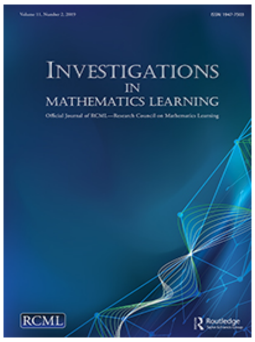

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Teaching of the associative property: A natural classroom investigation

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ABSTRACT

In this study we investigate the teaching of the associative property in a natural classroom setting through observation of classroom video of several elementary math classes in a large urban school district. Findings indicate that the associative property was often conflated with the commutative property during teaching. The role of the associative property in many computational tasks remained fully implicit, even after the property had been formally introduced. Most classrooms did not present the associative property with substantial justification in terms of concrete representations, especially those in which the abstract property was formally introduced - while a few classrooms did situate the property in rich concrete contexts, the property remained implicit in these classes, indicating a lack of linking between the concrete and the abstract when teaching the property. Instruction also did little to develop the notion of the associative property as a property of an operation conceptualized as a mental object, rather than as a rule governing the outcome of a procedure. Much of the instruction displayed a significant focus on computational strategies, which aggravated the challenge of providing a clear explanation of the nature and meaning of the associative property.

KEYWORDS

Arithmetic; associative property; early algebra; operation sense; teaching

Introduction

Properties of operations (e.g., the commutative, associative, and distributive properties) are fundamental mathematical ideas that have been systematically emphasized by the Common Core State Standards across elementary grades (Common Core State Standards Initiative, 2010). These properties undergird computation strategies (National Research Council & Mathematics Learning Study Committee, 2001) and serve as logical tools for proof (Schifter, Monk, Russell, & Bastable, 2008). Moreover, the incorporation of algebraic concepts into early math education is of great importance to students' ability to succeed in later algebra instruction (Blanton et al., 2007; Lins & Kaput, 2004), and the literature has increasingly emphasized the importance of arithmetic operations and their properties to the teaching of "early algebra." Operations form an important basis for the development of functional thinking (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; Schifter, 1997; Slavit, 1998), and their properties form an important basis for the development of algebraic content as "generalized arithmetic" (Carraher & Schliemann, 2007; Kaput, Carraher, & Blanton, 2017; Tent, 2006).

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However, teaching these properties can be difficult, and they are often the root of much confusion for students and teachers alike; the teaching of the associative property (AP) of addition and multiplication is particularly problematic. This study explores how elementary teachers teach the AP in a natural (in that the participating teachers have received no training or intervention) classroom setting.

While prior studies have examined the knowledge of elementary teachers (and pre-service teachers) in a university classroom setting (e.g., Ding, Li, & Capraro, 2013), few studies have examined the teaching of the AP by in-service teachers in an elementary classroom context. Using data from a multiyear cross-cultural research project, this study examines the in-classroom practices of expert teachers in a large mid-Atlantic school district based on video data of natural classroom instruction. We ask the following questions about AP instruction:

(a) Is teacher instruction of the AP mathematically correct? How do teachers disambiguate the AP and the commutative property (CP)?
(b) Do teachers make explicit the role of the AP in various computational strategies?
(c) Do teachers present the AP as an *operational* rule, or a *structural* property?
(d) Do teachers justify the AP with explicit and substantive links to concrete examples and/or fundamental quantitative concepts?

Literature review

This study focuses on the AP of addition and multiplication for three reasons: First, the AP is one of the defining features of both addition and multiplication—it is one of the central properties in defining the behavior of the real numbers, and its use is pervasive in common manipulations, not only in arithmetic, but also later in algebra and beyond. Second, of the three basic properties of arithmetic traditionally taught in elementary school (commutative, associative, and distributive), the AP presents several unique challenges—it is conceptually subtler than the commutative property yet less easily identifiable than the distributive property (which appears quite different by virtue of relating two different operations). Third, there is a dearth of research about the teaching of the AP—and hardly any on actual employed teaching methods in a natural classroom setting. Below, we will first review the importance of the AP, then common challenges faced during the teaching and learning of AP, and finally extract from these the theoretical lens through which we analyze our data.

Why is the AP important?

The AP of addition and multiplication is tremendously important to understanding and execution of arithmetic manipulations (National Research Council & Mathematics Learning Study Committee, 2001). The AP of addition is ubiquitous in common computation strategies such as "double plus 1" [e.g., $9 + 8 = (1 + 8) + 8 = 1 + (8 + 8) = 1 + 16 = 17$], "making a ten" [e.g., $9 + 8 = 9 + (1 + 7) = (9 + 1) + 7 = 10 + 7 = 17$, CCSSI, (2010)], or the standard "carry"-based addition algorithm. The AP of multiplication is similarly important as a tool for multiplication strategies such as the ubiquitous "adding a 0" strategy for multiplying by multiples of 10.

Moreover, while it is important to be able to use properties of operations (including the AP) implicitly in computational strategies, it is also important that students develop an explicit awareness of the role of these properties in justifying those strategies: Understanding the properties of arithmetic operations as a set of "rules" that justify mathematical manipulation forms the basis for the later learning of algebra as a generalization of those rules (Carraher & Schliemann, 2007; Kaput, Carraher, & Blanton, 2007; Tent, 2006).

Developing understanding of properties such as the AP is important for developing students' sense of binary operations as objects (Slavit, 1998), which is crucial to later algebraic thought. Research suggests that students progress from an "operational" to a "concretized" or "structural" understanding of mathematical concepts, with the process repeating and building on itself toward ever higher-order structures (Sfard, 1991). Understanding addition or multiplication as "objects" clearly requires a firm understanding of their properties.

Finally, properties of operations are important in a *definitional* capacity—the basic properties of arithmetic, in some sense, form the foundational basis of what those operations *are* (in that they are included in the standard axiomatization of the real field). As Roger Howe states, the properties of operations form the "rules of algebra" (Howe, 2005, as quoted by Carraher & Schliemann, 2007) which form the basis of algebraic structure. Thus, understanding of these properties is arguably necessary to understand fully what we *mean* by multiplication. Such formalization of mathematical concepts is an important part of the development of mathematical understanding and the transition from operational to structural understanding of mathematical concepts (Pirie & Kieren, 1994).

Why is the AP difficult to teach and learn?

The AP is commonly conflated with the commutative property (CP) by students, teachers, and even learning materials, likely because the two often co-occur in the same calculation (Fletcher, 1972; Tent, 2006). Moreover, the AP is notably more difficult to demonstrate graphically for students than the CP: For addition, the CP can be easily seen with a quick demonstration by permuting parts of a whole, while for multiplication, it can be easily seen from an array model; for both, no comparatively simple parallel exists for the AP (Fletcher, 1972; Howe, 2016).

Conflation of the AP and CP is not restricted to the lower grades. Even students in undergraduate-level math courses struggle to disambiguate the two properties. In fact, when Arthur Cayley (19th-century English mathematician and pioneer of "pure," or axiomatic, mathematics) first enumerated the axioms of a group, he felt the need to note explicitly that the group elements need be associative but *not necessarily commutative* (Kleiner, 1986, as quoted by Larsen, 2010). Both undergraduate mathematics students and doctorate students in educational programs struggle to distinguish between "types of ordering"—the CP involves ordering of *operands*, while the AP involves ordering of *operations* (Boyce, 2016; Larsen, 2010).

Moreover, the common notation for the AP relies on parentheses to indicate the order of operations [e.g., "$(5 + 3) + 2 = 5 + (3 + 2)$"]. However, students' first introduction to addition and multiplication in the early grades often omits parentheses entirely. For instance, we can write "$5 + 3 + 2$" without ambiguity because both possible ways of parsing of the expression yield the same sum. Thus, students are unlikely to understand the use of parentheses at all until the necessity of disambiguating the order of operations has been properly understood. However, Hsieh (1999) suggests that even in mathematically high-achieving countries such as Taiwan, students have trouble applying parentheses properly and tend to overapply the AP when it is not valid [e.g., $(5 + 3) \times 2 = 5 + (3 \times 2)$].

Difficulties with AP instruction are likely compounded by weak teacher subject knowledge. Teachers and student teachers struggle to produce accurate examples and counterexamples of the AP due to the above-mentioned overgeneralization and (as noted before) confusion with regard to the notion of "order" with respect to *operations* versus *operands* (Zaslavsky & Peled, 1996). Only a small fraction of pre-service teachers are able correctly to describe or provide an example of the AP of multiplication, and fewer still are able to produce correct concrete representations of it—a skill that is crucial for effective instruction (Ding et al., 2013). Additionally, pre-service teachers (like students) have difficulty disambiguating the AP from the CP (Ding et al., 2013).

These gaps in teacher content knowledge have clear effects on the ability of elementary school math teachers to justify properties of arithmetic operations, especially the AP. Teachers struggle to justify AP of multiplication with concrete representations, often by applying the "equal groups" definition of multiplication incorrectly or inconsistently, or by failing to produce two different computations to illustrate the meaning of the equivalent expressions given by the AP, or by conflating the AP and the CP (Ding et al., 2013). More generally, Lo, Grant, and Flowers (2008) report that pre-service math teachers fail to explain computation strategies adequately even outside the context of the AP—often, pre-service teachers struggle to understand the fundamental *meaning* of "justification," failing to distinguish the act of justifying *why* a computation produces the correct answer from simply describing the rote process used in the computation, and failing to understand the purpose of an exercise beyond merely "obtaining an answer." Research on in-service teachers (Hill, 2010; Ma, 1999) indicates that teachers struggle to identify examples that illustrate how the AP can be used to simplify calculation, and that teachers demonstrate weak conceptual knowledge of arithmetic operations and struggle to provide convincing mathematical justifications for various computational procedures. Thus, it seems these difficulties continue even after teachers have reached the classroom.

Theoretical framework

Considering the conceptual difficulties posed by the AP, and its importance as a steppingstone to algebraic reasoning, our analysis will use a theoretical framework focusing on four "dimensions" of AP instruction, corresponding to our research questions.

Correctness

First, we are interested in whether instruction relating to the AP is *mathematically correct*. As noted, prior research in a laboratory setting suggests that teachers and pre-service teachers struggle to describe, and produce examples of, the AP (Boyce, 2016; Ding et al., 2013; Larsen, 2010; Zaslavsky & Peled, 1996)—it is thus necessary to investigate whether the teaching of this property in the classroom displays similar struggles. Special attention is paid to the conflation of the AP with the commutative property, whose difficulty is a recurring theme in much of the literature.

Explicitness

Second, we investigate whether the AP itself is made *explicit*. Here we use "explicit" in a sense similar to that of Greeno (1987): A concept is "explicit" if it is something we may "discuss and observe"; understanding that cannot be directly referenced in this way (yet may still be used in service of solving problems), we call "implicit." Several authors point to the importance of explicitness in the process of mathematical understanding: Mason (1989) refers to the making explicit of knowledge as the step of "getting a sense of," serving as a crucial intermediate step between "manipulating" and "articulating;" Pirie and Kieren (1994) refer to "properly noticing" and "formalizing" in a similar capacity, and Warren (2003) notes the importance of explicit understanding of properties of arithmetic operations—in particular, to the ability of students to perform algebraic manipulations.

This notion of explicitness is of importance to the teaching of the properties of operations (and, by extension, the AP), as these properties later form the "rules of arithmetic" (Howe, 2005, as quoted by Carraher & Schliemann, 2007), which later are generalized to become the rules of algebra itself. Thus, the later ability of students to discuss and reason explicitly about algebraic manipulations clearly depends on the explicitness of their understanding of the properties of arithmetic operations.

Structuralness

Third, we investigate whether the AP is presented as an operational "rule" that governs the outcome of a procedure, or a structural "property" of an operation. As discussed previously, a crucial notion in the development of mathematical knowledge is the growth from "operational" to "structural" understanding, in a process Sfard (1991) calls "reification." This transition is of particular importance to the understanding of arithmetic operations, as the "reification" of a student's understanding of an operation (e.g., from a notion of "adding" as a *procedure* of putting quantities together to an *object* of "addition")—what Slavit (1998) calls "operation sense"—forms a crucial part of students' pre-algebraic development. This reification of arithmetic operations as objects serves to highlight their "general nature" and to reveal the "inherent algebraic character" of arithmetic (Carraher et al., 2006). The importance of structuralness to properties of operations is obvious: Without a structural conception of an operation, we cannot truly have a "property" so much as a "rule," as a property must be *of* something concrete and reified.

Concrete justification

Finally, we analyze the justification of the AP in terms of concrete representations. In addition to understanding mathematics on a formal level, it is also important that students develop a meaningful understanding of what they learn in mathematics classrooms. Unfortunately, this sense-making process is often lacking—elementary students are prone to developing "external justification schemes," and often ascribe the truth of mathematical statements to the authority of their source (Flores, 2002), and the introduction of abstract mathematical notation without a firm basis in the concrete can result in students performing symbolic manipulations without understanding of what the symbols *mean* (Stacey & Chick, 2004).

The firm linking of abstract properties such as the AP to concrete contexts serves to ground these properties in students' concrete intuitions, allowing them to make sense of these abstract ideas. Additionally, current educational and cognitive science research highly implicates the importance of firm linking between the concrete and the abstract to the learning of mathematics (Pashler et al., 2007). As discussed, the AP presents unique challenges in concrete representation, with concrete representation of the AP of multiplication being particularly subtle (Fletcher, 1972).

Method

To explore the research questions, this study conducted video analysis of classroom lessons. Following Chi's (1997) method of quantifying qualitative analysis of verbal data, we analyzed transcripts of the identified AP episodes based on a coding framework, which were enriched with typical examples.

Participants

The participants were experienced teachers in a large mid-Atlantic urban school district participating in the second year of a five-year cross-cultural research study. Except for one teacher, all had 10-year and above teaching experiences. All were female teachers who have either been national board certified, or else were highly recommended by either their school principals or another national board certified teacher. There were nine total participants—three fourth-grade teachers, and two each of third, second, and first grade. The textbooks used in this study included *Go Math, My Math*, and *Investigations*, all of which claimed themselves Common Core–aligned.

Each participant was filmed over four class periods, resulting in a total of 36 videotaped lessons. The four lessons were selected through a process of evaluation of the corresponding textbook sections and consultation with the teacher participants. At the time of data collection, the participants had received no intervention from the researchers, thus, the classes were filmed as they would usually be taught.

Teaching episode identification

All 36 videos were transcribed and each lesson was screened to identify "AP teaching episodes," which comprised the primary unit of analysis for our data coding. A "teaching episode" was defined as a single, complete class activity (either a worked example, individual work task, or group work task) whose underlying concept (either explicitly or implicitly) either involved or required AP, or else in which the teacher made explicit reference to AP. Exchanges between the teacher and individual students (or small groups of students) were also scored as episodes. Coding difficulties were encountered when considering activities that consisted of multiple related problems done one after another; in these cases, problems were grouped together and considered part of the same episode if each individual problem was sufficiently short and/or the problems were sufficiently closely related that, when separated, the resulting episodes would seem incomplete. Ultimately, cases that could be reasonably interpreted either way were decided by researcher discretion. Each "teaching episode" was then coded according to a framework designed to probe the four dimensions of our research question. In addition to counting each instructional episode, the start time and end time of each episode were recorded to obtain an additional metric of the "quantity" of instruction.

Coding framework and procedures

Based on the theoretical framework and screening of the project data, we developed the following coding framework (see Table 1). Our coding went through three passes. The first two passes judged the explicitness of AP and correctness of the instruction involved in the episodes. We excluded those episodes where discussion of AP was fully implicit or totally incorrect from the third pass of coding, which judged justification and structuralness. Table 1 illustrates detailed categories.

Table 1. Coding framework used in this study.

Pass	Dimension	0	1	2
1	Explicitness	The use of the AP is **fully implicit**—while the AP is needed to justify the manipulations performed, the instruction does not make any indication that anything requiring special attention, explanation, or justification has taken place.	The use of the AP is **partially explicit**. The formal rule is not stated precisely or entirely, but the manipulation requiring the AP is "singled out" as a specific step in the solution of the problem, and given some level of discussion/justification.	The use of the AP is **fully explicit**. The manipulation is clearly connected to the formal, abstract principle, either by recalling it by name or else providing a full statement of the definition.
2	Correctness	The episode contains **totally incorrect mathematics**. The computation performed is not mathematically correct, or an explanation given is mathematically incorrect, or the AP is incorrectly stated.	The episode contains **confused, muddled, or partially incorrect mathematics**. This includes episodes in which **AP and CP are confused or conflated**, and episodes in which **confusing, partially incorrect, or incomplete definitions of AP** are given.	The AP is invoked **correctly**. The given manipulation is valid mathematics, and is justified by the AP. If the AP is identified explicitly by name, then it is identified correctly.
3	Structuralness	Discussion of AP is **entirely operational**. Discussion is limited to that which is strictly necessary to compute the desired quantity. The AP is treated entirely as an "operational rule" in a calculation process. Language use in discussion of AP reflects only procedural notions, without hinting at structural ones (e.g., "do these first").	Discussion of the AP is **partially structural**. While emphasis remains on the AP as an operational rule, some phrasing refers to the AP as a structural "property of addition/multiplication." Language use in discussion of AP can be interpreted structurally (e.g., "the expressions are equivalent"), but emphasis may not fully engender students' structural understanding.	Discussion of the AP is **mostly or entirely structural**. The AP is treated as a property of "structuralized" operations, which are referred to as if they were objects rather than simply markers for a computational procedure. Language use in discussion of AP strongly reflects structural notions (e.g., "addition doesn't care which pair we add first").
	Justification	Use of the AP is unjustified. Discussion of the AP is restricted to surface-level details (such as computational rules), and the quantitative principles permitting the manipulation are not made clear. Surface-level student responses are accepted as is, without further rephrasing or prompting.	Discussion of the AP is partially or incorrectly justified. Underlying quantitative relationships are mentioned, but not discussed in depth. Surface-level details are still given significant focus, and underlying quantitative principles are mentioned, but not fully or correctly elaborated. Surface-level student responses may be rephrased or probed further, but this does not result in a deeper restatement.	Discussion of the AP is fully justified. The underlying quantitative relationships are explicitly mentioned, and sense is made of them through clear reasoning from the problem context. Surface-level details are de-emphasized. Surface-level student responses are rephrased or met with further probing questions to elicit deep responses.

Table 2. Example instruction of varying explicitness.

Fully implicit (G1) Problem: "Making a ten" and "doubles minus one," 6 + 4 + 3 = 10 + 3 = 6 + 7	Partly explicit (G2) Problem: "Making a ten", 37 + 24 = 40 + 21
Teacher 2: So what strategy did they use first? What strategy did they show us first? Student: Cubes. T: They showed us the cubes and they made a what? S: Together. T: They made them together. So they added numbers and they made a. . . . S: 10. T: 10. They made a 10. Then they had 10 + 3 so 10 + 3, Hillary. What's 10 + 3? S: 13. T: 13. So they came up with the sum of 13. Then, they added the addends in a different order. So they did Bill's 4 and Joe's 3 are 7. S: (inaudible). T: Well, I don't know why it's orange. And then Kelly with 6. So they used the doubles. . . . S: Minus 1. T: So they used the doubles minus 1. So 7 + 7 is? 14. Minus 1 is? S: 13.	Teacher 3: Why are they the same answer? Student: Cause it's the same thing but you're just changing it. T: What do you mean it's the same thing but I'm just changing it? What does that mean? S: You're changing it how you (inaudible) and how you took some of them away to make the ten and then the other one you didn't. T: Okay, does anyone want to add anything to that? Think about what he said, he said it's the same but you changed it. So how did we change it, but it was still the same? S: Because at the top you turned it to 40 + 21 = 61. You changed the number but that one, they still have the same number (inaudible). T: Okay good, so Jaiden's saying I didn't put in any new numbers, right? I just moved the numbers we had around. Did I add in any new tens or ones blocks? S: No. T: Did I take any away? S: No. T: No. I just regrouped them, right?"

During the "explicitness" coding, episodes in which it was not made clear that the AP undergirded the mathematical manipulation performed were coded as "fully implicit." Episodes that did not explicitly name or state AP, but regardless still "singled out" the manipulation using AP as deserving of special attention or justification, were coded as "partially explicit." Table 2 illustrates representative coding examples: Teacher 2's instruction did not make any indication at all of the role of the AP in the computational strategies; it was coded as "fully implicit" and was not subject to the third coding pass. Teacher 3, however, explicitly discussed the reason why each strategy yielded the same answer, eliciting student explanation based on regrouping the numbers. As such, this episode was coded as "partially explicit." Episodes that clearly mentioned AP, or wrote its algebraic formula, were classified as "fully explicit."

A second pass of coding determined basic "correctness" of the use of the AP for each episode: Episodes that contained significant mathematical errors (either computational or explanatory) were scored 0; episodes that contained mathematics that was not totally incorrect, but was confusing, muddled, or which conflated AP and CP, were scored 1, and episodes that contained no mathematical errors were scored 2. "Partially correct" episodes were further analyzed to ascertain the nature of the mathematical error, with special attention paid to conflation between the AP and CP. Both instructional instances in Table 2, for example, were coded as "fully correct."

The "structuralness" scale in the third pass of coding indicates whether an instructional episode treated AP as a property of a structural operation, or a rule constraining the outcome of an operational procedure. An episode was considered "structural" if the operation (either addition or multiplication) is referred to as if it were an *object*, with the AP being a property of that object. Examples of structural phrasing include referring to expressions involving the operation statically rather than dynamically (e.g., describing "(5 × 3) × 7" as "five multiplied by three, multiplied by seven," rather than describing the procedure of "multiply five by three first, and then multiply by seven"), referring to "pseudo-concrete" manipulation of the operations (e.g., describing the AP as "changing the order of the operations," in the same way one might describe the CP as "changing the order of the numbers," as numbers are concrete objects), and describing formal equivalence between mathematical expressions without computing their value. Examples of operational phrasing include describing the properties in terms of "doing" or "computing" (e.g., "do this first") and stating equivalence in terms of the outcome of a computational procedure (e.g., "we get the same thing both ways"). Instruction that contains a mixture of operational and structural phrasing, or that otherwise "hints" at structural descriptions but does not fully present them, was classified as "partly structural." Teacher 3's instruction in Table 2 provides an example of "fully operational" instruction.

The "justification" scale measured to what extent the abstract property is situated in and justified by concrete representations. In other words, by "justification," we mean justification *for the property*. If the property was merely named or stated, or explained entirely in terms of abstract symbol manipulation, the episode was scored 0 for "justification." If concrete justification for the AP was mentioned, but not thoroughly explained or illustrated (e.g., mentioning "regrouping" in the context of concrete quantities rather than numbers, but not illustrating the regrouping process with a concrete representation), then the episode was scored as "partly justified." In Table 2, Teacher 3 calls attention to the need for a justifying property (and thus achieves partial explicitness) by asking the crucial question, "Why are they the same answer?" Despite not explicitly naming the AP, Teacher 3 clearly links the computational manipulation to the concrete context, explaining that the sum must be the same because they did not "add in any new tens or ones blocks," and thus the instruction is fully justified.

Data analysis

Codes resulting from the analysis were tabulated and compared across grades. As the number of teachers at each grade level was not equal, and thus total quantity of instruction filmed varied across grades, codes were tabulated by percentage of total episodes rather than total number. Given the small size of the cohort and the qualitative nature of the analysis, codes were not subject to statistical testing. Episodes and textbook excerpts identified are presented together with results from the coding below.

Results

A total of 146 episodes, totaling approximately 633 minutes of instruction (average ~4.3 minutes per episode) were identified and coded across all four grades. As seen in Table 3, the amount of instruction relating to AP varied greatly across all four grades, both in terms of number of AP-related teaching episodes and in terms of instructional time.

The situation in grade 3 was unique. For one of the two teachers, none of the four lessons focused explicitly on the AP or on strategies using the AP, resulting in only a single AP-related instructional episode. The other grade 3 teacher formally introduced both AP of addition and AP of multiplication over the course of the four lessons—however, instruction generally did not involve the AP apart from worked examples during the formal introduction of the property, and practice problems directly following said formal introduction (which tended to be brief in nature).

Teaching of the AP: Explicitness

As seen in Figure 1a, most instructional episodes (99 out of 146) were fully implicit. The exception seen in grade 3, again, corresponds to the formal introduction of the property in one of the classes:

Table 3. Average instructional episodes and instructional time per teacher by grade.

	Grade 1	Grade 2	Grade 3	Grade 4
Number of episodes	28.5	21	5	12.3
Instructional time (minutes)	120.3	104.8	18.8	48.3
Average episode length (minutes)	4.2	5.0	3.8	3.9

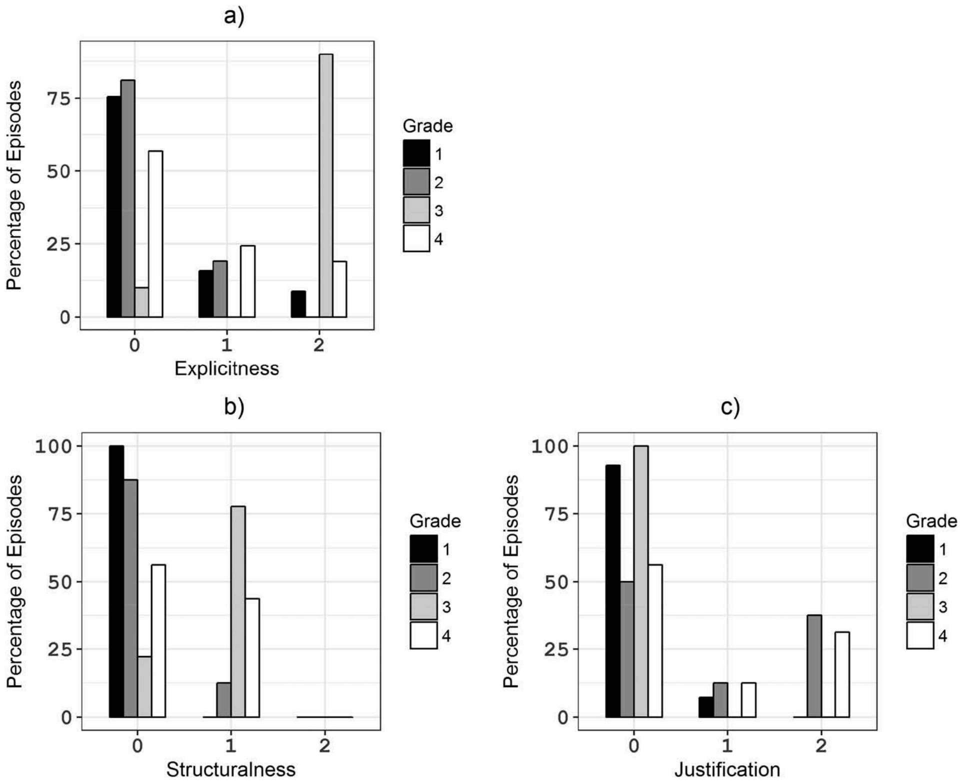


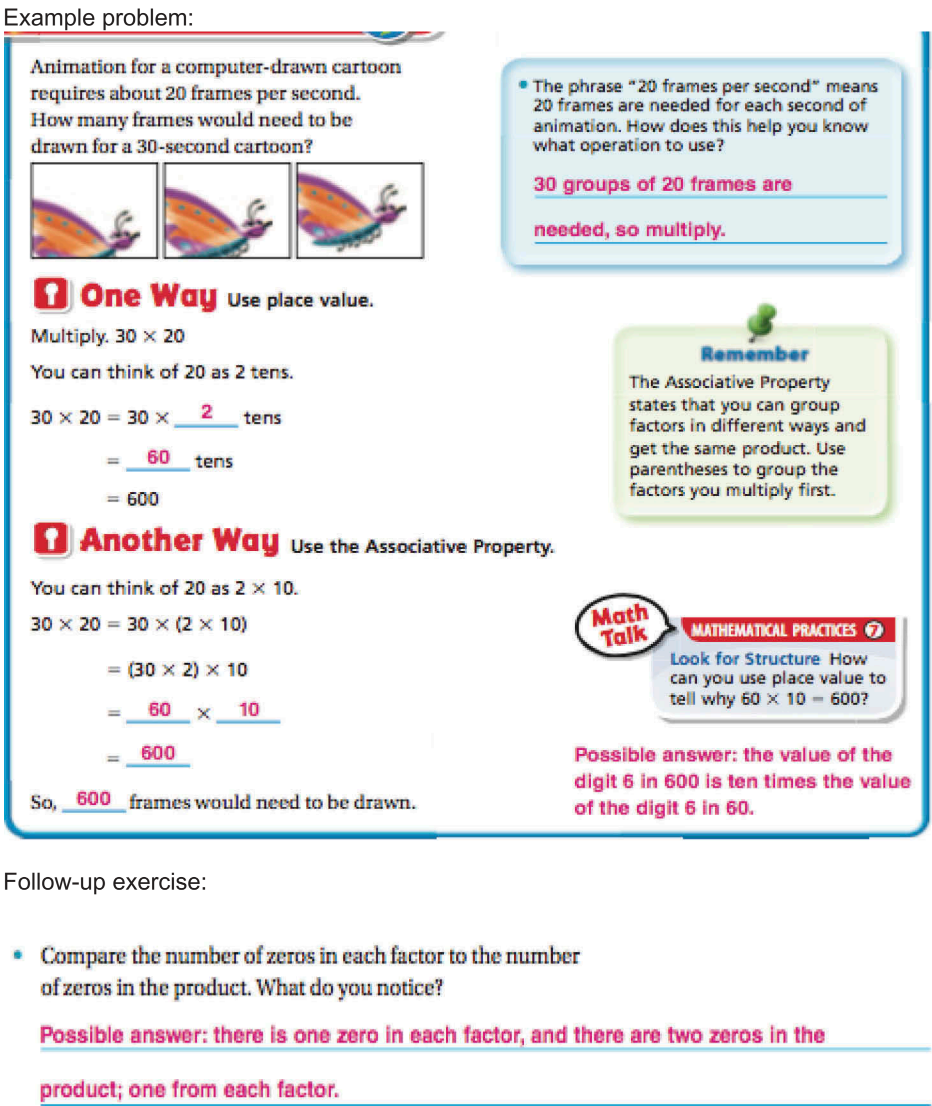
Figure 1. Results of coding. (a) Percentage of total episodes at each explicitness level, by grade. (b) Percentage of partially or fully explicit episodes at each structuralness level, by grade. (c) Percentage of partially or fully explicit episodes at each justification level, by grade.

> And finally the last property is the associative property… the associative property says that the way in which the addends, those numbers you're adding together, the way you group them, does not change the sum. The associative property is when we see these parentheses. (Teacher 6)

Most fully implicit instructional episodes (which, as seen, constitute the bulk of all instructional episodes) deal with the teaching of "strategies" such as "making a ten" or "near-doubles facts" (addition), or "doubling-and-halving" (multiplication). As previously explained, Teacher 2 in Table 2 discussed only computational strategies ("make a ten," "doubles facts"), without prompting students to consider why these different strategies resulted in the same answer. While the teaching of different computational strategies may contribute to students' computational fluency, discussion of computational strategies without connection-making to the underlying property cannot develop students' structural sense of the basic properties.

The lack of explicitness of AP episodes seems to relate to the corresponding textbook presentations. A representative example (Figure 2), found in the *GoMath!* textbook, resulted in highly confusing instruction in two separate grade 4 classes:

Both strategies presented in this two-part problem rely on the AP. In fact, they are precisely the same strategy, as "2 tens" is precisely the meaning of "2 × 10"—the example problem not only presents the AP purely as a computational strategy, but moreover explicitly (and erroneously) claims that a simple rewording of the same manipulation constitutes a different strategy. The obfuscation of the role of AP in the first "way" is typical of the instruction seen, and was present in the corresponding instruction in both classes. The textbook's follow-up exercise (also shown in



Example problem:

Animation for a computer-drawn cartoon requires about 20 frames per second. How many frames would need to be drawn for a 30-second cartoon?

- The phrase "20 frames per second" means 20 frames are needed for each second of animation. How does this help you know what operation to use?

30 groups of 20 frames are needed, so multiply.

One Way Use place value.

Multiply. 30 × 20

You can think of 20 as 2 tens.

30 × 20 = 30 × 2 tens

= 60 tens

= 600

Another Way Use the Associative Property.

You can think of 20 as 2 × 10.

30 × 20 = 30 × (2 × 10)

= (30 × 2) × 10

= 60 × 10

= 600

So, 600 frames would need to be drawn.

Remember

The Associative Property states that you can group factors in different ways and get the same product. Use parentheses to group the factors you multiply first.

Math Talk

MATHEMATICAL PRACTICES 7

Look for Structure How can you use place value to tell why 60 × 10 = 600?

Possible answer: the value of the digit 6 in 600 is ten times the value of the digit 6 in 60.

Follow-up exercise:

- Compare the number of zeros in each factor to the number of zeros in the product. What do you notice?

Possible answer: there is one zero in each factor, and there are two zeros in the product; one from each factor.

Figure 2. Typical instructional example involving AP from *GoMath!* textbook, with follow-up exercise.

Figure 2) continues the lack of explicitness—the suggested "possible answer" does not contain any reasoning about the underlying role of the AP in explaining *why* the number of zeroes remains constant, and the resulting in-classroom instruction fails to make any mention of it, instead finishing the exercise by merely listing the "strategies" covered so far.

Teaching of the AP: Correctness

Most of the instruction observed was mathematically correct; however, given the largely implicit use of the AP, "correctness" was usually limited to accurate computation, as seen in this representative excerpt from grade 4 instruction [$14 \times 30 = (2 \times 7) \times 30 = 2 \times (7 \times 30)$]:

> When I'm trying to do mental math to do 14 times 30, I can't do that in my head. I mean, I'm gonna do it in my head, but like, 14 times 30? I can't do it in my head, so I'm gonna take a look at which of those I can break

> apart into something I can work with... you can chop 14 in half. So, if I chop 14 in half, then I can say okay, it's going to be 14 divided by 2 is 7. If I take 7 and multiply by 30, I can do that in my head, right? 7 times 3 is 21, 7 times 30 is 210.... I want to find the product of 14 times 30, I have to double my answer. So, if I'm doubling 210, it would be 420. (Teacher 8)

Of 146 total episodes across all teachers and grades, 125 (86%) were scored "totally correct," 20 (14%) were "partially correct," and only 1 was "totally incorrect." Of the "partially correct" episodes, the majority (15 of 20) were incorrect due to conflation of AP and CP (elaboration on and examples of AP/CP conflation can be found in the next section), confirming that this is a significant cause of instructor error when teaching the AP. Other causes of "partial errors" include incorrect/confused usage of terminology regarding the AP [such as using "associative property" to refer to the expression "$6 \times (10 \times 10)$" rather than the rule that equates it to "$(6 \times 10) \times 10$"]. The lone "totally incorrect" episode was the result of a transcription error by the instructor ["$(10 + 9) \times 50$" became "$(10 \times 9) \times 50$" halfway through the problem] that remained uncaught for the entire exercise, and resulted in an incorrect answer.

However, correctness of instruction varied significantly with explicitness—instruction that *explicitly* teaches the AP was significantly more likely to contain mathematical errors than instruction that merely uses it implicitly in the course of a calculation. Of 47 instructional episodes that were scored as partially or fully explicit, 20 (43%) contained some degree of mathematical error. In fact, every single "partially correct" episode was at least partially explicit. As such, while instructors seem highly capable of *using* AP correctly during instruction (as in various computational strategies such as "making a ten" or "doubling and halving"), they appear to struggle with *teaching* the AP itself correctly with a high degree of explicitness.

Disambiguation between the AP and CP

As noted above, the data indicate that conflation between AP and CP is a significant cause of mathematically erroneous AP instruction. Moreover, while 15 of 146 total instructional episodes involved AP/CP conflation, this figure is somewhat misleading, as not every instructional episode offers equal opportunity for this error—episodes that involve only the AP, and not CP, do not present a significant opportunity for conflation. We might instead ask how often the AP and CP are properly disambiguated in episodes that involve the use of both properties.

Only 16 instructional episodes involved (implicitly or explicitly) the use of both the AP and CP. Of these, the two properties were explicitly disambiguated in only 3. Moreover, two of these three episodes, while offering explicit disambiguation, did so in a muddled and/or confusing manner, scoring only a 1 on the correctness scale. For example, the problem from Figure 3a was explained in the following way:

> Alright, now let's look at B. $4 \times 9 \times 250$. Alright, now we're going to use the commutative property and we're going to move our numbers around... . Oh no, they're giving you two different properties to use in one

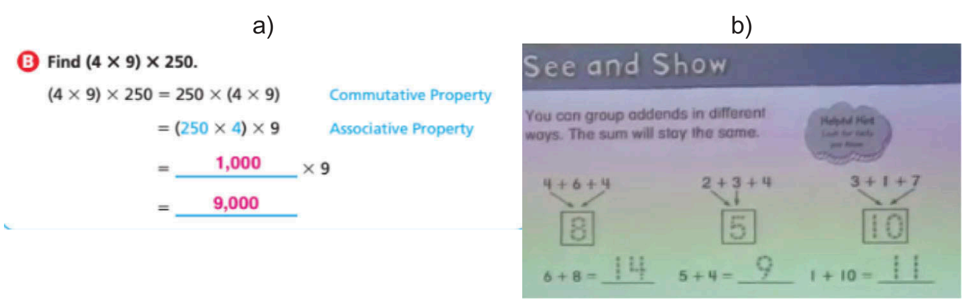


Figure 3. Instructional examples involving both CP and AP. (a) Instructional example from grade 4, displaying correct disambiguation of the properties in the textbook. (b) Instructional example from grade 2, displaying "any which way" rule.

> problem… . They're giving you two different ideas or strategies that you could use in order to solve the problem… . So if you look, the first way, well they didn't really commute too much, they just moved things around. So, they put the 250 here, times 4, times 9, okay? (Teacher 7)

In the above episode, the language used to distinguish between the AP and CP is highly imprecise ("they didn't really commute too much, they just moved things around"), and in no instructional episode were the differences between the AP and CP made explicitly clear in concrete terms. The lone fully correct episode involving both the AP and CP merely correctly noted the name of each property as it was used. Also note that, in this problem, the AP and CP are explicitly disambiguated by the textbook itself—it is unlikely, considering the rest of the data, that the teacher would have disambiguated the two properties correctly had this not been the case. Two additional instructional episodes involving only AP were still coded as conflating the AP and CP—despite the use of only one property to solve the problem, the justifying phrasing used implicated both the AP and CP (by way of "any which way" phrasing)—bringing the number of total "conflated" episodes to 15.

The most common manner of AP/CP conflation was "any which way" phrasing, with 11 of 15 conflated episodes being of this sort. All but one of the "any which way" episodes occurred in the earlier grades (grades 1 and 2), before the formal introduction of the property. Figure 3b illustrates an example of "any which way": During the class discussion of this example, the teacher explained, "As long as we are adding the same numbers together, the same three numbers, our sum will stay the same."

Teaching of the AP: Structuralness

Structuralness of the presentation of the AP varied across grade levels, but presentation was generally operational, with the emphasis on *doing* rather than *being*, as in the following instructional excerpt (corresponding to the problem in Figure 2):

> Here we have, 6 × 10, they've already put that in parentheses, times 10. And now they're showing me that another way that I can use it is using the associative property. (Teacher 7)

Figure 1b shows the coded structuralness of instruction across all four grade levels. Note that only those episodes that scored above 0 on the explicitness scale were coded for structuralness; however, with the exception of the instruction in grade 3, such episodes comprise a relatively small fraction of total instruction (Figure 1a).

As indicated by Figure 1b, grade 3 appears to have more structural instruction than any other grade—this is likely because the explicit episodes from grade 3 were from instruction that formally introduced the abstract definition of the AP, which did not occur in any of the grade 4 lessons. Thus, nearly all grade 3 instructional episodes contained partially structural descriptions of the AP in terms of the formal expressions, as in the following excerpt:

> The associative property tells us the way we group those numbers with the parentheses, it doesn't change the sum. You can group them however you want using the parentheses, it does not change the sum. (Teacher 6)

In contrast, AP instruction in grade 4 occurred almost entirely in the context of computational strategies, containing many fully operational descriptions. For instance, Teacher 8 states, "Now I'm going to use the associative property and I'm going to multiply 250 times 4 first."

Note that both examples above did not meet the category of "fully structural." In fact, no instructional episode in this entire study met the criteria for "fully structural." This may be due to the fact that the notion of an "operation" was never mentioned at any level of instruction, and thus the structural idea of "order of operations" (as opposed to the computation-oriented operational phrasing of "do this first") could not be introduced. For instance, the above grade 3 example from Teacher 6, while introducing the semistructural notion of the "grouping" of numbers, still frames the property within the operational context of different procedures carried out by the students (i.e., the outcome of grouping the terms one way c.f. the outcome of grouping the terms a different way, as opposed to the static "structures" of the different groupings representing equivalent quantities). The grade 4 example, on the other hand, contains no notion of a mathematical structure at all and refers only to the direct computational process.

Additionally, because so many of the episodes were computationally focused, almost every episode involved at least some amount of operational phrasing. Even in the grade 3 instruction, in which the formal definition of the AP was introduced to the students, the focus often remained on how the AP can be used to make computation easier. On some occasions, the act of computation forced instructors to rephrase partly structural statements of the AP (e.g., "you can group the factors in different ways and get the same product") in fully operational terms ("do this first"), rather than promoting them to fully structural statements of the AP (e.g., "multiplication doesn't care how the factors are grouped").

Teaching of the AP: Justification

As shown in Figure 1c, the majority of partly or fully explicit episodes (35 of 47) across all grades contained no substantive concrete justification for the AP. It appears that grades 2 and 4 contain a somewhat greater number of episodes containing concrete justification than grades 1 and 3. Bear in mind that episodes that scored 0 for explicitness (Figure 1a) were not coded for justification, and thus are not shown in Figure 1c. Interestingly, not a single grade 3 episode contained any level of concrete justification for the AP—as noted earlier, the relevant grade 3 episodes were almost entirely performed as part of a formal introduction to the property. While this instruction was fully explicit, it was highly flawed. Following is a representative example:

> Alright these are some big words. Alright the associative property says that the way in which the addends, those numbers you're adding together, the way you group them, does not change the sum. The associative property is when we see these parentheses. (Teacher 6)

In the above example, while the instruction does mention the notion of "grouping" and explain that "the parentheses tell me to add the numbers inside the parentheses first," this phrasing places undue emphasis on superficial details of notation (namely, the parentheses) which have little to do with the substance of the property. Moreover, the idea of "grouping" is never connected to any concrete representation; the teacher indicates that one should first add the terms that are "grouped together," but does not connect this notion to any concrete manipulation of objects, and so the use of the term "grouping" is robbed of the context that gives it meaning, and thus cannot be linked to previous implicit and concrete instances of the property seen in earlier grades.

Also of note is that every single episode with "full" justification for grade 4 occurred in the classroom of just one of the three teachers (teacher 9). Interestingly, this teacher's AP instruction was also all only partly explicit—the AP was never formally introduced by name. However, it was situated in rich concrete contexts, with especially strong support coming from the use of array models (Figure 4). Following is relevant class discussion:

> Teacher 9: Okay, so tell us about what you have here, what are we looking at?
> Student: This is a 4-by-8, and, so, to show the difference between these two, so, you know how, like, we noticed that these <points to equations> are doubling? Well, when you double them. . . .
> Student: <pointing> So, basically, by multiplying this by 2, it creates this plus that.
> Teacher: Okay, so, can we see that? So we shifted it, I think, so we could see it a little more clearly. . . . So, look in this array model for the 4 dimension. . . <pointing> It's right here, right, the 4 rows? Now in this smaller array. . . everyone look for and find the 8 dimension. . . <pointing to equation> So that is this, right? 4 times 8, or 8 times 4, that's 32. Okay? But then, look, she did a whole other 4 times 8, look, 4 times 8, right? So now, look at the 16 dimension. Do you see it up here? Yeah, because 8 plus 8 is 16. Go ahead.
> Student: The 4 is the dimension that's not changing.
> Teacher: Great, good, the 4 factor didn't change. But the 8 dimension doubled. So what does that do to the product? Look. What does that do to the product? Mmhmm, it doubles it, okay?

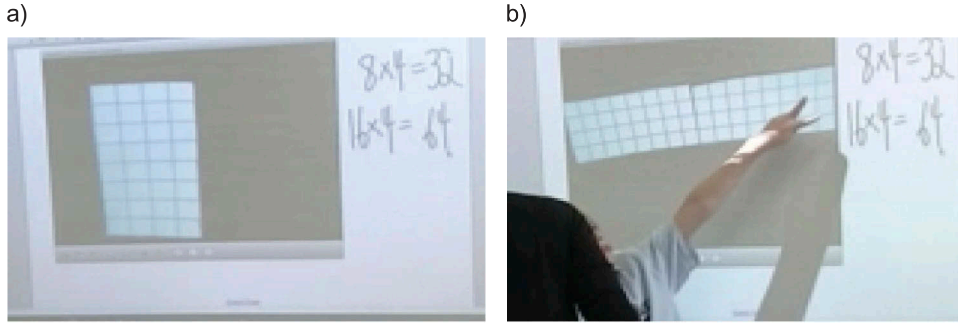


Figure 4. Concrete representation of "effect of doubling one factor on the product" employed by Teacher 9.

In the above episode, the teacher was leading the class to investigate the "effect of doubling a factor on the product," which clearly relies on the AP ["$(2 \times a) \times$ b" = "$2 \times (a \times b)$"]. Even though the AP is never mentioned by name, the teacher's instruction clearly reveals the concrete justification for this identity through the presentation of the student-generated array model, mapping the doubling of one factor to the doubling of one "dimension" of the array, and the subsequent doubling of the total quantity represented by the array. This is especially impressive, as the AP of multiplication is particularly difficult to represent concretely. In fact, in the next lesson, when the class was discussing the related "doubling-and-halving strategy," a student spontaneously generalized from the specific case of doubling and halving to the general property for any whole-number factor, which was not observed in any other class. This generalization could have been used as a natural point of departure for the introduction of the formal property, and provides support for the efficacy of situating AP instruction in substantive concrete contexts in developing student intuition and promoting discovery of the underlying property.

The "fully justified" episodes seen in grade 2, as well, all appear during one class of one of the two teachers (the other teacher having no "justified" episodes at all), and involve the notion of "regrouping" (AP of addition). Similar episodes from other teachers, especially in grade 1, focus on the same strategies but are often fully implicit and almost never justified concretely, even if they include concrete manipulatives or representations.

This can be seen in the contrasting instructional episodes presented earlier in Table 2. Though both episodes involved concrete representations (Teacher 1 utilized a digital "base ten blocks" tool to demonstrate the decomposing/regrouping process, while teacher 2's students had stacking blocks with which to model a story situation of determining the total number of birds seen by three different students), only in Teacher 3's instruction are the representations linked meaningfully to the mathematical concept being employed—in Teacher 2's instruction, the notion of combining the blocks is mentioned but is not developed, and provides no meaningful justification. Note also that the AP is actually used twice in the excerpt from Teacher 2's episode: once for the "making a ten" strategy, but also again in the "doubles minus one" strategy ["$6 + 7 = 7 + 6 = 7 + (7 - 1) = (7 + 7) - 1$"]. Thus, the two strategies presented actually share a common justifying property, and thus have fundamentally similar concrete justifications—a fact which likely would become apparent if the strategies were situated firmly in the concrete context.

Discussion

This study explores teachers' instruction of the AP in natural classrooms. Taken as a whole, we have identified three major trends in the data with implications for instruction and future research: overemphasis on computational "strategies," insufficient concrete–abstract linking, and CP/AP conflation in the early grades.

Overemphasis on computational strategies

Notable throughout the data is a pervasive instructional emphasis on "strategies"—particularly, strategies to obtain answers to computational problems. Apart from the grade 3 instruction, during which the AP was formally defined and introduced, the AP was almost never mentioned outside the context of computational strategies in any of the classes. Most of the coded instruction was "fully implicit" largely because the emphasis lay so heavily on computation, and not on fundamental concepts. Much of this emphasis is present in the textbook materials (such as that seen in Figure 2), which rarely distinguish between the concepts of "property" and "strategy," and may possibly be traced back to a surface implementation of the Common Core State Standards ("Apply properties of operations as strategies to multiply and divide," Common Core State Standards Initiative, 2010).

The proper role of strategies in early mathematics education is undecided, with questions remaining as to "exactly which methods should be taught, and in what order" (Straker, 1996, as quoted in Beishuizen & Anghileri, 1998). A balance is needed between the explicit teaching of strategies and the development of pupils' spontaneously generated strategies from their own mathematical experience and understanding (Beishuizen & Anghileri, 1998). Additionally, the "strategy"-focused instruction seen in the data was almost always seen in service of obtaining an *answer* (to the point where the word "sum" was defined as "the answer to an addition problem" in one of the third-grade classes). Such an emphasis might result in "missed opportunities" for early algebra learning by failing to promote reasoning about the mathematical expressions themselves and the relationships between their terms rather than the mere outcome of a procedure, and build misconceptions that must be "undone" in later algebra instruction (Carraher et al., 2006), such as viewing fundamental properties themselves merely as computational strategies.

While the AP does undergird many of the important computational strategies taught in early education, it is not itself a strategy. Students who are not taught to distinguish between the notions of "property" and "strategy" will likely struggle to transfer their knowledge of the properties to later mathematical subjects such as algebra, in which mathematical manipulations often are performed without any intent to obtain an "answer" and so the notion of "strategy" may lose relevance or even coherence. Additionally, when AP is introduced itself as a *strategy* rather than a *property*, its relation to other strategies may appear to students as parallel rather than hierarchical. In a hierarchical understanding, one sees that many different strategies "follow from" the AP, as many branches stemming from a common node in a tree. The connection between the abstract property and the numerous strategies is then thrown into relief: The commonalities in the concrete justifications for each strategy are, precisely, the substance of the property. This intuition may be hidden from students if they come to view the AP as residing "on the same level" as the strategies that rely on it for mathematical justification.

Thus strategies, which are closely tied to their original concrete problem contexts, may need to be "faded" into more abstract "properties" to promote effective transfer (Goldstone & Son, 2005). Moreover, presentation of properties-as-strategies may obscure the operation-specificity of the properties, as it becomes difficult to emphasize the importance of the AP being "*of* addition" or "*of* multiplication" when it is presented simply as a "way" to perform a computation. This may contribute to "overgeneralization" errors, as reported in elementary school students by Hsieh (1999), and in pre-service teachers by Zaslavsky and Peled (1996).

Framing the AP in purely in terms of computational strategies may also contribute to the lack of structuralness seen in much of the instruction. A "strategy" is a fundamentally operational concept: Strategies can only govern things that are *done*, not things that simply *are*. With the emphasis placed so heavily on how the property might be employed in the *process* of calculating an answer, little room is left for thinking about the relational *structure* the property imposes on different mathematical expressions. To develop flexible understanding that can be transferred and applied in new situations, students must be able to form "schemas" that structuralize the underlying structure of arithmetic expressions—this emphasis on strategies may shift student attention toward merely obtaining answers rather than thinking about "why" these strategies work, stunting their ability to build such schemas (Richland, Stigler, & Holyoak, 2012). Moreover, while the building of such structural knowledge may be promoted through the substantive comparison of computational strategies, the further emphasis on "obtaining an answer" resulted in most comparisons between strategies amounting to asking "which is easier," which fails to highlight the relevant analogical structure between strategies and thus does not promote structural knowledge.

We do not, however, advocate for simply presenting formal statements in a fully structural manner—the structural must develop *from* the operational, with structuralness gradually increasing in salience as the students gain familiarity with the operational procedures. This is not seen in the present data, where structuralness appeared to peak during the formal introduction of the properties in grade 3, and then fade back away as further instruction regressed back toward computational procedures. Teachers can orient students' thinking from the operational toward the structural by asking questions about the underlying quantitative relationships, such as "Why do both strategies yield the same answer?" (as seen in Teacher 3's instruction in Table 2).

Insufficient concrete–abstract linking

The grade 3 instruction was the only instruction in which students were introduced to a formal statement of the AP; however, not a single grade 3 instructional episode contained any concrete justification for the AP. This is representative of a greater trend of insufficient linking between formal and concrete representations of the AP.

This trend was not restricted to grade 3. Instructional episodes that were highly justified in all grades were without exception only "partially explicit"—that is, they did not provide either the name or the formal statement of the full property. Instructional episodes that were fully explicit, which occurred almost exclusively in the later grades (3 and 4), almost always contained no linking to concrete justification at all. Thus, the formal AP was almost never connected to a concrete, intuitive reason for its truth.

The linking of concrete and abstract representations is critical in developing understanding and thus strongly supported by the current literature on cognitive science (Pashler et al., 2007), and the observed failure to adequately link the abstract notion of the AP to concrete justifications poses three potential challenges for student learning: First, it risks students developing an "external justification scheme" for the truth of the AP, in which students gain confidence in the truth of the property not because they are able to convince themselves or others of it through rational argumentation, but instead simply because they encountered it from a source of authority (Flores, 2002). Second, it may hinder students' ability to see connections between related strategies that they have already learned—for example, between "doubles plus one" and "make a ten"—and thus may hinder transfer of understanding gained during partially implicit AP instruction to new contexts. Third, it may negatively impact students' grasp of the formal property itself when it is introduced, as without the ability to connect the new terminology back to previously built understanding, students may instead conceptualize the AP in terms of details of formal notation (e.g., "the parentheses move") rather than the underlying meaning of the property.

To better link the concrete and the abstract, teachers might employ strategies such as "concreteness fading" (Fyfe, McNeil, Son, & Goldstone, 2014) to lead gradually to abstractions from concrete reasoning. In the context of the AP of addition, for example, this may involve first introducing regrouping in concrete real-world contexts, which may be modeled with manipulatives (such as blocks or counters) or diagrams (e.g., linear models or tens frames). Next, these semi-abstract representations may be further faded out into abstract number sentences, algebraic notation, and formal properties.

AP/CP conflation: Problem or inevitability?

As reported by Larsen (2010) and Boyce (2016), confusion regarding the differences between the AP and CP often persists to the undergraduate level, and is present in pre-service teachers. Our study reveals that this conflation may begin early, in elementary mathematics instruction. AP/CP conflation was identified across all grade levels (with an exception for grade 3, in which there was little opportunity for conflation as no instructional episodes involved both properties at once). However, it was most prevalent in the early grades (1 and 2), and in these grades, it most often took the form of "any which way" phrasing (Howe, 2016).

However, it is not clear whether "any which way" phrasing is instructionally inappropriate in the early grades—while it does mix the AP and CP, the rule is still mathematically correct. Moreover, the need to disambiguate between the AP and CP is mathematically quite subtle (hinging crucially on the notion of addition as a *binary* operation); one may argue (as does Fletcher, 1972) that it is inappropriate to expect elementary students to be able to disambiguate the two, and thus that we possibly should not conclude from the presence of "any which way" conflation in the early grades alone that the observed instruction is deficient in its disambiguation of the AP and CP.

We believe that if "any which way" is appropriate for use in the early grades, at *some* point it still must be made clear that it is an amalgamation of two properties, rather than a single property—ideally, after the students have grasped the notion that the operations are binary and both properties have been formally taught. In fact, the idea that addition is "binary" was covered explicitly in some grade 1 lessons, which stressed that "to add three numbers, you add two at a time." In addition, both the AP and CP of addition were formally introduced in first and/or second grades. However, at no point in the instruction was such a "transition" from "any which way" to separate properties (AP and CP) observed. Thus, while teachers may justifiably choose to teach the "any which way rule," they should remain mindful of the two underlying properties, and, when the time comes to disambiguate them, should take care to "tease apart" students' intuitions. As noted by Fletcher (1972), it may be difficult to teach the precise meaning of commutativity and associativity in the absence of non-commutative and nonassociative operations; teachers might therefore consider presenting students with suitable counterexamples (such as subtraction).

Limitations and future direction

This study has several limitations. As the cohort was limited to nine teachers, all within the same school district, results may not generalize across all contexts. Further research is needed to determine to what extent the results are representative of other classrooms. The study was also limited to four class sessions per teacher, and it is likely that we did not encapsulate all of the relevant AP instruction that took place during the year, and so the "quantity of instruction" may not be accurate. However, as the lessons were selected after careful textbook analysis and consultation with teachers, we believe that most relevant instruction was captured. Moreover, we argue that our findings relating to the *nature* of AP instruction likely generalize to additional instruction within the same classes, and remains valid.

Additionally, we acknowledge that it is not entirely clear how advanced appropriate AP instruction in the early grades should be—particularly with regard to "structuralness" and "AP/CP conflation." Our analysis here should be taken to be descriptive, not necessarily normative; further research is required to establish a coherent picture of "ideal" instruction. Possibly, a comparative analysis of relevant textbook presentations or classroom instruction in mathematically high-achieving countries may provide insights in this regard. Finally, the present study focuses only on teacher instruction, and not on student understanding, and so the impact of any of the observed trends on student learning cannot be confidently ascertained. Further research on the student data is warranted to explore to what degree the instructional trends observed in this research affect student understanding of the AP.

Conclusion

This study provides us with a window into the current teaching of the AP in a natural classroom setting. It thus provides us with an opportunity to reflect on the praxis of current literature recommendations on the teaching of arithmetic and early algebra, and to direct future endeavors in both research and practice in light of what goes on in actual classrooms. While the instruction observed did contain much opportunity for improvement, examples were seen of high-quality AP-related instruction with deep links to substantive concrete justification, even capable of prompting spontaneous student generalizations. Future research may provide insight into how such instruction can be promoted on a wider scale, or shed more light on the trends identified in this study. The mathematics education field may also reflect on teacher practices, and as a framing of the many challenges of real, in-classroom implementation of rigorous, mathematically correct AP instruction that provides a firm foundation for mathematical learning in algebra and beyond.

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