Running Head: Inverse Relations

**Children’s Strategies to solving Additive Inverse Problems: A Preliminary Analysis**

Meixia Ding

[Meixia.ding@temple.edu](mailto:Meixia.ding@temple.edu)

Abbey Auxter

[Abbey.auxter@temple.edu](mailto:Abbey.auxter@temple.edu)

Temple University

**Acknowledgements**

This study is supported by the National Science Foundation (NSF) CAREER program under Grant No. DRL-1350068 at Temple University and the NSF grant DUE-0831835 at the University of Nebraska–Lincoln. Any opinions, findings, and conclusions in this study are those of the author and do not necessarily reflect the views of the National Science Foundation.

The inverse relations between addition and subtraction (additive inverses) are one of the most important fundamental mathematical ideas for lower elementary grades (Baroody, 1987, 1999; Carpenter, Franke, & Levi, 2003). Despite reports that many children possess an intuitive sense of inverse relations in preschool (e.g., Gilmore & Spelke, 2008; Sophian & Vong, 1995), overwhelming evidence shows that elementary school children generally lack formal understanding of inverse relations (Baroody, 1987, 1999; Bisanz & LeFevre, 1990; De Smedt, Torbeyns, Stassens, Ghesquière, &Verschaffel, 2010; Riley, Greeno, & Heller, 1983). The aim of the study is to examine children’s existing strategies when solving additive inverse problems so as to better develop their understanding of inverse relations in lower elementary grades.

There are two types of additive inverse: (a) the three-term inversion principle, a + b – b = a, and (b) the two-term complement principle, if a + b = c, then c - b = a (Baroody, Torbeyns, & Verschaffel, 2009). Past studies have reported preschoolers’ informal understanding of inverse relations based mainly on the three-term inversion principle. With approximate numbers involved (e.g., no actual number manipulation needed), children were able to provide correct directional responses (e.g., an increasing action will result in a larger quantity and a decreasing action will result in a smaller quantity, Gilmore and Spelke, 2008; Sophian, Harley, & Martin, 1995; Sophian & McGorgray, 1994; Sophian & Vong, 1995). Prior studies, focused on entering elementary children, examined understanding based mainly on the two-term complement principle. These studies found that elementary school children generally lack formal understanding (Baroody, 1987, 1999; Baroody, Ginsburg, & Waxman, 1983; Bisanz & LeFevre, 1990, 1992; De Smedt et al., 2010; Resnick, 1983; Riley, Greeno, & Heller, 1983). For example, Baroody et al. (1983) found that approximately 61% of first and second graders in their study could not use addition to solve subtraction problems (e.g., using 3+4=7 to solve 7–4). Although the inversion and complement principles are different and there is no consensus on the sequence of learning them, it was generally agreed that these principles are closely related and an understanding of one would contribute to the other (Baroody et al., 2009; Gilmore & Bryant, 2008). Overall, the literature findings indicate children’s potential and difficulties in learning additive inverse relations.

The gap between preschoolers’ informal understanding and elementary children’s lack of formal understanding of inverse relations suggests that there are potentially missed opportunities to teach inverse relations in elementary school. As reported, some teachers only stressed inverse-based procedures such as drawing a small arrow from the subtrahend to the minuend (Torbeyns et al., 2009), which was unrelated to students’ existing knowledge. Without knowing and grasping children’s available levels of understanding and existing strategies when solving inverse-based tasks, the teaching of inverse relations will become ineffective. As such, there is a need to identify the middle status between children’s informal and formal understanding so as to provide them with better opportunities to learn.

Prior research has pointed out that children’s protoquantitative schema of “part-whole” is the key to learning inverse relations (Piaget, 1952; Resnick, 1992). This type of schema may have enabled children to informally understand inverse relations even when they lacked the knowledge to manipulate numbers (Gilmore & Spelke, 2008). Canobi (2005) viewed the sophistication of children’s understanding of the part-whole relation as an indicator of students’ conceptual understanding of addition and subtraction knowledge. According to this researcher, children who can compute addition and subtraction in a precise manner, when lacking an understanding of the part-whole relation, only possessed a procedural knowledge. International studies also reported that high-achieving textbooks arranged the part-whole topic before formally teaching inverse relations (e.g., Ding, 2012; Zhou & Peverly, 2005). For instance, the Chinese textbooks expected students to first learn number composing and decomposing (e.g., 3 and 5 are composed to 8; 8 can be decomposed by 3 and 5), which demanded an understanding of part-whole relationships. With this part-whole understanding, students were then expected to learn addition and subtraction as well as the relationships between them (3 + 5 = 8; 5 + 3 = 5; 8 – 3 = 5, 8 – 5 = 3). As such, attention to bridging factors like the part-whole schema may lead to the identification of paths to developing elementary students’ understanding of inverse relationships. Children’s part-whole schema in existing classrooms, however, remains largely unknown.

Students’ inverse understanding is not an all-or-nothing phenomenon, first due to the cognitive developmental factor. Theoretically, students should gain more understanding as grade levels increase; however, empirical studies do not necessarily support this prediction. For example, Canobi (2005) found that as grade level increases, students’ computation accuracy improves; yet, their conceptual understanding of the part-whole relationship did not necessarily increase. The research explained that the improved computation accuracy is likely due to the repeated practice overtime. However, if the conceptual underpinning was not addressed at the beginning, students’ understanding may not necessarily improve automatically.

Students’ understanding of inverse relations also depends on the contexts to which they are exposed. Facing a task that is situated in a contextual or non-contextual setting, students’ understanding may or may not be demonstrated. Therefore, it is suggested to measure students’ understanding of inverse relations both contextual and non-contextual tasks be used (Bisanz & LeFevre, 1992; Bisanz, Watchorn, Piatt, & Sherman, 2009). Through both contextual and non-contextual tasks, students may be asked to evaluate, apply, and explain inverse relations, thus assessing both their procedural and conceptual understanding. Many times, children who provide correct answers may not be able to explain the underlying reasons. Additionally, those that are able to explain still differ in level of explanation, indicating different levels of inverse understanding.

Existing studies have already suggested that there is a middle status between students understanding and not-understanding inverse relations, which may be associated with both grade levels and the types of tasks. However, it is unclear what students actually know during this middle status. What is the proportion of students’ correct and partial understanding in terms of the correctness and explanation? In what ways does students’ partial understanding differ from not- and full-understanding? Even within this partial understanding, is there any developmental evidence? How may this partial understanding be related to different grade levels and different types of tasks? This study aims to explore these questions using the natural data from regular classrooms. It is expected that information based on such data will better inform teachers and researchers to develop opportunities for students to learn inverse relations.

**Methods**

**Participants**

A total of 281 kindergarten through third grade students participated in this study. There were 50 kindergarten students with mean ages of 5 years and 3 months (*SD* = 6 months); 74 first grade students with mean ages of 6 years and 1 month (*SD* = 4 months); 79 second grade students with mean ages of 7 years and 1 month (*SD* = 3 months); and 78 third grade students with mean ages of 8 years and 2 months (*SD* = 5 months). There were originally 194 third graders; however, for comparison, 78 were randomly selected as a representative sample. Overall, these students came from 35 different classroom teachers—9 kindergarten, 6 first grade, 7 second grade, and 13 third grade—who were participants of a large research project in the mid-west of United States. Teachers were invited to distribute a questionnaire to their students. By the time of distribution, no teacher had received any project training with regard to inverse relations. As such, students’ responses to this questionnaire indicate a natural status of children’s inverse understanding.

**Materials**

To measure student’s existing inverse relations, this study used four items modified from the literature. These items may be solved with inverse-based strategies involving complement and/or inversion principles, which together served as indicators of additive inverses. To help children ease into these items, contextual tasks were first presented before non-contextual tasks. Figure 1 illustrates the questionnaire, followed by elaborations.

|  |
| --- |
| 1. Ali had some Chinese stamps in his collection and his grandfather gave him 2, now he has 8. How many stamps did he have before his grandfather gave him the 2 stamps? Please show your work. 2. Ali had some Chinese stamps in his collection and gave 2 to his grandfather, leaving his collection with 6. How many stamps did he have before he gave his grandfather the 2 stamps? Please show your work. 3. 5 + 3 – 3 = ( ). How did you get this answer? 4. 6 + 3 = ( )   9 – 6 = ( ). How did you get the answer for 9 – 6? Did the addition problem help you solve this subtraction problem? |

*Figure 1.* The questionnaire used in this study

Question 1 (Q1) and Q2 are contextual tasks that were modified from Nunes et al. (2009). Both tasks are initial-unknown, change word problems. Q1 describes an increase in quantity (? + 2 = 8) but the unknown quantity can be effectively found by subtraction (8 – 2 = ?). In contrast, Q2 describes a decrease in quantity (? – 2 = 6) but the quantity can be found by addition (6 + 2 = 8). To solve both problems, students need to reverse a sequence of actions (e.g., putting the given-away stamps back, Briars & Larkin, 1984). During this reversing process, students may use their part-whole schema to transform a change problem to a combination model, which indicates an understanding that a – b = c implies c + b = a (Resnick, 1989). Arguably, this reverse process may indicate students’ understanding of the inversion principle (a – b + b = a) because they know that putting back the given-away stamps (a – b + b) could lead to the original number (a). Q3 and Q4 are non-contextual tasks. Q3 assesses students’ understanding of the inversion principle (a + b – b = a) while Q4 assesses the complement principle (if a + b = c, then c – a = b). In addition to finding answers for each problem, students were expected to explain their reasoning process. According to Barrody (1999), although students may provide correct answers to these problems, they may not see and appreciate the inverse relations between them (Baroody, 1999).

It should be acknowledged that this questionnaire only contained four items in part due to the consideration that the participants were only K-3 students of whose attention span are limited. Although these tasks were selected from the existing research, we must caution the lack of sufficient triangulation among these items and thus not overgeneralized. However, given the relatively large sample, our preliminary findings from this study may be expected to shed light on classroom instruction and follow-up research.

**Data Coding and Analysis**

Students’ responses (K-3) to each question were coded first for correctness. This process was straightforward. For reliability checking, we randomly selected 10% of student responses in each grade and the reliability reached 100%. The percentages of student correctness were summarized by tasks and by grades. To identify the differences across grades and among tasks, the one-way ANOVA along with the Bofferroni post hoc tests were calculated.

Next, we analyzed students’ explanations for each question. To develop codes, we selected 10% responses from each grade and both authors coded them independently with coding difficulties documented. Next, the two authors came together to compare their codes and discuss the coding difficulties and disagreements. This was a long-lasting process, which enabled the ongoing refinement of codes. Overall, students’ explanations were classified into three levels—0, 1, and 2—indicating no-, partial-, and full-understanding of inverse relations, respectively.

A closer inspection of the patterns of student explanations within each level revealed further categories. In particular, Level 0 (no-understanding) was subcategorized into two levels: (0a) no explanation, and (0b) wrong/irrelevant explanation. Level 2 (full-understanding) was broken down into two subcategories: (2a) full explanations involving concrete aids, and (2b) full explanations at an abstract level. Detailed examples are provided in Results.

The identification of subcategories for partial understanding (level 1) was most extensive, which underwent several rounds of analysis (e.g., combining existing categories, adding new codes). Given that part-whole schema is a key to learning inverse relations, we coded students’ explanations that did not involve inverse relation but showed part-whole relation (e.g., a part-whole picture, a number sentence that shows part-whole relation) as evidence of partial understanding. An inspection of these explanations indicated three categories due to different representation uses: (a) part-whole picture only, (b) part-whole picture and number sentence, and (c) number sentence only. Within each category, we found that students may or may not be able to identify the unknown quantity. We used “-”sign, no sign, and “+”sign to indicate students’ responses with “incorrect answer,” “no answer indicated,” and “correct answer,” respectively. As a result, we obtained a 3 × 3 subcategory matrix (see Table 1), resulting in 9 sublevels of partial understanding. Detailed examples are provided in Results.

Table 1. *Subcategories of Partial Understanding of Inverse Relations.*

|  |  |  |  |
| --- | --- | --- | --- |
|  | Incorrect answer | No answer indicated | Correct answer |
| Part-whole picture | 1a- | 1a | 1a+ |
| Part-whole picture and number sentence | 1b- | 1b | 1b+ |
| Number sentence only | 1c- | 1c | 1c+ |

**Results**

**Correctness of Student Response**

Students’ overall correctness is presented in Figure 1, summarized by tasks (left) and by grades (right), respectively.

|  |  |
| --- | --- |
|  |  |

*Figure 1.* Percentage of student correctness to all tasks.

As indicated by Figure 1 (left), while the computation accuracy did not exceed 85% for each task across grades, there is a growing pattern over time. The one-way ANOVA test shows that the overall change across grades for each task is significant (*P* = 0.00 for each task), *FQ1* (3, 277) = 12.379, *pQ1* = .000; *FQ2* (3, 277) = 5.141, *pQ2* = .002; *FQ3* (3, 277) = 20.373, *pQ3* = .000; and *FQ4* (3, 277) = 41.378, *pQ4* = .000. The Bofferroni post hoc test shows that for contextual tasks (Q1 and Q2, see Figure 1, left), significant changes include those differences between non-neighboring grades (e.g., G2-K, G3-G1) but not neighboring grades (e.g., G1-K, G2-G1). For non-contextual tasks (Q3 and Q4), significant changes include those from Kindergarten to the other grades; however, the changes between later grades (G2-G1 & G3-G2 for Q3; G3-G2 for Q4) are non-significant. This indicates that for the intended computation problems, there was a shift of students’ ability from Kindergarten to the other grades but not necessarily the later grades.

The second pattern is specifically related to the Kindergarteners (see Figure 1, right). These beginning learners performed much better on contextual tasks (Q1 and Q2, solid shaded) than non-contextual tasks (Q3 and Q4, pattern shaded). This is consistent with prior findings that children may reason upon contextual information even before they have mastered number manipulations.

The third pattern relates to the two types of inverse relations (see Figure 1, right). It seems that students at the beginning of schooling (K and G1) performed equally as well or slightly better on the three-term inversion principle (Q3) than on the two-term complement principle (Q4). When the grade level increases (G2 and G3), however, children seemed to demonstrate more fluency in the two-term complement principle that were frequently reported hard in the literature. This may additionally show that children continue to grow their computation accuracy over time. In brief, even though there is room for students to improve computation skills on inverse-based tasks, there is a pattern of linear growth across grades.

**Student Explanations to Each Question**

In comparison with their correctness in computation, students’ explanations appeared to be much poorer and fell into three levels: no-, partial-, and full-understanding. Table 2 presents typical examples using contextual-task (Q2) and non-textual task (Q4). Q2 described a decreasing situation but may be solved with addition; Q4 expected students to solve a subtraction fact using the related known addition fact. Both tasks called for students’ understanding of inverse relations. As indicated by Table 2, when students possessed no understanding of inverse relations (see Level 0 examples), they either provided no explanations (0a) or irrelevant explanations (0b, e.g., “I used fingers”). In contrast, when students possessed full understanding (see Level 2 examples), for Q2, they were able to add back what was taken-away “2” to find the original number of stamps (e.g., 6+2=8). For Q4, they highlighted the relationship between 9–6=3 and 6+3=9. These responses either involved concrete aids (2a, e.g., drawing arrows or a part-whole mat) or reached an abstract level (2b, e.g., stating that “it was a fact family”). However, some students’ understanding fell between these two levels (see Level 1 examples). On one hand, the responses did not demonstrate explicit understanding of inverse relations. For Q2, these students might have listed a subtraction sentence (8-6=2) that was directly aligned with the action of “decreasing;” for Q4, they might have computed 9-6 by crossing out 6 circles from 9. On the other hand, these responses showed understanding of the part-whole relations (e.g., drawing a part-whole picture), and thus classified as partial understanding. These responses also differed in the types of representations including using pictures only (1a), symbols only (1c) or a combination of both (1b).

Table 2. *Examples of Student Explanations that Show No and Full Understanding.*

|  |  |  |  |
| --- | --- | --- | --- |
| Level | Category | Q2 | Q4 |
| 0 - No understanding | 0(a) no explanation | photo 1.PNG |  |
| 0(b) wrong/irrelevant explanation | photo 1(1).PNG |  |
| 1 – Partial understanding | 1(a) Partial explanations at concrete level |  |  |
| 1(b) Partial explanations involving both concrete and abstract aids |  | N/A |
| 1(c) Partial explanations at abstract level |  | N/A |
| 2 - Full understanding | 2(a) full explanations involving concrete aids (e.g., pictures or verbal descriptions) | photo 1(5).PNG |  |
| 2(b) full explanations at an abstract level (e.g., using number sentences only). | photo 2(5).PNG |  |

A summary of the percentages of students’ explanations that fell into each category under each level is presented in Table 3.

Table 3. *Student Explanations to Each Question across Grades.*

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | Grade | Level-0 | | | Level-1 | | | | Level-2 | | | Missing data |
|  |  |  | 0a | 0b | Total | 1a | 1b | 1c | Total | 2a | 2b | Total |  |
| Contextual tasks | Q1 | K | 64 | 20 | 84% | 4 | 0 | 2 | 6% | 8 | 2 | 10% | - |
| 1 | 12 | 49 | 61% | 26 | 0 | 6 | 32% | 3 | 0 | 3% | 4% |
| 2 | 15 | 19 | 34% | 31 | 9 | 19 | 59% | 4 | 4 | 8% | - |
| 3 | 15 | 18 | 33% | 1 | 5 | 20 | 26% | 10 | 28 | 38% | 3% |
|  |  |  | | |  | | | |  | | |  |
| Q2 | K | 74 | 20 | 94% | 0 | 0 | 2 | 2% | 2 | 0 | 2% | 2% |
| 1 | 11 | 36 | 47% | 33 | 0 | 2 | 35% | 7 | 8 | 15% | 3% |
| 2 | 15 | 23 | 38% | 37 | 1 | 9 | 47% | 4 | 3 | 7% | 8% |
| 3 | 15 | 24 | 39% | 2 | 9 | 16 | 27% | 6 | 26 | 32% | 2% |
|  |  |  |  | | |  | | | |  | | |  |
| Non-contextual tasks | Q3 | K | 58 | 38 | 96% | 0 | 0 | 2 | 2% | 2 | 0 | 2% | - |
| 1 | 26 | 35 | 61% | 35 | 0 | 3 | 38% | 0 | 0 | 0% | 1% |
| 2 | 23 | 44 | 67% | 10 | 0 | 15 | 25% | 1 | 0 | 1% | 6% |
| 3 | 15 | 47 | 62% | 5 | 0 | 27 | 32% | 3 | 1 | 4% | 2% |
|  |  |  | | |  | | | |  | | |  |
| Q4 | K | 66 | 20 | 86% | 0 | 0 | 0 | 0% | 0 | 0 | 0% | 14% |
| 1 | 30 | 41 | 71% | 30 | 0 | 0 | 30% | 0 | 0 | 0% | - |
| 2 | 20 | 54 | 74% | 9 | 0 | 0 | 9% | 9 | 5 | 14% | 3% |
| 3 | 38 | 54 | 92% | 4 | 0 | 0 | 4% | 0 | 4 | 4% | - |

Note. The percentages were rounded to whole numbers, which may not total up to 100%.

As indicated by Table 3, most students provided explanations that showed no understanding of inverse relations. The highest percentage of full understanding occurred with third graders, which only reached 38%. Overall, students’ explained contextual-tasks (Q1 and Q2) better than non-contextual tasks (Q3 and Q4). While there were about 32–38% of third graders whose explanations of contextual tasks achieved full understanding, only 4% of them fully explained the computation tasks.

Even though the overall situation of student explanation was poor, across grades, there was a trend of growth in student explanations. First, at level-0 (no understanding) the Kindergarteners held the highest percentage for the first three tasks (84%, 94%, and 96%, respectively). Consistent with this observation, at level-2 (full understanding) the third graders held the highest percentage for the first three tasks (38%, 32%, 4% for the first three tasks, respectively). It is strange that for Q4, the third graders explained more poorly than the second graders (4% vs. 14%, respectively). This might be due to the fact that 92% of them provided level-0 explanations, including 54% wrong/irrelevant explanations. In addition, within Level-0, a comparison of students’ responses between levels 0a and 0b shows their effort on explanations when grade level increases. In kindergarten, there were more students who provided no explanations (0a) than those who provided wrong/irrelevant explanations (0b). This trend, however, was reversed for later grades (G1-3). Finally, the trend of student growth also appeared with students’ level-1 explanations. From kindergarten to the later grades there was an increasing number of students who demonstrate an understanding of part-whole relation (see Table 3).

The one-way ANOVA test indicates that there were significant differences between levels of explanations across grades except for the level-2 in Q3 (see Table 4).

Table 4. *One-way ANOVA test for differences in partial explanations at each level.*

|  |  |  |  |
| --- | --- | --- | --- |
|  | Df | F | Sig |
| Q1\_Level 0 | 3 | 16.753 | .000 |
| Q1\_level 1 | 3 | 15.431 | .000 |
| Q1\_level 2 | 3 | 18.096 | .000 |
| Q2\_Level 0 | 3 | 18.410 | .000 |
| Q2\_level 1 | 3 | 11.146 | .000 |
| Q2\_level 2 | 3 | 10.691 | .000 |
| Q3\_Level 0 | 3 | 7.464 | .000 |
| Q3\_level 1 | 3 | 7.714 | .000 |
| Q3\_level 2 | 3 | 1.124 | .340 |
| Q4\_Level 0 | 3 | 5.018 | .002 |
| Q4\_level 1 | 3 | 13.564 | .000 |
| Q4\_level 2 | 3 | 7.097 | .000 |

The Bofferroni post hoc test indicated differences between grade levels. The most interesting observation is that, even though there was no difference between many other grades and kindergarten at level-0 and level-2 explanations, when it comes to level 1 (partial understanding), those grades show significant differences from kindergarten. This suggests a closer inspection at students’ partial explanations (Level 1) that may bridge students’ no- and full-understanding.

**A Closer Inspection of Students’ Partial Explanations**

Zooming into students’ partial explanations at Level 1, we identified differences in students’ representation uses and unknown quantity recognition in their part-whole schema, which suggested possible paths to develop students’ understanding of inverse relations. For representation uses, it was found that students in grades 1 and 2 preferred using part-whole pictures to show the answers; however, students in grades 2 and 3 (especially in grade 3) tended to use number sentences to show their thinking (see Table 3). In addition, students’ explanations differed in recognizing the unknown quantity in the part-whole schema. Table 5 illustrates typical examples using Q1 and Q3. For instance, with regard to Q1, students’ responses at 1a-, 1b-, and 1c- contained part-whole pictures and/or correct number sentences; yet, the wrong number “8” was marked as the answer. In their responses at 1a, 1b, and 1c, student did not mark any answer. All these were likely due to their non-clarity or implicit awareness of the unknown quantity in the part-whole schema, which hindered their ability to solve problems.

Table 5. *Examples of Student Explanations that Show Partial Understanding*

|  |  |  |  |
| --- | --- | --- | --- |
| Category | | Q1: (Unknown quantity/correct answer is 6) | Q3: (Unknown quantity/correct answer is 5) |
| 1a- | Part-whole picture with incorrect answer | photo 1(2).PNG |  |
| 1a | Part-whole picture with no answer | photo 1.PNG | N/A |
| 1a+ | Part-whole picture with correct answer | photo 2(2).PNG |  |
| 1b- | Part-whole picture and number sentence with incorrect answer | Macintosh HD:Users:abbeyauxter:Library:Containers:com.apple.mail:Data:Library:Mail Downloads:8D85E3FC-6643-4C52-8453-C3A29CD2FD56:photo.PNG | N/A |
| 1b | Part-whole picture and number sentence with no answer | photo.PNG | N/A |
| 1b+ | Part-whole picture and number sentence with correct answer | 171.PNG | N/A |
| 1c- | Number sentence only with incorrect answer | 120.PNG | N/A |
| 1c | Number sentence only with no answer | photo.PNG | N/A |
| 1c+ | Number sentence only with correct answer | . photo 1(4).PNG |  |

Students’ overall situation in unknown quantity recognition in part-whole relation is summarized in Table 6.

Table 6. *Percentage of Students’ Non-clarity of Unknown Quantity*

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Partial Understanding |  |  | 1a |  |  |  | 1b |  |  |  | 1c |  |  |
|  | Grade | 1a-  (%) | 1a  (%) | 1a+  (%) | **Total** | 1b-  (%) | 1b  (%) | 1b+  (%) | **Total** | 1c-  (%) | 1c  (%) | 1c+  (%) | **Total** |
| Q1 | K | 0 | 2 | 2 | **4** | 0 | 0 | 0 | **0** | 0 | 0 | 2 | **2** |
|  | 1 | 7 | 1 | 18 | **26** | 0 | 0 | 0 | **0** | 1 | 5 | 0 | **7** |
|  | 2 | 3 | 23 | 5 | **30** | 1 | 3 | 5 | **9** | 0 | 8 | 11 | **19** |
|  | 3 | 0 | 1 | 0 | **1** | 0 | 5 | 0 | **5** | 0 | 14 | 6 | **21** |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| Q2 | K | 0 | 0 | 0 | **0** | 0 | 0 | 0 | **0** | 0 | 0 | 2 | **2** |
|  | 1 | 4 | 1 | 28 | **34** | 0 | 0 | 0 | **0** | 1 | 1 | 0 | **3** |
|  | 2 | 8 | 24 | 5 | **37** | 0 | 0 | 1 | **1** | 0 | 3 | 6 | **9** |
|  | 3 | 1 | 1 | 0 | **3** | 0 | 9 | 0 | **9** | 1 | 10 | 5 | **17** |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| Q3 | K | 0 | 0 | 0 | **0** | 0 | 0 | 0 | **0** | 0 | 0 | 2 | **2** |
|  | 1 | 3 | 0 | 32 | **35** | 0 | 0 | 0 | **0** | 0 | 1 | 1 | **3** |
|  | 2 | 0 | 0 | 10 | **10** | 0 | 0 | 0 | **0** | 0 | 0 | 15 | **15** |
|  | 3 | 0 | 0 | 4 | **4** | 0 | 0 | 0 | **0** | 0 | 0 | 27 | **27** |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| Q4 | K | 0 | 0 | 0 | **0** | 0 | 0 | 0 | **0** | 0 | 0 | 0 | **0** |
|  | 1 | 3 | 0 | 27 | **30** | 0 | 0 | 0 | **0** | 0 | 0 | 0 | **0** |
|  | 2 | 0 | 0 | 9 | **9** | 0 | 0 | 0 | **0** | 0 | 0 | 0 | **0** |
|  | 3 | 0 | 0 | 4 | **4** | 0 | 0 | 0 | **0** | 0 | 0 | 0 | **0** |

As indicated by Table 6, the first pattern is that students’ difficulties with unknown quantity recognition were more apparent with the contextual tasks. This made sense given the unknown quantity in non-contextual tasks was often already marked. However, for contextual tasks (Q1 and Q2), students across all grades appeared to have difficulties. For instance, with regard to Q1, when students drew the part-whole pictures to solve this problem, 23% of second graders did not mark the unknown quantity (1a). It was uncertain whether these students could identify the unknown quantity because there were cases that students could not (e.g., 7% first graders). When students enlisted a number sentence to solve this problem their chance of marking a wrong number as the unknown was lessened. Yet, there were still 8% of second graders and 14% of third graders that could not identify it. The situation was similar in Q2.

**Discussion**

Previous studies indicate that students who come to elementary school with informal understanding of inverse relations generally “lack” formal understanding of this mathematical relation (Baroody, 1987, 1999; Baroody et al., 1983; Bisanz & LeFevre, 1990, 1992; De Smedt et al., 2010; Resnick, 1983; Riley et al., 1983). This study takes a further step beyond “lack” to explore what students may “have” in their existing understanding, which may afford opportunities for classroom instruction. To access students’ understanding in a relatively complete fashion, our assessments involve both contextual and non-contextual tasks of inversion and complement principles, requiring both computation and explanation skills (Bisanz & LeFevre, 1992; Bisanz et al., 2009). Results show that students generally perform better in computation than explanation even though some students’ still could not compute in third grade. Most students who obtained correct computational answers did not utilize inverse relations to solve these tasks. Given that additive inverses is one of the first fundamental ideas in early grades (Baroody, 1987; Carpenter, Franke, & Levi, 2003) and has been emphasized by the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), the status quo of students’ understanding should draw immediate attention.

Our findings suggest that students’ computation skills may grow naturally across grades, likely due to opportunities of repeated practices (Canobi, 2005). However, their explanations of inverse relations may not grow linearly. This finding is consistent with many others that reported students’ difficulties in inverse relations, especially with the complement principle (e.g., Baroody, 1983, 1999; Baroody et al., 1983; De Smedt et al., 2010; Riley et al., 1983), which calls for meaningful and explicit instruction on inverse relations. This study contributes to the literature by identifying opportunities for instruction based on students’ existing strategies. As indicated by their partial explanations, many students do possess part-whole schema, which is the key to understanding inverse relations (Piaget, 1952; Resnick, 1992). However, these students’ part-whole schema appears to be non-sophisticated as it is mainly limited to direct thinking. For instance, with regard to an increasing situation, students tend to use “part + part = whole.” This is only a portion of the part-whole structure. In fact, our findings regarding students’ detailed levels of part-whole understanding in terms of representation uses and unknown quantity recognition display a rich picture of opportunities to teach.

With regard to representations, we found that students’ understanding of part-whole relation moves from concrete to abstract across grades. This observation is more apparent with contextual tasks. As reported, younger students preferred drawing part-whole pictures while students in later grades listed number sentences more frequently. The difference between using concrete pictures and abstract number sentences indicates that students’ understanding of part-whole relation is not an all-or-nothing phenomenon (Bisanz et al., 2009). In order to develop meaningful part-whole understanding, teachers may first grasp concrete aids (e.g., part-whole pictures) that come more naturally for students. Indeed, students’ improved performance in contextual tasks versus non-contextual tasks, especially in kindergarten, supports this assumption. These findings challenge existing instruction that focused more on number manipulation when teaching additive inverses (Author, in press). Given that students have the ability to reason abstractly and abstract thinking is an ultimate goal of mathematics education (Bruner, 1960), teachers should help students connect concrete and abstract representations (e.g., number sentences) so as to promote explicit understanding. For instance, if students demonstrated part-whole understanding using the part-whole picture, a teacher may guide them to generate a corresponding number sentence. More importantly, the teacher should prompt students to think reversely, linking one part of the part-whole structure (e.g., part + part = whole) to the other part (e.g., whole – part = part), based on both concrete and abstract representations.

In addition to the progression of representation uses, students in this study differed in their ability to recognize the unknown quantity in a part-whole schema. For instance, students used either direct thinking (6+2=8) or reversed thinking (8-2=6) to solve Q1. While multiple solutions should be encouraged, students with the direct thinking should recognize that “6” indeed refers to the unknown in Q1. As such, the number sentence should be listed as □ + 2 = 8 for clarity. This is why 8-2 = □ can be used solve the same problem, which shows an understanding of inverse relation. However, this seemingly trivial point is often neglected. For instance, some existing textbooks simply suggest two number sentences to solve the same problem without highlighting the unknown quantity (Author, in press). Overall, our findings about students’ varied level of part-whole understanding illustrate a picture about what students may “have” in their existing knowledge base of inverse relations when they demonstrate a “lack” of understanding. Teachers and teacher educators should build on these existing opportunities to prompt students one-step further so as to develop their understanding of inverse relations.

The current study has a few limitations, suggesting future research directions. First, the assessment tasks are limited. Therefore, findings should not be overgeneralized. For instance, even though we noticed some patterns related to students’ learning of both types of inverse relations (three-term inversion principle, and two-term complement principle), the sequence for learning both two principles cannot reach a conclusion. Nevertheless, our findings may contribute to the relevant body of literature (Nunes et al., 2009; Gilmore & Spelke, 2008) and suggest necessity for future exploration. Second, our findings are based on paper and pencil assessments. Even though students shared their thinking through explanation, their explanations may not necessarily represent their understanding. As reported, there is a strange phenomenon that third graders performed more poorly than the second graders. Is it possible that these tasks were too simple that third graders lost interests to explain them? Or is it possible that these third graders really lacked understanding of inverse relation? Future studies may conduct student interviews to identify the possible reasons. Regardless of the limitations of study, findings about the status quo of students’ understanding add to the existing literature and shed light on classroom opportunities for teaching and learning inverse relations.

**References**

Author (In press).

Baroody, A. J. (1987). *Children’s mathematical thinking: A developmental framework for preschool, primary, and special education teachers.* New York: NY: Teacher College Press.

Baroody, A. J. (1999). Children’s relational knowledge of addition and subtraction. *Cognition and Instruction, 17*, 137-175.

Baroody, A., Ginsburg, H., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education, 14,* 156-168.

Baroody, A. J., Torbeyns, J., & Verschaffel, L. (2009). Young children's understanding and application of subtraction-related principles: Introduction. *Mathematics Thinking and Learning, 11,* 2-9.

Bisanz, J., & LeFevre, J. (1990). Strategic and nonstrategic processing in the development of mathematical cognition. In D. Bjorklund (Ed.), *Children’s strategies: Contemporary views of cognitive development.* Hillsdale, NJ: Erlbaum.

Bisanz, J., & LeFevre, J. (1992). Understanding elementary mathematics. In J. I. D. Campbell (Ed.), *The nature and origins of mathematical skills* (pp. 113–136). Amsterdam: North Holland, Elsevier Science.

Bisanz, J., Watchorn, R., Piatt, C., & Sherman, J. (2009). On “understanding” children’s developing use of inversion. *Mathematical Thinking and Learning, 11,* 10–24.

Briars, D. J., & Larkin, J. H. (1984). An integrated model of skill in solving elementary word problems. *Cognition and Instruction, 1*, 245-296.

Bruner, J. S. (1960). The process of education. Cambridge, MA: Harvard University Press.

Canobi, K. H. (2005). Children’s profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology, 92,* 220–246.

Carpenter, T. P., Franke, L. P., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic & algebra in elementary school.* Portsmouth, NH: Heinemann.

De Smedt, B., Torbeyns, J., Stassens, N., Ghesquière, P., & Verschaffel, L. (2010). Frequency, efficiency and flexibility of indirect addition in two learning environments. *Learning and Instruction, 20,* 205-215.

Ding, M. (2012). Early algebra in Chinese elementary mathematics textbooks: The case of inverse relations. In B. Sriraman, J. Cai, K. Lee, L. Fan, Y. Shimuzu, L. C. Sam, & K. Subramanium (Eds.), *The first sourcebook on Asian research in mathematics education: China, Korea, Singapore, Japan, Malaysia, & India*. Charlotte, NC: Information Age Publishing.

Gilmore, C. K., & Bryant, P. (2008). Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill. *British Journal of Developmental Psychology, 26*, 301–316.

Gilmore, C. K., & Spelke, E. S. (2008). Children’s understanding of the relationship between addition and subtraction. *Cognition*, *107*, 932–945.

National Governors Association Center for Best Practices (NGA Center) & Council of Chief State School Officers (CCSSO) (2010). *Common core state standards for mathematics.* Washington, D.C.: Authors.

Nunes, T., Bryant, P., & Watson, A. (2009). *Key understandings in mathematics learning:* A report to the Nuffield Foundation. London: Nuffield Foundation.

Piaget, J. (1952). *The child’s conception of number*. London: Routledge and Kegan Paul.

Resnick, L. B. (1983). A developmental theory of number understanding. In H.P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109–151). New York: Academic Press.

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologists, 44,* 163-169.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic* *for mathematics teaching* (Vol.19, pp. 275–323). Hillsdale, NJ: Lawrence Erlbaum.

Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children’s problem-solving ability in arithmetic. In H. Ginsberg (Ed.), *The development of mathematical thinking* (pp.153-196). New York: Academic Press.

Sophian, C., Harley, H., & Martin, C. S. M. (1995). Relational and representational aspects of early number development. *Cognition and Instruction, 13,* 253-268.

Sophian, C., & McGorgray, P. (1994). Part-whole knowledge and early arithmetic problem solving. *Cognition and Instruction, 12*, 3-33.

Sophian, C., & Vong, K. I. (1995). The parts and wholes of arithmetic story problems: Developing knowledge in the preschool years. *Cognition and Instruction, 13,* 469-477.

Torbeyns, J., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2009). Solving subtractions adaptively by means of indirect addition: Influence of task, subject, and instructional factors. *Mediterrannean Journal for Research in* *Mathematics Education*, *8*(2), 1–30.

Zhou, Z., & Peverly, S. (2005). The teaching addition and subtraction to first graders: A Chinese perspective. *Psychology in the Schools, 42*, 259-272.