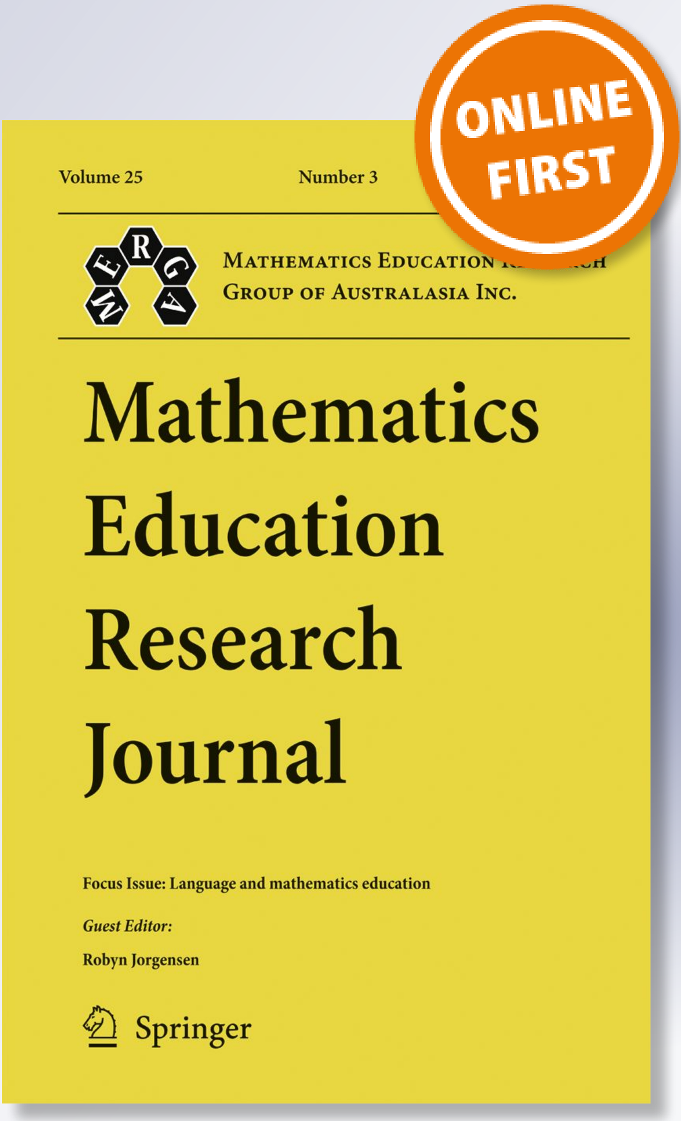
Children's strategies to solving additive inverse problems: a preliminary analysis

Meixia Ding & Abbey E Auxter

Mathematics Education Research Journal

ISSN 1033-2170

Math Ed Res J
DOI 10.1007/s13394-017-0188-4



Springer

Math Ed Res J
DOI 10.1007/s13394-017-0188-4

ORIGINAL ARTICLE

Children's strategies to solving additive inverse problems: a preliminary analysis

Meixia Ding[1] · Abbey E Auxter[2]

Received: 11 June 2016 / Revised: 15 November 2016 / Accepted: 5 January 2017

Abstract Prior studies show that elementary school children generally "lack" formal understanding of inverse relations. This study goes beyond lack to explore what children might "have" in their existing conception. A total of 281 students, kindergarten to third grade, were recruited to respond to a questionnaire that involved both contextual and non-contextual tasks on inverse relations, requiring both computational and explanatory skills. Results showed that children demonstrated better performance in computation than explanation. However, many students' explanations indicated that they did not necessarily utilize inverse relations for computation. Rather, they appeared to possess partial understanding, as evidenced by their use of part-whole structure, which is a key to understanding inverse relations. A close inspection of children's solution strategies further revealed that the sophistication of children's conception of part-whole structure varied in representation use and unknown quantity recognition, which suggests rich opportunities to develop students' understanding of inverse relations in lower elementary classrooms.

Keywords Inverse relations · Part-whole · Addition and subtraction · Children's strategy

The inverse relation between addition and subtraction (additive inverses) is one of the most important fundamental mathematical ideas for lower elementary grades (Baroody 1987, 1999; Carpenter et al. 2003). This relation, along with others, forms the basis for learning both arithmetic and algebra. For instance, children may use inverse relations

✉ Meixia Ding
meixia.ding@temple.edu

Abbey E Auxter
Abbey.auxter@temple.edu

1 Temple University, Ritter Hall 436, 1301 Cecil B. Moore Avenue, Philadelphia, PA 19122-6091, USA

2 Community College of Philadelphia, Philadelphia, PA, USA

Published online: 30 January 2017

for computation and checking (e.g., 81–79 = 2 because 79 + 2 = 81, Torbeyns et al. 2009) and use this relationship to solve algebraic equations (e.g., if $a + b = c$, then $b = c - a$, Carpenter et al. 2003). Longitudinal studies also have shown that students' performance on inverse tasks in second grade significantly predicted their algebraic achievement in 11th grade (Stern 2005). Indeed, inverse relations are involved in many other advanced topics such as functions and calculus, making these relationships an important building block of mathematics.

Despite the importance of inverse relations and the fact that many children possess informal understanding of inverse relations before entering elementary school (e.g., Gilmore and Spelke 2008; Sophian and Vong 1995), overwhelming evidence shows that elementary school children generally lack formal understanding of inverse relations (Baroody 1987, 1999; Bisanz and LeFevre 1990; De Smedt et al. 2010; Riley et al. 1983). What is missing behind children's "lack" of understanding is what they might already "have" in their existing conception. Knowing children's existing conception is critical because such information may serve as a basis for developing their understanding. The current study aims to fill this gap in the existing literature by identifying children's existing conception through an examination of their strategies when solving additive inverse problems.

Literature review

To situate our study in the existing literature, we first review prior research on children's understanding of inverse relations. Next, we review the part-whole structure, which is suggested as the key path to understanding inverse relations. Finally, we introduce research assertions on how to measure children's understanding of inverse relations. Together, these provide a theoretical foundation for the current study.

Children's understanding of inverse relations

Children's understanding of inverse relations is revealed by tasks involving two types of additive inverse principles: (a) the three-term inversion principle, $a + b - b = a$ and (b) the two-term complement principle, if $a + b = c$, then $c - b = a$ (Baroody et al. 2009). Although the inversion and complement principles are different and the sequence of learning them has not yet reached consensus, it is generally agreed that these principles are closely related and an understanding of one would contribute to the other (Baroody et al. 2009; Gilmore and Bryant 2008). Past studies have reported preschoolers' informal understanding of inverse relations based mainly on the three-term inversion principle. With approximate numbers involved (i.e., no actual number manipulation needed), children were able to provide correct directional responses (i.e., an increasing action will result in a larger quantity and a decreasing action will result in a smaller quantity, Gilmore and Spelke 2008; Sophian and Vong 1995; Sophian and Vong 1995). These directional responses may reflect children's primitive models (Fischbein et al. 1985) of part-whole structure and informal understanding of inverse relations. On the other hand, prior studies reported elementary school children's lack of formal understanding based mainly on the two-term complement principle (Baroody 1987, 1999; Baroody et al. 1983; Bisanz and LeFevre 1990, 1992; De Smedt et al. 2010; Resnick

1983; Riley et al. 1983). For example, Baroody et al. (1983) found that approximately 61% of first and second graders in their study could not use addition to solve subtraction problems (e.g., using $3 + 4 = 7$ to solve $7 - 4$). Other studies highlighted children' difficulties with the initial unknown change problems such as the following, "Ali had some Chinese stamps in his collection and his grandfather gave him 2; now he has 8; how many stamps did he have before his grandfather gave him the 2 stamps?" (Nunes et al. 2009, p. 5). According to Nunes et al., this initial unknown change problem is an inverse problem because the quantity in this situation increases while this problem is solved by subtraction. Nunes et al. pointed out that there is a consensus in the literature that inverse problems are more difficult for elementary children than are direct problems when the unknown is the result. These findings, however, indicate non-continuity of children's understanding of inverse relations from a younger age. For instance, in Sophian and McGorgray's (1994) study, the 4–6-year-olds demonstrated increased ability to provide correct directional responses to initial unknown change problems. Using the above stamps problem as an example, a child with correct directional responses would know that the initial quantity is less than "8" even if they may not necessarily know the exact answer. Preschoolers' directional responses indicate correct informal understanding of inverse relations, which contrasts with elementary children's lack of formal understanding.

The gap between children's informal and formal understanding of inverse relations suggests that there are potentially missed opportunities to teach inverse relations in elementary school. As reported, some teachers only stressed procedures such as drawing a small arrow from the subtrahend to the minuend (Torbeyns et al. 2009). Such procedural teaching styles are disconnected from children's prior knowledge that is often contextualized and concretely based. Indeed, prior research suggests a general growth of mathematical understanding moving from concrete image making to abstract structural knowing, although the abstract representations may be folded back to its primitive knowing that is relatively more concrete (Pirie and Kieren 1994). In other words, although the eventual goal is to grasp inverse relations at structural level, the connections back and forth between concrete and abstract representations seem to be a critical part of this knowledge. As Gilmore and Spelke (2008) suggested, children's existing conception of inverse relations brought to elementary school should be taken into consideration during formal classroom instruction.

Existing literature leaves a relatively incomplete picture of children's existing conceptions, such as the ones indicated by their existing strategies when solving inverse-based tasks. Arguably, the lack of information about children's existing conception of inverse relations might contribute to ineffective teaching of this important mathematical idea. As such, to provide children with better opportunities to learn, there is a need to identify possible degrees of construction between children's informal and formal understanding.

Part-whole: a path to inverse relations

Prior research has pointed out that children's protoquantitative notion of "part-whole" is a key to learning inverse relations (Canobi 2005; Gilmore and Spelke 2008; Piaget 1952; Resnick 1989, 1992; Sophian and McGorgray 1994). The part-whole structure is part of children's number development stages that starts from counting (Bobis et al.

2005; Young-Loveridge 2002). A series of early number development projects in the Australian region (Bobis et al. 2005; Clarke et al. 2006; Young-Loveridge 2002; Wright 1991, 1994), through task-based, one-to-one children interviews, have reported children's profiles and progression of number development including part-whole thinking. According to these researchers (e.g., Ellemor-Collins and Wright 2009), through counting activities, children may structure numbers into combination or partition relationships, which may further contribute to children's development of the part-whole structure. These views are aligned with Sfard (1991)'s dual nature of mathematical concepts. The counting process (e.g., counting from 1 to 6) is operational, but can eventually develop to a structural view (e.g., seeing 6 as an object or as a whole). The object of "6" can then be further operationalized on (e.g., combining 2 and 4 into 6, or decomposing 6 into 2 and 4), which, in turn, can also then develop into a structural view (e.g., seeing 2 and 4 as two parts of 6). According to Ellemor-Collins and Wright (2009), an automation of part-whole thinking will likely shift students from counting to non-counting strategies, enabling them to learn additive structures such as the inverse relations between addition and subtraction.

Likely due to the part-whole structure, children who lack the knowledge to manipulate numbers can still informally understand inverse relations (Gilmore and Spelke 2008). Canobi (2005) also viewed the sophistication of children's understanding of the part-whole relation as an indicator of students' conceptual understanding of addition and subtraction. According to this researcher, children who can compute addition and subtraction in a precise manner, when lacking an understanding of the part-whole relation, only possessed a procedural knowledge. International studies also reported that high-rated textbooks arranged the part-whole topic before formally teaching inverse relations (e.g., Ding 2012, 2016; Zhou and Peverly 2005). For instance, the Chinese textbooks expected students to first learn number composing and decomposing (e.g., 3 and 5 are composed to 8; 8 can be decomposed into 3 and 5), which demanded an understanding of part-whole relation. With this part-whole understanding, students were then expected to learn addition and subtraction as well as the relationship between them ($3 + 5 = 8$; $5 + 3 = 8$; $8 - 3 = 5$, $8 - 5 = 3$). As such, attention to bridging factors like the part-whole scheme may lead to the identification of paths to developing children's informal understanding of inverse relationships into formal understanding. While prior studies have highlighted the importance of part-whole structure, when it comes to learning inverse relations in existing classrooms, we do not yet know how this structure is adapted to deal with situations where using inverse operations is useful.

Measures of children's understanding of inverse relations

Students' inverse understanding is not an all-or-nothing phenomenon. Theoretically, students should gain more understanding as grade levels increase; however, empirical studies do not necessarily support this prediction. For example, Canobi (2005) found that as grade level increased, students' computation accuracy improved; yet, their conceptual understanding of the part-whole relationship did not necessarily increase. The research explained that the improved computation accuracy was likely due to the repeated practice overtime. However, if the conceptual underpinning is not addressed at the beginning, students' understanding may not improve automatically.

Students' understanding of inverse relations also depends on the contexts to which they are exposed. Facing a task that is situated in a contextual or non-contextual setting, students' understanding may or may not be demonstrated. By contextual tasks, we refer to those tasks that involve concrete objects or situations (e.g., pictures, blocks, or story problems). By non-contextual tasks, we refer to those tasks that are purely symbolic. It should be noted that even though contextual tasks are relatively more concrete while the non-contextual tasks are abstract, we do not view students' understanding demonstrated in non-contextual tasks as a higher level of understanding. This is aligned with Pirie and Kieren's (1994) theory that the growth of mathematical understanding is dynamic: leveled but non-linear. Students may fold back from abstract symbols to concrete images for sense making. Indeed, prior studies suggest measuring students' understanding of inverse relations using both contextual and non-contextual tasks (Bisanz et al. 2009). Through both types of tasks, students may be asked to evaluate, apply, and explain inverse relations, thus assessing their procedural and conceptual understanding (Bisanz and LeFevre 1992; Bisanz et al. 2009). Many times, children who provide correct answers cannot explain the underlying reasons, indicating a lack of explicit understanding. Of course, there may be occasions that children who understand the concept cannot explain it due to a lack of communication skills. Moreover, children who do explain still differ in quality of explanation, indicating different levels of inverse understanding.

To measure children's understanding of inverse relation, one may need to be cognizant of the complexity associated with children's satisficing for a solution. For example, even though Nunes et al. (2009) called a contextual task of the form $? - 2 = 6$ as inverse problem, one may argue that children may solve this problem through imagination. Instead of thinking $6 + 2 = ?$, a student may imagine giving away 2 from a set of 8 leaving with 6. In addition, students who have mastered the facts may directly recall $8 - 2 = 6$ and obtain the answer of 2. In both cases, children may solve this problem using direct or forward thinking. Children's lack of demonstration of inverse thinking does not necessarily indicate a lack of understanding. Rather, it may be the case that this task does not necessarily demand a use of inverse relations. Probably, a task involving large numbers such as $(\ ) - 79 = 2$ can better provoke students' understanding of inverse relations (Torbeyns et al. 2009) because images of 79 and 81 are unlikely to be available for children, and few children likely recall the fact of $81 - 79 = 2$. The complexity of children's performance on inverse tasks calls for careful interpretation of children's solutions and strategies on given measures.

Taken together, existing studies have suggested that there is a gap between students' understanding and non-understanding of inverse relations, which may be associated with both grade levels and the types of tasks. However, it is unclear what students actually know when they only possess partial understanding. What is the proportion of students' correct and partial understanding in terms of the correctness and explanation of solutions? In what ways does students' partial understanding differ from no and full understanding? How may this partial understanding be related to different grade levels and different types of tasks? This study aims to explore these questions. It is expected that our findings will inform teachers and researchers to better develop opportunities for students to learn inverse relations.

Method

To investigate the research questions, this study employs both quantitative and qualitative methods for analyzing data collected from a natural classroom. By "natural," we mean that students and teachers have not received any purposeful training related to inverse relations by our data collection. As such, student responses reflect their status quo of inverse understanding.

Participants

A total of 281 kindergarten through third grade students were recruited through a large project in the mid-west of the USA in which their teachers participated. Through this large project, the elementary teachers took graduate level courses with the goal to seek master's degrees concentrating on the teaching of elementary school mathematics. All teachers have sought parent consent for sharing their children's work if needed by the project. In the current study, there were 50 kindergarten students with mean ages of 5 years and 3 months ($SD = 6$ months); 74 first grade students with mean ages of 6 years and 1 month ($SD = 4$ months); 79 second grade students with mean ages of 7 years and 1 month ($SD = 3$ months); and 78 third grade students with mean ages of 8 years and 2 months ($SD = 5$ months). There were originally 194 third graders; however, for comparison, 78 were randomly selected as a representative sample. Overall, these students came from the classes of 35 different teachers—9 kindergarten, 6 first grade, 7 second grade, and 13 third grade. These teachers were invited to distribute a questionnaire to their students. As previously mentioned, by the time of data collection, no teacher had received any project training with regard to inverse relations. As such, students' responses to this questionnaire indicate a natural status of children's inverse understanding in existing classrooms.

Materials

To measure student's existing understanding of inverse relations, this study used four modified items from the literature. These items may be solved with inverse-based strategies involving complement and/or inversion principles, which together served as indicators of children's understanding of additive inverses. To help children ease into these items, contextual tasks were presented before non-contextual tasks. Figure 1 illustrates the questionnaire, followed by elaborations.

Question 1 (Q1) and Q2 are contextual tasks that were modified from Nunes et al. (2009). Both tasks are initial-unknown change problems. According to Nunes et al., Q1 describes an increase in quantity ($? + 2 = 8$) but the problem is solved by subtraction ($8 - 2 = ?$). In contrast, Q2 describes a decrease in quantity ($? - 2 = 6$) but the problem is solved by addition ($6 + 2 = ?$). Therefore, Nunes et al. (2009) refer to both tasks as "inverse problems." When solving both problems, students may reverse a sequence of actions (e.g., putting the given-away stamps back, Briars and Larkin 1984). During this reversing process, students may use their part-whole structure to transform a change problem to a combination (part-whole) model, which indicates an understanding that $a - b = c$ implies $c + b = a$ (Resnick 1989). Arguably, this reverse process may also indicate children's understanding of the inversion principle ($a - b + b = a$) because they

(1) Ali had some Chinese stamps in his collection and his grandfather gave him 2, now he has 8. How many stamps did he have before his grandfather gave him the 2 stamps? Please show your work.

(2) Ali had some Chinese stamps in his collection and gave 2 to his grandfather, leaving his collection with 6. How many stamps did he have before he gave his grandfather the 2 stamps? Please show your work.

(3) 5 + 3 – 3 = ( ). How did you get this answer?

(4) 6 + 3 = ( )

9 – 6 = ( ). How did you get the answer for 9 – 6? Did the addition problem help you solve this subtraction problem?

Fig. 1 The questionnaire used in this study

know that putting back the given-away stamps ($a-b+b$) could lead to the original number (a). The above reversing process indicates an understanding of inverse relations. However, due to children's primitive models (Fischbein et al. 1985) of directional responses, some children may be confused by the "reversing process." In fact, as acknowledged earlier, some children may not necessarily use inverse but direct thinking to model and solve these tasks [e.g., using "$(\ )-2=6$" to solve Q2]. Even in these cases, the semiotic demands for writing equations may still cause difficulties for children. This is because the lack of closure of ( ) in this equation may be confusing because the unknown occurs first in the equation but last in the story context. Such semiotic demands due to the inconsistent position of the unknown quantity in the context and in the equation may contribute child's difficulties in identifying the unknown (e.g., viewing a given quantity as the unknown, Riley and Greeno 1988).

Q3 and Q4 are non-contextual tasks. Tasks like Q3 ($5+3-3$) have been used in many previous studies (e.g., Baroody and Lai 2007; Bisanz, and LeFevre 1990; Gilmore and Bryant 2008; Stern 1992) to assess children's understanding of the inversion principle ($a+b-b=a$) involving children from preschoolers to fourth graders (ages 4–10). When presented with a problem like this, some children may use laborious, left-to-right procedure to find the answer (e.g., $5+3=8$, and then $8-3=5$), whereas others may answer it quickly without adding or subtracting (Bisanz, and LeFevre 1990). According to Bisanz and LeFevre (1992), the latter appear to use a shortcut based on the principle of inversion. Our follow-up question, "How did you get this answer?" was expected to elicit students' articulation of the procedures they used to solve this problem.

Q4 asked students to first solve a pair of problems ($6+3$ and $9-6$) and then explain their reasoning process. Such a task also has been used in many prior studies (e.g., Baroody 1999; Baroody et al. 1983; Canobi 2004) to assess children's understanding of the complement principle (if $a+b=c$, then $c-a=b$), involving children from kindergarteners to third graders (ages 5–9). On Q4, in addition to finding answers for each problem, students were expected to explain their reasoning process. According to Baroody (1999), although students may provide correct answers to these problems, they may not see and appreciate the inverse relations between them (Baroody 1999).

It should be noted that this questionnaire only contained four items. Although these tasks were selected from the existing research involving children with similar ages as in the current study, it could have been more reliable if this instrument contained more subtasks for each item. We also acknowledge that we did not pilot these items through a small student sample before asking K-3 teachers to work with their children. However, given the relatively large sample in each grade, our preliminary findings from this study may be expected to shed light on classroom instruction and follow-up research.

Procedures

An electronic copy of this instrument was sent to the participating teachers in summer as part of the instruction for the fall course preparation. Teachers were asked to administer this questionnaire with their students at the beginning of the fall semester. The purpose was to collect information about children's existing understanding of inverse relations that they brought to the classroom. Teachers were asked to provide 30 min and let children work through these four questions. For kindergarten teachers, we asked them to read the word problems and rephrase some words as needed to help their children understand the problem statement. However, we emphasized that, because this is an assessment rather than instruction, no hints for solution should be given to children. We asked the kindergarten teachers to tell their children not to worry about spelling because the researchers could figure out their writing. We also allowed kindergarten teachers to record children's explanations if they needed help. Children's responses were collected and brought back to the project by the participating teachers.

Data coding and analysis

Our data included both quantitative (e.g., student answers) and qualitative (e.g., student explanations) components. As such, both quantitative and qualitative analyses were involved in this study (Creswell 2014). Students' answers to each question were coded first for correctness. This process was straightforward. For reliability checking, we randomly selected 10% of student responses in each grade and the reliability reached 100%. The percentages of student correctness were summarized by tasks and by grades. To identify the differences across grades and among tasks, the one-way ANOVA along with the Bonferroni post hoc tests were calculated.

Next, we analyzed students' explanations for each question. These qualitative data were coded using the constant comparison method (Dye et al. 2000; Glaser 1965), which involves categorizing the responses through inductive analysis and constantly comparing incidents applicable to each category. As new responses were analyzed, the existing categories may be refined and new categories may emerge as necessary. To develop codes, we selected 10% of responses from each grade and both authors coded them independently. Next, the two authors came together to compare their codes and discuss the coding difficulties and disagreements. After a shared understanding had been reached, we combined our codes into broader categories. Once the initial categorization was completed, one author proceeded with coding the rest of data following the comparison method, which enabled ongoing enrichment and refinement of the categories.

Overall, students' explanations were classified into three macro levels (levels 0, 1, and 2) indicating (0) no evidence that shows understanding of inverse relations (simply, no understanding), (1) evidence that shows partial understanding of inverse relations (simply, partial understanding), and (2) evidence that shows full understanding of inverse relations (simply, full understanding), respectively. In particular, level 0 (no understanding) included two subcategories: (0a) no explanation and (0b) wrong/uninformative explanation. Since students who did not provide an explanation does not necessarily mean that they lack understanding (e.g., may be due to reading ability), it is worthwhile to separate them from students who did explain, but did so with a wrong or uninformative explanation. Level 2 (full understanding) also included two subcategories: (2a) full explanations involving concrete aids and (2b) full explanations at an abstract level.

Between levels 0 and 2, there were explanations that did not involve inverse relations but showed part-whole relations (e.g., a part-whole picture, a number sentence that shows part-whole relation). We coded these explanations as evidence of partial understanding (level 1) because prior research suggests that part-whole structure is a key to learning inverse relations. Under this category, there were three different situations in terms of representation uses: (a) part-whole picture only, (b) part-whole picture and number sentence, and (c) number sentence only. Within each situation, we found that students may or may not be able to identify the unknown quantity because some students may suggest the given quantity as the answer to the question (thus, a wrong answer). Prior studies (e.g., Riley and Greeno 1988) pointed out that some students had a harder time identifying the unknown quantity when its position was at "start," which indicated weak part-whole understanding. In this study, to track students' identification of the unknown quantity, we categorized student responses with "incorrect answer," "no answer indicated," and "correct answer," using the "− sign," "no sign," and "+ sign," respectively. As a result, we obtained a 3 × 3 subcategory matrix for partial understanding (see Table 1), resulting in nine sublevels of partial understanding. Detailed examples are provided in Results.

Results

In this section, we report findings on the correctness of student responses, followed by their explanations to each question. Further, we conducted a close inspection on

Table 1 Subcategories of partial understanding of inverse relations

	Incorrect answer	No answer indicated	Correct answer
Part-whole picture	1a−	1a	1a+
Part-whole picture and number sentence	1b−	1b	1b+
Number sentence only	1c−	1c	1c+

students' partial explanations, which suggests opportunities for instruction in leveraging their understanding toward a higher level.

Correctness of student responses

Students' overall correctness of responses is presented in Fig. 2, summarized by tasks (left) and by grades (right), respectively.

As indicated by Fig. 2 (left), while the computation accuracy did not exceed 85% for all tasks, there is a growing pattern over time. The one-way ANOVA test shows that the overall change across grades for each task is significant, F_{Q1} (3, 277) = 12.379, $p_{Q1} < .001$; F_{Q2} (3, 277) = 5.141, $p_{Q2} = .002$; F_{Q3} (3, 277) = 20.373, $p_{Q3} < .001$; and F_{Q4} (3, 277) = 41.378, $p_{Q4} < .001$. The Bonferroni post hoc test shows that for contextual tasks (Q1 and Q2), significant changes include those differences between non-neighboring grades (e.g., G2-K, $p_{Q1} < .001$; G3-K, $p_{Q1} < .001$; G3-G1, $p_{Q1} < .001$; G3-K, $p_{Q2} < .01$; G3-G1, $p_{Q2} < .05$) but not neighboring grades (e.g., G1-K, G2-G1). For non-contextual tasks (Q3 and Q4), significant changes include those from kindergarten to the other grades (e.g., G1-K, $p_{Q3} < .001$; G2-K, $p_{Q3} < .001$; G3-K, $p_{Q3} < .001$; G1-K, $p_{Q4} < .001$; G2-K, $p_{Q4} < .001$; G3-K, $p_{Q4} < .001$) and from G1 to later grades (G3-G1, $p_{Q3} < .05$; G2-G1, $p_{Q4} < .001$; G3-G1, $p_{Q4} < .001$) except for the change from G1 to G2 for Q3; however, the changes between G2 and G3 for both Q3 and Q4 are non-significant. Overall, our data indicates that for simple addition and subtraction problems, students quickly develop their computation skills from kindergarten to later grades.

The second pattern is specifically related to the kindergarteners (see Fig. 2, right). These beginning learners performed much better on contextual tasks (Q1 and Q2, solid shaded) than non-contextual tasks (Q3 and Q4, pattern shaded). This is consistent with prior findings that children may reason upon contextual information even before they have mastered number manipulations (Gilmore and Spelke 2008; Sophian et al. 1995; Sophian and Vong 1995).

The third pattern relates to the two types of inverse relations (see Fig. 2, right). It seems that students at the beginning of schooling (K and G1) performed equally well or slightly better on the three-term inversion principle (Q3) than on the two-term complement principle (Q4). When the grade level increases (G2 and G3), however, children seemed to demonstrate more fluency in the two-term complement principle that was frequently reported to be difficult in the literature. In brief, even though there is room for students to improve computation skills on inverse-based tasks, there is an overall pattern of linear growth across grades.

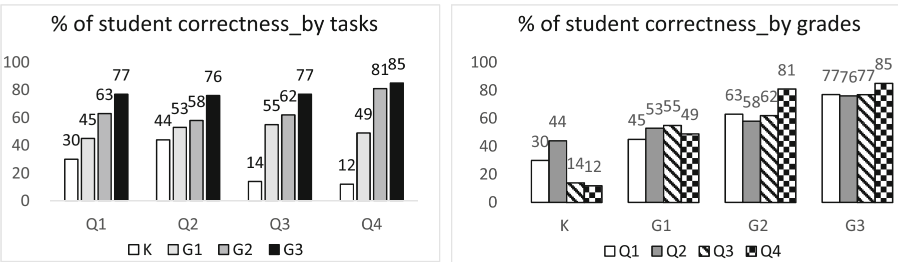


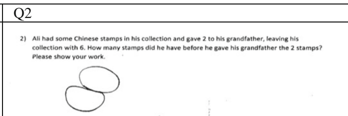
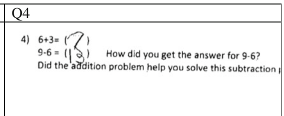
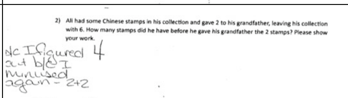
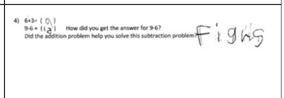
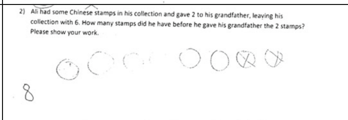
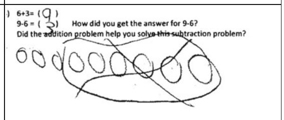
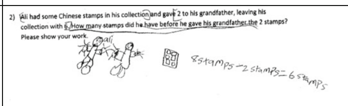
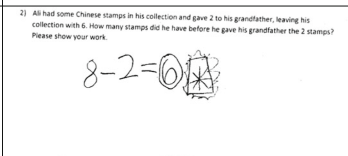
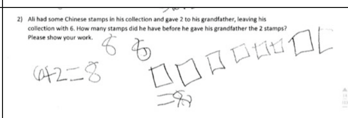
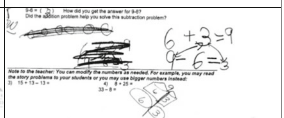
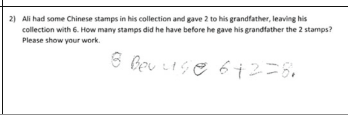
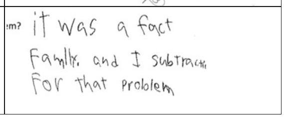
Fig. 2 Percentage of student correctness across tasks and grades

Student explanations to each question

In comparison with their correctness in computation, students' explanations appeared to be much poorer and fell into three levels: no, partial, and full understanding. Table 2 presents typical examples using contextual-task (Q2) and non-contextual task (Q4). As explained in Methods, Q2 described a decreasing situation but may be solved with addition (Nunes et al. 2009); Q4 expected students to solve a subtraction fact using the related known addition fact. Both tasks called for students' understanding of inverse relations. In Table 2, when students possessed no understanding of inverse relations (level 0 examples), they either provided no explanations (0a) or wrong/uninformative explanations (0b, e.g., "I used fingers"; "I minused again −2 + 2"). In contrast, when students possessed full understanding (level 2 examples), for Q2, they were able to add back what was taken-away to find the original number of stamps, 6 + 2 = 8; for Q4, they highlighted the relationship between 9 − 6 = 3 and 6 + 3 = 9. The above responses either involved concrete aids (2a, e.g., drawing arrows or a part-whole mat) or reached an abstract level (2b, e.g., stating that "it was a fact family").

However, some students' understanding fell between these two levels (see level 1 examples). On one hand, their responses did not demonstrate explicit understanding of

Table 2 Examples of student explanations that show no and full understanding

Level	Category	Q2	Q4
0 - No understanding	0(a) no explanations		
	0(b) wrong/uninformative explanations		
1 – Partial understanding	1(a) Partial explanations at concrete level		
	1(b) Partial explanations involving both concrete and abstract aids		N/A
	1(c) Partial explanations at abstract level		N/A
2 - Full understanding	2(a) full explanations involving concrete aids (e.g., pictures or verbal descriptions)		
	2(b) full explanations at an abstract level (e.g., using number sentences only).		

inverse relations. For Q2, these students might have listed a subtraction sentence ($8-6=2$) that was directly aligned with the action of "decreasing"; for Q4, they might have computed $9-6$ by crossing out 6 circles from 9. On the other hand, these responses showed understanding of the part-whole relations (e.g., drawing a part-whole picture), and thus classified as partial understanding. These responses also differed in the types of representations including using pictures only (1a), symbols only (1c), or a combination of both (1b).

A summary of the percentages of students' explanations that fell into each category under each level is presented in Table 3.

As indicated by Table 3, most students in this study did not provide an explanation (0a) or provided wrong/uninformative explanations (0b), indicating no evidence of understanding. The highest percentage of full understanding occurred with third graders, which only reached 38%. Overall, students' explained contextual-tasks (Q1 and Q2) better than non-contextual tasks (Q3 and Q4). While there were about 32–38% of third graders whose explanations of contextual tasks achieved full understanding, only 4% of them fully explained the computation tasks. This is possibly due to the fact that the third graders have mastered the basic facts like $5+3-3=(\ )$ and $9-6=(\ )$ and simply did a fact recall. Thus, they may be confused about how to explain these simple computation tasks. Alternatively, it may be that these computation tasks are too simple and the third graders lose interest in explaining the process of obtaining the answers.

Even though the overall situation of student explanation was poor, across grades, there was evidence of growth in student explanations. First, at level 0 (no

Table 3 Percent of student explanations to each question across grades

		Grade	Level 0			Level 1				Level 2			Missing data
			0a	0b	Total	1a	1b	1c	Total	2a	2b	Total	
Contextual tasks	Q1	K	64	20	84%	4	0	2	6%	8	2	10%	–
		1	12	49	61%	26	0	6	32%	3	0	3%	4%
		2	15	19	34%	31	9	19	59%	4	4	8%	–
		3	15	18	33%	1	5	20	26%	10	28	38%	3%
	Q2	K	74	20	94%	0	0	2	2%	2	0	2%	2%
		1	11	36	47%	33	0	2	35%	7	8	15%	3%
		2	15	23	38%	37	1	9	47%	4	3	7%	8%
		3	15	24	39%	2	9	16	27%	6	26	32%	2%
Non-contextual tasks	Q3	K	58	38	96%	0	0	2	2%	2	0	2%	–
		1	26	35	61%	35	0	3	38%	0	0	0%	1%
		2	23	44	67%	10	0	15	25%	1	0	1%	6%
		3	15	47	62%	5	0	27	32%	3	1	4%	2%
	Q4	K	66	20	86%	0	0	0	0%	0	0	0%	14%
		1	30	41	71%	30	0	0	30%	0	0	0%	–
		2	20	54	74%	9	0	0	9%	9	5	14%	3%
		3	38	54	92%	4	0	0	4%	0	4	4%	–

The percentages were rounded to whole numbers, which may not total up to 100%

understanding), the kindergarteners held the highest percentage for the first three tasks (84, 94, and 96%, respectively). Consistent with this observation, at level 2 (full understanding), the third graders held the highest percentage for the first three tasks (38%, 32%, 4% for the first three tasks, respectively). It is unexpected that for Q4, the third graders explained more poorly than the second graders (4 vs. 14%, respectively). As mentioned above, this might be due to the fact that basic facts like $9 - 6 = (\ )$ are too simple for the third graders and 38% of them did not provide an explanation to this task. Overall, student responses show growth even though the pattern did not show linearity.

Within level 0, a comparison of students' responses between levels 0a and 0b within a grade shows that students in G1-3 attempted to explain more frequently than did kindergarteners. In kindergarten, there were more students who provided no explanations (0a) than those who provided wrong/uninformative explanations (0b). This observation holds true for all four items (e.g., 64 vs 20% for Q1; 74 vs 20% for Q2; 58 vs 38% for Q3; and 66 vs. 20% for Q4). This trend, however, was reversed for later grades (G1-3) across all items. Possibly, students in kindergarten lacked the ability to read the items and some of them hesitated to ask for teacher's help, which caused their lack of responses. Alternatively, students in kindergarten demonstrated the lowest ability to explain their thinking.

Finally, the trend of student growth appeared with students' level 1 explanations. From kindergarten to the later grades, there was an increasing number of students who demonstrated an understanding of part-whole relation (see Table 3). The one-way ANOVA test indicates that there were significant differences between levels of explanations across grades except for the level 2 in Q3 (see Table 4).

The Bonferroni post hoc test indicated detailed differences between grade levels. The most interesting observation is that, even though there was no difference between kindergarten and many other grades at level 0 and level 2 explanations, when it comes to level 1 (partial understanding), those grades show significant differences from kindergarten. This suggests a closer inspection of students' partial explanations (level 1) that may bridge students' no and full understanding.

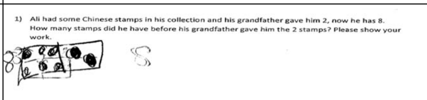
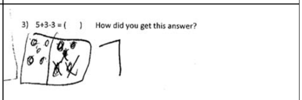
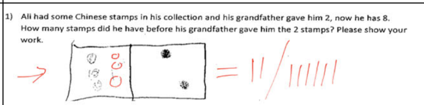
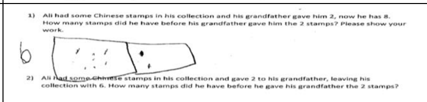
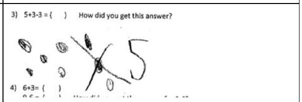
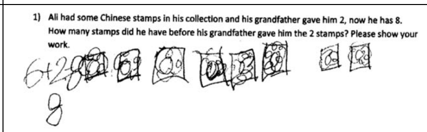
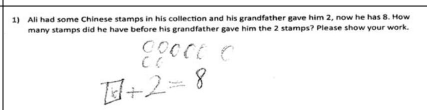
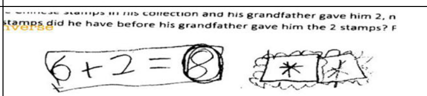
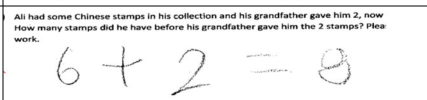
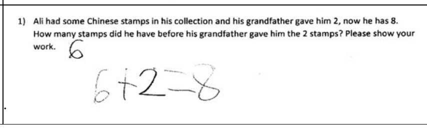
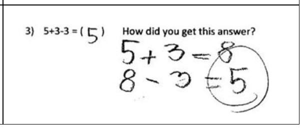
Table 4 One-way ANOVA test for differences in partial explanations at each level

	Df	*F*	Sig
Q1_level 0	3	16.753	.000
Q1_level 1	3	15.431	.000
Q1_level 2	3	18.096	.000
Q2_level 0	3	18.410	.000
Q2_level 1	3	11.146	.000
Q2_level 2	3	10.691	.000
Q3_level 0	3	7.464	.000
Q3_level 1	3	7.714	.000
Q3_level 2	3	1.124	.340
Q4_level 0	3	5.018	.002
Q4_level 1	3	13.564	.000
Q4_level 2	3	7.097	.000

A closer inspection of students' partial explanations

Investigating students' partial explanations at level 1, we identified differences in students' representation use and unknown quantity recognition in their part-whole structure, which suggested possible paths to develop students' understanding of inverse relations. For representation use, it was found that students in grade 1 preferred using part-whole pictures to show the answers, students in grade 3 tended to use number sentences to show their thinking, and students in grade 2 were split between the two (see Table 3). In addition, students' explanations differed in recognizing the unknown quantity in the part-whole structure. Table 5 illustrates typical examples using Q1 and Q3. With regard to Q1, students' responses across 1a−, 1b−, and 1c− marked the wrong number "8" as the answer regardless of these responses containing part-whole pictures

Table 5 Examples of student explanations that show partial understanding

Category		Q1: (Unknown quantity/correct answer is 6)	Q3: (Unknown quantity/correct answer is 5)
1a-	Part-whole picture with incorrect answer		
1a	Part-whole picture with no answer		N/A
1a+	Part-whole picture with correct answer		
1b-	Part-whole picture and number sentence with incorrect answer		N/A
1b	Part-whole picture and number sentence with no answer		N/A
1b+	Part-whole picture and number sentence with correct answer		N/A
1c-	Number sentence only with incorrect answer		N/A
1c	Number sentence only with no answer		N/A
1c+	Number sentence only with correct answer		

or number sentences. In their responses at 1a, 1b, and 1c, student did not mark an answer, which were likely due to their non-clarity or implicit awareness of the unknown quantity in the part-whole structure.

The percentage of students' overall situation in unknown quantity recognition under the part-whole structure is summarized in Table 6.

As indicated by Table 6, students' difficulties with unknown quantity recognition were more apparent with the contextual tasks (Q1 and Q2) than non-contextual tasks (Q3 and Q4). This made sense given the unknown quantity in non-contextual tasks was often already marked. With regard to contextual tasks (Q1 and Q2), students across all grades appeared to have difficulties. For instance, when students drew the part-whole pictures to solve Q1, 23% of second graders did not mark the unknown quantity (1a). It was uncertain whether these students could identify the unknown quantity because there were cases that students could not (e.g., 7% first graders; 3% second graders). When students employed a number sentence to solve this problem, their chance of marking a wrong number as the unknown was lessened (1% first graders). Yet, there were still 5% of first graders, 8% of second graders, and 14% of third graders that did not identify it (1a), leaving uncertainty whether these students had a clear understanding of the unknown quantity. The above situation was similar to Q2. With regard to non-contextual tasks (Q3 and Q4), some students performed all steps. For instance, in Q3, they either drew out each step (see Table 4, example 1a+) or actually computed each step (e.g., $5+3=8$, and $8-3=5$, see Table 4, 1c+). These responses do not show inverse understanding, but their work does indicate part-whole structure and is therefore considered as level 1 understanding. However, at this level, a few students still identified the wrong unknown quantity in both Q3 (see Table 4, example 1a−) and Q4 (see Table 2).

Table 6 Percentage of students' recognition of unknown quantity

Partial understanding	Grade	1a			Subtotal	1b			Subtotal	1c			Subtotal
		1a− (%)	1a (%)	1a+ (%)		1b− (%)	1b (%)	1b+ (%)		1c− (%)	1c (%)	1c+ (%)	
Q1	K	0	2	2	4	0	0	0	0	0	0	2	2
	1	7	1	18	26	0	0	0	0	1	5	0	6
	2	3	23	5	31	1	3	5	9	0	8	11	19
	3	0	1	0	1	0	5	0	5	0	14	6	20
Q2	K	0	0	0	0	0	0	0	0	0	0	2	2
	1	4	1	28	33	0	0	0	0	1	1	0	2
	2	8	24	5	37	0	0	1	1	0	3	6	9
	3	1	1	0	2	0	9	0	9	1	10	5	16
Q3	K	0	0	0	0	0	0	0	0	0	0	2	2
	1	3	0	32	35	0	0	0	0	0	1	1	3
	2	0	0	10	10	0	0	0	0	0	0	15	15
	3	0	0	5	5	0	0	0	0	0	0	27	27
Q4	K	0	0	0	0	0	0	0	0	0	0	0	0
	1	3	0	27	30	0	0	0	0	0	0	0	0
	2	0	0	9	9	0	0	0	0	0	0	0	0
	3	0	0	4	4	0	0	0	0	0	0	0	0

Discussion

Previous studies indicate that students who come to elementary school with informal understanding of inverse relations generally lack formal understanding of this mathematical relation (Baroody 1987, 1999; Baroody et al. 1983; Bisanz and LeFevre 1990, 1992; De Smedt et al. 2010; Resnick 1983; Riley et al. 1983). This study takes a further step beyond lack to explore what students may have in their existing conception especially related to the part-whole structure, which may afford opportunities for classroom instruction. To access students' understanding in a relatively complete fashion, our assessments involve both contextual and non-contextual tasks of inversion and complement principles, requiring both computation and explanation skills (Bisanz and LeFevre 1992; Bisanz et al. 2009). Our quantitative and qualitative analyses show that students generally perform better in computation than explanation even though some students still could not compute in third grade. Most students who obtained correct computational answers did not utilize inverse relations to solve these tasks. Given that inverse relations is one of the fundamental mathematical ideas in early grades (Baroody 1987; Carpenter et al. 2003) and has been systematically emphasized by the Common Core State Standards (National Governors Association Center for Best Practices and Council of Chief State School Officers 2010), the status quo of students' understanding should draw immediate attention.

Our findings suggest that students' computation skills may grow naturally across grades, likely due to the opportunities of repeated practices (Canobi 2005). However, their explanations of inverse relations, as revealed by our qualitative analysis, may not grow in a linear fashion. This finding is consistent with many others that reported students' difficulties in inverse relations, especially with the complement principle (e.g., Baroody et al. 1983; Baroody 1999; De Smedt et al. 2010; Riley et al. 1983). These findings call for meaningful and explicit support for elementary children's development of inverse understanding.

The strength of qualitative analysis in this study has enabled the identification of children' existing conception of inverse relations based on their explanations. As indicated by their solution strategies, many students do possess understanding of part-whole structure, which is a key to understanding inverse relations (Piaget 1952; Resnick 1992). However, these students' understanding of part-whole structure appears to be non-sophisticated as it is mainly limited to direct thinking, which only show partial understanding. For instance, with regard to an increasing situation (e.g., getting more stamps), students tend to use direct thinking that aligned with the direction of quantity changes, "part + part = whole." This is only a portion of the part-whole structure, which also contains the reversed aspect, "whole − part = part." A more complete understanding of the part-whole structure may provide flexibility when dealing with what is given and what is unknown, thus better problem solving with inverse problems. In fact, our findings regarding students' detailed levels of part-whole understanding in terms of representation uses and unknown quantity recognition portrays a rich picture of opportunities to teach.

With regard to representations, we found that students' understanding of part-whole relation generally moves from concrete to abstract across grades. This observation is more apparent with contextual tasks. As reported, younger students preferred drawing part-whole pictures while students in later grades listed number sentences more frequently. The difference between using concrete pictures and abstract number sentences indicates that students' understanding of part-whole relation is not an all-or-nothing phenomenon (Bisanz et al. 2009) and the have/have-not dichotomy is not helpful for supporting student learning. In order to develop meaningful understanding of part-whole relation, teachers may first employ concrete aids (e.g., part-whole pictures) that come more naturally for students. Indeed, students' improved performance in contextual tasks versus non-contextual tasks, especially in kindergarten, supports this assumption. These findings challenge existing instruction that focused more on number manipulation when teaching additive inverses (Ding 2016). Given that students have the ability to reason abstractly and abstract thinking is indeed an ultimate goal of mathematics education, teachers should help students abstract from context to promote shared, explicit understanding (Hershkowitz et al. 2007; Hershkowitz et al. 2001. For instance, if students in appropriate grades demonstrated part-whole understanding using the part-whole picture, a teacher may guide them to generate a corresponding number sentence. It should be noted again the growth of children's mathematical understanding is leveled but not necessarily linear (Pirie and Kieren 1994), and teachers should help students fold back from abstract to concrete representations for sense-making. As such, when a teacher prompts students' inverse understanding by linking one part of the part-whole structure (e.g., part + part = whole) to the other part (e.g., whole − part = part), it might be critical to help students see this connection based on both concrete and abstract representations.

In addition to the connection between concrete and abstract representations, students in this study differed in their ability to recognize the unknown quantity in a part-whole structure. In this study, students used either direct thinking ($6+2=8$) or reversed thinking ($8-2=6$) to solve Q1. While multiple solutions should be encouraged, students with the direct thinking should recognize that "6" indeed refers to the unknown in Q1. As such, the number sentence should be listed as $\square+2=8$ for clarity. This is why $8-2=\square$ can be used solve the same problem, which shows an understanding of inverse relation. However, this seemingly trivial point is often neglected in existing instructional environments. For instance, some existing textbooks simply suggest two number sentences to solve the same problem without highlighting the unknown quantity (e.g., $6+2=8$ and $8-2=6$ are suggested to solve the same story problem, Ding 2016). Even though it is obvious for an educated adult that an unknown quantity refers to a number that was not given in the original question, it might be unclear for some children who always treated the last number in an equation as the answer to the unknown due to semiotic demands for writing such equations. Such an expert blind spot (Nathan and Koedinger 2000) should draw teachers' attention. As Riley and Greeno (1988) pointed out, when the position of the unknown quantity is at the "start" instead of "result," some students have difficulty identifying it and suggest a given quantity as the answer, which indicates their weak part-whole understanding. As such, it seems beneficial to stress clear understanding of the known and unknown quantities so as to develop students' sophisticated part-whole structure. Overall, our findings about students' varied level of part-whole understanding illustrate what students may have in their existing conception when they appear to lack understanding of inverse relations. These findings highlight opportunities for teachers and teacher educators to prompt students one-step further so as to develop their understanding of inverse relations.

Conclusion

Disparities among children's mathematical understanding at school entry tend to remain or even increase over time when children advance through the school system (Bobis et al. 2005). As such, it is critical to address children's understanding gaps of key concepts from the very beginning of schooling. This study focuses on K-3 children's understanding of the inverse relations, potentially contributing to narrowing the understanding gap. We are aware of several limitations of this study, which suggests future research directions. First, the number of assessment tasks is limited. Therefore, our analysis is basically preliminary and findings should not be overgeneralized. For instance, even though we noticed some patterns related to students' learning of both types of inverse relations (three-term inversion principle, and two-term complement principle), the sequence for learning both principles cannot be concluded. Second, our findings are based on paper and pencil assessments. Even though students shared their thinking through explanation, their explanations may not necessarily represent their understanding. This may be particularly true for kindergarteners who may lack sufficient reading ability and third graders who may have mastered the basic facts and need more challenges. Future research may conduct child interviews using approaches similar to prior research (Bobis et al. 2005; Clarke et al. 2006; Young-Loveridge 2002; Wright 1991, 1994) so as to obtain clearer pictures. Regardless of the above limitations, our findings about the status quo of students' understanding based on a relatively large sample add to the existing literature (Nunes et al. 2009; Gilmore and Spelke 2008) and shed light on classroom opportunities for teaching and learning inverse relations. With a greater emphasis on students' complete and flexible part-whole structure at both concrete and abstract levels, students can be expected to achieve more sophisticated understanding of inverse relations.

Acknowledgements This study is supported by the National Science Foundation (NSF) CAREER program under Grant No. DRL-1350068 at Temple University and the NSF grant DUE-0831835 at the University of Nebraska–Lincoln. Any opinions, findings, and conclusions in this study are those of the author and do not necessarily reflect the views of the National Science Foundation.

References

Baroody, A. J. (1987). *Children's mathematical thinking: a developmental framework for preschool, primary, and special education teachers*. New York: NY: Teacher College Press.

Baroody, A. J. (1999). Children's relational knowledge of addition and subtraction. *Cognition and Instruction, 17*, 137–175.

Baroody, A., & Lai, M. (2007). Preschoolers' understanding of the addition–subtraction inverse principle: a Taiwanese sample. *Mathematical Thinking and Learning, 9*, 131–171.

Baroody, A., Ginsburg, H., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education, 14*, 156–168.

Baroody, A. J., Torbeyns, J., & Verschaffel, L. (2009). Young children's understanding and application of subtraction-related principles: introduction. *Mathematics Thinking and Learning, 11*, 2–9.

Bisanz, J., & LeFevre, J. (1990). Strategic and nonstrategic processing in the development of mathematical cognition. In D. Bjorklund (Ed.), *Children's strategies: contemporary views of cognitive development.* Hillsdale: Erlbaum.

Bisanz, J., & LeFevre, J. (1992). Understanding elementary mathematics. In J. I. D. Campbell (Ed.), *The nature and origins of mathematical skills* (pp. 113–136). Amsterdam: North Holland, Elsevier Science.

Bisanz, J., Watchorn, R., Piatt, C., & Sherman, J. (2009). On "understanding" children's developing use of inversion. *Mathematical Thinking and Learning, 11*, 10–24.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., & Young-Loveridge, J. (2005). Supporting teachers in the development of young children's mathematical thinking: three large scale cases. *Mathematics Education Research Journal, 13*(3), 27–57.

Briars, D. J., & Larkin, J. H. (1984). An integrated model of skill in solving elementary word problems. *Cognition and Instruction, 1*, 245–296.

Canobi, K. H. (2004). Individual differences in children's addition and subtraction knowledge. *Cognitive Development, 19*, 81–93.

Canobi, K. H. (2005). Children's profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology, 92*, 220–246.

Carpenter, T. P., Franke, L. P., & Levi, L. (2003). *Thinking mathematically: integrating arithmetic & algebra in elementary school*. Portsmouth: Heinemann.

Clarke, B., Clarke, D., & Cheeseman, J. (2006). The mathematical knowledge and understanding young children bring to school. *Mathematics Educational Research Journal, 18*(1), 78–102.

Creswell, J. W. (2014). *Research design: qualitative, quantitative, and mixed methods approaches* (4th ed.). Thousand Oaks: Sage.

De Smedt, B., Torbeyns, J., Stassens, N., Ghesquière, P., & Verschaffel, L. (2010). Frequency, efficiency and flexibility of indirect addition in two learning environments. *Learning and Instruction, 20*, 205–215.

Ding, M. (2012). Early algebra in Chinese elementary mathematics textbooks: the case of inverse relations. In B. Sriraman, J. Cai, K. Lee, L. Fan, Y. Shimuzu, L. C. Sam, & K. Subramanium (Eds.), *The first sourcebook on Asian research in mathematics education: China, Korea, Singapore, Japan, Malaysia, & India*. Charlotte: Information Age Publishing.

Ding, M. (2016). Opportunities to learn: inverse operations in U.S. and Chinese elementary mathematics textbooks. *Mathematical Thinking and Learning, 18*(1), 45–68.

Dye, J. F., Schatz, I. M., Rosenberg, B. A., & Coleman, S. T. (2000). Constant comparison method: a kaleidoscope of data. *The Qualitative Report, 4*(1), 1–10.

Ellemor-Collins, D., & Wright, R. (2009). Structuring numbers 1 to 20: developing facile addition and subtraction. *Mathematics Education Research Journal, 21*(2), 50–75.

Fischbein, E., Deri, M., Sainati Nello, M., & Sciolis Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*, 3–17.

Gilmore, C. K., & Bryant, P. (2008). Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill. *British Journal of Developmental Psychology, 26*, 301–316.

Gilmore, C. K., & Spelke, E. S. (2008). Children's understanding of the relationship between addition and subtraction. *Cognition, 107*, 932–945.

Glaser, B. G. (1965). The constant comparative method of qualitative analysis. *Social Problems, 12*, 436–445.

Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in context: epistemic actions. *Journal for Research in Mathematics Education, 32*, 195–222.

Hershkowitz, R., Hadas, N., Dreyfus, T., & Schwarz, B. (2007). Abstracting processes, from individuals' constructing of knowledge to a group's "shared knowledge". *Mathematics Education Research Journal, 19*(2), 41–68.

Nathan, M. J., & Koedinger, K. R. (2000). An investigation of teachers' beliefs of students' algebra development. *Cognition and Instruction, 18*, 209–237.

National Governors Association Center for Best Practices (NGA Center) & Council of Chief State School Officers (CCSSO). (2010). *Common core state standards for mathematics*. Washington, DC: Authors.

Nunes, T., Bryant, P., & Watson, A. (2009). *Key understandings in mathematics learning: a report to the Nuffield Foundation*. London: Nuffield Foundation.

Piaget, J. (1952). *The child's conception of number*. London: Routledge and Kegan Paul.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: how can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*, 165–190.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109–151). New York: Academic.

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologists, 44*, 163–169.

Resnick, L. B. (1992). From protoquantities to operators: building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (19; 275–323). Hillsdale: Lawrence Erlbaum.

Riley, M. S., & Greeno, J. G. (1988). Developmental analysis of understanding language about quantities and of solving problems. *Cognition and Instruction, 5*, 49–101.

Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. Ginsberg (Ed.), *The development of mathematical thinking* (pp. 153–196). New York: Academic.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36.

Sophian, C., & McGorgray, P. (1994). Part-whole knowledge and early arithmetic problem solving. *Cognition and Instruction, 12*, 3–33.

Sophian, C., & Vong, K. I. (1995). The parts and wholes of arithmetic story problems: developing knowledge in the preschool years. *Cognition and Instruction, 13*, 469–477.

Sophian, C., Harley, H., & Martin, C. S. M. (1995). Relational and representational aspects of early number development. *Cognition and Instruction, 13*, 253–268.

Stern, E. (1992). Spontaneous use of conceptual mathematical knowledge in elementary school children. *Contemporary Educational Psychology, 17*, 266–277.

Stern, E. (2005). Knowledge restructuring as a powerful mechanism of cognitive development: how to lay an early foundation for conceptual understanding in formal domains. *British Journal of Educational Psychology, Monograph Series II (Pedagogy–Teaching for Learning), 3*, 155–170.

Torbeyns, J., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2009). Solving subtractions adaptively by means of indirect addition: influence of task, subject, and instructional factors. *Mediterrannean Journal for Research in Mathematics Education, 8*(2), 1–30.

Wright, R. (1991). What number knowledge is possessed by children beginning the kindergarten year of school? *Mathematics Education Research Journal, 3*(1), 1–16.

Wright, R. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics, 26*(1), 25–44.

Young-Loveridge, J. (2002). Early childhood numeracy: building an understanding of part-whole relationships. *Australian Journal of Early Childhood, 27*(4), 36–42.

Zhou, Z., & Peverly, S. (2005). The teaching addition and subtraction to first graders: a Chinese perspective. *Psychology in the Schools, 42*, 259–272.