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Elementary Students' Understanding of Inverse Relations in the United States and China

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Concepts are important in that they are the mental units of thinking and communication (Carey, 2000). Carey further commented that the main barrier to successful learning is not what students lacked but what they have, in particular, their misconceptions. As Resnick (1982) pointed out, "Difficulties in learning are often a result of failure to understand the concepts on which procedures are based" (p. 136). Therefore, in order to better help students learn and understand mathematics, it is important to explore how core mathematics concepts and principles are understood at an early age. In fact, prior studies (Baroody, 1999; Torbeyns et al., 2009; Nunes et al., 2009) indicate that many students have difficulties in understanding core mathematical concepts during their early learning, which may cause difficulties in their later learning of more advanced topics such as algebra.

This study focuses on the understanding of one core mathematical concept- *inverse relations* that are aligned with the major topics of early algebra proposed by Kaput (2008). From an international perspective in this study, we explore Chinese and U.S. students overall performance and methods for solving problems designed to measure understanding of inverse relations. In the following sections, we review the importance and difficulties of understanding inverse relations, and suggest how this international comparison study might provide unique and alternative ways to understand students' thinking, problem solving strategies, difficulties, and misconceptions surrounding inverse relations.

The Prior Research on Importance of Understanding Inverse Relations

Among the four basic operations, by definition, addition and subtraction are inverses (Vergnaud, 1988). "Inverse relations" in this study refers to the *complement principle* (e.g., if $a+b=c$, then $c-b=a$), which can be initially learned through (a) fact family (e.g., 7+5=12, 5+7 = 12, 12-7=5, and 12-5=7) or(b) inverse word problems (the solutions form a fact family, Carpenter et al., 2003). The inverse relations between addition and subtraction (additive inverses) are one of the most important fundamental mathematical ideas for lower elementary grades and are also a basis for learning both arithmetic and algebra in later grades (Baroody, 1987, 1999; Carpenter, Franke, & Levi, 2003).

Furthermore, an understanding of inverse relations contributes to one's full comprehension of the four operations (Wu, 2011), algebraic thinking (Carpenter et al., 2003; Stern, 2005), and mathematical flexibility (Nunes et al., 2009). For example, students' awareness of "fact families" (e.g., 5+8=13, 8+5=13, 13-5=8, and 13-8=5) may allow them to see the structure of operations and develop their reasoning abilities, such as if $a + b = c$, then $b + a = c$, $a = b – c$, and $b = c – a$ (Carpenter, et al.; NCTM, 2000). As a result, the adequate understanding of inverse relations not only deepens students' understanding of arithmetic but also lays a foundation for their future learning of algebra (Carpenter et al., 2003; Schoefeld, 2008).

The Difficulties to Learn and Understand Inverse Relations

Including U.S. students, overwhelming evidence shows that elementary school children generally lack deep understanding of inverse relations (Baroody, 1999) and have difficulties in applying such thinking to solve problems (Baroody et al. 1983). For example, some students cannot spontaneously use addition to solve subtraction problems (Torbeyns et al., 2009a, b) or cannot use addition to check subtraction (Baroody, 1987). In addition, word problems that demand inverse understanding (e.g., start-unknown problems) have presented incredible challenges for elementary students (Nunes et al., 2009).

Why are inverse relations, a ubiquitous mathematical concept, so hard to learn? For one, the very nature of this concept may cause learning difficulties (Nunes et al., 2009). Students have some informal experiences of inverse thinking, as noted by Bryant, Christie, and Rendu (1999) who described the "identity of inversion" as adding and taking away the same objects results in the initial quantity. The second level of understanding is the "quantity of inversion." This entails adding and subtracting the same quantity, but not necessarily the same objects, leading to the original status. Significant learning difficulties may occur in this transition from reasoning upon objects to quantities (Bryant, Christie, and Rendu, 1999, also cited by Nunes et al. 2009). An understanding at both levels can lay a good foundation for inverse thinking.

The failure to understand inverse relations may be further attributed to classroom instruction and textbooks. In many U.S. classrooms, learning and instruction often emphasize the syntax of mathematics rather than semantics (Resnick, 1989). In the case of inverse thinking, the existing instruction is to teach inverse-based strategies rather than the underlying relation. For example, students in Baroody's study (1999) were led to think in the following way: "5 take away 3 makes what?" can be understood as "3 added to what makes 5?" It was not clear to students why this strategy worked. Similarly, the textbooks in Torbeyns et al (2009 b) taught a strategy named *indirect addition* (e.g., using 79 + 2 = 81 to solve 81 – 79) by drawing a little arrow from the subtrahend to the minuend. Focusing on procedural strategies (syntax) rather than the underlying relations (semantics) neither contribute to students' deep initial learning nor later transfer (Chi & VanLehn, 2012).

If students' learning difficulties with inverse relations is caused by syntax approaches, it is expected that instruction and textbooks with the focus on semantics should help students better understand inverse thinking. In this study, we selected U.S. expert teachers whose teaching approaches might be generally characterized as "semantics." Since prior studies indicate that Chinese textbook and instruction focuses on the semantics, Chinese expert teachers were also chosen for this study. Chinese students demonstrated high achievement and are engaged in classroom instruction focusing on semantics (Stevenson & Stigler, 1992; Ma, 1999). Furthermore, Chinese textbooks appear to present fundamental mathematical ideas in pedagogically and mathematically appropriate ways for students (Cai & Moyer, 2008).

Given that students' understanding of inverse relations at an early age plays an important role in learning, we are interested in exploring how U.S. and Chinese teachers in lower elementary grades apply inverse thinking in solving problems. In particular, we ask the following research questions:

1. How do Chinese and U.S. students improve their correctness when solving additive inverse problems after one year of instruction?
2. How do Chinese and U.S. students improve their inverse thinking when solving additive inverse problems after one year of instruction?

Method

This study is part of a large five year NSF-supported research project on expert teachers' knowledge for teaching early algebra.. The data used in this study were collected during the 2014-2015 school year, with the targeted mathematics topic being inverse relations.

Participants

All student participants were recruited from expert teachers' classrooms in the U.S. and China. We define expert teachers to be those who have at least 10-years of teaching experience and have received recognition for their teaching (e.g., earned teaching awards or national-board certificate) or have a remarkable teaching reputation (e.g., recommended by principals or school district).

US sample. A total of 113 U.S. students were recruited from 8 U.S. expert teachers' classrooms. This included students from four different schools within one large school district in Mid-Atlantic region The participants were in either first or second grade (N_{G1}=61, N_{G2}=52). This is a large high-needs urban school district and the majority of the student population is underrepresented (e.g., 76% were low-income, 62% were African American). According to the state assessment, only 43% of schools in this district make adequate yearly progresses. When students in this district arrive in the 11th grade, 52% of the students are below basic in mathematics.

Chinese sample. A total of 171 first and second grade Chinese students (N_{G1}=81, N_{G2}=90) were recruited from 8 Chinese expert teachers' classrooms. These classrooms were in four urban schools in Jiangsu province, China. This province is known to have high-quality education and ranked very top in light of economics and school quality.

Test Items

To analyze students' strategies and reasoning process in solving additive inverse problems, four items that require explanations were selected from both a pre- and posttest given during the 2014-15 school year (see Figure 1). These include both non-contextual and contextual items, which together may show a relatively complete picture of students' understanding of inverse relations. Each item was adapted from the literature (Broody, 1987, 1999; Nunes et al. 2009, Resnick, 1989, Torbeyns et al. 2009). It should be noted that all of these items may be solved with inverse-based strategies or with other strategies, but students were asked to explain their reasoning process for solving each problem.

Non-contextual items. *Q5, Q6, and Q7.* Research shows that students have difficulties applying inverse thinking to solve the *complement* problem $a + b = c$; $c - a = ?$ (Baroody, 1999; Baroody, Ginsburg, & Waxman, 1983; Baroody & Tiilikainen, 2003). Most commentaries on this problem (Resnick, 1983; Putnam, de Bettencourt, & Leinhardt, 1990) point out that c consists of two parts, a and b, and claim that the main reason for children's failure is due to the difficulty in grasping the relationship between parts and wholes. Another possibility that children fail to solve the problem is because solutions rest on understanding the inverse relation between addition and subtraction. That is, students may not grasp that because c is the result of adding b to a, subtracting b from c will cancel out the effects of that addition and will therefore restore the initial quantity a. Q5, Q6, and Q7 are designed for assessing whether students are able to identify and apply inverse relation to solve these problems .

Contextual item. *Q8.* These two problems are story problems. Therefore, students need to choose appropriate operations based on their understanding of the story situation and the meaning of each operation. Resnick (1989) reported that few students are able to solve the "Initial Unknown" problem before the age eight or nine. For this type of problem, the initial amount is unknown but the change and results of change are known. Students are asked to figure out the initial unknown amount. The solution to this type of problem did not match the problem statement directly. Resnick (1989, p. 165) provided an example to illustrate the difficulties.

> Ana went shopping. She spent \$3.50 and then counted her money when she got home. She had \$2.35 left. How much did Ana have when she started out?

The word "spent" may indicate that "subtraction" is needed. To solve this problem, the initial amount that Ana has is unknown. As a result, the students should think reversely: if Ana adds what she spent (\$3.50) to what she has left, she will have the initial amount. Therefore, addition should be used to find out the initial amount. Such thinking is very difficult for students. Q8a is the "initial unknown" problem. Ali gave 2 to his sister with 6 left. Students did not have good understanding of inverse thinking may use 6-2 to find the answer. On the other hand, students with good understanding of inverse thinking might know that to give back what he gave to his sister (2) will restore the initial amount, which lead to correct addition solution: 6+2. Q8b is similar in this regard. According to Resnick (1989), students in both Q8(a) and Q8(b) may use their part-whole schema to transform a change problem to a combination model, which indicates an understanding that $a - b = c$ implies $c + b = a$.

[INSERT FIGURE 1 ABOUT HERE]

Data Coding and Analysis

Two researchers developed a rubric for coding students' inverse thinking based on the literature. Then 10% of student work was coded in order to refine the rubric based on student's actual responses. Examples in which both researchers' scores agreed were incorporated into each category of the refined rubric. Using the refined rubric, students' responses to each question were first coded for overall correctness. Next, we analyzed and coded students' problem solving strategies for each question regardless of if the question was answered correctly(see Table 1 for the details). Students' strategies were coded as (1) using inverse thinking, (2) using direct computations, that is, students directly did the operations to get the answer, (3) using other conceptual strategies, and (4) no or an empty answer (e.g., I don't know).

We are particularly interested in how students use inverse thinking (1) in answering each problem. In order to qualify for this category, students' solutions and/or explanations should demonstrate clear use of inverse relations to solve the problems. For instance, to solve or check for a subtraction problem, students should use the corresponding addition number facts. All other problem solving strategies such as counting-on or counting-back, using a hundreds chart, or using a standard algorithm or basic facts, were not coded as "using inverse thinking."

Based on the above rubric, for non-contextual item Q5, if a student explained that he/she used the given addition fact (9+3=12) to obtain the answer for 12-3 = (   ), it would be coded as "using inverse thinking". Similarly, for Q6 and Q7, if a student provided a response such as " 5+6=11 so 11-6=5," it was coded as inverse thinking. In contrast, for the responses to Q5, if the student got correct answers by directly computing 12-3 but no relation between addition and subtraction was made explicit, their response was not coded as "inverse thinking." Similarly, strategies other than using addition to obtain or check the answer for Q6 and Q7 were not coded as "inverse thinking."

With regard to the contextual items, Q8(a) appears to be a subtraction problem because the problem involves a decreasing action, "He gave 2 of them to his sister" (?-2=6); yet, the initial unknown quantity most effectively can be found by using addition (6 + 2 = 8). That is, the students should think of "putting the given away back" (Briars & Larkin, 1984). As such, only when a student used addition strategies to solve this problem did we code it as "inverse thinking." Likewise, Q8(b) appears to be an addition problem (?+2=8) because the problem involves an increasing action, "his sister gave him 2"; yet, the initial quantity may be found by subtraction (8-2=6). That is, students should think of giving away the received, which demands inverse thinking. As such, only when a student used "subtraction" to solve this problem, was it coded as "inverse thinking."

[INSERT TABLE 1 ABOUT HERE]

Coding Reliability

One researcher coded all the data and a second coder coded 10% of all cases. The reliability was 94% that is percent of number of agreed cases out of the total cases. A third recorder independently recoded all the cases of disagreements. All three coders reached eventual agreement on all of the items.

Results and Discussion

We first report the correctness of students in both countries when solving each of these problems. We then report students' strategies used to solve these problems as indicated by their explanations. In particular, we report how students applied inverse thinking when solving these problems. Students' typical solutions were provided to illustrate the findings.

Chinese and U.S. Students' Correctness

How did students perform across countries? In light of the correctness, Chinese students performed much better than U.S. students at each grade level in both pre and posttests for each problem (US_{pre}=47%, US_{pst}=65%; $China_{pre}$=82%, $China_{pst}$=86%, see Table 2). For the U.S. sample, students' performance greatly improved from pretest to posttests (18% gain). In contrast, the students in the Chinese sample showed little progress (only 4% gains from pre to posttests). This leaves unanswered questions about whether the Chinese student data is illustrating the ceiling effect. That is, did they already reach their highest potential and thus have little room for improvement.

[INSERT TABLE 2 ABOUT HERE]

How did students perform across problems and grades? Students' performance varied in different problems. Q6 and Q8(b) are the most difficult for students in both countries. For Q6, 81–79 = (   ), 30% and 50% of U.S. students answered correctly in the pre and posttests respectively; 75% and 72% of Chinese students answered the same question correctly in the pre and posttests respectively. These are all below the respective average correctness across problems. For Q8(b) (Ali had some chocolate candies and his sister gave him 2, now he has 8), the percentage of U.S. students who correctly answered this problem increased from 34% to 53% from the pre to posttest. On the other hand, 61% of Chinese students answered this problem correctly in the pretest and 73% of them answered it correctly in the posttest.

When analyzing students' performance across grades, it appears that the U.S. sample improved their performance greatly from the pretests at grade 1 to the posttests at grade 2 (see table 3).

[INSERT TABLE 3 ABOUT HERE]

Chinese and U.S. Students' Use of Inverse Thinking

How did students use inverse thinking across countries? Chinese students overall applied inverse thinking to solve problems more frequently than the U.S. sample in both the pretest and posttests (US_{pre}=11%, US_{pst}=26%; $China_{pre}$=43%, $China_{pst}$=60%).

The typical inverse thinking that U.S. students demonstrated was to use addition to solve the subtraction problem as shown in the figures below for Q5, Q6, Q7(see figure 1), and Q8(b) (see figure 3). Overall US students used the strategies consistent across these problems and represent typical student solutions that used inverse thinking.

Q5

How did you get the answer for 12 – 3 = ( )?

I know that 9+3=12 so 12-3 must eqales 9

Q6

6. 81–79 = (2)

How did you come up with your answer?

I know that 79+2=81 so 81-79 would =2.

Q7

How can you check if this is correct or not?

5+6=11 so, 11-6=5

Figure 1. Typical US students inverse thinking strategies for Q5, Q6, and Q7

Similar to U.S. students, Chinese students typically used "addition" to solve the subtraction problems. However, when Chinese students were asked to explain how they solved this subtraction, many of them provided this consistent explanation: "Think of additions when I do subtractions." Compared to U.S. students who most often referred to specific facts (such as 9+3=12 or 79+2=81), Chinese students were able to refer to a broad principle: the relation between addition and subtraction.

5. 9 + 3 = (12)
12 – 3 = (9)

你是怎样得到 12 – 3 = ( ) 这个题的答案的？

我是用 zuò jiǎn fǎ, xiǎng 加 fǎ。

English translation:

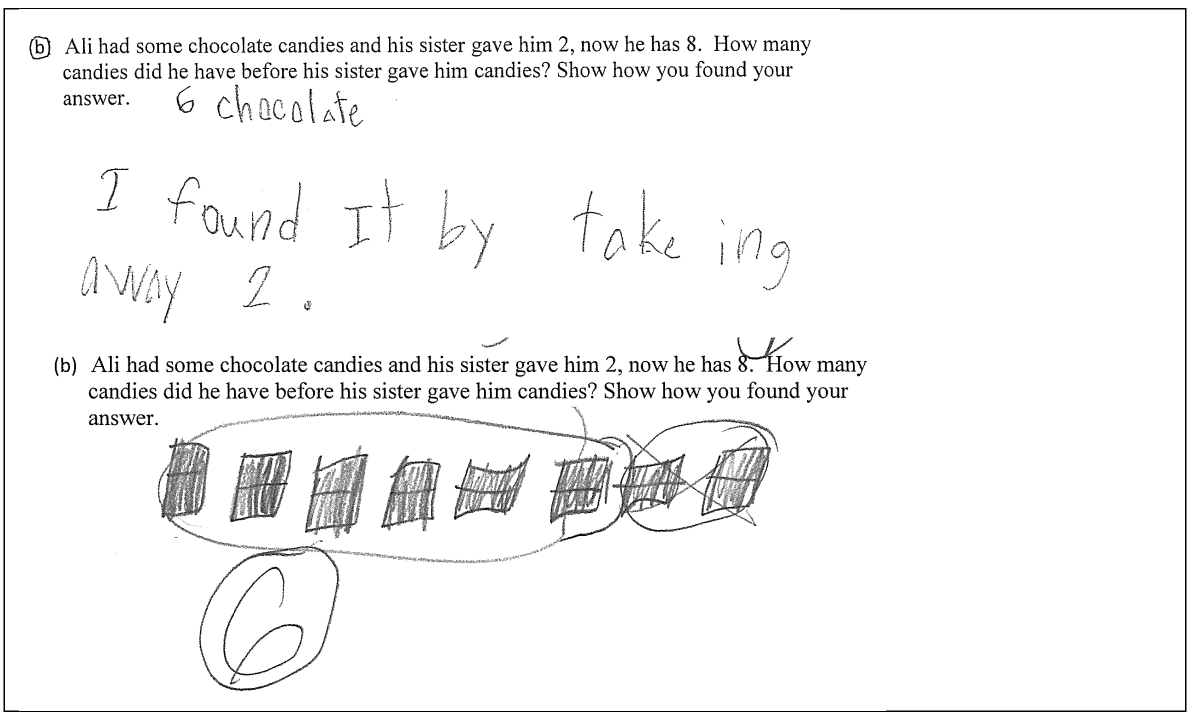
5. 9+3=(12)

12-3=(9)
How did you get the answer for 12 – 3 = ( )?
I used thinking of addition when do subtractions

Figure 2. Typical Chinese students inverse thinking strategies for Q5.

It is interesting that inverse thinking strategies were used the least when solving the problem 81-79 (for Q6, US_{pre}=3%, US_{pst}=6%; $China_{pre}$=13%, $China_{pst}$=14%).This problem is not structurally different from Q5. The major difference however lies in the fact that 9+3 =12 was provided for Q5, but79+2= 81 was not provided for Q6. In addition, Q6 involved two-digit subtraction while Q5 was involved one-digit. Although many Chinese students are able to articulate the general principle like “think of addition when do subtraction,” this indicates that many of them are unable to apply this principle into a situation when no clear hint is available or when numbers get bigger.

For Q8(b), there was a lot of variance in U.S. students solutions. Their solutions included: (1) the number sentence 8-2=6, (2) written explanations such as “I found it by taking away 2,” or (3) drew a picture (See figure 2). In contrast, Chinese students universally provided the number sentence 8-2=6 and no drawing or verbal explanations were used. This finding was consistent with Cai’s (2005) findings that Chinese students either use numerical solutions to solve a problem or just leave the solution blank, whereas U.S. students tend to use various ways to identify a solution.



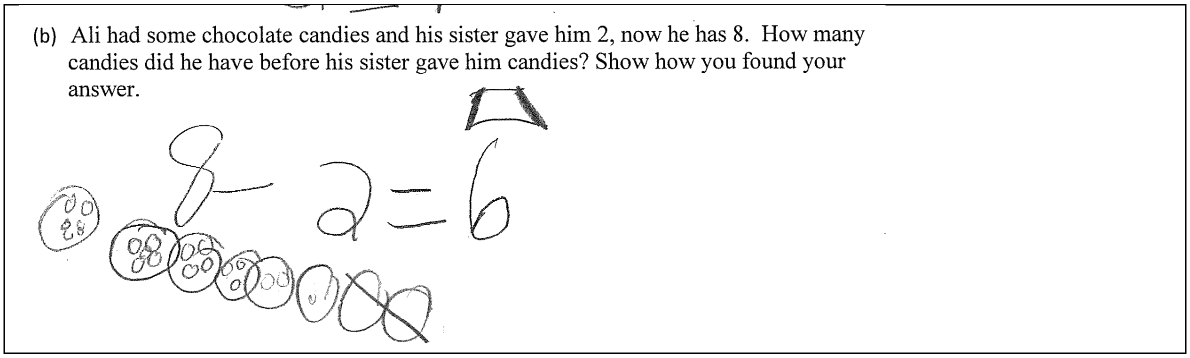


Figure 3. US students' typical inverse thinking strategies for Q8b.

How did students use inverse thinking across grades? One of the most surprising and interesting findings is that U.S. students demonstrated great progress with the use of inverse thinking from pretest to posttest in both grades. For example, for Q8(b) which is the most challenging type of initial unknown problem (Resnick 1989), the U.S. sample showed great progress both in grade 1 (US_{pre}=2%, US_{pst}=12 %) and in Grade 2 (US_{pre}=22%, US_{pst}=57%). In contrast, the Chinese sample showed a different pattern for this problem. In ghe Chinese sample, the best result was the posttest in grade 1; yet, little progress was made from pretests to posttests at grade 2 ($China_{pre}$=44%, $China_{pst}$=77 %, Grade 1; $China_{pre}$=62%, $China_{pst}$=67%, Grade 2). A huge achievement gap in the pretest at grade 1 was found (US_{pre}=2% VS $China_{pre}$=44%), but this gap was greatly reduced by grade 2 posttests (US_{pst}=57% VS $China_{pst}$=67%).

With regards to using inverse thinking, the other problems demonstrated a similar pattern (i.e., for Q5, US_{pst} =37%, $China_{post}$ =42%; for Q6, US_{pst} =10% VS $China_{pst}$ =21%; for Q8a US_{pst} =71% VS $China_{post}$ =94%). The biggest gap that exists was found to be on Q7 where students were asked to check their answer. Considering the grade 1 pretests, it is 0% or 2% U.S. samples can use inversing thinking (see table 2), their progress is impressive. That is, the U.S. sample made stable and great progress from pretests at grade 1 to posttest at grade 2 but the Chinese sample did the best at posttests grade 1. The Chinese sample showed adequate increase from pretests at grade 1 to posttests at grade 1; but little progress from pretests at grade 2 to posttests at grade 2 (See Figure 2 for more information). A possible interpretation is that Chinese students in second grade already start learning and focusing on multiplications rather than additive thinking.

Chinese and U.S. Students' Use of Other Thinking Strategies

The use of direct computations. Except for the use of inverse thinking, students in both countries used many other strategies to solve these problems. The most frequently used strategy was the use of direct computations (US_{pre}=9%, US_{pst}=17%; $China_{pre}$=25%, $China_{pst}$=24%). In other words, students' correct solutions were mainly based on computations, indicated by using "Counting back", "Hundred charts" or key words suggested by context problems (see Figure 3). Solutions to Q8(a) and Q8(b) demonstrate that students were able to understand the problems but had difficulties thinking reversely.

They tended to literately follow the story problems word-for-word and write the number sentence accordingly.

Q5

9 + 3 = (12)
12 – 3 = (9)

How did you get the answer for 12 – 3 = (9)?
I was at 12 I counted back 3.

Q6

6. 81–79 = (2)

How did you come up with your answer?
How I got the answer is I used tally-marks.

6. 81–79 = (18)

$$\begin{array}{r} 81 \\ -\ 79 \\ \hline 18 \end{array}$$

你是怎样得出这个答案的？
我是用立 shí fǎ suàn 的

English translation:
How did you get the answer?
I lined up the numbers vertically and computed the answer

Figure 3. The use of direct computations

Using other number facts. Students' use of other strategies varied.. For example, some used number facts in checking the answer for 11-6. Many looked for relations between number facts but this was limited to relations within subtraction (Figure 4).

To solve 11–6=?, Mary's answer is 5, is this correct? yes

How can you check if this is correct or not?
I know because if 10-5=5 then that means 6-11=5

5. 9 + 3 = (12)
12 – 3 = (9)

你是怎样得到 12 – 3 = (9) 这个题的答案的？ 12-2=10 10-1=9

English translation for Q5
How did you get the answer for 12-3=( )
12-2=10, 10-1=9

7. 计算 11–6=?, 小强得到的答案是 5。对吗？ √

你怎样才能验算这个答案是否正确?
6+6=12 nà不shì jiùsh 12-1=11 shuōyi
jiùshì 11-5=6 suǒ 我 Dàn duàn chū 3=11

English translation for Q7
7. To solve 11–6=? Mary's answer is 5, is this correct?
How can you check if this is correct or not?

6+6=12, 12-1=1, so that is 11-5=6

Figure 4. The use of other strategies

Discussion

Performance Differences and Adequate Progress

U.S. students made stable and great progress from pre- to posttests, from grade 1 to grade 2. Furthermore, Chinese students outperformed U.S. students both in the correctness and the frequencies of using inverse thinking strategies. However, we are careful to report such performance differences due to the fact that the two student samples are different regarding social economic status and quality of local education systems. Most Chinese students are from middle class families and they live in a city with high quality education system, whereas the U.S. sample is from a large urban district where about 76% are low-income families. It has been found in many studies that students' SES highly correlated with their academic achievement. However, the US sample made much more progresses from grade1 pretests to posttests compared to their Chinese counterparts.

How may we explain these differences? The common feature between these two groups of students is that their teachers are all expert teachers. With the low starting points, the U.S. expert teachers in this study were able to greatly improve their students' understanding of inverse thinking and significantly reduced the achievement gap with their Chinese counterparts.

The features of such quality teaching that may have improved students' inverse understanding as reported by researchers in this project (Chen & Ding, 2016; Ding et al., 2016; Hassler, 2016) include making connection to prior knowledge and asking comparison type questions.

How may U.S. classroom instruction and textbooks contribute to improving first graders' initial understanding of inverse relationship between addition and subtraction? Why did U.S. first grade students do much more poorly than the Chinese students in the pretests? It is interesting to note that while Chinese elementary schools start with grade 1, U.S. elementary school starts with kindergarten and thus the U.S. first grade students in this study were already in their second year of elementary school. As our data shows, there were plenty of Chinese first graders (33%, 9%, 33%, 66%, 44% for each problem) who were able to use inverse thinking in the pretests, which indicates that students at earlier grades do in fact have the capability to understand the concept of inverse relations. As such, U.S. educators should consider better ways to improve students' initial learning environments (e.g., classroom instruction, textbooks) in order to develop understanding of inverse relations at an earlier age.

Finally, even though our data echoes prior findings that inverse relations are a hard concept for both U.S. and Chinese students, our data shows that Chinese students have achieved a relatively satisfactory level within 1 year (overall about 70% students answered correctly in the posttests at grade 1). Even though their learning focus shifts to multiplicative thinking, Chinese students still maintain the level of additive inverse understanding obtained during grade 1. In contrast, U.S. students took two years to learn additive inverses and do not learn multiplication until the third grade. Even though they have made adequate progress over years, one may re-consider whether textbook presentation and classroom instruction can challenge and support U.S. students to learn additive inverse relations in a more efficient and effective way.

Strategies Choices and Inverse Thinking

Although the most effective way to solve these problems is to use the inverse thinking strategies (i.e., students can quickly find the answer for 12-9 given 9+3=12), no U.S. first graders in Q5, Q6, and Q7 of the pretest used this strategy (see Figure 2). For Q5, although no U.S. students at grade 1 used inverse thinking in the pretests, 41% of these students were able to use other strategies to get the correct answer for this problem. For each of Q6, Q7, Q8b, 0% to 2% of U.S. students used the inverse thinking during the pretests. This same struggle with retrieving inverse thinking strategies was also apparent in the Chinese sample. Clearly, for both the U.S. and Chinese samples, the use of inverse thinking does not top the list of strategy choices. Why do students not use the most efficient strategies? According to Siegler and Shrager (also cited by Resnick, 1989), although students may have a few strategies available to use, their choice of strategies are based on reaction time needed to find the answer. That is, students tend to select the strategy that requires the least effort and leads to the quickest answer (Resnick, 1989). On the other hand, as Figure 2 shows, the use of inverse thinking increased greatly over time, which indicated the exposition to quality learning did improve students' performance in this respect. Students decreased their reaction time to retrieve inverse thinking strategies when this inverse strategy is taught and highlighted.

General Principle and Fact Families in Inverse Thinking

Regardless of the observed similarities, Chinese and U.S. students demonstrated different thinking patterns. When students are asked to explain how they used inverse thinking, Chinese students repeatedly mentioned the general principle "think of addition when do subtraction." In contrast, U.S. students refer to specific facts. Principle knowledge can transfer across different situations (Bruner, 1960; Goldstone & Son, 2005), which may partially explain why Chinese students did relatively better in using inverse thinking across problems and grades. There are many advantages when students know fact families as US samples did; however, students should go beyond the fact families to a more general level of understanding. If teachers can ask students questions such as "what is the relationship between addition and subtraction," instead of surface level questions involving fact families, students might better be able to transfer their understanding of inverse thinking.

Concrete and Symbolic Solutions

Another difference between the two student samples lies in that the Chinese students tend to use symbolic solutions such as number sentences, and U.S. students used more concrete representations such as drawing pictures. This finding is consistent with that of the other researchers (e.g., Cai, 2005) who found that Chinese students in general in grades 4-6 tend to use symbolic ways and U.S. students use multiple ways including drawing.

More than 66% Chinese students used symbolic solutions, which also shows that students can reach a symbolic understanding of inverse thinking. However, teachers should be careful in using symbolic solutions because the use of symbolic solutions might become number manipulations without meaning. In fact, many Chinese students mis-manipulated numbers to respond to Q8(b) (e.g., 2+8=10), which shows that students at a young age might get confused easily without referring to contexts or concrete representations. Whenever possible, concrete representation such as pictures or word contexts should be used to accompany and make sense of the numerical sentences. For example, for Q8(b) when the number sentence 8-2 is used to find the answer, students should be encouraged to refer to the context of the problem. For instance, students can be asked if the 8 represents what Ali has in the beginning? One might also ask students if= the 8 has two parts: one part owned by Ali in the beginning and one part from his sister? A number line or a picture can also clearly show the relationship between part, part, and whole. Such concrete situations and drawings may help students to know which operation to use and thus avoid the mistakes like using 2+8=10 to solve Q8(b).

It is suggested that teachers should not rush to symbolic ways before students completely understand the meaning of operations and inverse relations. Concrete situations should be provided for students to make sense of number sentences. At the same time, 72% of Chinese students succeeded in using number sentences to solve Q8(b). This also suggests that if given proper instruction, it is very likely for students to reach the symbolic level even at the early grades.

Conclusion

This study shows promises to improving U.S. students' achievement in fundamental concepts, even for students from a district with mostly low income and minority families. Even compared to the Chinese sample, the U.S. sample showed adequate progress and comparable achievement by the end of grade 2. As we mentioned, the common feature between these two groups of students is that both had expert teachers. We hypothesize that these students' successes cannot be separated from the quality of instruction delivered by these expert teachers. In other words, the quality of instruction seems to matter in students' understanding of inverse relations (Stull et al., 2016). Other studies from this project will demonstrate how instructions by expert teachers develop and enhance students' understanding of inverse thinking.

Table 1. *Coding Rubric for Students' Solution Strategies.*

	1 – correct solution	0 – wrong solution
a – inverse	1a Correct solution based on inverse thinking with referring to additions to solve subtraction problems	0a Wrong solution that indicates inverse thinking
b – computation	1b Correct solution but mainly based on computation indicated by the number sentence such as using "Counting back", "Hundred charts" or key words suggested by context problems	0b Wrong solution that indicates computation
c – other conceptual	1c Correct solution based on other conceptual strategies (e.g., based on number relationships, drawing circles which does not tell inverse)	0c Other conceptual errors
d– no/empty	1d Correct answer with no or empty explanations (e.g.., blank, I am smart, I used my fingers, I searched my brain, I looked at the number grid)	0d Wrong solution with no or empty explanations

Table 2: *Students' Performance and the Use of Strategies*

		Strategies with Correct Solution					Strategies with Wrong Solution				
		1a-Inverse	1b-Direct	1c-Other	1d-Empty	Total	0a-Inverse	0b-Direct	0c- Other	0d-Empty	Total
US_Pre	Q5	9%	14%	5%	28%	55%	0%	2%	5%	38%	45%
	Q6	3%	8%	7%	13%	30%	0%	7%	7%	56%	70%
	Q7	6%	10%	11%	33%	59%	1%	0%	1%	39%	41%
	Q8a	24%	6%	13%	16%	58%	0%	1%	14%	27%	42%
	Q8b	12%	6%	11%	6%	34%	0%	0%	27%	38%	66%
	Average	**11%**	**9%**	**9%**	**19%**	**47%**	**0%**	**2%**	**11%**	**40%**	**53%**
US_Post	Q5	24%	22%	5%	24%	76%	1%	1%	7%	16%	24%
	Q6	6%	23%	7%	14%	50%	0%	14%	11%	26%	50%
	Q7	17%	21%	8%	35%	81%	0%	3%	7%	10%	19%
	Q8a	50%	7%	7%	5%	68%	0%	1%	21%	10%	32%
	Q8b	34%	10%	4%	6%	53%	0%	0%	38%	9%	47%
	Average	**26%**	**17%**	**6%**	**17%**	**65%**	**0%**	**4%**	**17%**	**14%**	**35%**
China_Pre	Q5	33%	39%	4%	22%	98%	0%	0%	0%	2%	2%
	Q6	13%	41%	2%	18%	75%	0%	3%	7%	16%	25%
	Q7	35%	35%	4%	16%	90%	0%	0%	1%	8%	10%
	Q8a	79%	7%	1%	1%	88%	1%	0%	4%	8%	12%
	Q8b	53%	4%	2%	1%	61%	0%	1%	23%	16%	39%
	Average	**43%**	**25%**	**3%**	**11%**	**82%**	**0%**	**1%**	**7%**	**10%**	**18%**
China_Post	Q5	57%	40%	1%	1%	99%	0%	0%	0%	1%	1%
	Q6	14%	56%	1%	1%	72%	2%	7%	1%	17%	28%
	Q7	66%	24%	2%	1%	94%	0%	0%	0%	6%	6%
	Q8a	94%	0%	1%	0%	94%	0%	0%	5%	1%	6%
	Q8b	72%	1%	1%	0%	73%	0%	0%	26%	1%	27%
	Average	**61%**	**24%**	**1%**	**1%**	**86%**	**0%**	**1%**	**6%**	**5%**	**14%**

Table 3. US and Chinse *Students' Correctness and the use of Inverse thinking strategies* across grades

		Correctness					**Inverse thinking strategies**				
		Q5	Q6	Q7	Q8a	Q8b	Q5	Q6	Q7	Q8a	Q8b
US	G1_Pre	40%	17%	35%	42%	12%	0%	0%	0%	15%	2%
	G1_post	58%	33%	67%	56%	37%	12%	2%	6%	29%	12%
	G2_pre	71%	43%	84%	75%	57%	18%	6%	12%	33%	22%
	G2_post	94%	66%	94%	80%	71%	37%	10%	27%	71%	57%
Chinse	G1_Pre	97%	56%	86%	84%	56%	33%	9%	33%	66%	44%
	G1_post	99%	49%	88%	94%	78%	73%	6%	70%	94%	77%
	G2_pre	99%	92%	94%	93%	65%	34%	17%	37%	92%	62%
	G2_post	99%	98%	99%	95%	69%	42%	21%	62%	94%	67%

Fill in the blanks:

5. 9 + 3 = (   )

   12 – 3 = (   )

   How did you get the answer for 12 – 3 = (   )?

   ____

6. 81–79 = (   )

   How did you come up with your answer?

   ____

7. To solve 11–6=?, Mary's answer is 5, is this correct?

   ____

   How can you check if this is correct or not?

   ____

8. (a) Ali had some chocolate candies. He gave 2 of them to his sister, and then he had 6. How many candies did Ali have before giving his sister candies? Show how you found your answer.

   (b) Ali had some chocolate candies and his sister gave him 2, now he has 8. How many candies did he have before his sister gave him candies? Show how you found your answer.

   you solve this subtraction problem?

Figure 1. The test items used in this study.

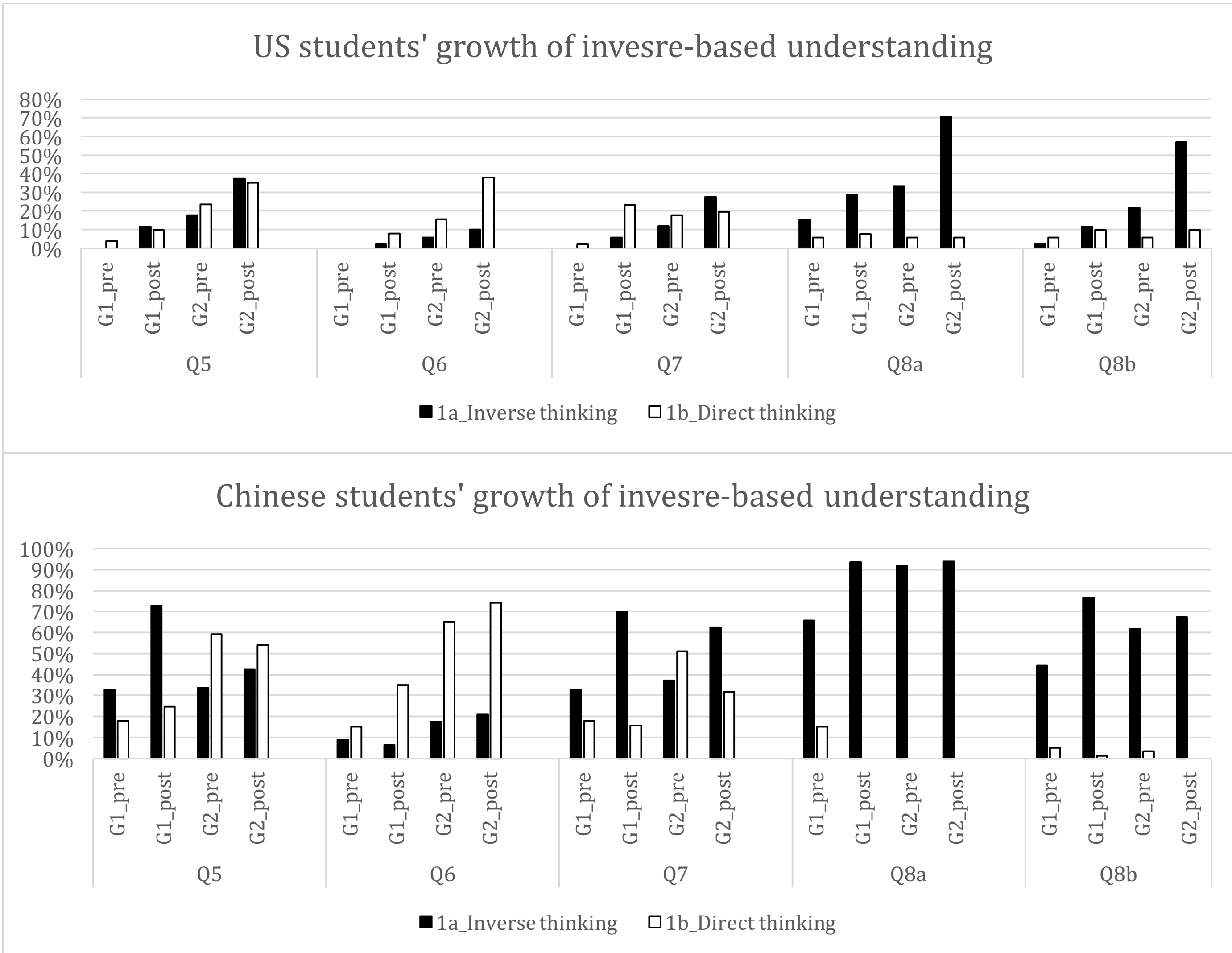


Figure 2. Inverse-Based Understanding

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