of a rectangular vegetable garden plot shows the length and width). The length is 64 m and the width is 26 m. What is the perimeter?
	(China, grade 6, vol. 11, p. 15) (A picture shows a rectangular prism) A box is in the shape of a rectangular prism with a length of 6 cm, a width of 5 cm, and a height of 4 cm. How many cm^2 of paper are needed to make the box?
Distance	*(China, grade 4, vol. 8, p. 91)* Xiaoming and Xiaofang walked toward school. Xiaoming walked at 70 meters/minute. Xiaofang walked at 60 meters/minute. They met each other after 4 minutes. What is the distance between their two homes?
	(China, grade 6, vol. 11, p. 6) Xiaoli and Xiaoming walked toward each other from a distance of 960 meters apart. Xiaoli's speed is 52 meters/minute while Xiaoming's is 62 meters/minute. How many minutes does it take for them to meet each other?
Multiplicative-comparison Involved	*(China, grade 3, vol. 5, p. 43)* A Pant is 28¥. The price of a coat is 3 times that of a pant. How much does one suit cost?
	(China, grade 4, vol. 8, p. 111) Students went to collect plant samples. The number of fourth graders' collections is a. The number of sixth graders' collections is 3 times the number of fourth graders' collections. Students in both grades together collected ( ). Fourth graders collected ( ) fewer than the sixth graders.
	(China, grade 6, vol. 11, p. 115) Students collected 60 plant and insect samples. The number of plant samples is 1.5 times the number of insect samples. How many samples does each collection have?

provides "52 × 986" and then suggests "Different ways to use the DP to find 52 × 986." Similarly, the SF-AW offers the solution "17 × 25" and then provides an array model to illustrate the partial products. In this process, the word problem contexts are actually discarded, and what students experience is almost the same as in computation problems. In contrast, although the Chinese text series only have two "one-step multiplication" problems, they may help students to understand the DP by making explicit the connection between the abstract and the concrete (Pashler et al., 2007). The given Chinese example (*buy 102 shirts with every shirt costing 32¥*) is a worked example. The Chinese text suggests three approaches concurrently. The first way

TABLE 4
The Frequency of Types of Problems Involving the DP Within Word Problem Context

Country	Period	Numbers Used	Multiplicative-Comparison Involved	Equal-Groups Involved: Common factor	Equal-Groups Involved: Area or Perimeter[2]	Equal-Groups Involved: Distance	One-Step Multiplication	Total
China	Before	Whole numbers	12	1	9	0	1	23
	During	Whole numbers	0	10	1	0	1	12
	After	Fractions, decimals, percents, expressions, and equations	108	17	2	14	0	141
	Total		**120**	**28**	**12**	**14**	**2**	**176**
US-HM	Before	Whole numbers	0	0	0	0	2	2
	During	Whole numbers	0	0	0	0	0	0
	After	Whole numbers, decimals	0	2	0	0	8	10
	Total		**0**	**2**	**0**	**0**	**10**	**12**
US_SF-AW	Before	Whole numbers	0	4	0	0	5	9
	During	Whole numbers	0	0	0	0	1	1
	After	Whole numbers	0	23	3	0	4	30
	Total		**0**	**27**	**3**	**0**	**10**	**40**

is to estimate: Since the cost of 100 shirts is 3200¥, the cost for 102 shirts will be more than 3200¥. The second way is the standard algorithm. The third way is mental math: since the cost for 100 shirts is 3200¥ while the cost for 2 shirts is 64¥, the total cost will be 3264¥. Based on the earlier discussion, students are asked to finish $32 \times 102 = 32 \times (100 + 2) = 32\times () + 32 \times () = \ldots$. The *Curriculum Analysis* suggests teachers assist students in understanding why 102 could be viewed as $(100 + 2)$ by referring to the word problem situation (e.g., 102 shirts are viewed as 100 shirts and 2 shirts). Such guidance likely allows students to see the meanings and reasonableness of breaking 102 up into $100 + 2$. After the computation, the text raises a question, "Why can we compute in this way?" This deep question is expected to elicit students' deep explanation (Chi et al., 1994) about the abstract principle (the DP) underlying the computation. Such a use of well-designed examples might help students learn the abstract principles (Calhoun et al., 2005).

With regard to the "equal-groups involved – common factor" problems (see Table 3 for examples), neither of the U.S. texts appears to use this type of problem as a means to develop students' understanding of the DP. Regarding the given example problem of the SF-AW text *(the total money for selling 6 cards and 3 cards at 8¢ each),* the SF-AW text presents two ways of thinking to find the "hidden questions," resulting in two solutions (a) $3 + 6 = 9$, $9 \times 8 = 72$, and (b) $3 \times 8 = 24$, $6 \times 8 = 48$, $24 + 48 = 72$. The underlying connection between the two solutions [e.g., $(3 + 6) \times 8 = 3 \times 8 + 6 \times 8$] is not at all mentioned, even though the DP has already been formally introduced. In fact, among all of the 30 "common factor" problems in the SF-AW text, the embedded DP is never made explicit. The HM only has two "common factor" problems. One problem (see Table 3*)* is treated in a similar manner as the SF-AW text, while the other problem (not listed in Table 3) focuses on computation [find $(5 \times 6) + (5 \times 4)$ using the DP] without an analysis of the problem situation. In contrast, the Chinese texts provide numerous word problems where students are able to experience the use of the DP through problem solving. The given example problem in Table 3 [*the total cost for 5 jackets (¥65 each) and 5 pants (¥45 each)*] is also a worked example during the formal introduction of the DP. The textbook reveals the DP through four steps. Similar to SF-AW, the Chinese text first guides students to solve this problem in two ways: (a) by first figuring out how much one suit costs resulting in the solution $(65 + 45) \times 5$, and (b) by first figuring out how much 5 pants cost and how much 5 shirts cost resulting in the solution $65 \times 5 + 45 \times 5$. However, getting a solution for this problem is not the final goal. Instead, as explained in the *Curriculum Analysis*, the context is used to help students discover the DP, sense its reasonableness, and understand its meanings. Therefore, after the problem is solved, the text asks students to write the above two expressions as one equation and to explain the relationship between both sides of the " = ." The *Curriculum Analysis* alerts teachers to refer this equation tightly back to the problem situation and then appropriately abstract the nature of the operations in this equation. Next, the textbook suggests that students post more examples of this sort and communicate their findings in small groups. Such opportunities for students' self-explanation of the DP are likely to increase student learning and their ability to transfer this knowledge to other contexts (Chi, 2000; Chi et al., 1994). Based on the above three steps, the textbook reveals, "If we use *a, b*, and *c* to represent these three numbers, we can write it as $a(b + c) = ab + ac$. This is the distributive property" (p. 54). As explained by the *Curriculum Analysis,* the purpose of using the above four steps with increasing levels of abstraction is to help students "construct" their understanding of the DP rather than "input" the property to students. The above worked example is a well-designed one because it allows students to map the structures of familiar situations onto the DP to better understand this principle. Such a mapping process is considered critical to human cognition (Gentner, 1983).

Using two different ways to solve a word problem, thus implicitly or explicitly illustrating/applying the DP, is a common approach in Chinese texts. In addition to the "common factor" problems, the other distinguishable problem types (e.g., perimeter/area, distance, multiplicative-comparison involved) also employ this approach. With regard to the perimeter problems, the Chinese text recommends both methods, $p = 2l + 2w$ and $p = 2(l + w)$, and raises a question, "What is the connection between the two solutions?" to highlight the DP. Such an approach is recommended by NRC (2001); however, it is missing from both U.S. textbooks. Throughout the elementary grades, both texts teach only $p = 2l + 2w$ to compute the perimeter of a rectangle, without asking for or showing students $p = 2(l + w)$. Thus, students may miss an important opportunity to make sense of the DP through such a concrete experience. The distance problem is first introduced in the Chinese fourth grade as an application of the DP (e.g., *two speeds 70 meters/minute and 60 meters/minute with a common time of 4 minutes*; see Table 3). After the problem situation is analyzed using both the number line and table representations, the textbook expects students to see two solutions, $70 \times 4 + 60 \times 4$ and $(70 + 60) \times 4$, which also illustrates the DP.

In summary, it seems that Chinese textbooks align well with the cognitive research that finds abstract principles may be learned through clear worked examples (Gentner et al., 2003) with comparison solutions (Rittle-Johnson & Star, 2007). The concrete and abstract representations were connected even within one problem context with apparent "concreteness fading" (Goldstone & Sakamoto, 2003; Goldstone & Son, 2005). In addition, students were prompted by deep questions to self-explain the key ideas. Given the effect of worked examples (Renkl, 2002; Zhu &

Simon, 1987), it is plausible that interleaving such well-designed examples with related practice problems could contribute to Chinese students' deep understanding of the DP.

Typical Problem Structures Across Grades in Chinese Texts. Both the "equal-groups involved" and "multiplicative-comparison involved" problems appear before, during, and after the DP is formally introduced. The same word problem structures, often illustrated and accompanied by number line diagrams, are used with increasing levels of abstraction across grades, offering enough spacing for learning the DP over time (Pashler et al., 2007). Since students have been experiencing both types of problems from the third grade, the Chinese fourth-grade texts formally introduce the DP using these problem structures. After the DP is formally introduced, the Chinese fourth-grade text introduces repeated variables. All types of word problems are utilized for practicing this topic, during which the DP may be strengthened. A worked example of a "common factor" problem is, "*Xiaohua made n triangles with sticks and Xiaofang made n squares with sticks. How many sticks did they use?*"[2] Students are again taught two ways of thinking, resulting in two solutions "$3n + 4n$" and "$(3 + 4)n$," the connection of which shows the computation process $3n + 4n = (3 + 4)n = 7n$. The textbook encircles the second step, asking "What property is actually used?" Such a deep question prompts student self-explanation of the DP, which in turn may deepen students' understanding of operations with repeated variables. Another example is a distance problem often presented along with number line diagrams. Given two different speeds (65 meters/minutes and 75 meters/minute) and the common walking time of "a" minutes, students are expected to figure out the total distance using "$(75a + 65a)$" or "$(75 + 65)a$." The introduction of repeated variables in the Chinese fourth grade may not only reinforce students' understanding of the DP, but may also extend the known problem structures to an algebraic context, thus opening a door for algebraic problem solving in the sixth grade. The distance problem examples in Table 3 (*the speeds are 52 meters/minute and 62 meters/minute, the time is an unknown, and the total distance is 960 meters)* clearly show how the same problem structure is systematically used across grades. The expected two algebraic solutions, $52x + 62x = 960$ and $(52 + 62)x = 960$, demonstrate the DP. Solving equations like the first one also demands an understanding of DP.

The "multiplicative-comparison involved" problems that are unique to Chinese texts ($n = 120$) also demonstrate how this problem structure is used in a hierarchical way. Before the DP is introduced, this type of problem is used in whole number arithmetic contexts ($n = 12$). The example in Table 3 (*A Pant is 28¥. The price of a coat is 3 times that of a Pant*) is from the chapter *Addition and Subtraction within 100* in the third-grade textbook. The text provides the first number line representing the price of a pant (*28¥*); it then asks students to draw a second line to show the price of a coat (the length should be 3 times that of the first line). Engaging students in the construction process of the number line diagram may help them see the deep problem structure (Pashier et al., 2007). After constructing the representation, the Chinese textbook guides students to consider two approaches—to first find out the price of a coat (28×3), or to first find out how many "28¥" one suit will cost $(3 + 1)$—which encourages an intuitive sense of the DP, $28 \times 3 + 28 = 28 \times (3 + 1)$. The same problem structure is also used for learning repeated variables in the

[2]The Chinese text uses "a" *triangles,* which is clear among Chinese characters. For clarity, we changed "a" to "n" in this article.

fourth grade (see Table 3, "multiplicative-comparison involved," fourth-grade example). In the sixth grade, the comparison multipliers are changed from whole numbers to fractions (e.g., the number of boys is 3/5 that of girls), decimals (e.g., the number of plant samples is 1.5 times the number of insect samples), and percents (e.g., the number of girls is 80% that of boys). Therefore, the same problem structure, represented by number line diagrams, is used across different topics; and the same problem structure is used for teaching students to solve problems with equations. With regard to the sixth-grade example in Table 3 *(The number of plant samples is 1.5 times the number of insect samples while the total collection is 60 samples)*, the textbook suggests it be solved with an equation ($x + 1.5x = 60$). In fact, recent research has found that students tended to solve such a double-reference problem using formal algebraic strategies, which seemed more effective than the informal ones (Koedinger et al., 2008). These algebraic strategies apply the DP, although it may not necessarily be made explicit.

Problem Types Involving the DP Within the Computation Context

Computation is the main problem context used by both the U.S. textbooks to teach the DP (HM: 67%; SF-AW: 88.9%) (see Table 2). Regardless of the difference in frequency ($n_{\text{HM}} = 77$, $n_{\text{SF-AW}} = 491$ $n_{\text{China}} = 129$), both U.S. texts tend to use similar problem types, while the Chinese text appears to use more diverse types. Table 5 provides typical examples of each problem type. A summary of the frequency for each problem type by country and period is shown in Table 6. Inspection of Table 5 and Table 6 reveals considerable cross-country differences in the text approaches and goals within the computation problem context. Since the approach to the last problem type "computing the perimeter/surface area" for given figures (instead of word problems) is similar to what we previously discussed, we will not discuss it further.

Multiple "Breaking Apart a Factor" Strategies. Both U.S. texts present various strategies such as "*the doubles strategy,*""*using known facts to find unknown facts,*" and "*breaking numbers apart to multiply.*" These strategies, in fact, share the same feature, that is, to break apart a factor for computing whole number multiplication facts or multiplication with larger numbers. Both texts appear to emphasize the strategies themselves rather than the basic property underlying the strategies. One example concerns "the doubles strategy" as previously mentioned in the Methods section. The HM third grade text first formally introduces the DP with an example of "multiplying in different ways" to find 5×6. It says, "Sean uses *doubles.* 5×6 is double 5×3. $5 \times 6 = (5 \times 3) + (5 \times 3) \dots$; Theresa uses the *Distributive Property*. She thinks of 6 as $4 + 2$. $5 \times 6 = 5 \times (4 + 2) \dots$; Bob also uses the *Distributive Property*. But he thinks of 6 as $5 + 1$. $5 \times 6 = 5 \times (5 + 1) \dots$ " (p. 239). The above text presentation reflects that the HM text views the doubles strategy as different from the DP, which is unlikely to lead students to a deep appreciation of the DP. In addition, asking students to learn various strategies without pointing out the underlying principle seems less efficient. This is because strategies learned in this way may be hard to retrieve when new problems share few surface features with the previous ones, even though they are governed by the same principle (Chi, Feltovitch, & Glaser, 1981). In contrast, the Chinese text has one instance of the doubles strategy introduced as a knowledge expansion during the formal introduction of the DP. After a brief introduction of the origination of the doubles strategy, it asks students, "Can

TABLE 5
Examples of Problem Types Involving the DP Within Computation Contexts

Problem Types	*Examples*			
Breaking apart a factor				
– The doubles strategy	$7 \times 6 = 7 \times 3 + 7 \times 3$			(HM)
– Using known facts to find unknown facts	How can you use 2×7 to find 3×7? Breaking 3 into 2 and 1, thus $2 \times 7 + 7 = 3 \times 7$			(SF-AW)
– Breaking numbers apart to multiply	$29 \times 4 = (20 + 9) \times 4$			(HM)
	$327 \times 5 = (300 \times 5) + (20 \times 5) + (7 \times 5)$.			(SF-AW)
– Breaking numbers apart to divide	To find $68 \div 2$, break 68 into 60 and 8, $60 \div 2 = 30$, $8 \div 2 = 4$, $30 + 4 = 34$			(SF-AW)
– Breaking arrays to show partial products	Similar to NCTM (2000) example (see Figure 1)			
– Using the DP to find a product	Use the DP to rewrite an expression for 25×365.			(HM)
	Use the DP to do mental math for 25(12).			(SF-AW)
– Using an array to model the DP	Similar to NCTM (2000) example (see Figure 1)			
Does the DP work with subtraction (division)	Does the DP work with subtraction? Try it with 9(50–1) and (20–4) × 5.			(SF-AW)
Comparing two number sentences				
– Computing in groups	$2 \times 5 + 5$ 3×5	$25 \times 30 + 25 \times 20$ $25 \times (30 + 20)$	34×21 $34 \times 20 + 34$	(China)
– Writing >, <, or =	1) $99 \times 99 + 199$ O 100×100 $999 \times 999 + 1999$ O 1000×1000 $9999 \times 9999 + 19999$ O $\square \times \square$ 2) $32 \times (30–2)$ O $32 \times 30 – 32 \times 2$ 3) $(3.2 + 0.8) \times 0.6$ O $3.2 \times 0.6 + 0.8 \times 0.6$			(China)
Using convenient ways to compute	16×401 $125 \times (8 + 16)$ $38 \times 7 + 62 \times 7$ $23 \times 134 - 34 \times 23$	$7.5 \times 0.45 – 0.45 \times 6.5$ $30 \times (1/5 + 1/3)$ $2/3 \times 5/7 + 2/3 \times 2/7$ $3/4 \times 1/9 + 1/4 \div 9$		(China)
Simplifying expressions and solving equations involving repeated variables	$4x + 5x$, $4a - 2a$, $27x + 31x = 145$; $x - 0.8x = 10$; $1.9x + 0.4x = 9.2$; $x + 60\%\, x = 48$; $x - 4/9x = 10/21$			(China)
	Evaluate the expressions given $a = 2/3$: $(10 \times a) + (2 \times a)$;	$a \times 3\frac{1}{6} - a \times \frac{7}{6}$		(HM)
Computing the perimeter/surface area	Compute perimeters and surface areas for given figures using two ways.			(China)

you explain the doubles strategy using the distributive property?" It seems Chinese texts focus more attention to the underlying principles by eliciting students' self-explanations (Chi, 2000) with deep questions (Craig et al., 2006) even in the context of computation.

Although the SF-AW text has about 7 times the number of computation problems as that of the HM, it has a similar focus on learning multiplication strategies rather than using the basic property to understand these strategies. The SF-AW fourth-grade text formally introduces the DP in the lesson "*using known facts to find unknown facts.*" The textbook begins with a sentence,

TABLE 6
The Frequency of Problem Types Involving the DP Within Computation Contexts

	China (n = 129)				US_HM (n = 77)				US_SF-AW (n = 491)			
Problem Types	Bf[1]	Du[1]	Af[1]	Tot[1]	Bf	Du	Af	***Tot***	*Bf*	*Du*	*Af*	***Tot***
Breaking apart a factor												
The doubles strategy	0	1	0	**1**	5	1	0	**6**	18	0	0	**18**
Using known facts to find unknown facts					0	0	1	**1**	40	0	0	**40**
Breaking numbers apart to multiply					1	8	3	**12**	18	36	112	**166**
Breaking numbers apart to divide									41	0	0	**41**
Breaking arrays to show partial products					0	0	14	**14**	33	0	100	**133**
Using the DP to find a product					0	0	22	**22**	0	0	90	**90**
Using an array to model the DP	0	1	0	**1**	0	0	16	**16**				
Does the DP work with subtraction (division)?									0	0	3	**3**
Comparing two number sentences												
Computing in pairs	18	5	4	**27**								
Writing >, <, or =	2	5	1	**8**								
Using convenient ways to compute	0	31	22	**53**								
Solving/evaluating expressions and equations	0	0	30	**30**	0	0	6	**6**				
Computing the perimeter/surface area	5	0	4	**9**								

[1]"Bf" denotes before; "Du" denotes during; "Af" denotes after; "Tot" denotes "total."

"How can I break apart facts? The Distributive Property shows that you can break apart facts to find the product" (p. 132). The follow-up practice problems only ask students to "use breaking apart to find each product" without mentioning the DP. In addition, after the DP is formally introduced, the SF-AW text continues to teach various multiplication strategies about "breaking numbers apart to multiply" ($n = 112$) and "breaking arrays to show partial products" ($n = 100$) but rarely bringing the DP to the forefront. The corresponding teacher guides do not at all encourage teachers to reveal the underlying property. Only one place revisits the DP as "enrichment" with 5 practice problems.

Comparing Number Sentences. A distinguishable computation problem type in the Chinese texts is *comparing number sentences* (see Table 5), revealing the Chinese texts' emphasis on understanding underlying principles through a constructive approach. As the *Curriculum Analysis* states, the teaching of the DP should focus on sense making, and students should be guided to discover this principle through comparison. This first type of comparison, "computing in a pair" is used across grades. For example, the Chinese second-grade text asks students to compute multiplication facts in a pair (e.g., $2 \times 5 + 5$ and 3×5), which is likely to prompt students to think of the relationship between two sentences. When this type of problem [e.g., $25 \times 30 + 25 \times 20$ and $25 \times (30 + 20)$] is used immediately before the formal introduction of the DP, the purpose of sensing and identifying the DP appears more obvious. In addition, during the formal introduction of this property, comparing two computations in a pair (e.g., 34×21 and $34 \times 20 + 34$) may not only encourage students to identify the DP (which is less apparent than $34 \times 20 + 34 \times 1$), but also recognize the powerfulness of using the DP.

Another type of comparison problem is "*writing >, <, or =* ." Although there are only 8 instances, these problems are carefully designed to encourage discovery. The first example of this type in Table 5 (a group of three problems) is challenging and interesting because students are supposed to identify a pattern and explore the underlying reasons for the pattern, which is embedded as a double use of the DP. For example, $99 \times 99 + 199 = 99 \times 99 + 99 + 100 = 99 \times (99 + 1) + 100 = 99 \times 100 + 100 = (99 + 1) \times 100 = 100 \times 100$. Such a problem deepens students' understanding of the DP. Other problems are used for investigating the applicability of the DP with subtraction [e.g., $32 \times (30 - 2)$ O $32 \times 30 - 32 \times 2$] or with decimals [$(3.2 + 0.8) \times 0.6$ O $3.2 \times 0.6 + 0.8 \times 0.6$]. The *Curriculum Analysis* points out, "Teachers should suggest students first calculate both sides before filling the box. Otherwise, the process is not to explore the applicability of the DP but to simply use the DP to rewrite a sentence, which is not consistent with the learning theories and not beneficial for a serious learning attitude." Similarly, there are three problems in the SF-AW sixth-grade text discussing the applicability of the DP with subtraction or division (e.g., "*Does the DP work with subtraction?*") However, the textbook simply suggests students try two examples without any guidance [e.g., "Try it with $9(50 - 1)$ and $(20 - 4) \times 5$"], and the teacher guide does not provide any direction either. In addition, the HM sixth-grade text has two instances regarding "*writing >, <, or =* " ["$1.4\ (a + b)$ O $1.4a + 1.4b$" and "$(a \times 6) - (a \times b)$ O $a(6 - b)$"]. However, these problems clearly ask for an application of the DP rather than a discovery through computation. As a result, these problems are classified into the definition context.

Using Convenient Ways to Compute. The most frequently used problem type in the Chinese text is "*using convenient ways to compute*" ($n_{\text{during}} = 32$, $n_{\text{after}} = 22$). Although some problems can be found in the U.S. texts (e.g., 16×401), the Chinese texts purposefully bring the power of the DP to the students' attention. In addition, some unique problems in the Chinese text (e.g., $7.5 \times 0.45 - 0.45 \times 6.5$; $3/4 \times 1/9 + 1/4 \div 9$) allow students explicitly to experience the benefit of using the DP beyond whole numbers (see Table 5 for more examples). The numbers used in these problems are often "non-friendly." Without the use of the DP, the computation would be complex. However, when the DP is used, the computation becomes extremely simple. For example, one does not need to really conduct fractional multiplication and division when the DP is applied in the following example:

$$
\begin{aligned}
&\frac{3}{4} \times \frac{1}{9} + \frac{1}{4} \div 9 \\
&= \frac{3}{4} \times \frac{1}{9} + \frac{1}{4} \times \frac{1}{9} \\
&= \left(\frac{3}{4} + \frac{1}{4}\right) \times \frac{1}{9} \\
&= 1 \times \frac{1}{9} \\
&= \frac{1}{9}
\end{aligned}
$$

This example demands a combination of two fractions into "1," thus using the DP in an opposite direction. This contributes to students' complete understanding of the DP, allowing them to use this property flexibly. As the *Curriculum Analysis* states, "The main purpose of learning operational properties is to use them to make computations easier. Therefore, the curriculum design principle is to guide students to experience the power of the properties, thus activating their inside needs for an automatic use of these properties to compute in convenient ways."

The above intention of showing the power of the DP to students is not obvious in the U.S. texts. Although both U.S. texts have instances explicitly asking for the use of the DP (e.g., "*using the DP to find the product*," $n_{\text{HM}} = 22$, $n_{\text{SF-AW}} = 90$), these problems are mainly limited to breaking apart a factor to find multiplication facts (e.g., 3×6) or multiplying with larger numbers (e.g., 25×365). However, a student who knows the fact $3 \times 6 = 18$ may have neither the motivation nor the need to break up "3" to perform $2 \times 6 + 1 \times 6$. Only a few variations are found in the SF-AW sixth-grade texts where students need to combine two factors first and do mental math using the DP in an opposite way [e.g., $(9 \times 17) + (9 \times 3)$, $(18 \times 4) + (2 \times 4)$]. However, such problems are rare and the numbers involved are exclusively whole numbers that are so accommodating that students may not appreciate the contrast between the difficulty of direct computation and the simplicity that would result from applying the DP.

Expressions (Equations) Involving Repeated Variables. As we mentioned previously regarding the word problem context, the fourth-grade Chinese text introduces expressions with repeated variables ($\mathrm{a}x \pm \mathrm{b}x$) after the DP is formally introduced, which seems to contribute to the ability to solve equations involving various numbers in the sixth grade ($\mathrm{a}x \pm \mathrm{b}x = \mathrm{c}$) (see Table 5 for examples). The underlying idea of simplifying such expressions or solving such equations involves the use of the DP. These problems are basically unique to the Chinese texts ($n = 30$). It is noticeable that the coefficients of variables across these problems have a great variability (e.g., whole numbers, decimals, fractions, and percents), which adds cognitive demands for students but may pay off in better understanding (Renkl, Stark, Gruber, & Mandl, 1998). The *Curriculum Analysis* alerts teachers that although students have already learned to use the DP to simplify $\mathrm{a}x \pm \mathrm{b}x = (\mathrm{a} \pm \mathrm{b})x$ in the fourth grade, teachers in the sixth grade should still ask questions to prompt students to explain what enables them to simplify an equation in a similar way and what is the purpose for doing so.

The SF-AW only introduces expressions and equations with a single variable (e.g., $x + 8$; $5 + x = 10$). Although the HM has a few instances ($n = 6$) involving repeated variables, these problems mainly ask students to "evaluate" rather than "simplify" the expressions. For example, the sixth-grade textbook (p. 143) asks students to evaluate $(10 \times a) + (2 \times a)$ given $a = 2/3$. This is a nice instance offering an opportunity to use the DP in an opposite direction. However, according to the previous text presentation of the worked example, students should first replace the letter with the number to get $(10 \times 2/3) + (2 \times 2/3)$, then calculate each multiplication, and finally add them together. In this way, the DP is not necessarily used to make the computation easier. A follow-up question on the same page confirms the above interpretation, "How could you have used the DP to make it easier to evaluate the expression $[(10 \times a) + (2 \times a)]$?" This is the only place that the text explicitly reminds students to use the DP to simplify algebraic expressions.

TABLE 7
Examples and Frequencies for Each Problem Type Involving the DP Within the Definition Context

			China_JSEP			*US_HM*			*US_SF-AW*		
Problem Type	*Example*		*Bf*[1]	*Du*[1]	*Af*[1]	*Bf*	*Du*	*Af*	*Bf*	*Du*	*Af*
Completing the number sentence	40 × ( ) = (40 × 5000) + (40 × 20) + (40 × 5)	(HM)		8	1			15			10
	5(17 + □) = (5 × 17) + (5 × 3)	(SF-AW)									
	3x + 2x = (□ + □) × □	(China)									
Comparing the equivalency	1.4 (a + b) O 1.4a + 1.4b	(HM)		4				6			3
	Which is equivalent to 8 × 52? (with 4 choices)	(SF-AW)									
	Check the box if the answers are the same: 40 × 50 + 50 × 90; 40 × (50 + 90) □	(China)									
Vocabulary	An example of the ______ is 4 × (5 + 6) = (4 × 5) + (4 × 6).	(HM)			1			5			8
	(A table asking for examples and algebraic expressions for given properties)	(China)									
Total				**12**	**2**			**26**			**21**

[1]"Bf" denotes before; "Du" denotes during; "Af" denotes after.

Problem Types Within the Definition Context

Within the definition context, we classified the problems into three types: *completing the number sentences, comparing the equivalency,* and *vocabulary.* Table 7 provides examples for each problem type and summarizes the corresponding frequencies by countries and periods.

Similarities of text approaches to definition problems are observed. All of the three texts use "*completing the number sentences*" most frequently ($n_{\text{HM}} = 15$, $n_{\text{SF-AW}} = 10$, $n_{\text{China}} = 9$). "*Comparing the equivalency*" is also a commonly used problem type across texts with somewhat different formats ($n_{\text{HM}} = 6$, $n_{\text{SF-AW}} = 3$, $n_{\text{China}} = 4$).

Differences regarding definition problems across countries are also identified. As previously mentioned, the Chinese texts mainly use the definition problems during the formal introduction of the DP, while both U.S. texts use them after the DP is formally introduced. It seems both U.S. texts emphasize knowing the definition of the DP throughout the elementary grades, while Chinese texts place more attention on the application rather than the definition once the property is formally introduced. Such a difference is more clearly reflected by the Chinese texts' low frequency of "*vocabulary*" problems, a type of problem directly targeting at defining what the DP is ($n_{\text{HM}} = 5$; $n_{\text{SF-AW}} = 8$; $n_{\text{China}} = 1$). In fact, the text approaches to "vocabulary" problems also vary considerably. Both U.S. texts tend to ask students to recall the term "distributive property" based on given examples, while the Chinese text requests examples as well as algebraic expressions for given properties. For example, the only instance of "vocabulary" in the Chinese text appears in sixth grade. At the end of the sixth-grade text, there is a chapter named *Organize and Reflect* that reviews all of the key knowledge pieces that students learned throughout the elementary grades (1–6). The first problem is to revisit three basic properties (the Commutative Property, Associative Property, and Distributive Property) with a table asking students to offer examples of arithmetic and then write corresponding algebraic expressions (which the Chinese texts express as "using letters to represent these properties"). This delayed review connects the concrete and abstract representations and demonstrates a higher expectation of student understanding of the DP.

Variability in the Use of the DP as Reflected by the Involved Directions and Numbers

In order to better understand the variability in using the DP across texts, we closely examined the "directions" of using the DP within computation and definition contexts. In addition, we examined the "numbers used" across all problem contexts. We first present the findings regarding directions followed by numbers used.

Directions. The directions in using the DP are classified three ways: regular, opposite, and dual. The regular direction is to expand an expression by using the DP in the ways of "$a(b + c) = ab + ac$" or "$(a + b)c = ac + bc$." The opposite direction is to combine the common factors as one, thus using the DP as "$ab + ac = a(b + c)$" or "$ac + bc = (a + b)c$." There are a few instances involving the use of the DP in both ways, thus referred to as "dual direction." An example is the previously mentioned "*using known facts to find unknown facts*" in the SF-AW text. Other examples are "computing in pairs" or "computing the perimeter/surface area" in the Chinese texts. Figure 2 sketches the striking differences in using the DP involving various directions across countries.

Detailed information about frequencies and percentages broken down by periods and problem contexts for each country is presented in Table 8. In general, regardless of the large difference in frequencies for each problem context, the majority of computation and definition problems in both U.S. texts use the DP in a regular direction (90.3% in HM and 93.2% in SF-AW). In contrast, the Chinese texts have more problems in the opposite direction than the regular one (56.6% are opposite while 30.8% are regular). The differences regarding the "regular versus opposite" are extremely clear in the computation problems after the DP is introduced: 8.4% versus 30.8% in the Chinese texts, as opposed to 57.3% versus 1.2% in the HM, and 58.4% versus 3.9% in the SF-AW.

It is not surprising to find this pattern considering the problem types used in the computation context. Both the U.S. texts mainly emphasize the approach of "breaking apart factors to multiply." Thus, the majority of problems both texts provide are whole number multiplication, which demands breaking up the factor into "$b + c$" (e.g., tens and ones) and performing the multiplication in the regular direction, $a(b + c) = ab + ac$. In contrast, the Chinese texts emphasize "computing in convenient ways" which possibly includes various problems, some of which require a combination of two factors into whole tens or hundreds or even one whole (e.g., $33 + 67 = 100$; $1/5 + 4/5 = 1$, $0.72 + 0.23 = 1$). These problems need to use the DP in the opposite direction, $ab + ac = a(b + c)$. In addition, problems such as "*simplifying expressions or solving equations involving repeated variables*" also demand an opposite use of the DP [e.g., $\mathrm{a}x + \mathrm{b}x = \mathrm{c} \Rightarrow (\mathrm{a} + \mathrm{b})\, x = \mathrm{c} \Rightarrow x = \mathrm{c}/(\mathrm{a} + \mathrm{b})$].

The definition context directly relates to "what the DP is." It is reasonable to expect problems involving both regular and opposite directions in this context. While the Chinese text series has

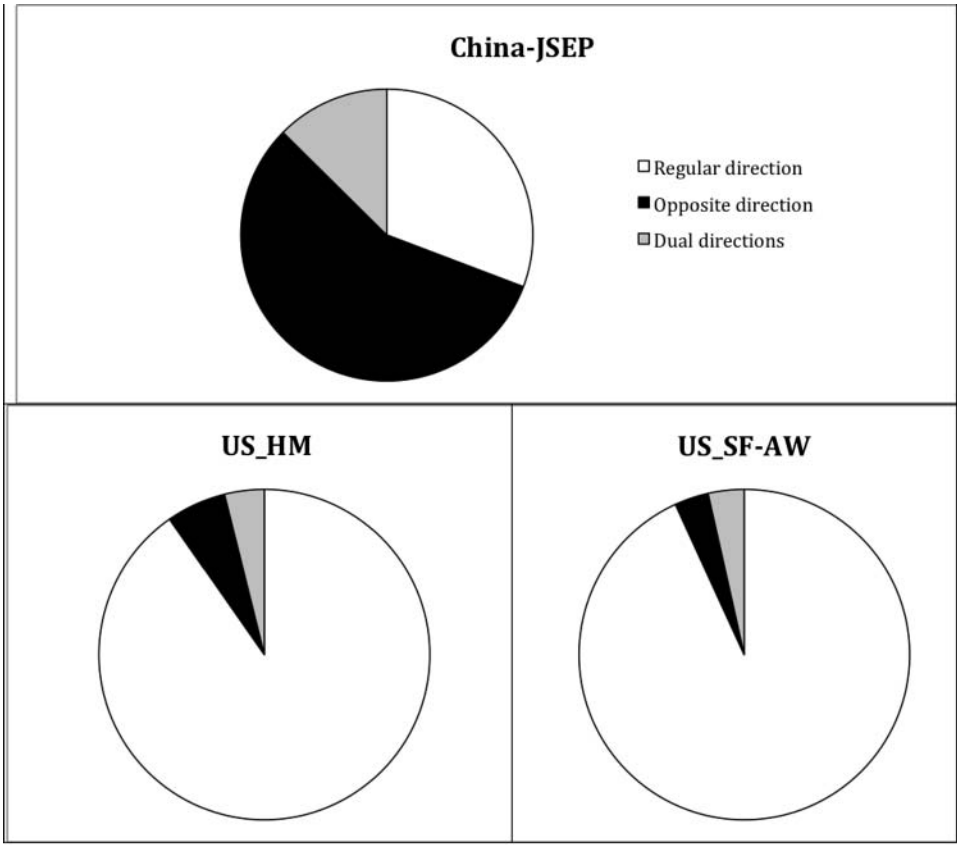

FIGURE 2 Percentage of different directions involving the use of the DP in each text series.

almost the same amount of problems involving either direction (8 regular and 6 opposite), both the U.S. texts mainly use problems involving the regular direction (HM: 20 regular and 2 opposite; SF-AW: 19 regular and 2 opposite). Evidence from the cognitive literature (e.g., Anderson, 1993) suggests that people tend to acquire just a single direction of a possibly dual directional operation if they experience a single direction much more often than the other. One might expect, therefore, that the U.S. elementary texts' overemphasis on the regular direction contributes to students' incomplete and superficial understanding of the DP.

Numbers Used. Figure 3 shows an overview regarding the types of numbers used in the instances across countries. Regardless of the difference in frequency, both U.S. texts again appear to have a similar style: that is, an arithmetic context dominated by whole numbers.

Table 9 provides detailed information regarding the frequency, percentage, and grade level of types of number presentations in each text. In the Chinese texts, instances, both arithmetic

TABLE 8
The Directions of Using the DP in the Computation and Definition Contexts

			Regular ⇒		*Opposite ⇐*		*Both ⇔*	
Country	*Period*	*Context*	*Frequency*	*Percent*	*Frequency*	*Percent*	*Frequency*	*Percent*
China_JSEP	Before	Computation	2	1.4	13	9.1	10	7.0
($n = 143$)		Definition	0	0	0	0	0	0
	During	Computation	22	15.4	18	12.6	3	2.1
		Definition	7	4.9	5	3.5	0	0
	After	Computation	12	8.4	44	30.8	5	3.5
		Definition	1	0.7	1	0.7	0	0
	Total		**44**	**30.8**	**81**	**56.6**	**18**	**12.6**
US_HM	Before	Computation	6	5.8	0	0	0	0
($n = 103$)		Definition	0	0	0	0	0	0
	During	Computation	9	8.7	0	0	0	0
		Definition	0	0	0	0	0	0
	After	Computation	58	56.3	4	3.9	0	0
		Definition	20	19.4	2	1.9	4	3.9
	Total		**93**	**90.3**	**6**	**5.8**	**4**	**3.9**
US_SF-AW	Before	Computation	127	24.8	5	1	18	3.5
($n = 512$)		Definition	0	0	0	0	0	0
	During	Computation	32	6.3	4	0.8	0	0
		Definition	0	0	0	0	0	0
	After	Computation	299	58.4	6	1.2	0	0
		Definition	19	3.7	2	0.4	0	0
	Total		**477**	**93.2**	**17**	**3.3**	**18**	**3.5**

and algebraic, involve all kinds of numbers. Whole number arithmetic problems only account for 43.3% of the total instances. The textbooks start to extend the DP to expressions (repeated variables) in the fourth grade; decimals in the fifth grade; and fractions, percents, and various algebraic equations in the sixth grade. In contrast, both the U.S. texts are dominated by whole number arithmetic operations (HM: 86.1%; SF-AW: 99.5%). Note in particular that, although the SF-AW text series has the highest frequency of instances involving the DP among the three texts ($n = 552$), it only has 3 instances that go beyond whole number arithmetic operations.

The above pattern regarding numbers used is consistent with the previously reported problem types. Both the U.S. texts emphasize whole number multiplication and only introduce algebraic expressions (equations) with a single variable. In addition, the U.S. upper elementary textbooks mainly teach one-step decimal or fraction computations, but not mixed operations, resulting in a rare application of the DP for rational number operations.

DISCUSSION

Although the U.S. texts integrate current research suggestions regarding various multiplication strategies involving the DP (e.g., Ball, 1988; Carpenter et al., 2003; NCTM, 2000; Schifter et al., 2008), the cross-national curriculum comparison indicates that the U.S. texts have a much more

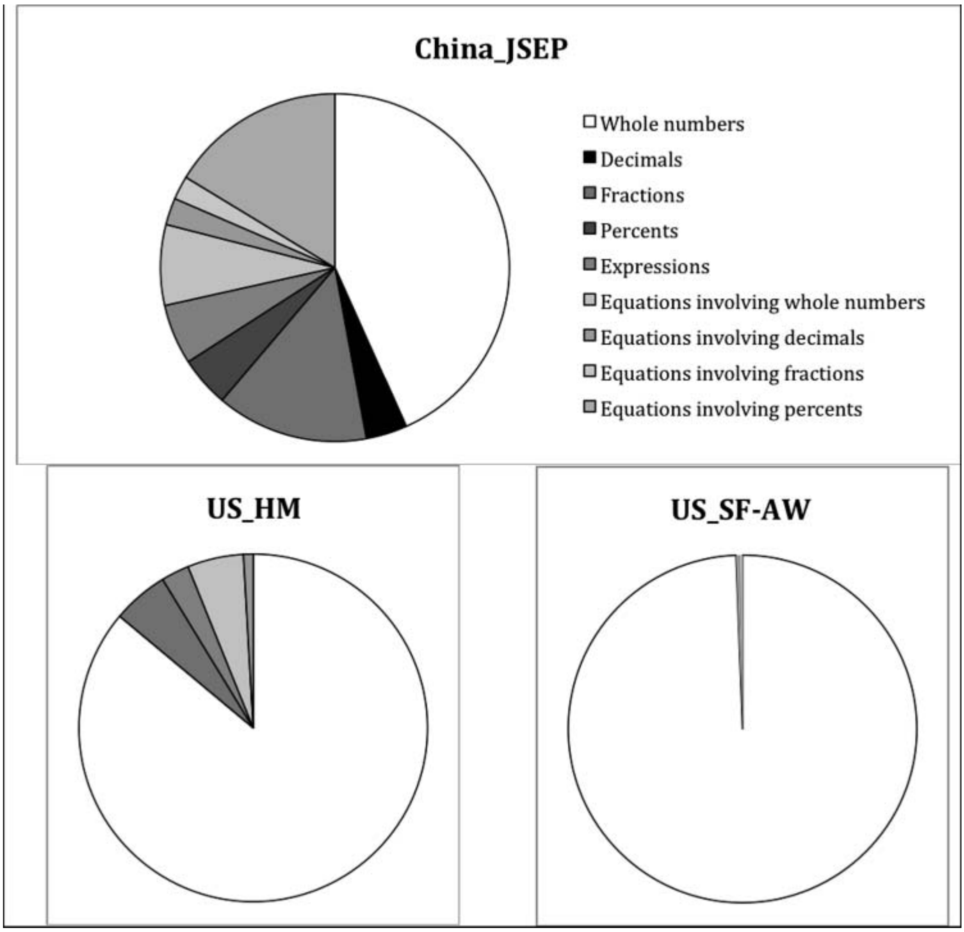

FIGURE 3 Percentage of the numbers used while involving the DP in each text.

narrow view of the DP. For example, the U.S. texts limit students' understanding of the DP to whole number contexts with a regular direction. They also focus more on surface strategies rather than the principle underlying these strategies. The word problems (often one-step) are not used to provide contextual support to learn the DP.

In contrast, Chinese text aligns well with the *Curriculum Focal Points* (NCTM, 2006), developing students' intuitive sense from the second grade, formally introducing the DP with a strong focus at the fourth grade, and then heavily applying it in various later learning. Four distinguishable features have been identified in Chinese texts. First, Chinese texts aim at students' conceptual understanding of the DP by using diverse word problem contexts and constructive approaches such as asking deep questions to prompt students to understand the DP. Second, Chinese texts strengthen students' understanding of the DP through a coherent text presentation, as indicated by systematic and hierarchical problem structures and problem sequences (e.g., the

TABLE 9
The Numbers Used in the Problems Involving the DP in All Problem Contexts

	China (n = *319)*			*US_HM* (n = *115)*			*US_SF-AW* (n = *552)*		
Numbers used	*Freq*	*%*	*Grade*	*Freq*	*%*	*Grade*	*Freq*	*%*	*Grade*
Arithmetic									
Whole numbers	138	43.3	2, 3, 4, 5, 6	99	86.1	3, 4, 5, 6	549	99.5	3, 4, 5, 6
Decimals	12	3.8	5, 6				1	0.2	6
Fractions	45	14.1	6	6	5.2	6			
Percents	15	4.7	6						
Algebra									
Expressions	18	5.6	4	3	2.6	6	1	0.2	6
Equations involving whole numbers	24	7.5	6	6	5.2	4, 6	1	0.2	5
Equations involving decimals	8	2.5	6	1	0.9	6			
Equations involving fractions	7	2.2	6						
Equations involving percents	52	16.3	6						

same word problem structures are used in both arithmetic and algebraic contexts with increased complexity and concreteness fading). Third, Chinese texts aim at an automatic use of the DP by making the power of the DP explicit by using various kinds of numbers and directions across grades. Fourth, the Chinese texts introduce repeated variables as an application of the DP, which subtly connects arithmetic and algebra, and opens a door to more challenging mathematics topics in later grades. These Chinese text approaches are consistent with suggestions from cognitive research that robust student learning is supported by contextual interferences, spaced practice, and encoding variability (see Pashler et al., 2007). Given that Chinese teachers are required to study textbooks and teacher guides intensively (Li et al., 2008; Ma, 1999), it is reasonable to hypothesize that the Chinese approaches to the DP contribute considerably to teachers' sound understanding of this property (Howe, 1999; Ma, 1999), which may in turn contribute to students' later algebraic thinking (Cai, 2004). In the following sections, we elaborate the above Chinese-like text approaches.

Making Sense of the DP Through Word Problems With Particular Structures

Chinese textbooks heavily utilize word problems, especially two-step problems, to help students make sense of the DP. The situational contexts of word problems and problem-solving approaches have been emphasized in previous research on learning algebra (e.g., Bell, 1996; Hall, Kibler, Wenger, & Trnxaw, 1989; Koedinger et al., 2008; Koedinger & Nathan, 2004; Lannin, Barker, & Townsend, 2007; Nathan, Kintsch, & Young, 1992; Resnick et al., 1987). The word problem contexts provide concrete and meaningful situations for students to bootstrap their real world knowledge to understand abstract principles as well as their formalizations (Resnick et al., 1987; Schifter, 1999). Impressively, the Chinese texts' approach to the perimeter problems [both $p = 2l + 2w$ and $p = 2(l + w)$] exploits this feature of word problems and is consistent with what was suggested by the NRC (2001). However, the U.S. texts in this study have not adopted the NRC suggestion, even though both texts are recent editions. The other distinguishable problem types in Chinese texts such as "distance" and "multiplicative-comparison involved" problems also enrich students' understanding of the DP. The SF-AW texts do include a certain number of two-step word problems. However, the SF-AW texts could go further by posing deep questions (Craig et al., 2006), similar to what the Chinese texts do, to guide students to self-explain the underlying principles (Chi, 2000; Chi et al., 1994). In fact, even with one-step problems, the Chinese text refers the abstract computation process back to the concrete problem situation to help students understand why a number can be broken up. However, both U.S. texts only focus on computation, without using the one-step problems in such a meaningful way.

The Chinese texts systematically and hierarchically use two-step word problems with typical problem structures. The problem structures, unchanged across increasingly challenging topics, offer enough spacing for learning the underlying principle (Cepeda et al., 2006) and provide "concreteness fading" that promotes the transfer of learning (Goldstone & Son, 2005; Goldstone & Sakamoto, 2003). By providing or asking for two solutions to these problems, or by using equations, students' understanding of the DP is reinforced and deepened. This more sophisticated understanding of the DP is then likely to assist students to come up with multiple solutions, thus developing students' problem-solving skills. For example, after the Chinese sixth- grade textbook presents the first solution for a word problem, $45 - 45 \times 5/9$, it asks students, "What is another way to solve this?" Students who have a sophisticated understanding of the DP are likely to figure out the second solution $[45 \times (1 - 5/9)]$ with ease. As previous studies (Cai, 1995; Cai, 2004) have pointed out, Chinese students are skillful at algebraic problem solving and tend to solve problems multiple ways. Our findings suggest that the Chinese text approach to the DP may play an important role in this and demonstrate how an understanding of the DP could pay a big dividend in the upper elementary grades (Carpenter et al., 2003).

It is worth pointing out that Chinese approaches, similar to other high achieving countries (Cai & Moyer, 2008; Murtura, 2008), systematically utilize the number line diagram in co-ordination with word problems. Although this type of representation may be non-transparent to students initially (Koedinger & Terao, 2002; Rittle-Johnson & Koedinger, 2001), the Chinese texts consistently use this tool across grades, thus increasing students' familiarity with this type of representation. The Chinese texts also involve students in the process of constructing such representations (e.g., asking students to complete the diagram by drawing a second line). These constructions, along with consistent and frequent exposure, may reinforce the strengths of schematic representations (to illustrate problem structures) while attenuating the limitations (non-transparency).

Focusing on the Basic Property and Making its Power Explicit for Computation

The dominant context for the DP in both U.S. texts is computation. This finding is consistent with Murata's (2008) conclusion about the SF-AW text based on a cross-national curriculum comparison on other topics. With regard to the computation problems, both U.S. texts focus more on presenting various computing strategies than on making the underlying principles explicit. This phenomenon is possibly due to the U.S. texts' overemphasis on the preparation stage thus delaying the opportunity to reveal the key principles and their formal algebraic treatments (Cai et al., 2005). In contrast to the U.S. texts' focus on computation, the Chinese texts guide students to discover and abstract the principle of the DP through many comparison problems (e.g., compute two problems in a pair). This practical use of comparison problems is supported by experimental findings (Gentner et al., 2003, 2004) that show that comparison can lead to better learning, transfer, and retrieval of principles from long-term memory to improve problem solving (Piroli & Anderson, 1985). The comparison problems in Chinese texts may also facilitate students' procedural knowledge so they may be more likely to use powerful methods and be more able to justify the use of a particular method (Rittle-Johnson & Star, 2007).

For Chinese curriculum designers, the final goal of teaching the DP is to enable students to use this property automatically to make computation easier. Findings from cognitive and education psychology (e.g., Anderson, 1995; Brown & Bennett, 2002; Rittle-Johnson et al., 2001; Sweller, 1988, 1989), as well as mathematics education (e.g., Hiebert & Wearne, 1996; NRC, 2001), support the notion that "automaticity" and "procedural fluency" are associated with conceptual understanding. The Chinese texts consistently ask students to "compute in convenient ways" involving complicated numbers such as fractions and decimals. These contexts dramatically show the power of the DP, and accumulated experiences can reasonably be expected to motivate students spontaneously to use this property and perhaps even enable them to see the beauty of mathematics. The automaticity resulting from procedural fluency can also reduce cognitive load thus saving working memory for higher-order activity such as problem solving (Brown & Bennett, 2002). In contrast, the U.S. texts mainly teach students to break apart a factor to perform whole number multiplication. Although these strategies are useful, important, and do involve the DP, they do not show the power of the DP. As a result, they hardly cultivate students' appreciation or automatic use of the DP, and do not hint at the flexible transition of this knowledge to other contexts.

Varying the Use of DP With Particular Attention to the Opposite Direction

There is little disagreement that a deep understanding of a concept entails a flexible use of it in multiple ways and various contexts (Hiebert et al., 1997; Stigler et al., 1986). In this study, our findings reveal a striking difference of variability in using the DP across countries. While Chinese texts use the DP in both regular and opposite directions (more opposite directions) with different kinds of numbers (whole numbers, fractions, decimals, percents, expressions, and equations), both U.S. texts limit the use of the DP to the regular direction and whole numbers. As previously noted, limiting the use of the DP to a pure whole number context is representative of U.S. texts' avoidance of presenting repeated variables and mixed operations with decimals (fractions). The U.S. texts' lower expectation for student learning is not a new finding of our study (Cai et al., 2005; Murata, 2008; Schmidt, Wang, & McKnight, 2005; Stigler et al., 1986). Even within the whole number contexts, however, we suggest that U.S. textbooks pay increased attention to an opposite use of the DP. As Carpenter et al. (2003) illustrated, using the DP in an opposite direction [e.g., $50 + 30 = (5 + 3) \times 10$] was beneficial for algebraic thinking. In fact, a dominant use of the DP in a regular direction may result in students' superficial understanding and be detrimental to their mathematical thinking. Anecdotally, when the first author selected several problems from the Chinese sixth-grade text (e.g., $2/3 \times 5/7 + 2/3 \times 2/7$) as a review of the DP in her math methods class, all of the preservice teachers were frustrated and confused by the transformation "$2/3 \times 5/7 + 2/3 \times 2/7 = 2/3\ (5/7 + 2/7)$." As one preservice teacher commented, "We never thought of using the DP in an opposite direction." Such a comment was truly consistent with our curriculum examination findings.

Introducing Repeated Variables as a Natural Bridge between Arithmetic and Algebra

The DP is a basic idea underlying both arithmetic and algebraic learning. A key to naturally bridging arithmetic and algebra through the DP in Chinese texts is the introduction of repeated variables. The Chinese fourth-grade text introduces this type of expression right after the DP is formally introduced, thus quickly setting the stage for later work with equations. In contrast, neither of the U.S. elementary texts introduced repeated variables, but only expressions and equations with a single variable ($\mathrm{a}x \pm \mathrm{b}$; $\mathrm{a}x \pm \mathrm{b} = \mathrm{c}$). One possible reason for postponing the learning of repeated variables is curriculum designers' concerns about students' inadequate preparation (Cai et al., 2005). However, current research suggests that this topic is beneficial for students' early algebraic thinking and can be learned by elementary students (e.g., Carpenter et al., 2003). Indeed, without a deep understanding of the DP, even if the topic of repeated variables is postponed to later grades, students will still struggle with it. The difficulty many American college students have working with repeated variables was confirmed by a recent study reported by Koedinger et al. (2008): Although 80% of 153 college students (in Experiment 1) could perform correctly on an equation with a single variable ($6\mathrm{x} + 46 = 65.50$), only 29% of them correctly solved an equation with repeated variables ($x - 0.15x = 38.24$). If these students were able to use the DP to simplify "$x - 0.15x$" as "$(1 - 0.15)x$," thus transforming the original equation to $0.85x = 38.24$, the correctness rate might have been greatly improved. In fact, another group of college students in this study (in Experiment 2) who had stronger prior knowledge solved this same problem ($x - 0.15x = 38.24$) with a much higher correctness (75%). Koedinger et al.'s study seems to support the positive effects of a deep and flexible understanding of the DP on later algebraic understanding. As mentioned earlier, Koedinger et al.'s study also revealed the benefits of using algebraic equations (involving repeated variables) to solve complex word problems. We believe it is safe to conclude that the move from arithmetic to algebra may be facilitated through an introduction of repeated variables, along with targeted practice, after a formal and focused presentation of the DP.

FUTURE STUDY

Our study focusing on the DP serves as a window showing the cross-cultural differences in textbook approaches to mathematical principles. The marked differences observed in textbook presentation may be one of the sources of the gap between U.S. and Chinese elementary students' understanding of the key concepts and principles of mathematics (e.g., Li et al., 2008), both in terms of initial learning and lateral transfer. Our cross-cultural findings suggest alternatives for improving the U.S. elementary text approaches and potentially inspiring U.S. teachers to develop better classroom activities. In addition, our findings suggest that it is worthwhile for cognitive psychology and mathematics education researchers to conduct future experimental and analytical studies, to further ascertain how the approaches to learning, as exemplified by the Chinese texts in the current study, may contribute to students' arithmetic and algebraic thinking and problem-solving skills.

ACKNOWLEDGMENTS

We are grateful to the editor and reviewers for their constructive comments on an earlier version of this article. Thanks also go to Becky Kromminga, Kelsey Campfield, and Melinda diSessa for helpful assistance.

REFERENCES

Anderson, J. R. (1993). *Rules of the mind.* Hillsdale, NJ: Lawrence Erlbaum.

Anderson, J. R. (1995). *Cognitive psychology and its implications* (4th ed.). New York: Freeman.

Baek, J. M. (2008). Developing algebraic thinking through explorations in multiplication. In C. E. Greens & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 141–154). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education* (Unpublished doctoral dissertation). Michigan State University, East Lansing.

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is—Or might be—the role of curriculum materials in teaching learning and instructional reform? *Educational Researcher*, *25*, 9, 6–8, 14.

Bell, A. (1996). Problem-solving approaches to algebra: Two aspects. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 167–186). Dordrecht, The Netherlands: Kluwer Academic.

Brown, S. W., & Bennett, E. D. (2002). The role of practice and automaticity in temporal and nontemporal dual-task performance. *Psychological Research*, *66*, 80–89.

Bruner, J. S. (1960/1977). *The process of education.* Cambridge, MA: Harvard University Press.

Cai, J. (1995). A cognitive analysis of U.S. and Chinese students' mathematical performance on tasks involving computation, simple problem solving, and complex problem solving. *Journal for Research in Mathematics Education Monograph*, *7*, 1–160.

Cai, J. (2004). Why do U.S. and Chinese students think differently in mathematical problem solving? Impact of early algebra learning and teachers' beliefs. *Journal of Mathematical Behavior*, *23*, 135–167.

Cai, J., Lew, H. C., Morris, A., Moyer, J. C., Ng, S. F., & Schmittau, J. (2005). The development of students' algebraic thinking in earlier grades: A cross-cultural comparative perspective. *ZDM: The International Journal on Mathematics Education*, *37*(1), 5–15.

Cai, J., & Moyer, M. (2008). Developing algebraic thinking in earlier grades: Some insights from international comparative studies. In C. E. Greens & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 169–180). Reston, VA: National Council of Teachers of Mathematics.

Carpenter, S. K., Pashler, H., Cepeda, N. J., & Alvarez, D. (2007). Applying the principles of testing and spacing to classroom learning. In D. S. McNamara & J. G. Trafton (Eds.), *Proceedings of the 29th Annual Conference of the Cognitive Proceedings of the 29th Annual Conference of the Cognitive* (p. 19). Nashville, TN: Cognitive Science Society.

Carpenter, T. P., Fennema, E., Franke, M., Levi, L., & Emspon, S. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Franke, L. P., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic & algebra in elementary school*. Portsmouth, NH: Heinemann.

Cepeda, N. J., Pashler, H., Vul, E., Wixted, J. T., & Rohrer, D. (2006). Distributed practice in verbal recall tasks: A review and quantitative synthesis. *Psychological Bulletin*, *132*, 354–380.

Charles, R. I., Crown, W., Fennell, F., Caldwell, J. H., Cavanagh, M., Chancellor, D. et al.. (2004). *Scott Foresman–Addison Wesley mathematics* (Student edition, grades K–6; Teacher edition, grades 3–6). Glenview, IL: Pearson Education.

Chi, M. T. H. (2000). Self-explaining: The dual processes of generating and repairing mental models. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 161–238). Mahwah, NJ: Lawrence Erlbaum.

Chi, M. T. H., de Leeuw, N., Chiu, M., & LaVancher, C. (1994). Eliciting self-explanations improves understanding. *Cognitive Science*, *18*, 439–477.

Chi, M. T. H., Feltovitch, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science*, *5*, 121–152.

Colhoun, J., Gentner, D., & Loewenstein, J. (2005). Learning abstract principles through principle-case comparison. In B. C. Love, K. McRae, & V. M. Sloutsky (Eds.), *Proceedings of the 30th Annual Conference of the Cognitive Science Society* (pp. 1659–1664). Austin, TX: Cognitive Science Society.

Craig, S. D., Sullins, J., Witherspoon, A., & Gholson, B. (2006). The deep-level-reasoning-question effect: The role of dialogue and deep-level-reasoning questions during vicarious learning. *Cognition and Instruction*, *24*, 565–591.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science*, *7*, 155–170.

Gentner, D., Loewenstein, J., & Thompson, L. (2003). Learning and transfer: A general role for analogical encoding. *Journal of Educational Psychology*, *95*, 393–408.

Gentner, D., Loewenstein, J., & Thompson, L. (2004). Analogical encoding: Facilitating knowledge transfer and integration. In K. Forbus, D. Gentner, & T. Regier (Eds.), *Proceedings of the 26th Annual Conference of the Cognitive Science Society* (pp. 452–457). Chicago, IL: Cognitive Science Society.

Goldstone, R. L., & Sakamoto, Y. (2003). The transfer of abstract principles governing complex adaptive systems. *Cognitive Psychology*, *46*, 414–466.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *Journal of the Learning Sciences*, *14*, 69–110.

Greenes, C., Larson, M., Leiva, M. A., Shaw, J. M., Stiff, L., Vogeli, B. R., & Yeatts, K. (2005). *Houghton Mifflin* (Student edition, grades K–6; Teacher edition, grades 3–6). Boston, MA: Houghton Mifflin.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York: Macmillan.

Hall, R., Kibler, D., Wenger, E., & Trnxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. *Cognition and Instruction*, *6*, 223–283.

Hamann, M. S., & Ashcraft, M. H. (1986). Textbook presentations of the basic addition facts. *Cognition and Instruction*, *3*, 173–192

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., et al.. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding and skill in multidigit addition and instruction. *Cognition and Instruction*, *14*, 251–283.

Horizon Research. (2002). *The 2000 national survey of science and mathematics education: Compendium of tables*. Retrieved July 15, 2008, from http://2000survey.horizon-research.com/reports/tables/tables_complete.pdf

Howe, R. (1999). Knowing and teaching elementary mathematics. *Notices of the American Mathematical Society*, *46*, 881–887.

Kaminiski, J. A., Sloutsky, V. M., & Heckler, A. F. (2006). Effects of concreteness on representation: An explanation for differential transfer. In R. Sun & N. Miyake (Eds.), *Proceedings of the 28th Annual Conference Proceedings of the 28th Annual Conference* (pp. 1581–1586). Mahwah, NJ: Cognitive Science Society.

Koedinger, K. R., & Terao, A. (2002). A cognitive task analysis of using pictures to support pre-algebraic reasoning. In C. D. Schunn & W. Gray (Eds.), *Proceedings of the 24th Annual Conference of the Cognitive Science Society* (pp. 542–547). Mahwah, NJ: Lawrence Erlbaum.

Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2008). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science*, *32*, 366–397.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representation on quantitative reasoning. *Journal of the Learning Sciences*, *13*, 129–164.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2007). How students view the general nature of their errors. *Education Studies in Mathematics*, *66*, 43–59.

Li, X., Ding M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and in the United States, *Cognition and Instruction*, *26*, 195–217.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 57–94). New York: Lawrence Erlbaum.

Mayer, R. E. (2001). *Multimedia learning*. New York: Cambridge University Press.

McNeil, N. M., & Alibali, M. W. (2005). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development*, *6*, 285–308.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., et al.. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction*, *24*, 367–385.

McNeil, N. M., & Jarvin, L. (2007). When theories don't add up: Disentangling the manipulatives debate. *Theory into Practice*, *46*, 309–316.

Moreno, R., & Mayer, R. C. (1999). Multimedia-supported metaphors for meaning making in mathematics. *Cognition and Instruction*, *17*, 215–248.

Murata, A. (2008). Mathematics teaching and learning as a mediating process: The case of tape diagrams. *Mathematical Thinking and Learning*, *10*, 374–406.

Nathan, M., Kintsch, W., & Young, E. (1992). A theory of algebra-word-problem comprehension and its implications for the design of learning environments. *Cognition and Instruction*, *9*, 329–389.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

National Research Council. (2004). *On evaluating curricular effectiveness: Judging the quality of K–12 mathematics evaluations* (J. Confrey & V. Stohl, Eds.). Washington, DC: National Academy Press.

Pashler, H., Bain, P. M., Bottge, B. A., Graesser, A., Koedinger, K., McGaniel, M., et al.. (2007). *Organizing instruction and study to improve student learning* (NCER 2007–2004). Washington, DC: National Center for Education Research.

Piroli, P. L., & Anderson, J. R. (1985). The role of learning from examples in the acquisition of recursive programming. *Canadian Journal of Psychology*, *39*, 240–272.

Porter, A. (1989). A curriculum out of balance: The case of elementary school mathematics. *Educational Researcher*, *18*(5), 9–15.

Renkl, A. (2002). Worked-out examples: Instructional explanations support learning by self-explanations. *Learning and Instruction*, *12*, 529–556.

Renkl, A., Atkinson, R. K., & Große, C. S. (2004). How fading worked solution steps works—A cognitive load perspective. *Instructional Science*, *32*, 59–82.

Renkl, A., Atkinson, R. K., Maier, U., & Staley, R. (2002). From example study to problem solving: Smooth transitions help learning. *Journal of Experimental Education*, *70*, 293–315.

Renkl, A., Stark, R., Gruber, H., & Mandl, H. (1998). Learning from worked-out examples: The effects of example variability and elicited self-explanations. *Contemporary Educational Psychology*, *23*, 90–108.

Resnick, L. B., Cauzinille-Marmeche, E., & Mathieu, J. (1987). Understanding algebra. In J. Sloboda & D. Rogers (Eds.), *Cognitive processes in mathematics* (pp. 169–203). Oxford: Clarendon.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology* (vol. 3, pp. 41–95). Hillsdale, NJ: Lawrence Erlbaum.

Rittle-Johnson, B., & Koedinger, K. (2001). Using cognitive models to guide instructional design: The case of fraction division. In J. Moore & K. Stenning (Eds.), *Proceedings of the 23rd Annual Conference of the Cognitive Science Society* (pp. 857–862). Mahwah, NJ: Lawrence Erlbaum.

Rittle-Johnson, B., Siegler, S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*, 346–362.

Rittle-Johnson, B., & Star, J. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology*, *99*, 561–574.

Schifter, D. (1999). Reasoning about operations: Early algebraic thinking in grades K–6. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K–12* (pp. 62–81). Reston, VA: National Council of Teachers of Mathematics.

Schifter, D., Monk, S., Russell, S. J., Bastable, V. (2008). Early algebra: What does understanding the laws of arithmetic mean in the early grades? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 413–448). New York: Lawrence Erlbaum.

Sheng, C. (2008). *Curriculum analysis (Grades 2–6)*. Retrieved October 25, 2008 from http://www.xxsx.cn/item.asp?SmallClassID=2&ItemID=571

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*, 525–559.

Stevenson, H. W., & Stigler, J. W. (1992). *The learning gap: Why our schools are failing and what we can learn form Japanese and Chinese education*. New York: Simon & Schuster.

Stigler, J., Fuson, K., Ham, M., & Kim, M (1986). An analysis of addition and subtraction word problems in American and Soviet elementary mathematics textbooks. *Cognition and Instruction*, *3*, 153–171.

Su, L., &Wang, N. (2005). *Elementary mathematics textbook* (vols. 1–12). Nanjing: Jiang Su Education Press.

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science*, *12*, 257–285.

Sweller, J. (1989). Cognitive technology: Some procedures for facilitating learning and problem solving in mathematics and science. *Journal of Cognitive Psychology*, *81*, 457–466.

Sweller, J. (1999). *Instructional design in technical areas*. Victoria, Australia: Australian Council of Education.

Trends in International Mathematics and Science Study. (1999). *Mathematics and science achievement of eighth-graders in 1999*. Retrieved June 1, 2009 from http://nces.ed.gov/timss/results99_1.asp

Trends in International Mathematics and Science Study. (2003). *Highlights from the trends in international mathematics and science study (TIMSS) 2003*. Retrieved June 1, 2009 from http://nces.ed.gov/pubs2005/2005005.pdf.

Trends in International Mathematics and Science Study. (2007). *Highlights from TIMSS 2007: Mathematics and science achievement of U.S. fourth and eighth-grade students in an international context*. Retrieved June 1, 2009 from http://nces.ed.gov/pubs2009/2009001.pdf.

Uttal, D. H., Liu, L. L., & Deloache, J. S. (1999). Taking a hard look at concreteness: Do concrete objects help young children learn symbolic relations? In C. S. Tamis-LeMonda (Ed.), *Child psychology: A handbook of contemporary issues* (pp. 177–192). Philadelphia: Psychology Press.

VanLehn, K., Jones, R. M., & Chi, M. T. H. (1992). A model of the self-explanation effect. *The Journal of the Learning Sciences*, *2*(1), 1–59.

Zhu, X., & Simon, H. A. (1987). Learning mathematics from examples and by doing. *Cognition and Instruction*, *4*, 137–166.