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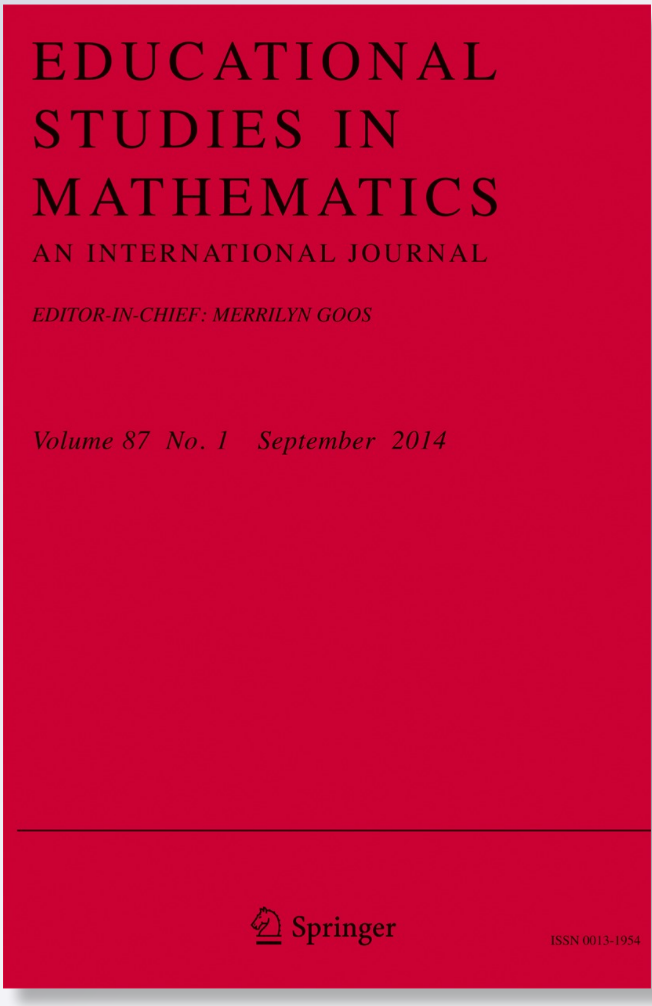
Educational Studies in Mathematics
An International Journal

ISSN 0013-1954
Volume 87
Number 1

Educ Stud Math (2014) 87:103-121
DOI 10.1007/s10649-014-9558-y



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Transition from concrete to abstract representations: the distributive property in a Chinese textbook series

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Published online: 13 May 2014
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Abstract Through examining a representative Chinese textbook series' presentation of the distributive property, this study explores how mathematics curriculum may structure representations in ways that facilitate the transition from concrete to abstract so as to support students' learning of mathematical principles. A total of 319 instances of the distributive property were identified. The representational transition among these instances was analyzed at three tiers: within one worked example, from the worked example to practice problems within one topic, and across multiple topics over grades. Findings revealed four features that facilitate the transition process in the Chinese textbook series. First, it situates initial learning in a word problem context, which serves as a starting point of the transition process. Second, it sets up abstract representations as an ultimate goal of the multi-tier transition process. Third, it incorporates problem variations with connections in carefully designed tasks that embody the same targeted principles. Fourth, it engages students in constant sense making of the transition process through various pedagogical supports. Implementations and future research directions are also discussed.

Keywords Concrete representation · Abstract representation · Representational transition · Chinese textbooks · The distributive property

Mathematical principles like the distributive property are extremely powerful, but notoriously difficult to learn, because these principles, by nature, are abstract and lack close relevance to learners' lives. To tackle this difficulty, one of the pedagogical traditions is to ground the learning of abstract knowledge in concrete contexts (Bruner, 1966; Piaget, 1952). However, how students might be supported to make transitions from concrete to abstract representations remains largely unknown. As the National Mathematics Advisor Panel (2008) pointed out: "Students must eventually transition from concrete (hands-on) or visual representations to internalized abstract representations. The crucial steps in making such transitions are not clearly understood at present and need to be a focus of learning and curriculum research" (p. 29).

The purpose of this study is to explore, from curriculum perspectives, how concrete and abstract representations might be structured in ways that facilitate the process of transition so

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as to support students' learning of mathematical principles. Specifically, we focus on the case of the distributive property presented in a representative Chinese elementary textbook series to explore this question. Below, we review the literature to provide a theoretical basis of this study.

1 Review of literature

1.1 Abstract and concrete representations

Abstract representations in this study referred to using symbols to represent mathematical concepts and ideas. Such representations eliminate detailed perceptual properties and are often arbitrarily linked to referents (McNeil & Fyfe, 2012). The notion of abstractness is relative in the sense that specific abstract representations [e.g., $(65+35)\times5=65\times5+35\times5$] may be further manipulated to get a sense of a more general representation [e.g., $(a+b)c=ac+bc$] that is more abstract for students (Mason, 2008). Abstract representations, in comparison to concrete ones, are powerful because they can transcend over contexts for reasoning and problem solving (Kaminski, Sloutsky, & Heckler, 2008). Recent studies (e.g., Cai, 2004; Koedinger, Alibali & Nathan, 2008) found that abstract representations have advantages over concrete representations in solving complex problems. However, simply having students learn mathematical principles in abstract contexts may be ineffective because students may only obtain inert knowledge from abstract representations (Goldstone & Son, 2005). Therefore, it is critical to ground the learning of mathematical principles in concrete representations to enable sense making.

Concrete representations in this study refer to the use of physical objects (e.g., manipulatives) or visual images (e.g., diagrams) to represent mathematical concepts and ideas, and/or the conceptualization of abstract ideas in real-world situations (e.g., word problems). We consider word problem contexts as concrete representations because word problems "have the potential to offer memorable imagery that can act as a touchstone for teachers and learners in building and discussing abstract concepts" (Gerofsky, 2009, p. 36). Like the notion of abstractness in this study, we view concreteness as relative as well (McNeil & Fyfe, 2012). For example, actual physical objects compared to their images, pictures of real objects compared to schematic representations, and word problem with illustrations compared to pure word problems may offer richer real-life references. Prior studies have demonstrated that when students' real-world knowledge is activated, students would have a better chance to solve problems and make sense of the abstract mathematical ideas (Baranes, Perry, & Stiegler, 1989; Palm, 2008). However, over-relying on concrete representations may also hinder learning. This is because the perceptually rich but irrelevant information may distract learners' attention or may be interpreted as an essential part of the intended concepts (Uttal, Scudder, & DeLoache 1997). Therefore, it is necessary to transit from concrete to abstract representations.

1.2 Transition from concrete to abstract representations

The idea of making transition from concrete to abstract representations can be traced back to Piaget (1952), which is only one of the views about connecting the concrete and the abstract. In fact, Davydov (1988) has suggested a reverse direction, the ascent from the abstract to the concrete. The present study takes the Piagetian view. Piaget (1952) suggested four cognitive development stages: sensory motor, preoperational, concrete operational, and formal operational, which indicated a transition from concrete to abstract. Bruner (1966) held a similar

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position that children's learning included three modes: physical, iconic, and symbolic. For both Piaget and Bruner, young children should construct their understanding of mathematical concepts starting from physical manipulations. In fact, concrete representations in this study are also referred to word problems that may activate students' informal real-world experience. In this sense, the process of transition from concrete to abstract is also aligned with the theory of Realistic Mathematics Education (Freudenthal, 1991; Gravemeijer, 1994) that emphasized mathematizing and reinventing students' informal situation-specific knowledge into formal and general understanding. As such, one may see the close connection between the notion of transition from concrete to abstract representations and the notions of transition from informal to formal (Carpenter, Franke, & Levi, 2003) and from specific to general understandings (Mason, 2008).

Empirical studies have shown that making the transition from concrete to abstract representations is not easy. Resnick and Omanson (1987) found that a child who was able to use Dienes blocks to solve three-digit additions could not solve simpler two-digit additions with written symbols. Moreover, children who performed best with the Dienes blocks did the worst in written subtraction problems. Furthermore, research on word problems reported students' difficulties in activating real-life knowledge based on word problem situations, leaving the word problems functioning as confusing mathematical puzzles (e.g., Baranes, et al., 1989; Palm, 2008). The difficulties in making the transition from concrete to abstract representations may be understood from the perspective of analogy (Holyoak & Koh, 1987). During the process of transition, the concrete and abstract representations may be considered as the base and the target, respectively. Given the key to analogy is structural mapping from the base to the target, a successful transition from concrete to abstract representations demands the same process, which in turn requires that students see the structural similarities between both. Unfortunately, structural connections are always hidden from direct observation, which makes it hard for novice students to detect them.

Recently, cognitive psychologists (e.g., Goldstone & Son, 2005) proposed a method named *concreteness fading* to tackle this issue. Goldstone and Son defined concreteness fading as "the process of successively decreasing the concreteness of a simulation with the intent of eventually attaining a relatively idealized and decontextualized representation that is still clearly connected to the physical situation that it models" (p. 70). Although this method was originated from Bruner (1966), there were two distinguishing features as compared to the prior learning theories. First, representations used in concreteness fading target the "same" concept. Second, the process of representational changes is "successive" with viable connections. These features seem to be critical because they contribute to maintaining the structural similarities (Holyoak & Koh, 1987) between concrete and abstract representations. In fact, the idea of "progressiveness" was advocated by Realistic Mathematics Education (Freudenthal, 1991; Gravemeijer, 1994), which viewed mathematics learning as a process of guided reinvention. Returning to concreteness fading, recent laboratory experiments have confirmed its effect on learning scientific principles (Goldstone & Son, 2005) and mathematical rules (McNeil & Fyfe, 2012).

Although the above studies have enhanced our understanding of representational transition, little is known about how textbooks may embody these ideas to facilitate the learning of mathematical principles. Given the important role of textbooks in improving teaching and learning (Ball & Cohen, 1996), this study explores representational transition in textbooks through an examination of the distributive property in a representative Chinese textbook series.

1.3 The distributive property and Chinese textbook presentation

The distributive property, symbolically represented as $(a+b)c=ac+bc$, is one of the most important mathematical principles because it allows great flexibility for computations, serves

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as one of the fundamentals for solving algebraic equations, and provides a foundation for generalizations and proofs (Bruner, 1960; Carpenter et al., 2003). Knowing the distributive property as an underlying idea of the partial products [e.g., $48\times37=(40+8)\times(30+7)=40\times(30+7)+8\times(30+7)=40\times30+40\times7+8\times30+8\times7$] may contribute to students' understanding of arithmetic algorithms and algebraic learning such as binomials [e.g., $(a+b)\times(c+d)=ac+ad+bc+bd$, Carpenter et al., 2003]. In fact, to solve algebraic equations such as $x-0.15x=38.24$, instead of memorizing the procedure of simplifying "$x-0.15x$" as "$(1-0.15)\,x$," students should also understand the distributive property (Ding & Li, 2010). Finally, the distributive property plays an important role in learning linear functions. For example, to transform functions of *point–slope forms* to *slope–intercept forms* involves the use of the distributive property: $y-y_1=m\,(x-x_1)\rightarrow y-y_1=mx-mx_1\rightarrow y=mx+(y_1-mx_1)$ or $y=mx+b$.

The distributive property, $(a+b)\,c=ac+bc$, like many other mathematics principles, appears to be very simple. However, many students tend to simply manipulate the symbols without being able to use it flexibly in different contexts. Indeed, some elementary teachers also had considerable difficulty in activating and applying this property to relevant topics (Ma, 1999). To help elementary students learn this principle in meaningful ways, concrete contexts, such as word problems and pictures, may be used to model this property. Because there were several types of multiplication (e.g., equal groups, multiplicative comparison, area, and combination, Carpenter et al., 2003), there were different ways to model the distributive property. The National Council of Teacher of Mathematics (2000) suggested that teachers draw an area model (a rectangle with the length of "$a+b$" and width of "c") to illustrate the distributive property. Ding and Li (2010) reported that Chinese textbooks mainly used word problems involving equal groups and multiplicative comparison models to illustrate this property. Prior research on the distributive property provides a foundation for this study to explore representational transition in elementary textbooks, which has not been studied in prior research.

We chose to examine a Chinese textbook series because Chinese students consistently outperform their counterparts in international mathematical assessments (e.g., Cai, 2004; Li, Ding, Capraro, & Capraro, 2008), which is directly related to the textbooks they used (Li et al., 2008; Li & Huang 2013; Wang, Han, & Lee, 2004). Since a sophisticated understanding of mathematical principles entails spacing learning over years (Pashler et al., 2007) and Chinese textbook series have displayed long term learning processes (Li, Chen, & Kulm, 2009), it is meaningful to examine a full textbook series. In fact, textbook studies that involve Chinese curriculum have brought many insights to the field (e.g., Ding & Li, 2010; Li et al., 2008; Sun, 2011), which might otherwise be impossible if one limited one's gaze to only one's own nation.

2 Methods

2.1 Textbook selection

We selected the Chinese JSEP textbook series (Su & Wang, 2005) because this was one of the main Chinese textbook series. This textbook series was found to possess overall merits in supporting students' learning of key concepts and principles (e.g., Li et al., 2008) including the distributive property (Ding & Li, 2010). The JSEP was a reform textbook series, with its development based on the new National Mathematics Curriculum Standards (Ministry of Education, 2001). All of the grade 1–6 student textbooks (12 volumes) and corresponding teacher guides (or *Curriculum Analysis*) were selected.

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2.2 Data coding

Coding instances The JSEP text series blocked the teaching of each mathematical topic in one chapter containing several interrelated lessons. In general, each lesson included one worked example (a problem with solutions given), followed by corresponding practice problems. All of the worked examples and practice problems in the textbook series were examined. A worked example would be coded as an instance if its solution either (a) explicitly used the distributive property or (b) implicitly involved the distributive property with a clear possibility to develop students' intuition of this property. A practice problem would be coded as an instance if students were explicitly asked to use the distributive property. If clear requirements did not exist, our decisions on coding practice problems were then referred back to the corresponding worked example. The first author coded all of the textbook pages and then recoded the entire textbook series 3 months later. A few missed instances were added. After this, the second author examined each page of Chinese second, fourth and sixth grade curricula using the same coding framework. Cohen's kappa was computed to check the interrater reliability (usually kappa should be 0.7; Leech, Barrett, & Morgan, 2008). The average kappa of 0.83 indicated high agreement between the two coders.

Coding types of representations For each identified instance, we classified its overall nature as either a concrete or an abstract representation. An example of a concrete representation is a fourth grade worked example that asked students to find the total cost for five jackets priced at ¥65 each and five pants priced at ¥45 each. The textbook provided two solutions $(65+45)\times 5$[1] and $65\times 5+45\times 5$ to this word problem, which together illustrated the distributive property $(65+45)\times 5=65\times 5+45\times 5$ (elaborated upon later). Given this worked example grounded the learning of the distributive property in a real-world context, its overall nature was coded as concrete representation. Examples of abstract representations were "Using convenient way to solve 16×401," "$1.9x+0.4x=9.2$," "$3x+2x=(\square+\square)\times\square$," and "Can you provide an algebraic expression for the distributive property?" These problems only involved symbolic manipulations and thus were classified as abstract. Both authors independently classified the instances and the agreements reached 100 %.

Coding levels of concreteness/abstractness Because concreteness and abstractness are viewed as relative in this study, for each instance, we further coded the levels of concreteness/abstractness, when possible. For concrete contexts such as word problems, we differentiated several types of problem formats including (a) picture plus keywords, (b) a complete word problem with pictures, and (c) a complete word problem only. From (a) to (c), there was a decreasing level of concreteness because the visual information became less involved. In addition, when solutions to a word problem were given, we differentiated the arithmetic from algebraic solutions. For abstract contexts such as computation problems, we differentiated the types of numbers/contexts. We viewed whole numbers and arithmetic contexts as relatively less abstract than rational numbers and algebraic contexts because the former might be more familiar to students. In fact, based on Son and Goldstone's (2009) extended notion of "concreteness"—"degree of specificity of contextualization" or "how much learning is embedded in a specific domain or situation" (p. 52)—a transition from a specific arithmetic operation [e.g., $6\times(35+65)=6\times 35+6\times 65$] to a general algebraic equation [e.g., $a(b+c)=ab+ac$] indicates decreased concreteness and thus increased abstractness (Goldstone, personal communication, September 26, 2010). This is also supported by the connection

[1] "65×5" represents "5 groups of 65" in this Chinese text.

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between the notions of "from concrete to abstract" and "from specific to general" (Mason, 2008).

2.3 Data analysis

After the above three-level coding, we analyzed the transition from concrete to abstract representations based on a framework involving worked examples and practice problems. For all of the instances, we first counted the frequency of concrete and abstract representations under worked examples and practice problems, respectively. We then conducted fine-grained analyses at three tiers: (1) within one worked example, (2) from worked examples to practice problems within one topic, and (3) across multiple topics over grades. At tier 1, we examined the representational changes and connections between the problem statement and its solutions. At tier 2, we only examined the chapter that formally instructs the distributive property. At tier 3, we examined, across worked examples, the changes of problem formats and solution representations. We also examined, across practice problems, the changes of types of numbers in abstract contexts. At each tier, we analyzed whether there were noticeable pedagogical techniques used to facilitate the transition, and how well those changes and connections might support student learning of the distributive property. Throughout the data analyses, both authors were engaged in constant conversations. Thus, the results reported in this study were agreed upon by both authors.

3 Results

The general quantitative results are presented in Table 1.

Table 1 indicates that JSEP textbooks presented 319 instances of the distributive property across various mathematical topics (e.g., whole number multiplication, using letters to represent numbers, solving algebraic equations) from grades 2 to 6. The formal introduction of this property occurred in grade 4 (n=67). Among 319 instances, there were a total of 16 worked examples and 303 practice problems. Several patterns of representation uses were revealed. First, the majority of the worked examples (15 out of 16) were presented in concrete contexts, which were word problems. Second, practice problems involved both concrete (n=161) and abstract (n=142) representations. Third, the proportion of concrete to abstract representations in practice problems varied across informal learning (20–25), formal learning (10–43) and later revisits (131–60). The above pattern indicates that the transition from concrete to abstract representations in JSEP textbook series was not a simple linear progression in the manner from word problems to computation problems. The complexity of representation uses suggests the need for fine-grained analyses as reported below.

3.1 Transition within one worked example

Among 15 worked examples that were situated in word problem contexts, there were common patterns in presenting each example: (1) presenting a word problem involving "key words and pictures" or "pure word problems with/without accompanying pictures"; (2) analyzing the quantitative relationships either using schematic diagrams (e.g., number lines) or "thinking bubbles" (highlights of structural relationships in the format of a child's thinking); and (3) presenting arithmetic or algebraic solutions that embodied the distributive property. The above steps indicated that the concreteness or specificity of the contextualization has been gradually decreased, which aligns with the process of concreteness fading (Goldstone & Son, 2005). A

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Table 1 Number of instances of the distributive property across different dimensions

Grade (volume)	Mathematical topic	Worked example		Practice problem	
		Concrete	Abstract	Concrete	Abstract
2 (v.3)	Ch2. Multiplication facts (Kou Jue)				12
3 (v.5)	Ch4. Addition and subtraction (<100)	1		7	
3 (v.5)	Ch6. Rectangle and square	1		7	5
3 (v.5)	Ch7. Multiplication (3-digit×1-digit)			3	1
3 (v.6)	Ch4. Multiplication (2-digit×2-digit)	1		1	2
4 (v.7)	Ch3. Mixed four operations				2
4 (v.8)	Ch1. Multiplication (3-digit×2-digit)			1	1
4 (v.8)	Ch4. Mixed operations			1	2
	$n_{\text{total informal learning}}=48$	3		20	25
4 (v.8)	Ch7. The basic laws of arithmetic (DP)	2		10	55
	$n_{\text{total formal learning}}=67$	2		10	55
4 (v.8)	Ch11. Problem solving (Distance)	1		6	
4 (v.8)	Ch13. Using letters to represent numbers	1		8	9
4 (v.8)	Ch14. Organize and review			2	4
5 (v.9)	Ch9. Decimal multiplication and division		1		8
6 (v.11)	Ch1. Equation (algebraic)	1		18	12
6 (v.11)	Ch2. Rectangular prism and cubes	1		1	4
6 (v.11)	Ch7. Mixed four operations of fractions	3		30	7
6 (v.11)	Ch10. Organize and review			3	2
6 (v.12)	Ch1. The application of percent	3		49	8
6 (v.12)	Ch8. Organize and reflection			14	8
	$n_{\text{total revisit}}=204$	10	1	131	62
$n_{\text{total}}=319$		15	1	161	144

closer examination of the worked examples revealed two noticeable pedagogical techniques that seemed to facilitate the representational transition. Elaboration follows.

Constructing and analyzing a schematic diagram Among the 15 worked examples, eight of them (53.3 %) used number line diagrams to represent the word problem situations. Number line diagrams are schematic representations that illustrate the problem structures and bridge the concrete to abstract representations, which, however, are often nontransparent for students (Pashler et al., 2007). The JSEP textbooks employed three approaches to engage students in the process of constructing and analyzing the diagrams. Table 2 presents examples.

As indicated in Table 2, the first approach was to have students draw part of a diagram (n=2). The third grade example involved multiplicative comparison. The given number line indicated the price for a single pair of pants. Students were then instructed to draw the second line for the price of a coat. This drawing process may prompt students to map the relevant information (the price of a coat is three times that of a pair of pants), onto the second line, which may bring embedded comparative relationships to students' attention. The second approach was to have students label key quantities on a line (n=3). The sixth grade example also involved multiplicative comparison. The textbook presented two lines with the first line labeled as x (boys). Students were asked to label the quantity for the second line (girls). This

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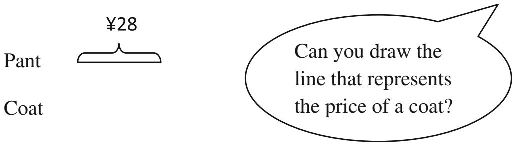
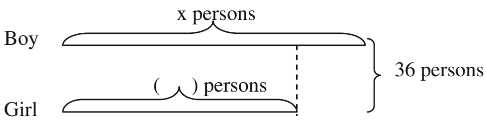
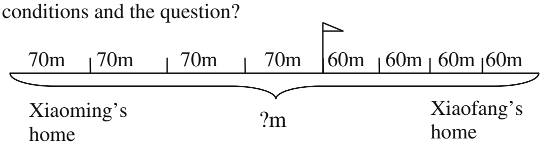
Table 2 Three approaches of using number line diagrams

Approach	Example	Number Line Diagram
1. Have students draw part of a diagram ($n = 2$)	A pair of pants is ¥28 (in picture). The price of a coat is 3 times the price of the pants. How much does one suit cost? *(3rd grade).*	
2. Have students label key quantities on a line ($n = 3$)	Zhaoyang's elementary school art team has a total of 36 students. The number of girls is 80% of that of boys. How many girls and boys are there respectively? *(6th grade)*	 The number of girls is 80% of the boys Number of ( ) + Number of ( ) = total people of art team
3.Pose specific questions to guide discussions of a diagram ($n = 3$)	Xiaoming and Xiaofang walked toward school. Xiaoming walked at 70 meters per minute. Xiaofang walked at 60 meters per minute. They met each other after 4 minutes. What is the distance between two homes? *(4th grade)*	Can you use number line diagram or list a table to sort out the conditions and the question?  According to this diagram, what can we compute first?

might prompt students to process and map the information, "the number of girls is 80 % of that of boys," onto an algebraic expression "80 % x," a key step to generate a double-reference equation, $x+80\ \%\ x=36$. The third approach was to pose specific questions to guide discussions of a diagram ($n=3$). The fourth grade distance problem in Table 2 involved the equal-groups meaning of multiplication. The textbook suggested using both number line and table to represent this problem, leading to two different solutions. After the number line diagram was presented, students were asked, "According to this diagram, what can we compute first?" This question was to orient students' attention to the number line, thus generating the expected first solutions, $70\times4+60\times4$, as illustrated by the diagram.

Comparing two solutions This is the second technique used to facilitate representational transition within one worked example. Among the 15 worked examples, 12 of them (80 %) employed this approach to reveal the distributive property. Figure 1 presents three examples, all of which involved the equal-groups meaning of multiplication.

Figure 1a illustrates the aforementioned fourth grade "jackets and pants" word problem that formally introduced the distributive property. The representational transition included four steps: (a) analyzing and solving this problem in two ways: $65\times5+45\times5$ and $(65+45)\times5$; (b) comparing the two solutions that lead to an instance of the distributive property: $(65+45)\times5=65\times5+45\times5$; (c) asking students to generate more arithmetic examples of this sort; and (d)

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(a) Formal introduction of DP – Grade 4

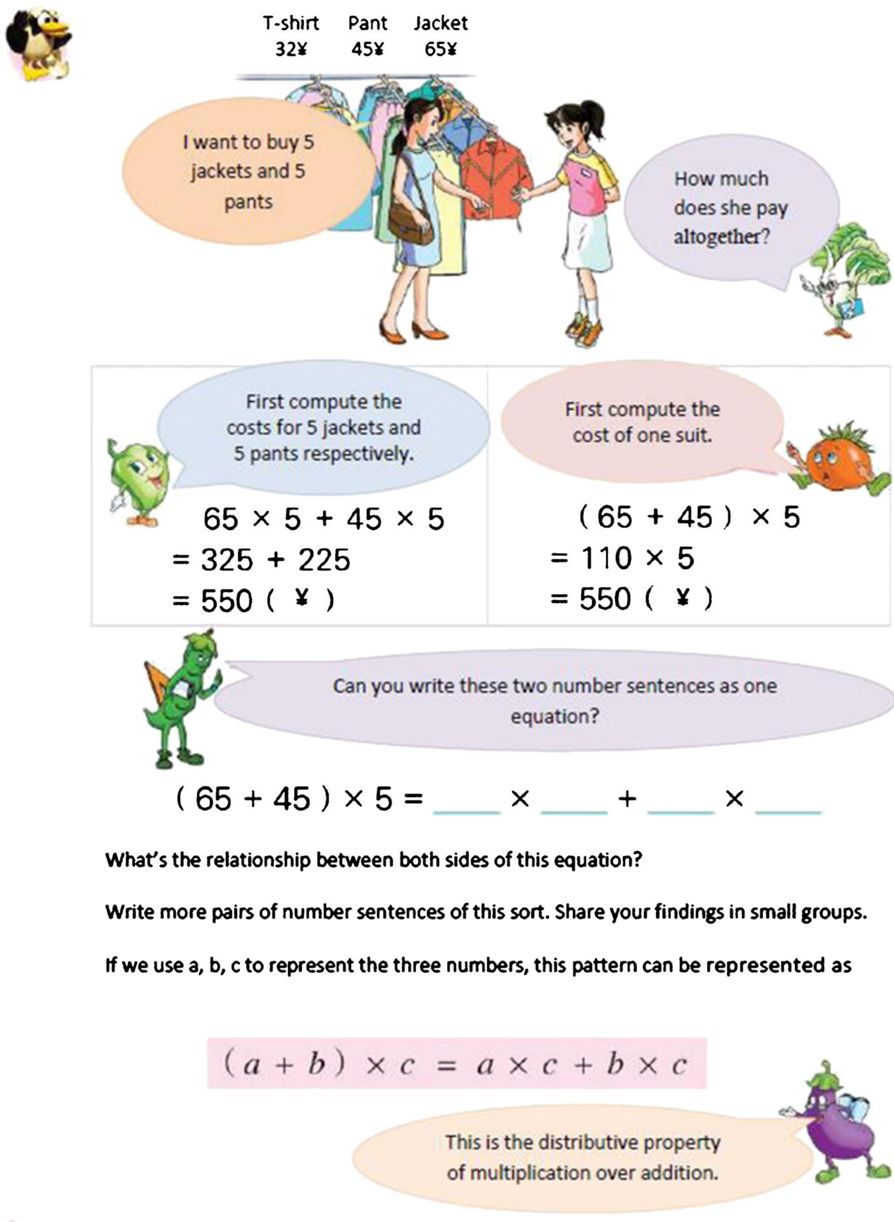

Fig. 1 Translated Chinese textbook pages of typical worked examples that involve DP

formally introducing the distributive property through generalization: $(a+b)\,c=ac+bc$. Among the above four steps, the second step, comparing two solutions, was critical. Without this step, even though this word problem was solved, the opportunity to mathematize students' situation-specific experience to learn the underlying property might be missed. Indeed, the *Curriculum*

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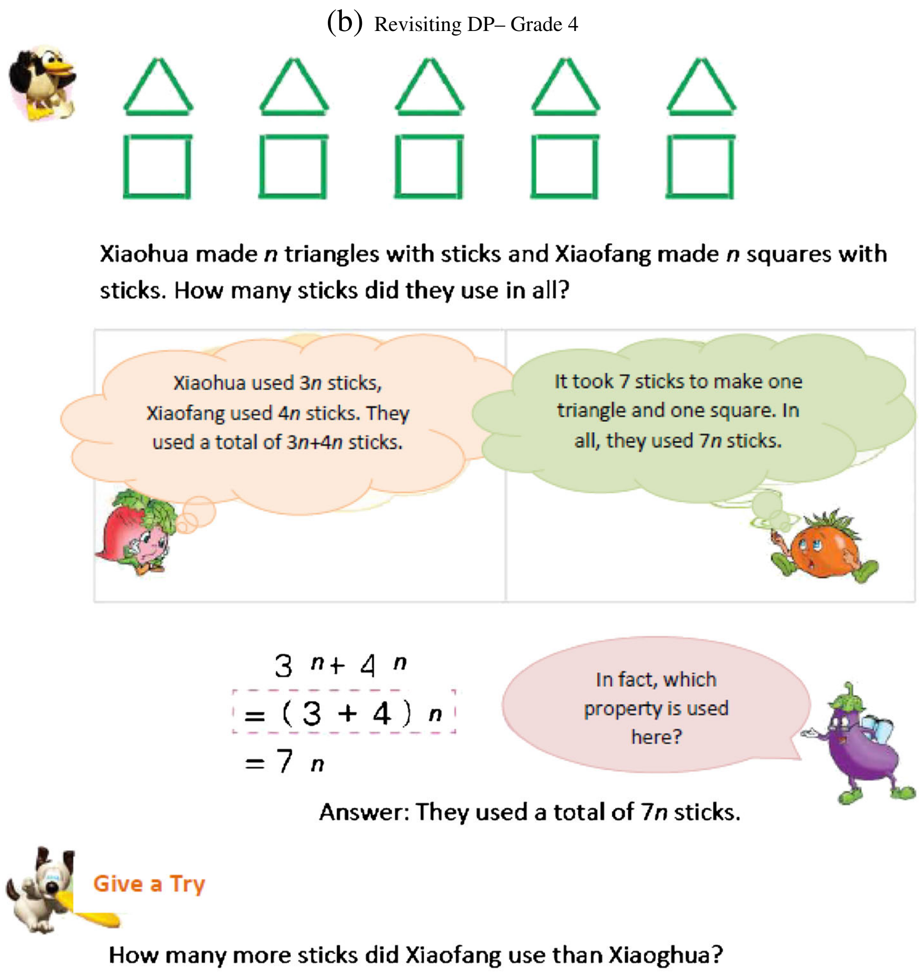

Fig. 1 (continued)

Analysis clearly suggested comparing and contrasting the two expressions at both sides of the equation, first referring back to and then generalizing beyond the problem situation, thus "promote students' understanding of the distributive property toward an abstract level."

The comparison approaches were also observed in worked examples during the later revisits of the distributive property. In a fourth grade lesson (see Fig. 1b), the comparison of two solutions to a problem exposed students to the procedure of simplifying an algebraic expression, $3n+4n=(3+4)\ n=7n$. Along with this procedure, the deep question – "What property is actually used?" – drew students' attention to the underlying distributive property. Similarly, in the sixth grade example (see Fig. 1c), comparison was made between two solutions (a) $2/5\times18+3/5\times18$, or (b) $(2/5+3/5)\times18$, through a set of deep questions: "What are the connections between the two solutions? Which solution is easier? Can the basic laws of arithmetic with whole numbers also be used with fractions?" These prompts potentially enable students to recognize the distributive property and its power in the new context of rational numbers.

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(c) Revisiting DP–Grade 6

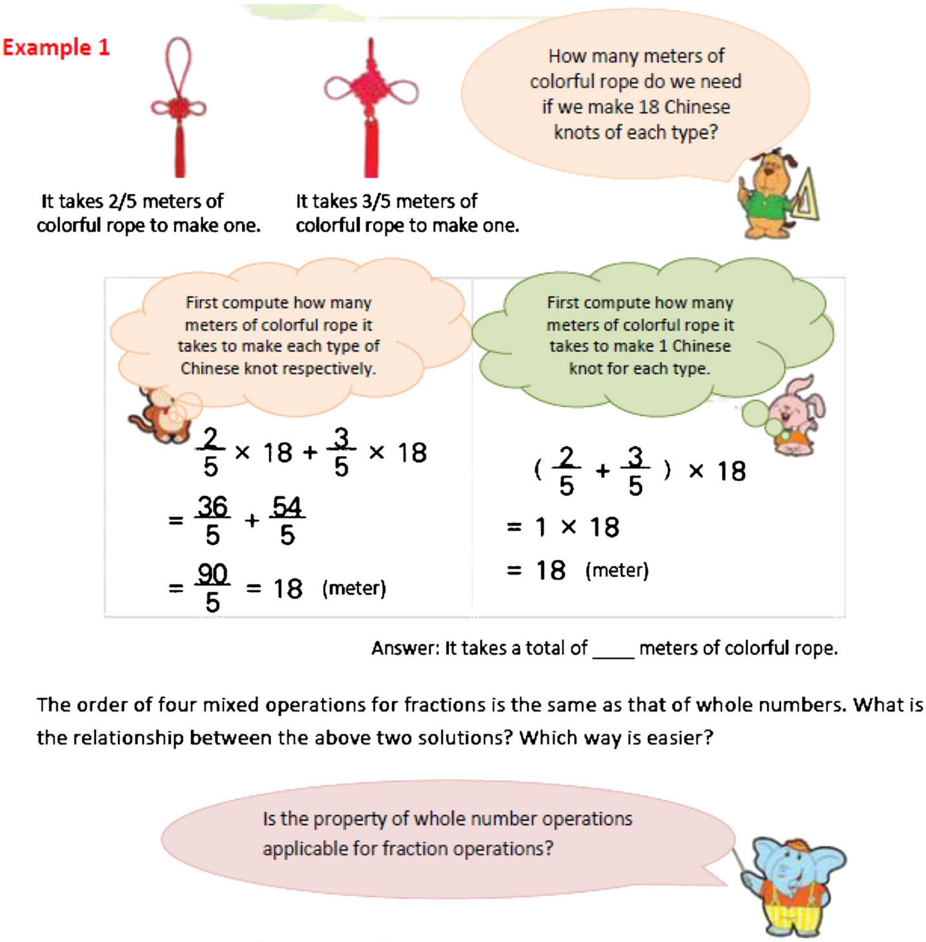

Fig. 1 (continued)

3.2 Transition from worked examples to practice problems within one topic

We elaborate on the representational transition within one topic using the chapter of formal introduction of the distributive property in fourth grade. This chapter was comprised of three consecutive lessons. According to the *Curriculum Analysis*, Lessons 1 and 2 targeted the meaning and applications of the distributive property, respectively. Lesson 3 was a review lesson. Table 3 shows the representation uses in each lesson.

As indicated by Table 3, each of the first two lessons included only one worked example situated in concrete contexts. The worked example in lesson 1 was the aforementioned "jackets and pants" word problem. The worked example in lesson 2 was about computing the total cost for 102 shirts priced at ¥32. Using this context, the textbook helped students make sense of the application of the distributive property, $32\times102=32\times(100+2)=32\times100+32\times2$, that is, the costs for 102 shirts can be thought of as the costs for 100 shirts and 2 shirts. Since lesson 3 was a review lesson, it did not contain worked examples. With regard to practice problems in all

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Table 3 Frequency of representations during formal introduction of the distributive property

Lessons	Worked examples		Practice problems	
	Concrete	Abstract	Concrete	Abstract
1 (new)	1	0	4	10
2 (new)	1	0	2	18
3 (review)	0	0	4	27
Total	2	0	10	55

three lessons, both concrete (n=10) and abstract (n=55) representations were included. Among the 10 concrete representations, nine were word problems involving equal-groups meaning and one was an area model. Among the 55 abstract representations, most of them asked students to do computation or fill in numbers/signs, thus applying the distributive property. Overall, there was a trend of fading concreteness into abstract within one lesson or one topic. In addition, we observed a technique of using contrasting cases in practice problems to effectively deepen or extend what was taught in the worked example. Below are examples.

Compare to deepen understanding In lesson 1, the worked example about jacket and pants was solved by (65+45)×5=65×5+45×5 (see Fig. 1a). The corresponding practice problem contained a group of contrasting cases:

$$\begin{array}{l}(42+35)\times 2 = 42\times\ \square\ +35\times\ \square\\ 27\times 12+43\times 12=(27+\square)\times\ \square\\ 15\times 26+15\times 24=\square\ \mathrm{O}(\square\ \mathrm{O}\ \square)\\ 72\times(30+6)=\square\ \mathrm{O}\ \square\ \mathrm{O}\ \square\ \mathrm{O}\ \square\end{array}$$

The surface features of these problems were different from the worked example in two aspects: (a) the direction of using the distributive property and (b) the position of the common factor (simply "direction" and "position"), as indicated below:

$$\begin{array}{l}\text{Example}: \ (a+b)c=ac+bc\\ \text{Practice 1}: (a+b)c=ac+bc\\ \text{Practice 2}: ac+bc=(a+b)c\\ \text{Practice 3}: ab+ac=a(b+c)\\ \text{Practice 4}: a(b+c)=ab+ac\end{array}$$

The first practice problem was a literal application of the distributive property embodied by the worked example. The second and third problems changed the direction in using the distributive property, while the third problem additionally changed the position of the common factor. Finally, the fourth problem possessed both similarities (in position) and differences (in direction) to the third problem. As such, from worked example to practice problems, the use of contrasting cases with progressive variations may deepen students' understanding of the distributive property.

Compare to extend understanding In lessons 1 and 2, both worked examples presented the distributive property as "multiplication over addition" [e.g., (65+45)×5=65×5+45×5]. The corresponding practice problems reinforced this structure. However, lesson 3 introduced a new

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case of the distributive property, "multiplication over *subtraction*," through a group of computation problems followed by a deep question, "What have you discovered?"

$$32 \times (30–2) \bigcirc 32 \times 30–32 \times 2$$
$$(40–4) \times 25 \bigcirc 40 \times 25–4 \times 25$$

For each problem, students were expected to first compute the number sentences at both sides of "O" and then compare the results in order to fill in "=." The *Curriculum Analyses* alerted teachers that students may simply fill in with an "=" without computing and comparing, which should be discouraged because it may conflict with the spirit of discovery.

3.3 Transition across multiple topics over grades

Across worked examples: changes in concrete contexts With regard to representational transition across topics, we examined worked examples and practice problems over grades, respectively. Figure 2 presents changes of problem formats and solution representations across worked examples that were situated in word problems.

As indicated by Fig. 2a, the problem formats over grades demonstrated a decreasing level of concreteness. In grade 3 (n=3), all worked examples were formed by "pictures plus key words." In grade 4 (n=4), all of the worked examples still involved pictures. However, half of them were presented in complete word problem formats along with pictures. In grade 6 (n=5), only 37.5 % of the worked examples involved pictures, while 62.5 % of them were stated purely in words. With regard to solution representations, while the arithmetic solutions were found across grades, it was the sole solution format in grade 3 and it decreased across grades (see Fig. 2b). In contrast, algebraic solutions occurred in grade 4 and increased from grade 4 to 6. In particular, the algebraic story problems in grade 6 were double-reference problems, which were difficult for many students (Koedinger et al., 2008).

Across practice problems: changes in abstract contexts Figure 2c indicates changes of number uses across practice problems. Across grades 2–6, there was a shift from the whole numbers (e.g., $64\times8+36\times8$), to algebraic expressions involving whole numbers (e.g., $4a–2a$), to rational numbers ($2/5\times18+3/5\times18$), and algebraic equations involving all kinds of numbers ($x-0.8x=10$). While whole numbers have been consistently involved, it was the only type of number used before the formal introduction of the distributive property. After this, various contexts and numbers were incorporated. The increasing abstractness indicated by the number uses may result in less transparency in using the distributive property. For instance, it will be relatively easier for students to activate the distributive property when they face $64\times8+36\times8$ than $\frac{6}{5}\times\frac{6}{7}-\frac{1}{5}\div\frac{7}{6}$. However, when the distributive property is used with the latter one (first transforming "$\div7/6$" into "$\times6/7$"), this seemingly complex problem becomes extremely easy, which may promote students' appreciation of and spontaneous uses of this property.

4 Discussion

As illustrated in this study, the representative Chinese textbook series has made systematic transitions from concrete to abstract representations when presenting the distributive property, which aligns with the existing theory of cognitive development (Bruner, 1966; Piaget, 1952)

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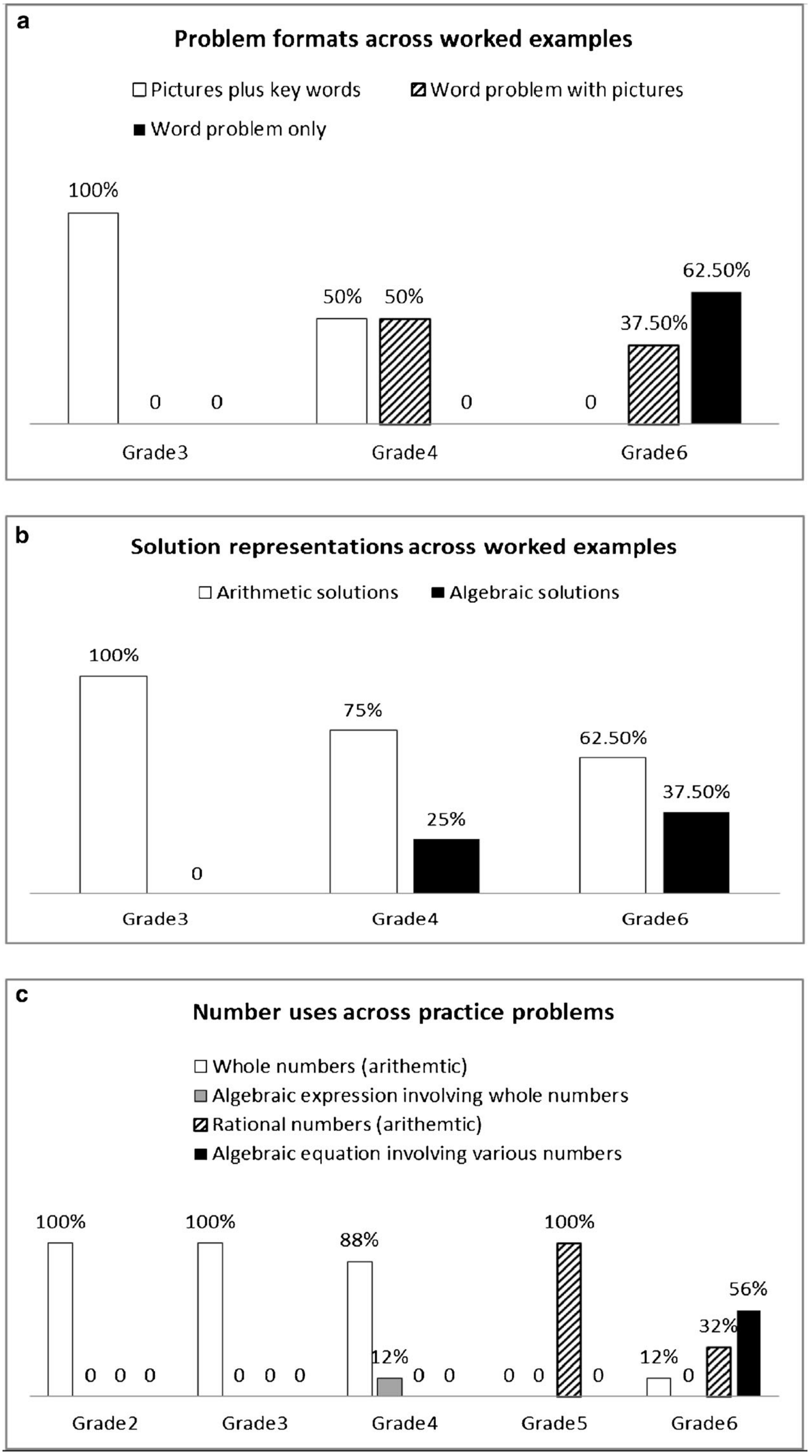

Fig. 2 Changes of representation uses across multiple topics over grades

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and the concreteness fading method (Goldstone & Son, 2005). This is not surprising because this reform Chinese textbook series is developed based on the new Chinese standards (Ministry of Education, 2001) that has incorporated some ideas of western learning theories and standards (Wang et al., 2004). The Chinese textbook presentation shows that there is no single "magical" and quick way to make representational transitions. To support students' learning of abstract mathematical principles like the distributive property, the Chinese textbook series coordinates the use of multiple strategies to be discussed below, which is much more complex than the treatment in a laboratory setting of the concreteness fading method (Goldstone & Son, 2005; McNeil & Fyfe, 2012). As such, our study has complemented laboratory findings from perspectives of curriculum. Figure 3 summarizes the multi-faceted representational transitions in the Chinese textbook series, followed by elaborations of four critical features.

4.1 Situating new learning in word problem contexts

The Chinese textbooks consistently situate new learning (the worked examples) in word problem contexts, which serves as a starting point of the transition process (see Fig. 3). This finding enriches existing theories, including the concreteness fading method, that mainly suggest starting with physical manipulations but not word problems (Bruner, 1966; Goldstone & Son, 2005; McNeil & Fyfe, 2012; Piaget, 1952). In fact, word problems can serve as real-world contexts in which mathematics is both developed and applied (Gerofsky, 2009). While the latter function (to apply knowledge) is commonly used in the field, the former function (to develop knowledge) is largely ignored. This is likely due to a common belief of teachers and curriculum designers who are informed by students' frequent failures with problem solving, that is, word problems are more difficult than computations (Nathan, Long, & Alibali, 2002). Such a belief, however, is inconsistent with the actual learning process where students tend to activate their familiar real-world knowledge to solve problems

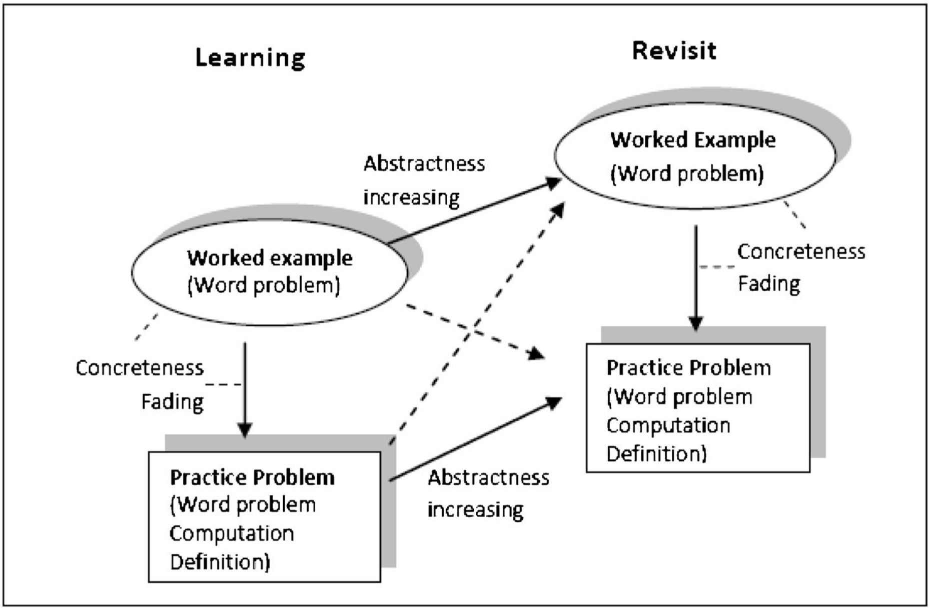

Fig. 3 Representational transition in Chinese textbooks

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(Koedinger et al., 2008). In fact, some researchers call for using word problem contexts to help students make sense of abstract ideas (Gerofsky, 2009; Palm, 2008) or situate new learning in situation-specific contexts that is real for students (Freudenthal, 1991; Gravemeijer, 1994). As such, the Chinese textbooks' systematic use of word problems as worked examples is insightful, which suggests rethinking the role of word problems in support of learning. It should be noted that all word problems selected in the Chinese textbooks as worked examples clearly embody the targeted mathematical principle without adding unnecessary cognitive difficulties.

The types of word problems used as Chinese worked examples, however, reveal a drawback. All of the problems are limited to equal-groups and multiplicative comparison meanings of multiplication. The area model suggested by NCTM (2000) and widely used in Europe and US elementary textbooks (Ding & Li, 2010) only occurred once in practice problems (but not in worked example). Chinese textbooks could have presented more instances of the area model when illustrating the distributive property.

4.2 Setting up a clear ultimate goal as abstract representations

Although word problems are used as concrete contexts for new learning, the concreteness has been intentionally and gradually faded out both within a worked example or with a lesson/topic, which is consistent with the concreteness fading style (Goldstone & Son, 2005, see Fig. 3). During the process of concreteness fading, Chinese textbooks have focused on the essential problem structure or the underlying mathematical principle embodied by the concrete situations. According to Chi and VanLehn (2012), focusing on the essential structure of the source problem will result in deep initial learning, which is a key to transfer. In addition, we observed decreasing concreteness/increasing abstractness across worked examples and across practice problems over grades (see Fig. 3). The above multi-faceted transitional processes suggest that abstraction seems to always serve as an ultimate goal for Chinese textbooks, whereas the concrete representations and the process of fading only serve as a necessary means toward the goal of abstraction. Chinese textbooks' constant transition from concrete to abstract representations may partially explain findings about Chinese teachers' and students' representational beliefs and competence in abstract thinking (Cai, 2004).

4.3 Incorporating problem variations with connection to prompt transition process

The representational transition in the Chinese textbook series is facilitated by careful task design, which appears to be consistent with prior findings about Chinese indigenous *variation* practice (Gu, Huang, & Marton, 2004; Sun, 2011). We observed both "problem variations with solution connection" and "problem variations with concept connection" (Sun, 2011, p. 73). As reported, there were carefully designed word problems solved with two solutions, which were further compared to introduce or revisit the distributive property (see Fig. 1). There were also simultaneously presented contrasting cases that were compared to deepen or extend students' understanding. Across grades, there were word problems with the same structures (e.g., equal-groups meaning, multiple comparisons) but involving different numbers and contexts (see Table 2). The number and context changes over grades were more obvious with computation problems (e.g., $226\times13-26\times13\rightarrow4a-2a\rightarrow x-0.8x=10$). The Chinese textbooks' problem variations indicate a progressive change in representation, which is consistent with the critical feature of concreteness fading that potentially keeps the "structural similarity" visible so as to promote representational transition (Goldstone & Son, 2005; McNeil & Fyfe, 2012). In fact, the progressiveness is also critical for mathematizing students' specific experiences into

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increasingly abstract and formal understanding of the distributive property (Freudenthal, 1991).

4.4 Engaging students in making sense of the transition process

Presenting carefully designed tasks alone may not ensure a successful representational transition, just as a good vehicle cannot guarantee arriving at the destination. This is because pedagogical tools, such as number lines, used for transition are nontransparent (Pashler et al., 2007) and students tend to make arbitrary connections between representations (Resnick & Omanson, 1987). Chinese textbooks provided scaffolding support by engaging students in the process of sense making. For example, when a number line diagram is used, students may be engaged in constructing lines, labeling the key quantities, or analyzing the diagrams. The textbooks also frequently pose deep questions and suggest comparisons for making effective connections. Indeed, deep questions are powerful tools that prompt students' self-explanations, which could lead to deep learning and effective transfer of the underlying principles (Craig, Sullins, Witherspoon, & Gholson, 2006). Recently, comparisons also drew renewed attention of the field as this method facilitates analogical reasoning and computation flexibility (Richland, Stigler, & Holyoak, 2012; Rittle-Johson & Star, 2007). Interestingly, while Richland et al. (2012) found that East Asian classrooms use visual representations as cognitive supports for effective comparison, we found that Chinese textbooks constantly use comparison as a critical aid to facilitate representational transition. Taking together, the above pedagogical supports may help students activate their relevant knowledge, enabling them to reason about upon the essential structure and structural similarities, which may lead to successful mapping between concrete and abstract representations (Chi & VanLehn, 2012; Holyoak & Koh, 1987).

Implementation, limitations, and future directions It should be mentioned again that transition from concrete to abstract is only one of the views in making connections between both representations (e.g., Piaget, 1952; Davydov, 1988) and the present study works with the Piagetian views. Our findings about Chinese textbooks' insights and alternatives provide rich resources for textbook designers and classroom teachers to refer back to. However, we caution against simple generalization of the findings due to the limitations of this study. First, our study has only focused on examination of textbooks rather than the effects of the identified Chinese approaches. Future classroom interventions may test these findings (e.g., concreteness fading within a worked example, different ways to use number line diagrams, comparisons, and representational transition) using the topic of the distributive property and beyond. Second, our findings have not yet been tested empirically in other cultural contexts. While the suggested approaches appear to be beneficial in the Chinese setting, there may be cultural factors that hinder the effective application of these approaches. Further studies may explore possible obstacles in implementing these approaches, thus informing modifications to enable a better fit with the actual settings.

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