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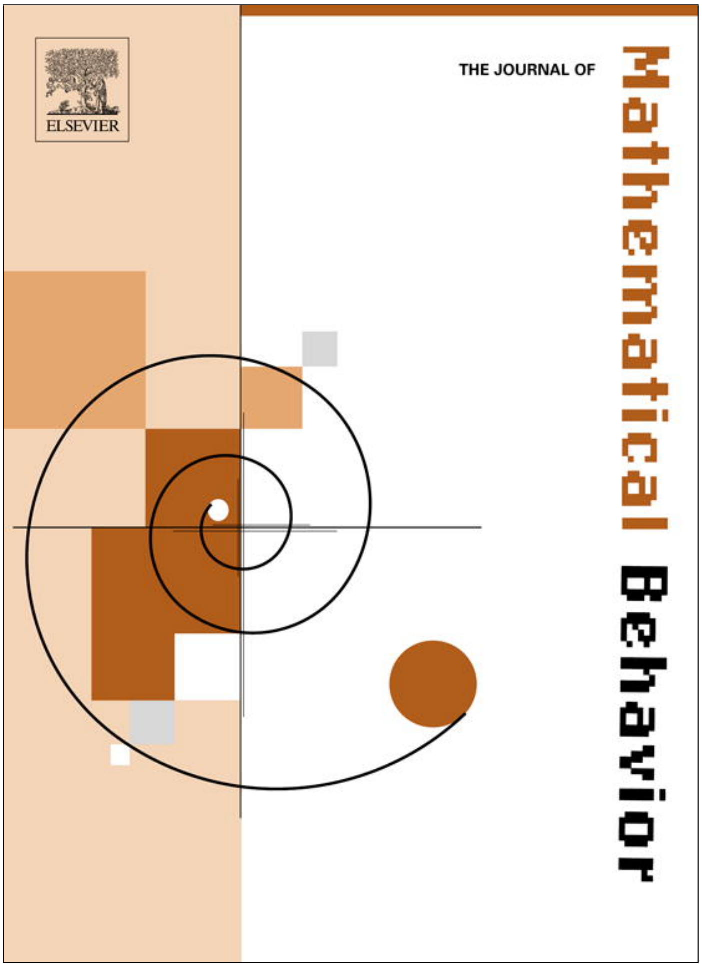

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Preservice elementary teachers' knowledge for teaching the associative property of multiplication: A preliminary analysis☆

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ABSTRACT

This study examines preservice elementary teachers' (PTs) knowledge for teaching the associative property (AP) of multiplication. Results reveal that PTs hold a common misconception between the AP and commutative property (CP). Most PTs in our sample were unable to use concrete contexts (e.g., pictorial representations and word problems) to illustrate AP of multiplication conceptually, particularly due to a fragile understanding of the meaning of multiplication. The study also revealed that the textbooks used by PTs at both the university and elementary levels do not provide conceptual support for teaching AP of multiplication. Implications of findings are discussed.

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1. Introduction

The understanding of fundamental principles in any domain is highly emphasized and viewed as an ultimate goal (Bruner, 1977; Goldstone & Sakamoto, 2003). The associative property (AP), commutative property (CP), and distributive property (DP) are fundamental mathematical principles that are taught at the elementary level (National Mathematics Advisory Panel [NMAP], 2008). Explicit understandings of fundamental properties are critical for students' algebraic learning, such as meaningful manipulation of algebraic expressions. Thus, Carpenter, Franke, and Levi (2003) have recommended that elementary teachers engage children in making, articulating, and refining conjectures about fundamental properties and identify important ideas for students to make conjectures about. However, typical U.S. computational lessons often emphasize steps leading to quick answers without a deep understanding of the underlying properties (Schifter, 1999; Thompson, 2008). Such procedurally-based teaching may stem largely from teachers' own weak mathematical knowledge for teaching. Very few studies have focused on elementary teachers understanding of fundamental mathematical properties possibly due to the fact that these principles are too commonly used to be noticed by professionals (Mason, 2008). The present study takes a first step, focusing on AP of multiplication [$(ab)c = a(bc)$], to explore what *preservice elementary teachers* (PTs) may bring to teacher education and what obstacles PTs may have in their existing conceptions. Since textbooks are potential resources that may support teacher learning and change (Ball, 1996; Davis & Krajcik, 2005), this study also briefly examines textbooks used by PTs. It is expected that findings will inform future work on preparing teachers to teach fundamental ideas more meaningfully.

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2. Rationale and theoretical framework

2.1. Why does this study focus on the AP of multiplication?

Our narrowed focus on the AP of multiplication in this study is based on the following considerations. First, AP is one of the three basic laws of arithmetic. Its significance warrants the necessity of this study (elaborated upon below). Second, in comparison to the other basic properties, research on AP is scarce. For example, there were studies about CP (e.g., Baroody, 1982; Baroody & Gannon, 1984) and DP (Ding & Li, 2010), however, no studies have focused exclusively on AP. Third, although AP can be applied to both addition and multiplication, we focus on the AP of *multiplication* because elementary students have more difficulties justifying this property (Carpenter et al., 2003). Thus, it is vital for teachers to help students make sense of it. Fourth, our focus was inspired by findings from a pilot research project on PTs' algebraic readiness where three participants demonstrated confusion about AP. Finally, it is our ultimate goal to use the case of AP to illustrate why PTs' understanding of basic mathematical ideas that are seemingly not complicated should be given adequate attention.

The use of AP of multiplication (along with other properties) provides tremendous flexibility for computation. For example, when computing $(3 \times 4) \times 25$, one may use AP to first compute the later two numbers $3 \times (4 \times 25)$ and find the answer easily (National Research Council [NRC], 2001). When students calculate 2×80, students may first compute $2 \times 8 = 16$ and then add one "0." This strategy also uses AP, $2 \times 80 = 2 \times (8 \times 10) = (2 \times 8) \times 10$ (Carpenter et al., 2003). Although AP, embodied by the algorithm in the latter example, is less transparent, procedures are seen "as possessing structures, as obeying constraints, as being related to each other in systematic ways" (Resnick & Omanson, 1987, p. 42). Thus, an understanding of the logic behind the procedures (in this case, AP) has been valued because it serves as a necessary condition to justify the algorithms (Resnick & Omanson, 1987). Being cognizant of the underlying property not only is indicative of sound understanding of AP but also allows students to transfer that understanding to novel contexts. For instance, when solving an equation like $\frac{1}{3}x = 11$, students may multiply both sides of the equation by 3 (the inverse of 1/3) to simplify it. An understanding of AP of multiplication will enable students to make sense of the rewriting process $3(\frac{1}{3}x) = 3 \times 11 \rightarrow (3 \times \frac{1}{3})x = 33$. In addition, AP of multiplication can serve as a powerful tool for reasoning and proof (Schifter, Monk, Russell, & Bastable, 2008; Wu, 2009). For example, to prove "when you multiply an even number times any whole number, you get an even number," one can use "$2n$" and "m" to represent any even number and whole number respectively and then use AP to obtain $(2n)m = 2(nm)$ (Carpenter et al., 2003). In fact, sound understanding of AP also plays a critical role in students' later learning of advanced concepts such as elementary group theory in university courses (Larsen, 2010).

Because of its significance, we focus on AP of multiplication in this study. We are cognizant that the AP and CP usually work together. For example, when one reasons $(5 \times 2) \times 3 = (3 \times 2) \times 5$, one may draw on both AP and CP [e.g., $(5 \times 2) \times 3 = (2 \times 5) \times 3 = 3 \times (2 \times 5) = (3 \times 2) \times 5$]. In order to exclude the possibility of conflating, we isolated AP from CP, focusing teachers' attention on AP only. For example, we expect that one use of AP would be $(5 \times 2) \times 3 = 5 \times (2 \times 3)$ without changing the order of these numbers. Below, we review the necessary knowledge components for teaching AP, which forms a conceptual framework for this study.

2.2. What do teachers need to know to teach AP of multiplication meaningfully?

2.2.1. Knowing what AP of multiplication is about

In order to recognize and discuss fundamental properties like AP involved in students' work, teachers should, at minimum, possess a clear understanding of this property. As Carpenter et al. (2003) stated, "Children have a great deal of implicit knowledge about fundamental properties in mathematics, but it usually is not a regular part of mathematics class to make that knowledge explicit" (p. 47). This was observed in Schifter et al. (2008) where third and fourth graders were engaged in reasoning about the statement, "a factor of a number is also a factor of that number's multiples" (p. 433), but the underlying idea of AP was never brought to the students' attention. Without teachers' explicit guidance, students' intuitive uses of the underlying properties do not indicate that they have discovered those properties (Baroody & Gannon, 1984). This, in turn, demands teachers' own explicit understanding about AP of multiplication.

One may describe AP of multiplication in natural language similar to the following, "When you multiply three numbers, it does not matter whether you start by multiplying the first pair of numbers or the last pair of numbers" (Carpenter et al., 2003, p. 108). AP of multiplication can also be formally represented as $(ab)c = a(bc)$ and indicated by arithmetic examples such as $(3 \times 4) \times 25 = 3 \times (4 \times 25)$. These knowledge components (definition/description, formula, arithmetic examples) are prerequisites for a teacher to present AP in mathematically precise and pedagogically comprehensible ways (Ball & Bass, 2000; Wu, 2010).

Very few studies have reported preservice *elementary* teachers' knowledge about AP of multiplication. However, we have identified a few studies (Tirosh, Hadass, & Movshovitz-Hadar,1991; Zaslavsky & Peled, 1996) that reported secondary mathematics teachers' (both inservice and preservice) confusion between AP and CP. These teachers commonly believed that AP and CP were dependent because both were related to changing the order. They did not realize that CP changes the order of elements while AP changes the order of operation. In addition, some teachers thought that CP and AP were identical except for the fact that CP referred to two elements while AP referred to three elements (Tirosh et al., 1991). Little empirical data, however, shows whether the above misconceptions about AP is shared by elementary PTs who have been reported to

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possess weak knowledge about various topics such as place value, fractions and decimals (Ball, 1988; Borko et al., 1992; Son & Crespo, 2009; Stacey et al., 2001). Such information is needed in order to provide better development for future teachers.

Simply knowing what AP of multiplication is (e.g., definition, formula, example) does not guarantee that a teacher will use it to teach for understanding. Elementary teachers should possess specialized content knowledge (Ball, Thames, & Phelps, 2008) such as using representations to illustrate AP of multiplication in meaningful ways. This is because definitions and formulas are abstract and generalizable facts (also called declarative knowledge) and need to be reconstructed by learners in order to apply them. Cognitive psychologists have also pointed out abstract representations of principles "offer little by way of scaffolding for understanding, and they may not generalize well because cues to resemblance between situations have been stripped" (Goldstone & Wilensky, 2008, p. 479). As such, teachers should use concrete contexts to assist students in the encoding of declarative knowledge (Hall, 1998) or to help elementary students make sense of abstract principles during their initial learning (Goldstone & Son, 2005; Goldstone & Wilensky, 2008).

2.2.2. Knowing how to illustrate AP of multiplication

2.2.2.1. Pictorial representations. One frequently recommended concrete context is pictorial representation. When a diagram is added to a text presentation, it enhances student learning (Mayer, 2001; Moreno & Mayer, 1999) because diagrams group information together thus supporting learners to make perceptual inferences and retrieve problem-relevant operators from memory (Larkin & Simon, 1987). In fact, the use of diagrams has been suggested as a powerful tool for teaching abstract mathematical ideas (e.g., Cai & Moyer, 2008; Murata, 2008; Ng & Lee, 2009; Simon & Stimpson, 1988). The significance of AP of multiplication has been acknowledged in the nation's leading documents (Common Core State Standards [CCSS] Initiative, 2010; National Council of Teachers of Mathematics [NCTM], 2000, 2006); these documents did not offer ways to illustrate AP pictorially. A few existing studies suggested a volume model. For example, Carpenter et al. (2003) offered a three-dimensional array. Yet, no discussions were provided in terms of how a volume diagram may be used to illustrate AP of multiplication. The National Research Council (NRC, 2001) did illustrate this process by decomposing a volume box into several groups of blocks. However, the given pictures and the corresponding number sentences did not match in meaningful ways (see Fig. 1):

In Fig. 1a, a volume box (middle) was decomposed in two ways (a) 3 groups of 5×4 blocks (left), and (b) 4 groups of 3×5 blocks (right). Decomposing a volume box into equal groups of blocks is a pedagogically and developmentally sound move because the equal groups meaning serves as children's preliminary model for multiplication (Fischbein, Deri, Nello, & Marino, 1985). Since multiplication is essentially a binary operation that composes two quantities into a third derived one (Schwartz, 1988), NRC's two pictures could be denoted as $3 \times (5 \times 4)$ (left) and $4 \times (3 \times 5)$ (right), based on the U.S. convention of multiplication – "a groups of b" is denoted as "$a \times b$." These two number sentences together, however, do not show AP. The NRC correctly labeled the left picture but arbitrarily labeled the right picture as $(3 \times 5) \times 4$, without following the conventions of multiplication consistently in the same context. It should be noted that there are different conventions in representing multiplication across cultures (e.g., "3×2" means "3 groups of 2" in the U.S. but "2 groups of 3" in Japan), it does not matter which convention has been adopted but it should be represented consistently within one's own culture and, of course, within the same context. As Schwartz (2008) emphasized, if a teacher sometimes uses 3×5 to represent "three groups of five" and then other times to represent "five groups of three," children will have difficulties constructing an understanding of multiplication and mathematics will become spurious and meaningless.

A correct illustration of AP of multiplication based on a volume model was found in Beckmann (2008) (see Fig. 1b). A 4-inch-high, 2-inch-long, 3-inch-wide box shape was decomposed into two ways. One way was to decompose the box into 4 groups of 2×3 blocks (left picture). The second way was to further decompose each 2×3 block into 2 groups of 3 blocks, resulting in 4×2 groups of 3 blocks (right picture). Based on U.S. convention of multiplication, these two pictures can be represented as $4 \times (2 \times 3)$ and $(4 \times 2) \times 3$ respectively, which together illustrate AP of multiplication. However, teachers should be cautious that a volume model may be too complex for initial learning of AP because a volume box may be decomposed in various ways, some of which do not demonstrate AP (as indicated by the NRC example).

2.2.2.2. Word problem situations. Another type of concrete context is word problems. Compared to formalisms or pure arithmetic examples, word problems may provide a familiar situation that allows students to activate their prior knowledge or real-world knowledge to make sense of abstract mathematical ideas (Koedinger & Nathan, 2004; NRC, 2001). To our best knowledge, no existing studies have discussed how AP of multiplication may be illustrated through a word problem situation. However, existing literature has suggested approaches to illustrating other mathematical principles that may be informative for the present study. Resnick, Cauzinille-Marmeche, and Mathieu (1987) discussed that "$a - (b + c) = a - b - c$" may be illustrated through two situations: (a) Mary had 30 candies. She gave 20 of them to her friends, 8 to Sandra, and 12 to Tom [representing $30 - (8 + 12)$]; and (b) Mary had 30 candies. She gave 8 of them to Sandra. Later she gave 12 candies to Tom (representing $30 - 8 - 12$). Ding and Li (2010) reported that a Chinese textbook introduced the distributive property, $(a + b)c = ac + bc$, through a word problem context – A pair of pants is 45¥, and a jacket is 65¥. Someone buys 5 jackets and 5 pants. How much do they pay altogether? This problem was solved two ways, $(65 + 45) \times 5$[1] and $65 \times 5 + 45 \times 5$, together illustrating this property.

[1] Based on Chinese convention, the total cost of 5 pants with 45¥ each is represented as 45×5.

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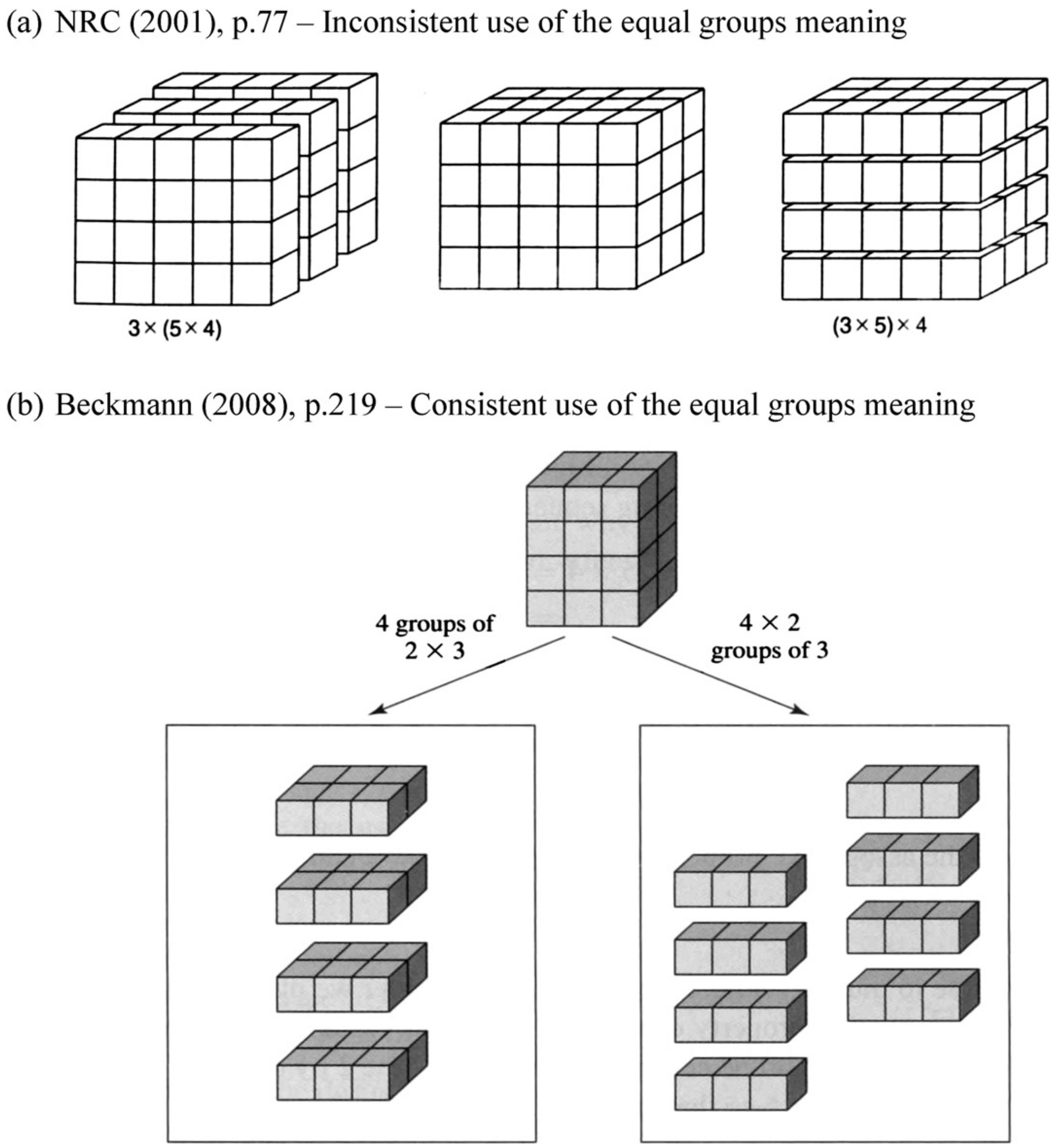

Fig. 1. Illustrations for AP of multiplication using a volume model.

The above approach – solving a word problem in two ways that together shows a fundamental principle – may be used to illustrate AP of multiplication. A widely used U.S. elementary textbook series, *Houghton Mifflin* (Greenes et al., 2005, 4th grade), does introduce AP of multiplication in a word problem context, "Upright bass strings come in sets of 4. Suppose one box holds 2 sets of strings. If a musician orders 3 boxes, how many strings will there be?" (p. 100). Such a story problem context, compared with a volume model, may be more familiar to students. The problem structure is "3 boxes of 2 sets of 4 strings" as illustrated by Fig. 2.

This problem structure can be viewed in two ways: either as 3×2 sets of 4 strings or 3 boxes of 2×4 strings. A teacher may first guide students to combine 3 boxes and 2 sets/box to obtain the total number of sets ($3 \times 2 = 6$) and then calculate 6 sets of strings ($6 \times 4 = 24$), resulting in the first solution $(3 \times 2) \times 4$. A teacher may also guide students to first find combinations of 2 sets and 4 strings/set to obtain the total number of strings in one box ($2 \times 4 = 8$) and then calculate 3 boxes of 8 strings/box ($3 \times 8 = 24$), resulting in the second solution $3 \times (2 \times 4)$. These two solutions together illustrate AP of multiplication [$(3 \times 2) \times 4 = 3 \times (2 \times 4)$, see Fig. 2]. However, the elementary textbook did not use the word problem context this way (to be elaborated later). A teacher should be careful to consistently stress the meaning of multiplication, that $a \times b$ means a groups of b. Otherwise, students will obtain a blurred understanding of multiplication (Schwartz, 2008) and their solutions to the above word problem will be random combinations of 3, 2, and 4 (e.g., $2 \times 3 \times 4$, $4 \times 3 \times 2$, $4 \times 2 \times 3$), which does not illustrate AP. A teacher should also be aware of the unit designation (e.g., boxes, sets, strings) and clearly understand which two quantities are related to which third quantity.

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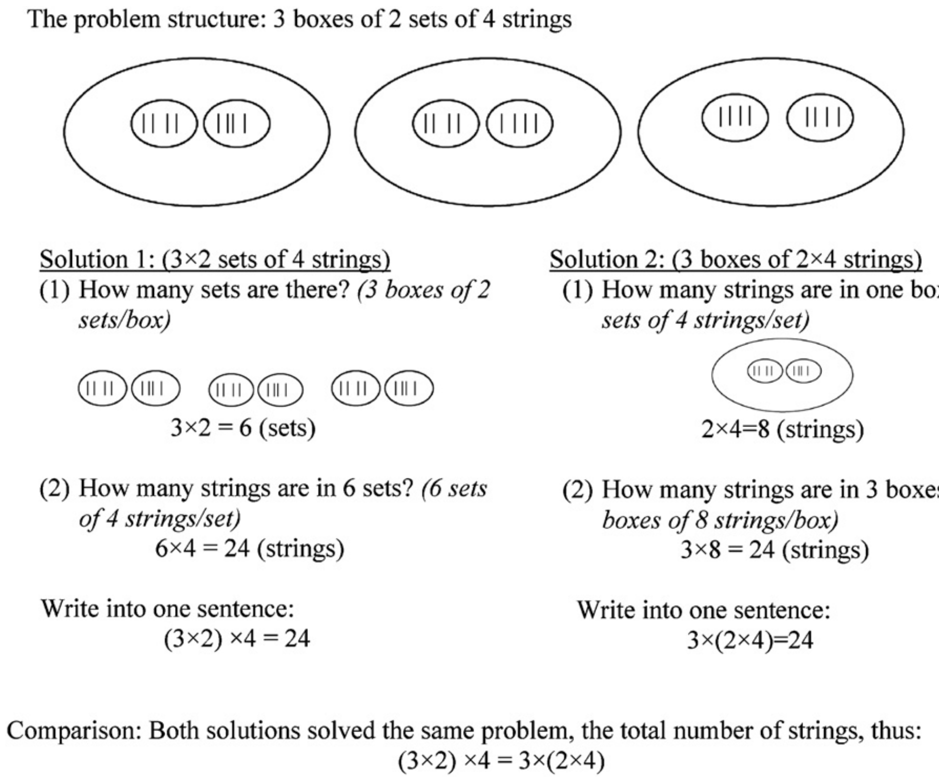

Fig. 2. The problem structure of a 4th grade word problem and the two solutions that together illustrate AP of multiplication.

In summary, regardless of using pictures or word problem contexts, teachers need to pay attention to two aspects in order to illustrate AP of multiplication: (a) using the meaning of multiplication consistently to enable correct explanations, and (b) viewing a picture or solving a story problem two ways that will together show AP. It is not a trivial task to teach AP for understanding because it demands more than knowing what the property is. Using the meaning of multiplication consistently may be particularly challenging for teachers including PTs, as indicated by the NRC (2001) example. Indeed, prior studies have reported PTs' difficulties in justifying multiplicative thinking (Lo, Grant, & Flowers, 2008; Simon & Blume, 1994). For example, in Simon and Blume (1994), when a given rectangular area is evenly divided into rows and columns, PTs suggested that they should multiply the number of rows by the number of columns to find the total number of units. However, when being prompted to think why multiply, most of PTs responded with "that's the way we've been taught," and "It's a mathematical law." Almost no PTs were able to view the first row as one group and the first column as an indicator of the number of groups. Thus, most PTs could not use the equal groups meaning of multiplication to justify why a rectangular area can be calculated using length × width (Simon & Blume, 1994). The complexity of teaching AP of multiplication, therefore, calls for explicit supports for PTs' knowledge growth including the potential support of textbooks.

2.3. Textbooks: potential support for teacher learning

Textbooks play an important role in teaching and learning (McNeil et al., 2006; Reys, Reys, & Chávez, 2004; Schmidt et al., 2001) by shaping teachers' instructional practices and curriculum planning (Borko & Shavelson, 1990; Nathan, Long, & Alibali, 2002). Evidence has been reported that preservice elementary teachers can improve their knowledge for teaching through the use of elementary textbooks during field experiences (Beyer & Davis, 2011; Forbes, 2011; Nicol & Crespo, 2006) and through discussions of them in university classes (Lloyd, 2009). In addition, PTs as university students can also directly learn from mathematics content and/or methods textbooks. As such, textbooks may serve as educative curriculum materials in supporting teachers' learning and changes (Ball, 1996; Davis & Krajcik, 2005; Drake & Sherin, 2006). In fact, teachers in mathematically high-achieving countries such as China consistently reported that studying textbooks was the most effective way to improve their mathematical knowledge for teaching (Cai & Wang, 2010; Ding, Li, Li, & Gu, 2012; Ma, 1999).

2.4. The present study

This study focuses on AP of multiplication, exploring what levels of understanding PTs may bring to teacher education and how existing textbook resources may provide support for their learning. In particular, we ask: (1) How do PTs understand what AP of multiplication is? (2) How do PTs illustrate AP of multiplication using concrete contexts? What are their difficulties and barriers in doing so? (3) What are the textbook potentials in preparing PTs to teach AP of multiplication meaningfully?

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1) What is the Associative Property of Multiplication?
(a) In words
____
(b) In a formula
____

2) How will you use the Associative Property to multiply three numbers, 2, 4, and 5?

3) Can you draw pictures to *show and explain* why the Associative Property of Multiplication works? Please use numbers, 2, 3, and 4. (*Clue: Connect your explanation with the basic meanings of multiplication.)*

4) How will you use the following word problem to teach third or fourth graders the idea of the Associative Property?
<u>Pencils come in sets of 4. Suppose one box holds 2 sets of pencils. If a teacher orders 3 boxes, how many pencils will there be?</u>

Fig. 3. The survey questionnaire.

3. Methods

3.1. Participants

PTs who registered in a *Math Semester* in a large mid-western four-year research university in the U.S. participated in this study. This Math Semester has been regularly offered for PTs each Spring and Fall semester. It includes a mathematics methods course, a math content course, a general pedagogy course, and field experiences. These courses are the first and also last set of required classes about learning to teach mathematics before student intern as student teachers and graduate as certified teachers. PTs in this study took the mathematics methods and content courses in a blocked time on Monday and Wednesdays, and conducted teaching practices at elementary schools on Tuesday and Thursdays. The textbook used for the methods course was Reys, Lindquist, Lambdin, and Smith (2009) and in the content course PTs used Sowder, Sowder, and Nickerson (2007). For field experience at elementary schools, they used the aforementioned *Houghton Mifflin* (Greenes et al., 2005).

A total of 56 PTs were recruited from three sessions of the Math Semester where the first author taught the methods course for one session. The participation rate was 84.8% (56 out of 66). Among the 56 K-6 PTs, the majority of them were females and Caucasian. There were 4 male and 4 non-white PTs. The knowledge survey was administered before teachers' learning of AP during the Math Semester. The purpose of conducting a survey at this early stage was to identify possible missing aspects of PTs' existing knowledge base for teaching AP of multiplication, which may inform subsequent teacher preparation. Since this study does not aim to investigate the instructional effect on PTs' knowledge growth (a different story), post-instructional assessment data was not collected.

3.2. Survey materials

The survey questionnaire included four questions (see Fig. 3) that were aligned with the reviews of literature.

The first two questions were used to examine whether PTs know what AP of multiplication is. Question 1 (Q1) asked for a description or a definition of AP of multiplication, using words (Q1a) or a formula (Q1b). Q2 asked for computing an arithmetic example using AP of multiplication. This question is a triangulation with Q1. In retrospect, we acknowledged that Q2 could have been better worded by directly asking for an example of AP of multiplication using the numbers 2, 4, and 5. This is because when being asked to "apply" AP, a PT may only provide one solution to find the answer [e.g., $2 \times (4 \times 5) = 20$], which does not necessarily show AP of multiplication. In later data analysis, we paid particular attention to this issue (to be elaborated upon later). The last two questions required teachers to use concrete contexts such as pictures (Q3) or a word problem situation (Q4) to illustrate AP of multiplication. In fact, Q4 was a literal modification of the previously discussed word problem presented by *Houghton Mifflin* (Greenes et al., 2005, see Fig. 2). Although the above four problems were designed with different intentions, they were not completely independent because one could not illustrate AP of multiplication if she/he did not know what this property is.

The above questionnaire was piloted in the aforementioned research project with three PTs. The PT who obtained the highest score demonstrated the best understanding during the follow-up interview while the PT who received the lowest score had fragile understanding. Some unclear wording in the original questionnaire was addressed. However, the possible wording issue in Q2 did not occur with the participants on the pilot test. In addition, we caution that this questionnaire should not be viewed as a direct measure of teachers' knowledge for teaching because such an instrument demands many

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more structured questions placed through a more rigorous testing and refinement process. Nevertheless, we expect an analysis of PTs' responses to the four questions will provide useful information for teacher educators.

3.3. Data analysis

3.3.1. Analyzing PT's knowledge about AP of multiplication: Q1a, Q1b, Q2

A three-scale rubric, 2, 1, and 0, indicating "correct," "partially correct," and "wrong" respectively, was used to rate PTs' responses to each item. For PTs' knowledge about what AP of multiplication is, we mainly focused on their responses to Q1a, Q1b, and Q2. Two points were assigned to the statements of AP as grouping either the first two or the last two numbers with the product not changing. For a few descriptions that were quite formal (e.g., involving words like "the quantity of $a \times b$"), as long as there were no mathematical mistakes involved, were coded as correct. Similarly, correct understanding was assigned if a PT provided a formula like $(ab)c = a(bc)$ for Q1(b) or an example like $(2 \times 4) \times 5 = 2 \times (4 \times 5)$ for Q2. In contrast, 0 points were assigned if a response did not explain AP of multiplication. Examples of wrong responses in Q1(a) include descriptions like "It doesn't matter the order you multiply your numbers because the answer will remain the same." For Q1(b), responses like "$ab = ba$," "$(a \times b) \times c = (c \times b) \times a$," "$(a + b) + c = a + (b + c)$," "$a(b + c) = ab + ac$" and "3 groups of $4 = 12$, $3 \times 4 = 12$" were coded as wrong because they actually stated other properties or did not pertain to AP. For Q(2), examples like $(2 \times 4) \times 5 = 2 \times (5 \times 4) = 4 \times (2 \times 5)$ received 0 points because they did not represent AP. We credited a teacher's partial understanding (1 point) for responses such as using a specific example [e.g., $(2 \times 3) \times 4 = 2 \times (3 \times 4)$] as a formula or providing a formula that partially included AP of multiplication [e.g., $(ab)c = a(bc) = (bc)a$].

After coding Q1a, Q1b, and Q2, we classified the types of incorrect responses and identified PTs' most common confusion. To obtain a more reliable picture of PTs' understanding of what AP of multiplication is, we compared each PT's responses across Q1(a), Q1(b) and Q2 and triangulated them with Q3 and Q4, with their common confusion in mind. This type of analysis distinguished PTs' overall understanding about AP of multiplication into four levels: (a) understands, (b) likely to understand (c) not likely to understand, and (d) clearly does not understand (to be elaborated upon later).

The above triangulation process also enabled a determination of the extent to which the wording of Q2 may have affected the PTs' responses. Among 56 PTs, 46 provided two or multiple solutions (regardless of the correctness), thus demonstrating that they were not affected by the wording in Q2. With regard to the other ten PTs who only provided a one-way solution, we compared their responses to that of Q1. We found that only PT15 was misled by Q2 wording because she correctly responded to Q1 but provided a one-way solution in Q2, $2 \times (4 \times 5) = 40$. The rest of the PTs' mistakes were simply indicative of their own thinking as consistently appeared in Q1. For example, PT50 equalized AP of multiplication with multiplication, describing AP as "multiplication is adding groups of numbers together" (Q1a) and offering a formula as "3 groups of $4 = 12$, $3 \times 4 = 12$" (Q1b). Consequently, this PT in Q2 chose two numbers from "2, 4, 5" to multiply together, resulting in a response "$2 \times 4 = 8$, $4 \times 5 = 20$." The above triangulation process, although did not change our coding for Q2, improved the validity in analyzing PTs' knowledge about what AP of multiplication was.

3.3.2. Analyzing PT's knowledge for illustrating AP of multiplication: Q3 and Q4

Two points were assigned when PTs (a) analyzed the diagrams or word problems based on the meaning of multiplication, and (b) solved these problems two ways that together show AP. In Q3, PTs were asked to draw pictures to show and explain AP using 2, 3, and 4. One may draw a picture about "2 groups of 3 sets of 4 objects" to show $(2 \times 3) \times 4 = 2 \times (3 \times 4)$ or draw a picture about "3 groups of 2 sets of 4 objects" to show $(3 \times 2) \times 4 = 3 \times (2 \times 4)$, etc. Each picture can be viewed two ways using a similar approach explained in Fig. 2. For example, the picture in Fig. 2 may be viewed either as "3×2 sets of 4 objects" (Way 1) or "3 groups of 2×4 objects" (Way 2). A volume model would be acceptable for Q3 but a volume box should be decomposed meaningfully as in Beckmann (2008). With regard to Q4, the word problem structure (3 boxes of 2 sets of 4 pencils) is the same as the one illustrated in Fig. 2, only changing "strings" to "pencils." Thus, the two solutions may either be "3×2 sets of 4 pencils" or "3 boxes of 2×4 pencils" (see Fig. 2 for the details). It should be noted that although we hoped PTs would draw pictures to represent the word problem situation in Q4, we did not require such an illustration during our coding of Q4.

Zero points were assigned if a PT could not illustrate AP of multiplication using a picture (Q3) or the word problem (Q4). Examples of wrong responses include the following: a PT drew a picture showing DP but not AP, a PT solved Q3 or Q4 one way which did not show AP, a PT provided two-way solutions which showed CP but not AP, a PT's two-way solutions did show AP of multiplication but all of the steps could not be explained based on the concrete context. We assigned partial credit (1 point) to those two-way solutions that showed AP of multiplication but one of the steps did not have a meaning based on the concrete contexts.

To obtain a clearer picture of the obstacles that may have hindered PTs' abilities to illustrate AP of multiplication, we re-coded Q3 and Q4 focusing on PTs' level of understanding in terms of each of the two aspects: (a) understanding the meaning of multiplication, and (b) solving a problem two ways. With regard to aspect (a), we expected PTs to correctly and consistently use the meaning of multiplication across both steps. Regardless of the correctness of an overall response in Q3 and Q4, we classified PTs' understanding of the meaning of multiplication into three levels: L2 – both steps showed understanding; L1 – the 2nd step did not show understanding; and L0 – both steps did not show understanding. With regard to aspect (b), we also differentiated PTs' understanding into three levels: L2 – solving a problem two ways; L1 – solving a problem one way; and L0 – no number sentences and unable to tell. Table 1 provides examples of our recoding:

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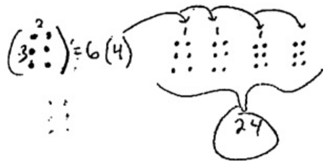
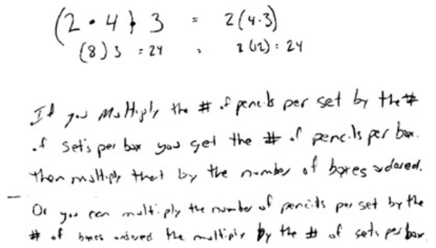
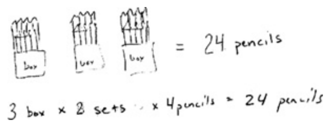
Table 1
Examples of how to recode Q3 and Q4.

Student response	Understand the meaning of multiplication	Solve a problem two ways
Q3 (0 point) 	L0 *Comment*: This picture does not show 2 groups of 3 as 2×3.	L2 *Comment*: Despite the confusion between AP and CP, the PT provides a two-way solution.
Q3 (0 point) 	L1 *Comment*: Even though we consider the first step as correct, the second step was wrong because this picture shows 4 groups of 6 for 6(4).	L1 *Comment*: This PT mistakenly wrote a running equation, $3 \times 2 = 6 \times 4 = 24$. Thus, this problem was actually solved one way, $(3 \times 2) \times 4$, which does not show AP.
Q4 (1 point) 	L1 *Comment*: The first step in first solution showed correct meaning based on the word problem context. The explanations are not accurate.	L2 *Comment*: This PT attempted to solve this problem two ways showing a correct approach.
Q4 (0 point) 	L2 *Comment*: The numbers are multiplied in a way that showed the meaning of multiplication in both steps (the units are not accurate, e.g., 2 sets should be 2 sets/box).	L1 *Comment*: This PT only solved this problem one way.

With regard to the above two aspects, PTs' understanding of the meaning of multiplication appeared to be more complex. We further analyzed PTs' use of multiplication and classified them into three typical cases. The purpose of the above in-depth analyses was to identify sources of PTs' difficulties in illustrating AP of multiplication.

3.3.3. Reliability checking

The first author independently coded all the surveys using the constant comparison method (Gay & Airasian, 2000). For example, after the PT's response was assigned to a scale (2, 1, 0), it was either then classified into an existing category or added as a new category if it did not fit into any existing category. Recoding of all of the items was conducted several months later to ensure a consistent use of the data coding and analysis rubric. A few changes were made. Moreover, the second author checked 20% of the PTs' responses using the finalized categories. The reliability for coding each of the five problems reached approximately 93%, 92%, 95%, 96%, and 94%. All the discrepancies were resolved before counting frequencies. In addition, the reliability for recoding Q3 and Q4 reached 100%. The pattern identification went through ongoing discussions among the authors.

3.4. Examining the textbooks used by the PTs

To examine the textbook potential in supporting PT's knowledge growth, we analyzed the methods and content textbooks (Reys et al., 2009; Sowder et al., 2007) used in the university courses and the elementary textbook used in PTs' field experiences (Greenes et al., 2005). The same conceptual framework was used for analysis, including (a) what is AP of multiplication (description, formula, arithmetical example) and (b) how to illustrate AP of multiplication (pictorial representation and word problem). The coding and analysis included three steps. First, we read through each textbook page and located instances about the AP of multiplication. Second, we analyzed how AP of multiplication was presented in terms of the above five knowledge components. Third, we analyzed whether the textbook presentations had the potential to address PTs' confusions and obstacles as indicated in the teacher survey.

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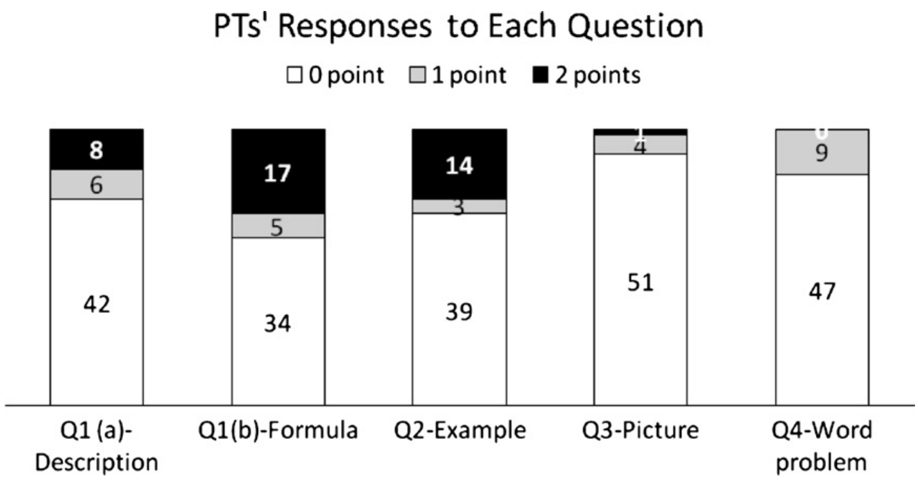

Fig. 4. Overall findings of teachers' knowledge for teaching AP of multiplication.

Table 2
A common confusion between AP and CP of multiplication across questions.

	# of responses examined[a]	# of PTs confused	% of PTs confused
Q1(a) – Description	42	23	55
Q1(b) – Formula	34	18	53
Q2 – Computation	39	22	56
Q3 – Diagram	37	31	83.8
Q4 – Word problem	19	17	89.5

[a] For Q1(a), Q1(b) and Q2, only the 0 points responses were examined; for Q3 and Q4, only those responses that used two-way solution were examined.

4. Results

4.1. General findings about PT's knowledge

Fig. 4 displays the overall results of the PTs' responses to each item surveyed.

As indicated by Fig. 4, the PTs in the current study had greater knowledge about what AP of multiplication was (Q1a, Q1b, Q2) than how to illustrate AP (Q3, Q4). However, PTs' understanding about this property was not ideal. Only about 8 PTs (14%) described this property correctly, 17 (30.4%) provided a correct formula, and 14 (25%) offered a correct arithmetic example. With regard to PTs' knowledge for illustrating AP of multiplication, no PTs attempted to use a "volume" model in Q3, but instead drew pictures based on equal groups meaning. However, in both Q3 and Q4, almost no PT successfully illustrated AP of multiplication using either pictures or word problems. In fact, 51 PTs (91.1%) in Q3 and 47 PTs (83.9%) in Q4 had no idea of how to illustrate this property.

4.2. Knowing what AP Is: common confusion

An inspection of PTs' incorrect responses reflected a common confusion between AP and CP. When being asked to describe AP in words (Q1a), many PTs responded in ways such as "numbers can be rearranged in any order." When asked to write a formula for AP of multiplication (Q1b) or use AP to multiply three numbers (Q2), many of them randomly switched the letters or numbers. Across these questions, PTs appeared to view AP as changing the order of numbers, which is indeed CP. Among wrong responses receiving 0 points, more than half (55%, 53%, and 56% in Q1a, Q1b, and Q2, respectively) were based on this confusion. When triangulated with Q3 and Q4, the common confusion between AP and CP continued. Table 2 provides detailed results.

Table 2 indicates that a high percentage of PTs who attempted to solve Q3 and Q4 in two ways were confused between AP and CP (83.8% in Q3 and 89.5% in Q4). This is not surprising because knowing "what is AP" is a prerequisite for illustrating this property. Because of this common confusion, many PTs switched numbers to multiply without being able to refer the symbols to concrete contexts.

With this common confusion in mind, the across-item triangulation indicates four levels of understanding about what AP of multiplication is. Fig. 5 provides examples for each level.

As indicated by Fig. 5, PT 38 was assigned as "understands AP." Although this PT's description of AP (Q1a) was not pedagogically sound, it was not confused with CP. PT38 also provided a correct formula (Q1b) and example (Q2). In addition, the expressions involved in Q3 [$2 \times (3 \times 4)$ and $(2 \times 3) \times 4$] and Q4 [$2 \times (4 \times 3)$ and $(2 \times 4) \times 3$] correctly embodied AP. PT53 was assigned as "likely to understand AP." This PT's description, "the grouping of the numbers does not matter," likely contained some confusion with CP. However, by referring to this PT's responses to Q1(b), Q2, Q3, and Q4, such confusion

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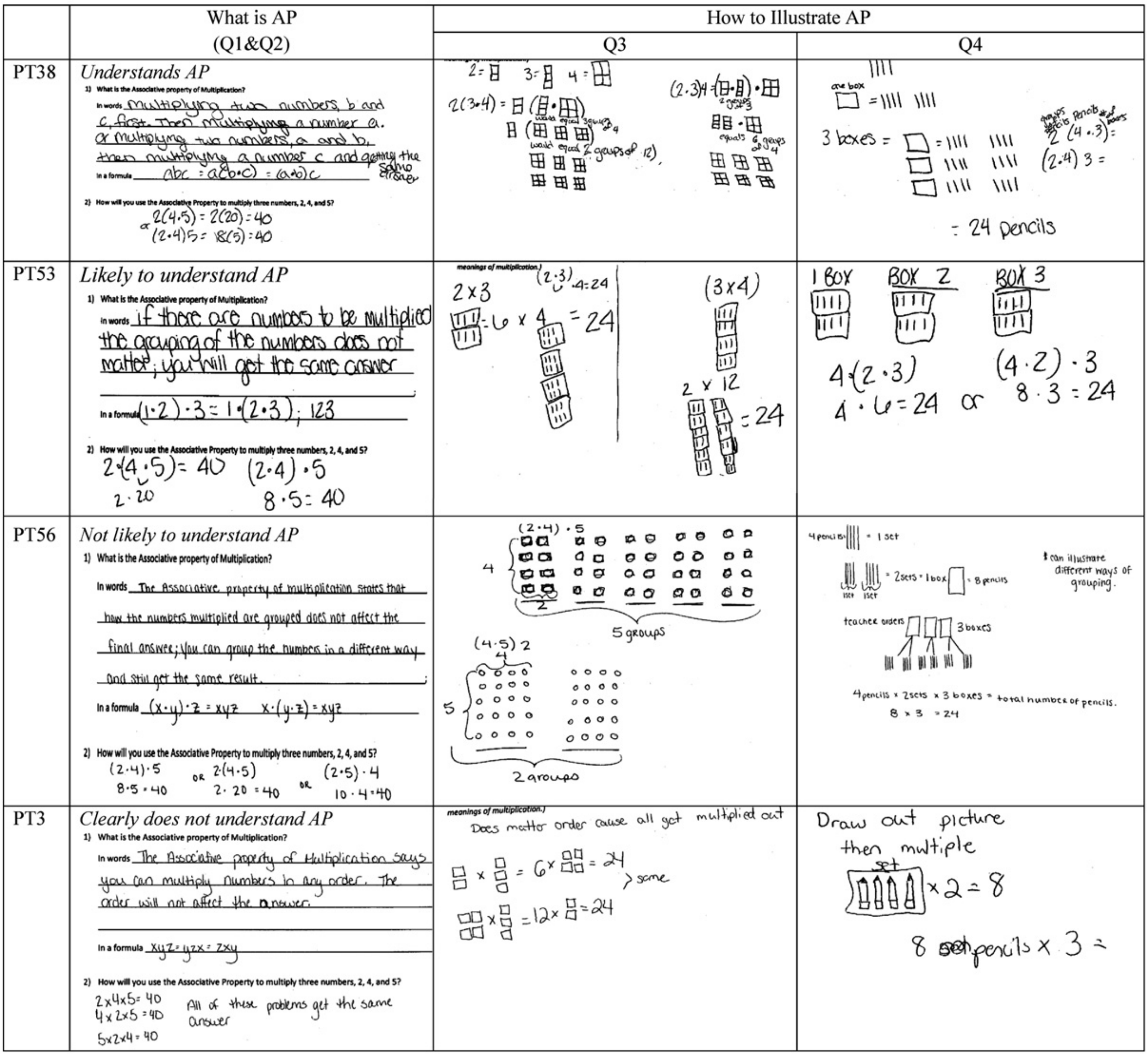

Fig. 5. Examples for each level of understanding of AP of multiplication.

did not appear (note that the formula was only a particular example, and the use of "123" for "$1 \times 2 \times 3$" was inaccurate). An example of "not likely to understand AP" came from PT56. Although this PT's formula was correct, her description about "how the numbers multiplied are grouped does not affect the final answer" also might involve some confusion with CP. This likelihood was indeed confirmed by her responses to Q2 [$(2 \times 4) \times 5 = 2 \times (4 \times 5) = (2 \times 5) \times 4$] and Q3 [$(2 \times 4) \times 5 = (4 \times 5) \times 2$]. Finally, PT3 was assigned as "clearly does not understand AP." Across these items, this PT consistently showed confusion between AP and CP. Based on the above classification, 9% of PTs ($n = 5$) clearly understood AP, 11% ($n = 6$) likely understood this property, 25% ($n = 14$) were not likely to understand it, and 55% ($n = 31$) clearly did not understand AP.

4.3. Knowing how to illustrate AP of multiplication using concrete contexts: two barriers

PTs who lacked understanding of what AP of multiplication was could not correctly illustrate this property (see Fig. 5, the cases of PT3 and PT56). However, PTs who appeared to understand AP of multiplication still had difficulties illustrating this property, likely due to a weak understanding of the two aspects for illustrating AP: Solving a problem two ways and understanding the meaning of multiplication. Table 3 illustrates PTs' understanding of these two aspects in Q3 and Q4.

Regardless of the correctness of their solutions, 66.3% of the PTs on Q3 and only 33.9% on Q4 attempted to solve the problem two ways to illustrate AP of multiplication. However, 28.5% of the PTs on Q3 and 53.5% on Q4 provided a one-way solution, which did not demonstrate AP. Particularly with Q4, although PTs were asked to use the word problem context to teach AP, the PTs' attention was focused mainly on solving this problem itself (thus one solution would work) rather than using it to illustrate AP of multiplication (thus one solution would not work).

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Table 3
Two difficulties in PTs' knowledge for teaching AP of multiplication.

	Level of understanding	Q3		Q4	
		%	Freq	%	Freq
Solving a problem two ways	L2 – Solving a problem two ways	66.3	37	33.9	19
	L1 – Solving a problem one way	28.5	16	53.5	30
	L0 – No number sentences or unable to tell	5.4	3	12.5	7
Understanding the meaning of multiplication	L2 – Both steps show understanding	3.6	2	1.8	1
	L1 – The 2nd step does not show understanding	48.2	27	46.4	26
	L0 – Both steps do not show understanding	48.2	27	51.8	29

With regard to "understanding the meaning of multiplication," PTs' responses demonstrated more serious issues. About half of the PTs (48.2% in Q3 and 51.8% in Q4) were unable to represent the meaning in either step. Approximately another half (48.2% in Q3 and 44.6% in Q4) were confused on the second step. Only three PTs (PT24 and PT38 on Q3; PT22 on Q4) showed correct and consistent understanding on both steps. However, since PT24 misunderstood AP as CP, her illustration did not show AP. Since PT 22 only solved Q4 in one way, she also failed to illustrate AP. As a result, PT38 was the only one who received full credit for Q3 even though her picture still contained shortcomings (see Fig. 5, elaborate upon later). A further inspection of PTs' understanding of the meaning of multiplication on Q3 and Q4 revealed three typical cases, as elaborated below.

4.3.1. *A further look at PTs' understanding of the meaning of multiplication*

Fig. 6 summarizes three cases. In Case 1, some PTs solely focused on the literal meaning of three numbers rather than the meaning of multiplication. For example, the PT on Q3 simply used a "×" sign to connect the pictures representing each of the numbers (2, 3, and 4). However, these illustrations did not demonstrate the meaning of 2×3 and 6×4. On Q4, another PT simply labeled the meaning of each quantity (4, 2, and 3) but did not show the relationship between these quantities (e.g., 2×3, 4×6, 4×2 and 8×3). In other words, they did not pay attention to the meaning of the third quantities derived from multiplication as a binary operation. In fact, each step above (e.g., 2×3, 4×6, 4×2 and 8×3) cannot be explained based on the U.S. convention of the meaning of multiplication.

In Case 2, some PTs started by representing each number on Q3 (similar to Case 1) but then considered the relationship between numbers or the meaning of multiplication. However, this often resulted in a change of the role of a number from "group size" to "number of groups," an issue that was also observed by Lo et al. (2008) with PTs in their study. As indicated by the first example in Fig. 6, the PT in the first step changed either "2 objects" or "3 objects" into "groups" and then in the second step changed either "6 objects" or "4 objects" into "groups." In the second example, another PT in the second step changed 4 objects into 4 groups (left side) and 2 objects into 2 groups (right side). No such errors were found in Q4 because the meanings of numbers were predetermined by the story problem context.

In Case 3, PTs were capable of representing the first step based on an equal groups meaning. However, they were unable to carry over this understanding into the second step – a new context – indicating an unsophisticated understanding of the meaning of multiplication. In Fig. 6, the PT on Q3 denoted "2 sets of 3" as "2×3" (1st step) which was consistent with the U.S. convention. Based on this understanding, the second step of this same picture (4 groups of 6) should have been represented as 4×6 but was, however, represented in the opposite way (6×4). In addition, in both examples, both PTs in the second step confused "3 groups" with "groups of 3," representing it as either "times 3" or "×3." Such an inconsistent use of the meaning of multiplication in the same context was identified with 46.4% PTs ($n = 26$) for both Q3 and Q4 (see Table 3).

4.3.2. *Understanding the meaning of multiplication matters*

Without a sophisticated understanding of the meaning of multiplication, even PTs who appeared to understand AP failed to meaningfully illustrate the property. The cases of PT38 and PT53 (see Fig. 5) are examples of such cases. Both PTs did not confuse AP with CP, and both used two ways to solve Q3 and Q4. However, because of a lack of solid understanding of the meaning of multiplication, both PTs made mistakes in either one of the steps or provided number sentences that could not be explained based on the concrete context. As previously mentioned, PT38 was the only one who received full credit on Q3 because of her written explanations (e.g., 3 groups of 4; 2 groups of 12) correctly matched the illustration even though this illustration still had shortcomings similar to Case 2. However, PT38 on Q4 simply labeled the meaning of each quantity (issue of Case 1) and the number sentences indicated inconsistent use of the meaning of multiplication (issue of Case 3). For example, in the second solution, $(2 \times 4) \times 3$, this PT correctly represented "2 sets of 4" as "2×4" (the 1st step), but represented "3 boxes of 8" as "8×3" (the 2nd step). Similarly, PT53's responses reflected inconsistent use of the meaning of multiplication. In the first solution to Q3, this PT correctly illustrated "2×3" as "2 groups of 3" but then used 6×4 to represent "4 groups of 6." In addition, when solving Q4, PT53 used $4 \times (2 \times 3)$ and $(4 \times 2) \times 3$. Based on the U. S. conventional

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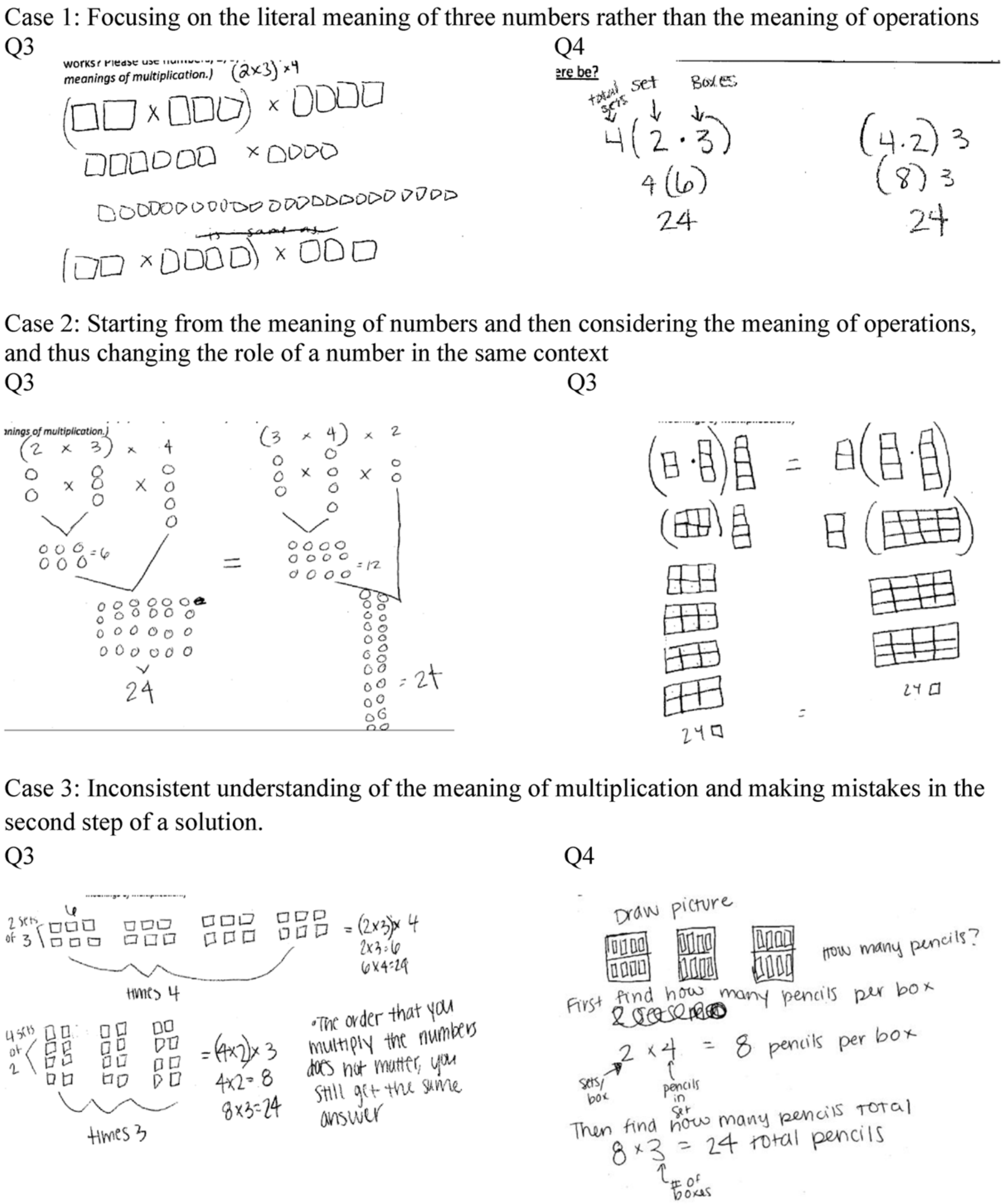

Fig. 6. Three problematic cases of PTs' understanding of the meaning of multiplication.

meaning of multiplication, it would be wrong to explain what "2 × 3" and "4 × 2" actually meant because one could not find 2 groups of 3 and 4 groups of 2 in this word problem context.

4.4. Results from textbook examination

Examination of the textbooks indicates that minimal opportunities were provided for PTs' learning to teach AP of multiplication. We first discuss university textbooks used in the Methods and Content courses and then discuss the elementary textbooks used in teaching practices.

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Table 4
Frequency of the appearance of AP of multiplication in Houghton Mifflin across grades.

AP	Grade 3	Grade 4	Grade 5	Grade 6
Definition	1	4	1	1
Formula	0	0	1	1
Arithmetic example	7	9	4	5
Pictorial representation	1	1	0	0
Word problem	1	1	0	0

4.4.1. Methods and content textbooks used in the university courses

Both textbooks included only one short paragraph about AP of multiplication. The methods book (Reys et al., 2009) listed AP together with CP, DP, and the identity property in a table. The given formula $(ab)c = a(bc)$ was explained by the following statement, "When more than two numbers are being multiplied, combinations that make the task easier can be chosen; for example, $37 \times 5 \times 2$ can be done as $37 \times (5 \times 2)$ or 37×10 rather than $(37 \times 5) \times 2$" (p. 193). The above statement together with the arithmetic examples may be clear for those who understand AP of multiplication. However, since many PTs in the present study referred "grouping" or "associating" numbers to "switching numbers around," the above statement "combinations that make the task easier can be chosen" could be misinterpreted as CP (e.g., $5 \times 37 \times 2 = 5 \times 2 \times 37$).

The mathematics content textbook (Sowder et al., 2007) stated AP of multiplication in the following way, ". . . for numbers p, q, and r, $(pq)r = p(qr)$. This property tells us that when multiplying, we do not need parentheses, because it does not matter which product we find first" (p. 70). The given example for AP of multiplication was "$(3 \times 2) \times 8 = 3 \times (2 \times 8)$ or $6 \times 8 = 3 \times 16$" (p. 70). Using Sowder's approach by taking away the parentheses, it is less risky for PTs to switch around numbers. However, similar to Reys et al. (2009), Sowder's (2007) brief statement "it does not matter which product we do first" could still be misinterpreted by PTs who are confused between AP and CP.

With regard to illustrating AP of multiplication, both the methods and content textbooks did not provide this type of instruction (neither pictures nor word problem contexts), although both textbooks provided pictorial illustrations for CP and DP. Interestingly, Sowder et al. (2007) included an exercise problem, "How could you illustrate, without a drawing, the associative property of multiplication? [Hint: $(a \times b) \times c = a \times (b \times c)$]" (p. 73). The emphasis on "without a drawing" may suggest PTs provide arithmetic examples only.

4.4.2. Elementary textbooks used in teaching practices

Table 4 summarizes the sightings of AP of multiplication in *Houghton Mifflin* (Greenes et al., 2005), the textbook series used by the PTs in their teaching practice.

As shown in Table 4, this textbook series formally introduced AP of multiplication in third grade using a word problem context with 3 tables of 2 plates of 5 mangos. The fourth grade textbook re-taught AP by using an almost identical context: the previously discussed word problem about 3 boxes of 2 sets of 4 strings (see Fig. 2). Accompanying pictures were provided for both word problem contexts. In both grades, AP of multiplication was described as "to group factors together. The parentheses show which factors to multiply first." Correct arithmetic examples were provided. In grades 5 and 6, AP of multiplication was continuously reviewed through recalling the definition, revealing the formula, and applying it to computation including fractional contexts. Based on the above information, the elementary texts demand PTs' knowledge for teaching AP of multiplication and also appear to have potential for supporting PTs' knowledge growth.

However, an inspection of the word problem contexts presented in the grades 3 and 4 texts reveals two issues that may hinder rather than support PT's knowledge for teaching AP. We illustrate both using the fourth grade string word problem (see Fig. 7).

(HM, grade 4, p.100)

Upright bass strings come in sets of 4. Suppose one box holds 2 sets of strings. If a musician orders 3 boxes, how many strings will there be?

Multiply. 4×3×2=

You can use the Associative Property to group factors together. The parentheses show which factors to multiply first.

You can multiply 4x3 first	You can multiply 3x2 first
4×3×2=	4×3×2=
(4×3) ×2=	4× (3×2)=
12×2=24	4×6=24

Fig. 7. An example of elementary textbook presentation for teaching AP of multiplication.

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Issue 1: Concrete contexts were not used for conceptual support. As previously analyzed, one can reason for this problem situation, 3 boxes of 2 sets of 4 strings, and then solve this problem two ways that together illustrate AP of multiplication, why $3 \times (2 \times 4)$ equals $(3 \times 2) \times 4$ (see Fig. 2 for the details). However, the elementary textbook did not use this word problem as anticipated. Rather, it directly presented $4 \times 3 \times 2$ and then told students that they could use AP of multiplication to find the answer two ways. Such a text presentation led to a procedurally-based focus on AP, that is, how to apply through computation rather than making sense of AP through a concrete context. As a result, the existing word problem was actually not used as conceptual support for learning AP of multiplication. Rather, this was essentially a computational context under the pretext of a word problem.

Issue 2: The meaning of multiplication was used imprecisely and inconsistently. Although the elementary textbook provided two multiplication sentences, $(4 \times 3) \times 2$ and $4 \times (3 \times 2)$, that together demonstrated AP, these two sentences did not contain a meaning based on this word problem context. In the *Houghton Mifflin* textbook series, multiplication was first formally introduced in third grade and "$a \times b$" was defined as "*a* groups of *b*," which was consistent with the U.S. convention. However, in this word problem, "4" was labeled as the number of strings in one set, and "3" as the number of boxes. Thus, "4×3" (denoting 4 groups of 3) did not have any referent in this word problem. Similarly, neither 12×2 nor 4×6 had meaning because one could not find "12 groups of 2" or "4 groups of 6" in this context. Without producing the numerical sentences based on the meaning of multiplication, the elementary textbook actually minimized its functional use of the word problem. Such a textbook presentation was surprisingly consistent with our PTs' existing knowledge for teaching AP of multiplication.

5. Discussion

5.1. Challenges in preservice elementary teachers' knowledge for teaching AP

This study explores PTs' existing knowledge for teaching AP of multiplication as they enter elementary education programs. The PT sample in our study had imprecise understandings about what AP of multiplication was. The majority of PTs possess a common confusion between AP and CP, which is consistent with that of secondary mathematics teachers and some undergraduates in prior studies (Larsen, 2010; Tirosh et al., 1991; Zaslavsky & Peled, 1996). According to Tirosh et al. (1991), such a misconception likely results from one's own past learning experiences where AP and CP were used together but rarely differentiated. Our findings about future teachers' common confusion call for attention in order to break down the unfortunate teaching–learning cycle. With blurred understanding of AP, teachers may at most teach procedures for computation but lack the ability to recognize and discuss the underlying principles and help students develop mathematical reasoning abilities (Thompson, 2008).

The PTs in this study had difficulties illustrating AP. One of the possible sources of difficulty is teachers' lack of sophisticated understanding of the equal groups meaning of multiplication ($a \times b$ means a groups of *b*), especially when it is continuously used twice. Although the equal groups meaning has limitations in transferring the learning of multiplication from whole numbers to other contexts (e.g., fractions and decimals with non-integer multipliers, Nesher, 1992), it has served as students' preliminary model for multiplication (Fischbein et al., 1985) and thus can provide conceptual support for justifying AP of multiplication. Our findings, echoing prior studies (Lo et al., 2008; Simon & Blume, 1994), further indicate the necessity of stressing the meaning of multiplication in university PT education settings. While Simon and Blume reported that PTs could not justify their multiplicative thinking with area models using the basic meaning of multiplication, our study reveals more striking findings that PTs did not possess a clear understanding of the basic meaning of multiplication itself. Our findings suggest teacher educators first ensure PTs' understanding of the most basic and foundational concepts such as the meaning of multiplication before delving into more sophisticated topics. As demonstrated by this study, when PTs lack sophisticated understanding of the meaning of multiplication, they are prevented from producing meaningful number sentences to demonstrate AP. In contrast, if PTs possess a solid understanding of the meaning of multiplication, they will have an easier time reasoning about concrete situations, which potentially lead to the illustration of AP. As such, it is safe to conclude that understanding the meaning of multiplication will contribute positively to PTs' ability to illustrate AP of multiplication.

One may have noticed that some PTs on Q4 produced number sentences such as $(4 \times 2) \times 3$ that may be more reflective of another culture (e.g., Japan). If the meaning of multiplication is consistently used across all the items (only reflected in one PT), we acknowledge that there is a possibility that these PTs have been taught multiplication in this manner and thus they may understand the meaning of multiplication from that perspective. However, since our PTs are future U.S. teachers, we expect them to generate number sentences that can be explained based on a conventional U.S. meaning so as not to confuse their future students. In fact, the PTs in our pilot project also generated sentences like $(4 \times 2) \times 3$ and they explained that the order of the three numbers 4, 2, 3 coincided with the order that these numbers appeared in that word problem. The PTs in this present study may possess the same reasoning as indicated by their attention to the meaning of numbers/quantities but not the relationship between numbers/quantities (Nunes, Bryant, & Watson, 2009). Further interviews may confirm or disconfirm our interpretation.

With regard to the second source of difficulty in illustrating AP of multiplication (solving a problem two ways), PTs tend to solve a problem using just one way, which in fact circumvents the possibility of illustrating AP. PTs seem to have difficulties understanding how word problem contexts can be utilized to illustrate AP. Since concrete contexts can provide meaningful

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support for learning abstract ideas (Goldstone & Son, 2005; Goldstone & Wilensky, 2008) and current elementary textbooks, indeed, provide sound word problem contexts, our PTs' knowledge for using word problems for conceptual teaching needs to be addressed and developed.

5.2. Enhancing textbook potential to support PT learning

Textbooks have the potential of supporting teachers' learning and changes, including PTs' knowledge growth (Ball, 1996; Davis & Krajcik, 2005). However, our findings reveal that the university textbooks used by our PTs did not treat AP as a significant topic as indicated by the very brief presentations of this property. In fact, the "order" issue is not clarified via the brief textbook presentations. Thus, this may become a source of PTs' confusion between CP and AP (Zaslavsky & Peled, 1996). The methods textbook also does not provide examples of how to model AP (but CP and DP) nor how to use word problem contexts to help elementary students make sense of this property. As such, there is a gap between what current university textbooks provide and what is needed for PTs' future teaching. Additionally the elementary textbook, the *Houghton Mifflin* text series treats this property with a procedural focus. Thus, the concrete contexts provided are not used fully and the number sentences provided cannot be explained based on the problem situation. Given PTs' own weak knowledge of AP and the meaning of multiplication, these textbook presentations are likely to reinforce PTs' difficulties when using these textbooks during field experiences. Thus, the textbook resources used by our PTs do not seem to function adequately as educative materials (Ball, 1996; Davis & Krajcik, 2005; Drake & Sherin, 2006) to support PTs' knowledge growth in terms of teaching fundamental principles like AP.

We acknowledge that it is not only the textbooks but also the use of textbooks that makes a difference in PTs' learning. In this study, the first author observed one math content course and also had informal conversations with two methods course instructors of the participating sessions. It was found that these instructors followed literately what was presented in the textbooks about AP with little modification. None of these instructors discussed how to illustrate AP of multiplication through concrete situations like the one presented in the elementary textbooks in this study.

5.3. Implications of this study

Our findings suggest that university teacher educators and mathematics professors devote more time and discussions to fundamental properties like AP of multiplication. The short paragraphs currently found in methods and content textbooks need to be expanded to address PTs' common confusion about AP and CP, focusing on the differences between changes in the order of elements and of operations (Tirosh et al., 1991; Zaslavsky & Peled, 1996). Approaches to illustrating AP of multiplication may be supplemented by university instructors because such knowledge is needed for elementary teaching but is currently missing from university textbooks and from PTs' existing knowledge base. In fact, the elementary school textbooks present rich word problems contexts that lend themselves to illustrations for AP of multiplication. While these word problems are not sufficiently utilized by the elementary textbooks themselves, we think such word problem contexts (combining pictorial representations) should be analyzed in university courses, especially methods courses. This echoes Lloyd's (2009) suggestion regarding using school mathematics curriculum materials in university courses to support PTs' mathematical knowledge for teaching.

In fact, the first author, as a methods course instructor, has conducted a follow-up study focusing on the development of PTs' knowledge for teaching AP through a one-semester long effort. After the class has reached an agreement that "a groups of b" is represented as "$a \times b$" in the class, she directly posed a word problem from the *Houghton Mifflin* third grade textbook. PTs were prompted to represent and solve this word problem in order to illustrate AP of multiplication. Because the word problem context demands a successive application of the meaning of multiplication, the PTs' demonstrated similar misconceptions and difficulties as in the present study. The first author then used these counterexamples as opportunities (Zaslavsky & Peled, 1996) to clarify and solidify the PTs' understanding of the meaning of multiplication and to differentiate AP from CP. Through such discussions of elementary word problems and a few delayed practices, PTs in this class gained much improved knowledge for teaching AP of multiplication. Some PTs were also able to spontaneously and critically identify the issues of the elementary textbook presentation. As such, we think through incorporating relevant elementary word problems in university courses, mathematics educators may be able to effectively deepen PTs' knowledge for teaching fundamental properties, enhance their knowledge for analyzing and using curriculum, and help them obtain critical thinking skills that will together support their use of elementary textbook resources in meaningful ways. Similarly, we suggest that university textbooks designers integrate typical elementary textbook examples and provide discussions about how to teach and illustrate fundamental ideas meaningfully.

5.4. Limitations and future directions

This study has several limitations. First, the questionnaire could be improved by revising the second question, "How will you use the associative property to multiply three numbers, 2, 4, 5?" which may possibly be misunderstood by only eliciting students' partial responses [e.g., $2 \times (4 \times 5)$]. Future studies may directly ask, "Please give an example of AP of multiplication using the three numbers 2, 4, and 5." The second limitation was that we did not interview the PTs. Although we have triangulated PTs' responses across items, we could have obtained clearer images of PTs' understanding of AP of

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multiplication if interviews were conducted, especially centering on those blurred responses. Future studies may address this limitation. Nevertheless, our preliminary analysis serves as an outlet for future research in this topic. First, replication studies may be conducted with different participants using a refined questionnaire. Second, one may conduct interventions explicitly addressing the weakness of PTs' knowledge for teaching AP of multiplication and then document the effects of these interventions.

The learning and understanding of fundamental mathematical ideas is an ultimate goal of mathematics education (English, 2008). Focusing on the AP of multiplication, our study serves as a window illustrating PTs' difficulties that are often overlooked and trivialized by the field. We call for greater attention to fundamental mathematical ideas like AP of multiplication and a better use of curriculum materials to prepare PTs for teaching with conceptual understanding in their future classrooms.

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