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A Comparative Analysis of Inverse Relations in U.S. and Chinese Elementary Mathematics Textbooks

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Abstract

This study, focusing on inverse relations, examines how representative U.S. and Chinese elementary textbooks may provide opportunities to learn fundamental mathematical ideas. Findings from this study indicate that both of the U.S. textbook series (grades K-6) in comparison to the Chinese textbook samples (grades 1-6), presented more instances of inverse relations, while also containing more unique types of problems; yet, the Chinese textbooks provided more opportunities supporting meaningful and explicit learning of inverse relations. In particular, before presenting corresponding practice problems, Chinese textbooks contextualized worked examples of inverse relations in real-world situations to aid in sense making of computational or checking procedures. The Chinese worked examples also differed in representation uses and made connections between concrete and abstract through concreteness fading. Finally, the Chinese textbooks spaced learning over time through a systematic emphasis on structural relations, especially inverse quantities relationships. Findings based on textbook presentations of both countries suggest that textbook designers and classroom instructors consider these approaches so as to support meaningful and explicit learning of fundamental mathematical ideas in elementary school.

Key words: Opportunities to Learn, Inverse Relations, Textbook Analysis, Comparison Study

It is widely agreed that students should learn and understand fundamental mathematical ideas such as the basic concepts, relations, properties, and structures that may transcend over contexts (Bruner, 1960; Common Core State Standards Initiative [CCSSI], 2010). International tests, such as the Trends in International Mathematics and Science Study (TIMSS) or the Programme for International Student Assessment (PISA), have consistently revealed that many students have difficulties with understanding and utilizing fundamental mathematical ideas. Students' difficulties may be largely attributed to the quality of their learning environments including textbooks that provide opportunities to learn (Thompson, Kaur, Koyama, & Bleiler, 2013). This study, focusing on the fundamental mathematical idea of inverse relations, examines learning opportunities presented in representative U.S. and Chinese elementary textbooks. Given that inverse relations are a critical concept in which students often find hard to grasp (CCSSI, 2010; Nunes, Bryant, & Watson, 2009), it is expected that this study will contribute to the development of students' understanding of this concept and beyond.

Review of Literature

Mathematical Significance of Inverse Relations

Inverse relations are important relationships throughout mathematics. For instance, there exist functions and inverse functions in algebra and differentiation and integration in calculus. Historically, mathematicians developed logarithms by finding the inverse of exponentials (Hobson, 2012). As such, it is important in all aspects of mathematics to understand and be able to reason with inverse relations (Baroody, Torbeyns, & Verschaffel, 2009; Carpenter, Franke, & Levi, 2003; Piaget, 1952; Resnick, 1983, 1992; Nunes et al., 2009). Because of this mathematical

significance, the Common Core State Standards for Mathematics in the U.S. have emphasized that inverse relations should be learned in each grade of elementary school (CCSSI, 2010).

In elementary school, inverse relations refer to the relationships between addition and subtraction (additive inverses) and between multiplication and division (multiplicative inverses; Baroody et al., 2009; Carpenter, et al., 2003; Nunes et al., 2009). Resent research on inverse relations (Baroody et al., 2009; Gilmore & Bryant, 2008) pointed out that arithmetic inverse relations mainly included two closely related but different principles: the two-term *complement principle* (e.g., if $a + b = c$, then $c - b = a$; if $a \times b = c$, then $c \div b = a$) and the three-term *inversion principle* (*e.g.,* $a + b - b = a$; $a \times b \div b = a$). Due to the desire for in-depth research within a limited scope, the current study focuses only on the two-term complement principle (simply called "inverse relations" hereafter).

An understanding of inverse relations is necessary to fully comprehend the four basic arithmetic operations and to develop essential algebraic reasoning skills (Baroody, 1987; Nunes et al., 2009; Vergnaud, 1988). Carpenter et al. (2003) suggested two ways to demonstrate inverse relations: fact family (e.g., 3 + 4 = 7, 4 + 3 = 7, 7 – 3 = 4, and 7 – 4 = 3) or a group of related word problems, the solution of which can form a fact family (termed as "inverse word problems" in this study). Students also need to use inverse relations to flexibly compute [e.g., 81 - 79 = ( ) can be thought of 79 + ( ) = 81, Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009], check computations (e.g., verifying that 35 – 17 = 18 by computing 18 + 17 = 35, Baroody, 1987), and solve hard word problems such as initial unknown change problems in which a quantity may increase but the solution may involve subtraction (e.g., Ali had some Chinese stamps in his collection and his grandfather gave him 2; now he has 8; how many stamps did he

have before his grandfather gave him the 2 stamps? Nunes et al., 2009). Moreover, an understanding of inverse relations at a structural level could contribute to students' algebraic thinking such as equation solving (e.g., if $a + b = c$, then $a = c - b$; if $a \times b = c$, then $a = c \div b$, Carpenter et al., 2003). Longitudinal studies also have shown that second graders' performance on inverse tasks significantly predicted their algebraic achievement when they were in eleventh grade (Stern, 2005).

Difficulties of Student Learning with Inverse Relations

Although it is important in all aspects of mathematics to understand and be able to reason with inverse relations, prior research has demonstrated that elementary school children generally lack a formal understanding of this relation (Baroody, Ginsburg & Waxman, 1983; De Smedt, Torbeyns, Stassens, Ghesquière, &Verschaffel, 2010; Resnick, 1983). For example, Baroody et al. (1983) found that about 61% of the sampled first- and second-graders could not use addition to solve subtraction problems (e.g., using 3 + 4 = 7 to solve 7 – 4). In a later study (Baroody, 1999), it was found that the K–1 graders who were trained to solve subtraction using addition did not acknowledge this relation. In addition, Baroody (1987) observed that a child practiced a whole page of problems using addition to check subtraction but could not spontaneously apply this procedure in new tasks. Recently, Torbeyns et al. (2009) in their study with Belgium children, found that when solving multi-digit subtraction problems with small differences (e.g., 81 – 79), most second-to-fourth graders could not call upon an effective strategy based on inverse relations (e.g., 79 + ? = 81), even though it had been explicitly taught. Similar findings were obtained from third graders who had experienced six weeks of instruction that aimed to develop inverse understanding (De Smedt et al., 2010).

With regard to the inverse relation between multiplication and division, children exhibit a parallel lack of understanding. Grossi (1985) reported that students who used $P \div M = N$ to check whether P is a multiple of M failed to recognize that obtaining $M \cdot N = P$ should produce the same conclusion (as cited in Vergnaud, 1988). Likewise, Thompson (1994) reported elementary students possessed difficulties in multiplicative reasoning, especially in the development of the concept of speed (including the relations between speed, time, and distance) and its relation to the concept of rate. Such conceptual difficulties extend beyond integers to include more complex numbers such as fractions, decimals, and proportions (Greer, 1994).

The above findings about elementary students' lack of formal understanding of inverse relations sharply contradict the reports on preschoolers' (from 3 to 6 years old) surprising level of informal understanding of this relation (Gilmore & Bryant, 2008; Klein & Bisanz, 2000; Sherman & Bisanz, 2007; Sophian & McGorgray, 1994). For instance, children could provide correct directional responses to addition and subtraction story problems (e.g., for an addition story problem, children provided a "bigger" number than the given quantities; Sophian & McGorgray, 1994) and demonstrate inverse understanding when "approximate numbers" were involved (Gilmore & Bryant, 2008). These findings may be explained by Resnick's (1992) assertion that children possessed protoquantitative schemas of "increasing-decreasing" and "part-whole," which are the keys to understanding inverse relations. Why then, is there a gap between students' informal and formal understanding? Why is inverse relation, a ubiquitous mathematical concept, so hard to learn? These questions call for reconsideration of the existing learning environments that students are exposed to with inverse relations.

Limitations of Existing Learning Environments with Inverse Relations

The term “learning environment” in this study refers to situational characteristics and cultural factors (e.g., instructional methods, textbooks) that influence students’ learning (Lizzio, Wilson & Simons, 2002). An analysis of literature reveals two limitations in the existing learning environments that hinder students’ opportunities to learn inverse relations. First, instruction on inverse relations tends to focus on number manipulations without using concrete contexts to activate students’ informal knowledge (e.g., Baroody, 1999; De Smedt et al., 2010; Torbeyns et al., 2009). Under this situation, students may obtain only inert knowledge and generate mistakes such as $7 \div 35 = 5$, and $5 \div 35 = 7$ (Ding & Carlson, 2013). As such, some researchers argued for “relational” (as opposed to “numeral”) calculation during which, concrete contexts were used for quantitative reasoning (Nunes, Bryant, Evans, Bell, & Barros, 2012; Thompson, 1994). Nunes et al. found that when students made sense of inverse relations, they outperformed their peers in difficult story problems that demanded inverse understanding (e.g., initial unknown change problem), which was also shown to be transferrable to numeral contexts.

The second limitation of current learning environments is procedural focus on strategies rather than the underlying principles (e.g., De Smedt et al., 2010; Torbeyns et al., 2009). In Baroody’s (1987) study, the teacher stressed using addition to check subtraction but did not help the student understand why this strategy worked. Similarly, the graduate assistant in Baroody’s (1999) study only trained students to think in the following way: “5 take away 3 makes what?” can be understood as “3 added to what makes 5?” It was not made clear to the students why they could think in this way. In addition, the Belgium textbook in Torbeyns et al. (2009) presented procedures such as drawing a little arrow from the subtrahend to the minuend without further explanation. Focusing on procedures rather than the underlying principles fails to facilitate

students' automatic use of the strategies (Torybens et al., 2009), possibly because the part-whole schema is loosely connected in children's minds (Baroody, 1999).

Students' difficulties resulting from the existing learning environments call for a better support for students' learning of inverse relations. How to create a supportive learning environment, however, has not been established by the current body of research (National Mathematics Advisory Panel, 2008). This study aims to narrow the research gap by exploring alternative opportunities from the perspective of textbook comparisons between the U.S. and China.

Exploring Opportunities to Learn: Why International Textbook Comparisons?

Most teachers in the U.S. and China use textbooks as a main resource for mathematics teaching. It was reported that Chinese mathematics teachers study textbooks frequently and implement them with fidelity (Ding, Li, Li, & Gu, 2013). Likewise, Malzahn (2013) reported that 85% of grade K-5 mathematics classes in the U.S. used commercially published textbooks, particularly for guidance of overall structure and content emphasis. Even though teachers implement textbooks in classrooms to different extents, textbooks as intended curricula provide opportunities to learn, thus serving as a critical component of students' learning environments (National Research Council, 1999; Thompson et al., 2013).

Prior studies based on the TIMSS's curriculum data (e.g., National Research Council, 1999; Schmidt, Wang, & McKnight, 2005) revealed that U.S. textbooks generally lacked coherence and focus, in comparison to those mathematically high-achieving countries. International textbook comparisons (e.g., Cai, Lew, Morris, Moyer, Ng & Schmittau, 2005; Murata, 2008; Stigler, Fuson, Ham, & Kim, 1986) have brought many alternatives and insights

to the field for improving the teaching and learning of fundamental mathematical ideas. For instance, Ding and colleagues compared U.S. and Chinese textbook presentations on the distributive property (Ding & Li, 2010), the associative property (Ding, Li, Capraro, & Capraro, 2012), and the concept of equivalence denoted by the "=" (Li, Ding, Capraro & Capraro, 2008). It was found that Chinese textbooks uniquely situated the initial learning opportunities of a fundamental idea in concrete situations for sense-making, which were gradually faded into increasingly abstract and challenging contexts in practice problems. The above approach was well aligned with the cognitive and educational research recommendations on teaching fundamental concepts and principles (Pashler et al., 2007); however, it remains unknown whether this approach exists when presenting inverse relations. In addition, Zhou and Peverly (2005) studied Chinese first grade textbook's presentation and briefly reported that additive inverse was set up as one of the learning goals, which was achieved through teaching composing-decomposing and the part-whole relationships. Since the mastery of fundamental ideas takes time (Baroody, 1999), it remains unknown how inverse relations may be presented beyond first grade. Moreover, prior studies on inverse relations overall focused more on additive inverses as opposed to the equally important multiplicative inverses (Nunes et al, 2009). To address the above research gaps, this study compares typical U.S. and Chinese textbooks on both additive and multiplicative inverses across elementary grades.

Conceptual Framework for Textbook Examination

To examine textbook presentations of inverse relations, this study uses a three-aspect framework drawn from the *Institute of Education Sciences* (IES) recommendations for organizing instruction and study to improve student learning (Pashler et al., 2007). The first

aspect is interweaving worked examples and practice problems. Worked examples (solutions provided) play a critical role in student initial learning, which determines the likelihood of later transfer (Chi & VanLehn, 2012). This is because examples help students acquire schemas for solving new problems (Sweller & Cooper, 1985). Examples with greater variability can better support the encoding and extracting of the abstract principles underlying these problems (Renkl, Atkinson, Maier, & Staley, 2002). In addition, worked examples were found to be more effective when interweaved with practice problems (no solutions provided, Renkl et al., 2002).

The second aspect is making connections between concrete and abstract representations. Abstract representations in this study refer to the use of symbols to represent mathematical ideas, which eliminates detailed perceptual properties and is often arbitrarily linked to referents (McNeil & Fyfe, 2012). Concrete representations refer to using physical objects (e.g., manipulatives) or visual images (e.g., diagrams) to represent mathematical ideas, or the contextualization of mathematical ideas in real-world situations (e.g., word problem contexts, Ding & Li, 2014). We consider word problems as concrete because the real-world contexts provided by such problems "have the potential to offer memorable imagery that can act as a touchstone for teachers and learners in building and discussing abstract concepts" (Gerofsky, 2009, p. 36). Empirical studies also demonstrated that students were more capable of solving simple algebra word problems than the mathematically equivalent equations (Koedinger & Nathan, 2004). During initial learning, concrete representations may activate students' familiar experiences for sense-making (Resnick & Omanson, 1987); however, concrete representations often contain irrelevant information that may blind students from seeing the underlying principles. This may hinder later transfer (Kaminski, Sloutsky, & Heckler, 2008). As such, it was

suggested to fade the concreteness into abstract representations to promote learning and transfer (also called "concreteness fading," Goldstone & Son, 2005).

The final aspect is spacing learning over time. The course of developing fundamental ideas may span several years (Vergnaud, 1988). In order for students to retain information, key concepts should be reviewed after the initial teaching, but not in the form of a simple repetition (National Academy of Education, 2009). Based on the above conceptual framework, this study examines opportunities to learn inverse relations presented in representative U.S. and Chinese textbooks centering on the following questions: (1) What is the degree of coverage of inverse relations in each textbook series? (2) How are worked examples and practice problems interweaved to support the learning of inverse relations? (3) How are concrete and abstract representations connected to present inverse relations in each textbook series? And (4) How is the learning of inverse relations spaced over time in each textbook series?

Method

Textbook Selection

One of the three main Chinese mathematics curricula (Grades 1-6), *Jiang Su Educational Press textbook* (JSEP, Su & Wang, 2005), was chosen for this study. Given that the Chinese educational system is centralized and all textbooks were developed based on the new Chinese curriculum standards (Ministry of Education, 2001), there was little qualitative difference among the three main textbook series (Ding & Li, 2010; Li et al., 2008). For comparisons, we chose two widely used U.S. textbook series (Grades K-6): *Houghton Mifflin* (HM, Greenes et al., 2005) and *Everyday Mathematics* (EM, University of Chicago School Mathematics Project, 2007). While HM is a commercially published curriculum, EM is a curriculum supported by the National

Science Foundation (NSF). Recent survey data on the frequency-of-usage of textbooks show that, EM was one of the three most frequently used textbooks in Grades K-5 while HM was one of the three most frequently used textbooks in Grades 3-5 during the 2011-12 school year (Dossey, Halvorsen, & McCrone, 2012; Malzahn, 2013).

For all three textbook series, curriculum materials specifically designed for teaching were coded because these materials suggested the minimum opportunities a teacher may use in classrooms. For the U.S. textbook series, the teacher editions of mathematics textbooks were examined. According to Dossey et al. (2012), the teacher editions were the central materials that U.S. teachers heavily relied on. These materials contained student textbook pages along with the detailed teaching steps (e.g., introduction, develop, practice, and assess and close). The difference between HM and EM was that, a volume of HM teacher edition contained all textbook pages of a student edition (e.g., p.89), together with the corresponding explanation pages (e.g., p.89A, p.89B, p.89C). However, an EM teacher edition contained the selected pages of various student resources (e.g., Math Journal, Assessment Handbook, Math Matters etc.), with none named as "student edition." It should be noted that during the suggested teaching processes, both U.S. teacher editions offered additional tasks beyond the student edition/resources.

The Chinese textbook series did not differentiate between the teacher and student editions. Rather, teachers and students would use the same textbooks. For each textbook, there was a corresponding teacher manual explaining tasks design but not the detailed teaching steps. The Chinese teacher manuals also did not suggest extra tasks beyond the textbooks. For equivalency, we coded Chinese textbooks plus teacher manuals, which were equivalent to the U.S. teacher editions. For all of the textbook series, there were two volumes for each grade except EM

kindergarten. As such, a total of 12 volumes of Chinese textbooks and 27 volumes of U.S. textbooks were examined for this study.

Instance Coding

All of the problems in each textbook series, including the worked examples and practice problems, were coded. Based on the literature, a worked example was coded as an instance of inverse relations if it involved any of the following:

- A group of related facts [e.g., 4 + 6 = 10, 10 – 6 = 4, HM, Grade 1, p.465] or a fact family (e.g., 4 + 6 = 10, 6 + 4 = 10, 10 – 4 = 6, 10 – 6 = 4, EM, Grade 1, p.554),
- Computing using inverse relations (e.g., to solve 28 ÷ 7 = 4, one may think of 4 × 7 = 28, China, Grade 2, p.65) or checking using inverse relations (e.g., to check if 439 – 275 = 164 is correct, one may think of 164 + 275 = 439, HM, Grade 2, p.631) ,
- A group of inverse word problems [e.g., (1) There are 7 groups of children playing. Each group has 5 children. How many children are there? (2) There are 35 children playing. If 5 children form one group, how many groups can be formed? (3) There are 35 children playing. If they are divided evenly into 7 groups, how many children will be in each group?, China, Grade 2, p. 69],
- One problem solved with inverse operations [e.g., to solve "Lucy had 7 pennies. She lost 3 pennies. How many pennies does Lucy have left?" one may use 7 – 3 = 4 and 3 + □ = 7, EM, Grade 1, p.213], and
- Solving an algebraic equation (9 + x = 15) or filling in the missing numbers using inverse relations [e.g., Solving 9 + ( ) = 15 by think of 15 – 9, HM, Grade 2, p. 65].

In particular, if a worked example in the textbooks contained a pair/group of inverse operations, it was considered as one instance. If a worked example contained several different pairs/groups of inverse operations (e.g., $n = 4$), it would be coded as several instances (e.g., $n = 4$).

With regard to practice problems, if there were clear suggestions for using inverse relations to solve a set of problems, all of them would be coded as instances. If there was no such request in the textbooks, the coders would then refer back to the corresponding worked examples. If the worked example contained various solutions that went beyond the inverse-based strategy and the practice problem did not specifically require using inverse relations, such a practice problem would not be coded due to uncertainty. With the above operational definitions, the Chinese and U.S. textbook pages were coded; however, some difficulties did occur especially for cases that were different from the literature.

Coding Challenges. The first coding challenge was related to number composing and decomposing in the Chinese first grade textbooks (e.g., 4 is decomposed into 3 and 1; 3 and 1 are composed into 4, see Figure 1a). This was a specific chapter presented before formal introduction of addition and subtraction. Such a task indicated an inverse relation; yet, no operations were involved. After referring to the Chinese teacher manual, these tasks were coded as instances because the teacher manual pointed out that these tasks were intended to develop students' understanding of inverse relations. In Figure 1a, the worked example was coded as 3 instances because the composing and decomposing of "4" involved three pairs of numbers (3 and 1, 2 and 2, 1 and 3).

The second coding challenge, related to "separate problems linked with similarity," appeared in all three textbook series (see Figure 1b). These separate word problems were either

linked by the similar real-world situations (e.g., the monkey-peach situation in the Chinese textbook) or diagrams (e.g., the multiplication and division diagram in EM text, and the part-whole model in HM text), which can be solved with inverse operations. The dilemma was, in such a group of problems, the quantities were not matched. A similar coding difficulty was related to "one problem involving inverse situations" (see Figure 1c). In this case, a problem involved sub-problems, the solutions to which indicated inverse operations. Yet, these sub-problems, again, involved non-matched quantities. As indicated in Figure 1c, the Chinese table contained three sub-problems (e.g., refrigerator, washer, TV), with one of the three key quantities (e.g., original, sold, leftover) missing in each sub-problem. The HM and especially EM, contained many rate tables (also called "in-out activity") situated in real-world contexts (e.g., weight, length). To complete these tables, students need to understand "$x \times \text{rate} = y$" and "$y \div \text{rate} = x$." Similar to the rate table, EM frequently contained a type of problem named "frames and arrows" (see Figure 1c). In order to fill in missing numbers in a frame, a student may think both forward and backward based on the given rule (e.g., - 6). To resolve the above difficulties and uncertainties, the coders referred to the teacher manuals/teacher editions for explanations. As a result, these related problems with non-matched quantities were coded as instances because "inverse relations" were explicitly mentioned as task design purposes. For example, EM in "Unit Organizer" section clearly stated that the multiplication and division diagram "helps reinforce the inverse relationship between the two operations" (Grade 3, p.402). In particular, a group of related problems (e.g., the Chinese word problems in Figure 1b) was considered as one instance.

(INVERST FIGURE 1 ABOUT HERE)

Coding Procedures and Reliability Checking. The author coded all of the textbook pages with additive and multiplicative instances in a separate manner. The coding process started with the Chinese textbook series, followed by the U.S. curricula. Instances on each textbook page were recorded using an Excel spreadsheet, including the title of a lesson where an instance was identified from, the textbook page, the typical example, and the frequency of a type of instance on a particular textbook page.

For reliability checking, the author re-coded all instances four months later and the reliability reached 95.3%. In addition, two coders were invited and trained for independent coding of 25% of the textbook pages, using the same procedure. The coder who was fluent with both Chinese and English coded the Chinese textbooks, while the other coder who only knew English coded the two U.S. textbook series. For additive inverses, all of volume 1 of the textbooks in the first, third, and fifth grades were coded page by page; for multiplicative inverses, all of volume 1 of the textbooks in second, fourth, and sixth grades were coded page by page. Both coders compared their codes with the author. The percentage of the consistency was computed. The reliability for the Chinese and the U.S. textbooks both exceeded 91%. Coding disagreements were mainly related to the above coding difficulties. In addition, there were a few other inconsistencies in coding. For instance, the EM Kindergarten texts suggested activities such as adding or taking away objects from a pocket to help students learn the meaning of addition and subtraction. One coder included these activities due to its potential in developing students' intuitive sense of inverse relations; the other excluded it because the numbers involved in these situations were uncertain. After discussion, we agreed with the latter. In addition, the HM teacher edition offered plenty of extra practices (named "leveled practices"), including

enrichment, re-teach, and intervention. Given that these were extra resources that went beyond the lesson development and overlapped with the existing worked examples and practice problems in the teacher edition, we agreed not to code them. All disagreements were resolved and the raw data were re-examined before data analysis.

Data Analysis

After all instances were coded, the author counted the frequency of instances for additive and multiplicative inverses in each grade for each textbook. Given that each textbook series contains different amount of pages, the average number of instances per page was computed based on the method of Li et al. (2008), which provided a general sense of the proportion of inverse relations that occurred in each textbook series. The author then classified an instance as either a worked example or a practice problem. For both the Chinese and the HM textbooks, it was straightforward. In contrast, the EM textbooks were not obvious at first glance. With a close inspection, the author found that in the section of "teaching this lesson," the teacher edition did list example problems. Consequently, the rest of the tasks were considered as practice problems. Next, the author classified the representation use of each instance as either concrete (physical, visual, contextual) or abstract (symbol). For instance, the aforementioned inverse word problems on children playing (see Method - Instance Coding Section) grounded the learning of inverse relations ($7 \times 5 = 35$, $35 \div 5 = 7$, and $35 \div 7 = 5$) in a real-world context; therefore, the overall nature of this instance was considered as concrete. If the contextual support was removed, this fact family would be coded as abstract.

After each instance was classified into a certain category, the author analyzed the problem types (e.g., fact family, inverse word problem, computing, checking) involved in both

the concrete and abstract contexts. This was complicated because the author tried to maintain the appropriate level of detail, but at the same time still limit the number of problem types. As a result, some problem types in the U.S. textbooks were combined. For example, EM frequently presented a type of flashcard named "fact triangle," expecting students to generate a fact family based on the numbers arranged at each corner of a triangle (e.g., 2, 4, and 8). This task was an opportunity to learn fact family/related facts and thus was combined into this problem type. Similarly, fact tables and part-whole mats were also combined into fact family/related facts due to the same reason. After a separate analysis of additive and multiplicative instances, the author compared the patterns across additive and multiplicative inverses and matched the similar problem types within and across textbook series.

Next, the author conducted a close inspection based on the conceptual framework of this study, leading to an identification of the cultural differences under each aspect. First, for interweaving worked example and practice problems, the author computed the percentages of worked examples and practice problems in each textbook series for additive and multiplicative inverses. This provided a sense of the space each textbook series devoted to worked examples that explain mathematical ideas and to unexplained practice problems for students to solve on their own. In addition, the author examined the purposes of worked examples and how they were connected to the corresponding practice problems. For making connections between concrete and abstract representations, the author examined the percentages of worked examples and practice problems under concrete and abstract contexts for additive and multiplicative inverses. The rationale was to find the pattern of representation uses during initial learning opportunities and later practice opportunities. In particular, the author analyzed how worked examples differed

in making connections between concrete and abstract in U.S. and Chinese textbook series. Finally, for spacing learning over time, the author counted the instances of additive and multiplicative inverses in each volume of each textbook. The author particularly analyzed how the knowledge structure was built into curriculum units to foster a longitudinal coherence over grades. Overall, the above analysis aimed to identify how well each textbook series intended to develop students' meaningful and explicit understanding of inverse relations.

Results

Degree of Coverage of Inverse Relations in Each Textbook Series

All three textbook series provided opportunities to learn inverse relations. Indeed, the U.S. textbooks contained about 3-5 times as many instances of inverse relations as that of the Chinese textbooks (EM: $n_{+/-} = 421$, $n_{\times/\div} = 400$; HM: $n_{+/-} = 686$, $n_{\times/\div} = 1144$; China: $n_{+/-} = 141$, $n_{\times/\div} = 201$). Given that both EM and HM textbooks were greater in length than the Chinese textbooks, the average number of instances per textbook page showed that when the inverse relations were initially presented, the Chinese textbooks indeed contained a larger proportion of instances (Grade 1 for additive) or about the same (Grade 2 for multiplicative) as the U.S. textbooks (see Table 1). In addition, the percentages of additive and multiplicative inverses in each textbook series indicated a common shift of focus from additive to multiplicative inverses over time.

(INSERT TABLE 1 ABOUT HERE)

All instances were classified into certain problem types involving concrete or abstract representations (see Table 2). Instances of "fact family/related facts" and "computing or checking using inverse relations" were classified as either concrete (with contextual support) or abstract (without contextual support). In addition, concrete representations contained four types

of word problem contexts. These included the ones mentioned in the literature ("inverse word problems," and "one problem solved with inverse operations") and the ones with coding difficulties ("separate problems linked with similarity," and "one problem involving inverse situations"). The Chinese textbook also presented number composing and decomposing. For abstract representations, there were types of instances similar to "one problem involving inverse situations" with concrete situations removed (e.g., in-out activity/frames and arrows, filling in structural tables). Other types of problems included finding the missing numbers and solving algebraic equations. Finally, HM textbook uniquely presented definition problems asking students' to recall or explain the terms of "related facts" and "fact families."

(INSERT TABLE 2 ABOUT HERE)

When highlighting the most frequent problem types involving concrete and abstract representations, it appears that there was a consistency of frequently used problem types between additive and multiplicative inverses for each textbook series, especially under the abstract contexts (see Table 2). Interestingly, although the three textbook series involved common problem types (e.g., fact family, checking/computing using inverse relations), each textbook series also contained unique ones. For example, both U.S. textbooks (especially EM) suggested that students should solve the same word problem in two ways using inverse operations, which rarely occurred in the Chinese textbook series. Both U.S. textbook series presented in-out activities ("what is my rule"), which were known as "rate tables" or "function tables" in the literature. Both U.S. textbooks also presented "find the missing numbers," which was related to solving algebraic equations. In contrast, the Chinese textbooks asked students to fill in structural tables that highlighted the key relationships (e.g., minuend, subtrahend, and difference) and

presented number composing and decomposing as mentioned above. In addition to the overall patterns, an inspection of textbook presentations revealed interesting cross-cultural differences in interweaving worked example and practice problems, making connections between concrete and abstract representations, and spacing learning over time.

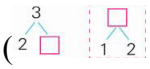
Interweaving Worked Examples and Practice Problems of Inverse Relations

Overall differences in using worked examples and practice problems. Figure 2 shows the percentages of worked examples and practice problems for additive and multiplicative inverses, respectively, in each textbook series.

(INSERT FIGURE 2 ABOUT HERE)

As indicated in Figure 2, for additive inverses, both U.S. textbooks had much smaller percentages of worked examples than the Chinese textbooks (EM: 9.0%, HM, 5.7%, Chinese: 24.1%). When the focus was shifted from additive to multiplicative inverses, both U.S. textbooks slightly increased the proportion of worked examples, while the opposite occurred in the Chinese textbooks (EM: 12.0%, HM, 6.8%, Chinese: 9.5%).

Cross-cultural features in interweaving worked examples into practice problems. Within a lesson, instructions/explanations in the Chinese textbooks were generally faded out from worked examples to practice problems with variations. This feature was not apparent in either US textbooks. Figure 1a illustrates the first lesson of number composing and decomposing in Chinese textbooks. From worked example to practice problems, the real-world situation was faded out from vivid peaches to manipulatives (blocks). The number was also changed from “4” in the worked example to “5” in the guided practice and then to “2” and “3” in the independent practices. In addition, instructions were gradually faded out to elicit more students’ self-

explanations. For instance, the worked example taught decomposing and asked a question on composing, “Do you know what and what can be composed into 4?” The guided practice then expected students to consider both decomposing and composing, “5 can be decomposed into what and what? What and what can be composed into 5?” Finally, the independent practice expected students themselves to complete math mountains in both ways (

).

Across lessons, Chinese textbooks used worked examples to stress sense-making. This laid a foundation for relevant practice problems. The second grade textbook situated the following worked example about checking subtraction in a real-world context (see Figure 3a), “There are 335 books on a bookshelf (picture). 123 of them have been lent out. How many are left?” After this problem was solved using “335 – 123 =,” the textbook suggested students think in the following way, “If we combine the number of lent-out books and the leftover, it should equal the original amount of books” (see the thinking bubble of the little creature in Figure 3a). The teacher manual explained that the purpose of using real-life experiences (The lent-out + the leftover = the original) was to help students make sense of the general way of checking. Consequently, the teacher manual suggested that teachers should help students realize the correspondence between “the lent-out” and “subtrahend” and between “the leftover” and “difference.” This led to a general checking method presented at the bottom of this textbook page, “We can add the difference and subtrahend to check subtraction.” A similar approach to worked examples was found with checking division (see Figure 3b). In Figure 3b, question 2 was about checking division with a remainder. After students solved a problem about “using ¥65 to buy chocolates with the unit price of ¥3” (65 ÷ 3 = 21 R3), students were reminded to think about “Each chocolate is ¥3, I bought 21 of them, which costs ¥63. Plus the leftover ¥2, it is exactly

¥65," which corresponded to the checking procedure of 21 × 3 + 2 = 65 (see the thinking bubble of little creature at the bottom of Figure 3b). The Chinese teacher manual further explained, "When people do shopping, they always asked themselves, 'Did I pay the correct money? Did I get the correct changes?' In mathematics, this is checking." After meaningful initial learning opportunities through worked examples, the Chinese textbooks presented 18 additive instances of checking (12.8%) and 68 multiplicative instances of checking (33.8%) across grades.

(INSERT FITURE 3 ABOUT THERE)

In contrast, both HM and EM textbooks did not stress sense-making through worked examples. Rather, worked examples mainly taught procedures that were reinforced in practice problems. HM presented a worked example for checking division through the concrete context of marbles (54 marbles stored in 4 bags, solved by 54 ÷ 4 = 13 R2). Without using this contextual support, it directly taught the following procedures: (1) Multiply the quotient by the divisor, 4 × 13 = 52, (2) Add the remainder. 52 + 2 = 54, and (3) The sum equals the dividend, so the answer is correct. For checking subtraction, HM taught students to use an arrow to link between "subtrahend" and "sum" in two related vertical number sentences. The second grade HM teacher edition stated, "Rewrite the numbers from the bottom. Then add. Draw a line from the sum to the top number in the subtraction problem. Tell children that addition and subtraction are opposite operations. Since they are opposite, one undoes the other" (p.357). Based on these procedural explanations/instructions through worked examples, across all grades HM provided 217 additive (31.6%) and 745 multiplicative (65%) practice problems that involved checking. EM presented three instances of checking (out of 10) that contained components of sense making. However, for the rest of instances, it only taught procedures similar to the ones in HM.

Making Connections between Concrete and Abstract Representations of Inverse Relations

Overall differences in using concrete and abstract representations. With regard to representation uses, the Chinese textbook series possessed an overall larger percentage of concrete representations (Figure 4, dark filled) than the U.S. textbooks (Figure 4, light filled). For additive inverses, Chinese textbooks contained 51.1% of concrete instances (n = 72) while both U.S. textbooks contained less than 20% concrete ones [EM: 17.3% (n = 73), HM: 14.6% (n = 100)]. For multiplicative inverses, concrete representations in Chinese textbooks decreased to 28.9% (n = 58), which was still a higher percentage than both U.S. textbooks (EM: 25.8% (n = 103), HM: 10.1% (n = 115), also see Table 2).

In particular, with worked examples (Figure 4, solid filled) the U.S. textbooks involved both concrete and abstract representations, whereas the Chinese textbooks used solely concrete representations due to the fact that the Chinese textbook series situated the initial learning opportunities completely in real-world contexts. In fact, the percentages of concrete worked examples (Figure 4, solid dark) in Chinese textbooks were higher than both U.S. textbooks. While the Chinese textbook contained 24.1% (n = 34) additive and 9.5% (n = 19) multiplicative concrete examples, EM contained 4.5% (n = 19) additive and 6.0% (n = 24) multiplicative ones, and HM contained 3.8% (n = 26) additive and 4.5% (n = 52) multiplicative concrete examples. With regard to practice problems (Figure 4, pattern filled), all textbooks included both concrete and abstract instances with the U.S. textbooks containing higher percentages of abstract ones. Analyzing the additive inverses presented in earlier grades, EM and HM respectively contained 78.1% (n = 329) and 83.5% (n = 573) abstract practice problems. In contrast, the Chinese

textbooks only contained 48.9% (n=69) such instances. Overall, both U.S. textbooks favored using abstract representations more than the Chinese textbooks.

(INSERT FIGURE 4 ABOUT HERE)

Cross-cultural features in connecting concrete and abstract within a worked example. In addition to the overall differences in the percentages of concrete and abstract representations, the U.S. and Chinese worked examples differed in making connections between concrete and abstract. More specifically, the sequence and completeness of representation uses within a worked example were quite different. Figure 5 presents the first formal presentation of worked examples in each textbook series, which illustrates differences in representational sequence.

(INSERT FIGURE 5 ABOUT HERE)

In Figure 5, the Chinese worked example started with a real-world situation of a swimming pool. The teacher manual pointed out that this story situation could be viewed two ways "5 boys and 3 girls" or "5 children inside and 3 outside of the pool." This story situation then led to four number sentences that formed a fact family. More importantly, the teacher manual suggested that teachers, based on this worked example, teach the general part-whole relationship – adding two parts to get a whole, and taking one part from the whole to get the other part, which is the key to inverse relations. Overall, the sequence of representation used in this worked example indicated concreteness fading (Goldstone & Son, 2005). This was a typical representational sequence of Chinese worked examples (see another example in Figure 1a about number composing and decomposing).

In contrast, the EM example in Figure 5 started with the domino diagram. The teacher edition explained, "The inverse relationship of addition and subtraction forms the basis of the

study of the subtraction facts. This relationship is first established by observing patterns on dominoes (Lesson 2.6) … (p.92).” In comparison to the Chinese swimming pool situation, the domino diagram had little contextual support to activate students’ familiar experience for learning. The EM teacher edition also did not expect students to move beyond the surface component of an example, in order to understand the underlying part-whole relationships. With regard to the HM example (see Figure 5), it immediately introduced an abstract term, “related fact,” which was illustrated by a part-whole mat with cubes. One may notice that there was a vivid kitten picture arranged at the bottom of this worked example. However, likely due to its location, this picture tended to be missed by teachers (Ding & Carlson, 2013). The HM teacher edition also did not suggest using the kitten situation to illustrate 6 + 3 = 9 and 9 – 3 = 6. As such, representations in HM were not well sequenced.

In addition to representational sequence, Chinese worked examples demonstrated a complete use of representations. In Figure 1b (Separate problems linked with similarity), the Chinese examples were drawn from two lessons. The first story situation described the problem in words along with pictures of two monkeys picking peaches. "The monkeys have picked 23 peaches off the tree. There are 5 left on the tree. How many peaches were originally on the tree?" On the left, a little creature said, "How could we figure out how many peaches were originally on the tree?" The little creature on the right offered the solution in words ("add the 23 peaches you have picked with the 5 that are left on the tree"), which was subsequently translated into the symbolic form: 23 + 5 = 28. The above problem solving process contained rich representations (e.g., contextual, visual, and symbolic). Indeed, the little creatures that occurred throughout the Chinese worked examples might have potential to add memorable imagery for class discussion

(Gerofsky, 2009), and thus might have positive impact on student learning. In fact, the Chinese worked examples not only contained rich representations but also guided students to think inversely based on the quantitative relationship, “taken away + leftover = original,” which transformed a difficult “change” problem to an easy “part-whole” model (Nunes et al., 2009; Resnick, 1989, 1992). The paired subtraction story situation indicated that there were originally 28 peaches. After the monkeys ate some, there were 6 peaches left. Students were asked to figure out how many peaches were eaten by the monkeys. This worked example used the same method to demonstrate the connections among contextual, visual, and symbolic representations in solving the problem (28 – 6 = 22). Similarly, the Chinese textbook drew students’ attention to the quantitative relationship, “original – leftover = taken away (eaten).” Taking together, this pair of problems presented "complete" explanations of the inverse relation even though the numbers did not match (23 + 5 = 28 and 28 – 6 = 22). This is the most salient feature of the Chinese instructional method: contextualization of the learning of a mathematical idea in a familiar real-world situation so that concrete and abstract representations are interconnected.

In comparison, the sample HM lessons demonstrated an incomplete use of representations (see Figure 1b). The lessons used words to state the problems, but there were no pictures or formal symbolic forms to represent step-by-step problem solving processes. The presentations did not emphasize the meaningful explanation of problem solving strategies. There were no descriptions of the procedures other than the general statement in the end: "I add the parts to form the whole (1st problem);" "I know the whole and one of the parts. I can subtract to find the other part (2nd problem)." Likewise, EM textbooks used a model named multiplication and division diagram (see Figure 1b). The teacher edition explained, “The number in the right

column is the product of the other two. The number in the left column (or middle column) can be found by dividing the number in the right column by the number in the middle column (or left column)" (Grade 4, p.393). Even though a diagram could have potentially connected the concrete story situation and abstract number sentences, the EM multiplication and division diagram only served as a means to organize information in the story situation and was treated with a focus on numerical but not relational reasoning. EM presented additive and subtraction diagrams in the same nature. In brief, both U.S. textbooks presented incomplete representations with a focus on numerical calculation, which was in contrast to Chinese textbooks' emphasis on rich-connected representations of quantitative relationships and problem solving strategies.

Spacing the Learning of Inverse Relations over Time

Overall differences in spacing learning over time. Across grades, instances in each textbook demonstrates different trends of spacing learning over time. Figure 6 shows the percentages of instances across grades for each textbook series, which indicates three cross-cultural differences. First, given that U.S. elementary school begins at the Kindergarten level while Chinese school starts at Grade 1, the Chinese textbook series addressed inverse relations much earlier (first half of Grade 1 for additive; first half of Grade 2 for multiplicative) than the U.S. textbooks (first/second half of Grade 1 for additive; second half of Grade 2/first half of Grade 3 for multiplicative). Second, over time, the Chinese textbooks series indicated a greater shift of focus from additive to multiplicative inverses than the U.S. curricula. In the later grades, the EM and HM textbook series presented both additive and multiplicative inverses, whereas the Chinese textbook series focused mainly on multiplicative inverses. Third, as grade level

increased, the Chinese textbooks decreased presentation of inverse relations to a greater extent than either U.S. textbook series.

(INSERT FIGURE 6 ABOUT HERE)

Cross-cultural features in spacing learning over time. A close inspection of how each textbook spaced learning over time revealed different formats. The Chinese textbooks often revisited inverse relations with hierarchical connections, which was achieved either through purposeful topic arrangement or underlying structural relations. An example of purposeful topic arrangement was additive inverse relations. From the initial presentation of number composing and decomposing in first grade (Chapter 7, see Figure 1a), the worked example modeled how to think sequentially (e.g., decomposing 4 peaches into 3 and 1, 2 and 2, and 1 and 3) and inversely (e.g., 4 is decomposed into 3 and 1; 3 and 1 are composed into 4). This was different from both U.S. textbooks where EM decomposed 10 beans into 8 and 2, 5 and 5, 6 and 2 and 2, and HM decomposed 9 cube trains into 4 and 5, 3 and 6, 7 and 2. Both U.S. textbooks did not stress sequential thinking and both only taught decomposing, not composing. As such, the Chinese textbooks attended to inverses relations from the very beginning. According to the teacher manual, the purpose of arranging number composing and decomposing as a specific chapter was to lay a foundation for students' understanding of the inverse relationship between addition and subtraction using numbers 1-10. As seen in the follow-up chapter (Chapter 8), additive inverse relations were formally presented through fact families involving one picture with four number sentences (see Figure 5, the swimming pool example). Interestingly, this chapter did not immediately jump to one picture with four number sentences. Rather, there were several prior lessons including one picture with one number sentence (addition or subtraction, using numbers

from 1-5, see Figure 1b) and one picture with two number sentences (two addition or two subtraction number sentences, using numbers of 6 and 7). These naturally led to one picture with four number sentences (using numbers of 8, 9, and 10). The above task design (chapters 7 and 8) illustrates how additive inverse relations were gradually laid out from informal to formal learning in the Chinese first grade textbook. After this initial presentation, additive inverses were revisited through addition and subtraction up to 100 (Grade 1), then up to 1,000 (Grade 2), further up to 10,000 (Grade 3), and eventually extended to decimal addition and subtraction (Grade 5).

Another feature of Chinese textbooks' spacing learning was to stress the underlying structural relations over time. As introduced in the section of Method – Instance Coding, the Chinese second grade textbook presented a group of inverse word problems involving quantities of 7 groups, 5 children each group, and 35 children. According to the teacher manual, the purpose of this group of inverse tasks was to further students' understanding of the inverse quantitative relationships among "number of groups × group size = total," "total ÷ number of groups = group size," and "total ÷ group size = number of groups," which was expected to strengthen students' understanding of the inverse relations between multiplication and division. This emphasis was continued through introducing other special relationships such as "price, amount, and total cost" (Grade 3) and "speed, time, and distance" (Grade 4). Indeed, the Chinese textbooks stressed inverse quantitative relationships through various types of problems across grades. As illustrated in Figure 1c, the Chinese textbook examples of "one problem involving inverse situations" demanded students to understand inverse quantitative relationships among "original, sold, and leftover" (Grade 1 table), "speed, time, and distance" (Grade 4 table), and

“original, output, and rate” (Grade 6 table). In fact, built on these inverse quantitative relationships, the Chinese sixth grade textbook formally presented advanced topics such as percentages (e.g., involving “original price,” “reduced price,” and “discount”) and direct and inverse proportions (e.g., for direct proportion, “price” or “speed” is fixed; for inverse proportion, “total cost” or “distance” is fixed).

The above Chinese textbook features in spacing learning over time – purposeful topic arrangement and stressing structural relations – were not evident in both U.S. textbooks. Even though the U.S. teacher editions suggested a few instances of inverse word problems (none in the student editions), the HM teacher edition only suggested students act out the story problems and the EM teacher edition suggested students compare the relevant problems so as to see that “They use the same 3 numbers.” Such instruction focused only on numerical calculation but not inverse qualitative relationships. In fact, HM’s definition tasks ($n_{+/-} = 22$, $n_{\times/\div} = 11$) further demonstrated such a focus. For instance, “related facts” were referred to through the metaphor, “people in a family are related and sometimes we call them relatives” (Grade 1, p.153); A fact family was defined as “a set or related facts” (Grade 2, p.63) or “a group of number sentences that use the same numbers” (Grade 3, p.132). All these explanations targeted surface similarities (e.g., the same three numbers) rather than the structural relations (e.g., part + part = whole, whole – part = part) that are the key to inverse relations.

Both U.S. textbooks demonstrated different ways to space learning. For EM textbooks, one way was to re-teach. For instance, “fact family” frequently served as the lesson title across several grades. While the range of numbers changed, the main idea of the lesson remained the same as in previous years. The other way that the EM textbooks spaced learning was to

frequently arrange an inverse task as practice problems, even in lessons with non-relevant topics. For instance, among the 238 additive and 216 multiplicative instances of fact family/related facts (see Table 2), 83% and 75% of them, respectively, were fact triangles. This idea of fact triangles was revisited in later lessons such as "Pattern-block and template shapes" (Grade 1, Lesson 7.3), "Exploring area, polygons, and geoboard fractions" (Grade 2, Lesson 10.7), "Exploring estimates and polygons" (Grade 3, Lesson 5.4), and "Rectangular arrays (Grade 5, Lesson 1.2). The HM textbook series shared the above strategy of re-teaching. This textbook series also stressed high frequency of practice to reinforce students' learning. As shown in Table 2, "checking using inverse relations" was the first or second most frequent instance in HM ($n_{+/-}$ = 217; $n_{\times/\div}$ = 745). In many occasions, one practice problem (e.g., Divide and Check) included 20-40 sub-problems on the same page (e.g., Grade 3: p.624, p.626; Grade 4: p.215, p229, 231; 239; Grade 5: p.113, p.119; Grade 6: p.37, p.163). Overall, both U.S. textbook series spaced learning over time through repetition across grades.

Weak connections between inverse relations and algebraic equation solving. According to Carpenter et al. (2003), students' understanding of inverse relations in arithmetic will contribute to their later solving of algebraic equations. However, this vertical connection was not evident in all textbook series, except for a few instances of filling in the missing numbers in U.S. textbooks (EM: $n_{+/-}$ = 4, $n_{\times/\div}$ = 2; HM: $n_{+/-}$ = 5, $n_{\times/\div}$ = 2) and a few algebraic equation solving problems in all textbooks (EM: $n_{+/-}$ = 3, $n_{\times/\div}$ = 5; HM: $n_{+/-}$ = 2, $n_{\times/\div}$ = 12; China: $n_{+/-}$ = 4, $n_{\times/\div}$ = 2). Most of the relevant tasks that were not coded were due to the textbooks' emphasis on direct thinking, rather than using inverse relations. One example was the third grade HM task, 2 × ( ) = 6. The textbook reminded students to think about 2 × 3 = 6, thus the ( ) = 3. Similarly, the EM

second grade teacher edition suggested a "count-up" strategy to solve 32 + ( ) =50: "A child might reason as follows: Which number, added to 2, will give me 10? It's 8, so 32 + 8 = 40. Which number, added to 40, will give me 50? 40 + 10 = 50. Finally, 8 + 10=18" (p.50). For algebraic equation solving, both U.S. textbooks presented them in a format of a$x \pm b = c$ and continuously stressed direct thinking. For example, the EM grade 4 textbook taught students to solve $12 + x = 55$ by thinking of 12 + (43) = 55 (Lesson 3.11). It also presented a "broken calculator activity," where students were suggested to solve $452 + x = 735$ without using the broken minus key. Indeed, the EM grade 6 textbook presented three ways to solve an equation without involving inverse relations. The first method was "trial-and-error," which was a count up strategy. The second method was the "cover-up" method that also involved direct thinking (e.g., $2m + 5 = 17$, covering $2m$ and think about ? + 5 = 17. As 12 + 5 = 17, $2m = 12$). The third method was named "equivalent equation method." In this method, to solve $2m + 5 = 17$, one may add "- 5" to both sides of the equation based on the property of equality. Cancelling out "+5" on the left side involved the three-term inversion principle ($a + b – a = b$) rather than the two-term complement principle (if $a + b = c$, $a = c – b$), the focus of this study. HM shared the same pattern as EM.

Likewise, the Chinese textbook series formally presented equation solving in fifth grade in both formats of $ax \pm b = c$ and $ax \pm bx = c$. According to the teacher manual, students were traditionally taught to solve equations using the two-term inverse relations. For example, to solve $3x – 4 = 16$, students can view $3x$ as one quantity, which equals 16 + 4. However, the new curriculum standards (Ministry of Education of China, 2001) suggested reforming this part using the property of equality (e.g., adding 4 to both sides of $3x – 4 = 16$) in order to better align with

the middle school curricula, which was similar to both U.S. textbooks. The Chinese fifth grade textbook, however, presented one type of problem, "Filling a sign in the O and filling a number in the □," which suggested that after students understood the method of using the property of equations, they might take a shortcut by not writing down the transformation steps. This may strengthen students' understanding of the two-term inverse relations.

$3.6 + x = 5.7$	$x - 20 = 30$	$0.6x = 4.2$	$x \div 20 = 5$
$x = 5.7$ O □	$x = 30$ O □	$x = 4.2$ O □	$x = 5$ O □

Discussion

This study explores the differences and similarities between U.S. and Chinese textbook presentation of the inverse relations. It should be noted that while these textbooks are representative of U.S. and Chinese curricula, they are not used exclusively, nor are textbooks even mandated in other nations, such as in Australia, Belgium, and Singapore. In addition, this study has a narrowed focus on the two-term complement principle, which is arbitrary due to the close relationship between the two- and three-term inverse relations. Nevertheless, given that the goals of teaching and learning of fundamental mathematical ideas such as inverse relations is global, findings from this study may shed light on improving existing learning environments by offering students more meaningful and explicit learning opportunities.

Meaningful Initial Learning of Fundamental Mathematical Ideas: Contextual Support

Initial learning affects later transfer (Chi & VanLehn, 2012). The key to meaningful initial learning is tying a new concept to students' existing knowledge (Ausubel, 1968; Bransford, Brown, & Cocking, 1999; Piaget, 1952). With regard to inverse relations, given that children already possess relevant preliminary schemas (Resnick, 1992) and informal knowledge (Gilmore

& Bryant, 2008; Klein & Bisanz, 2000; Sherman & Bisanz, 2007), it is important to activate their informal understanding to influence new learning. In this study, the initial learning opportunities (worked examples) of inverse relations in both U.S. and Chinese textbooks involved concrete representations; however, the nature and purpose of using concrete representations appear to be different. In particular, the nature of concrete representations in Chinese worked examples was found to be contextual (e.g., word problems with illustrations). In fact, all worked examples in Chinese textbooks were situated in real-world contexts, which was in direct contrasts to U.S. textbooks' small portion of such contexts. The main type of concrete representations in U.S. worked examples was found to be physical or visual without contextual support (e.g., dominos, cubes, and diagrams). Therefore, the Chinese textbooks may have a greater likelihood to activate students' personal experiences and informal understanding to aid in learning (Ding & Li, 2014; Goldstone & Son, 2005; Resnick & Omanson, 1987). Moreover, the purpose of using concrete contexts in Chinese textbooks (e.g., shopping) is clearly for sense- making of computation or checking procedures; yet, the U.S. textbooks mainly used them as a pretext for computation with a focus on procedures. HM's strategy of using arrows to stress checking procedures is similar to the presentation of Belgium textbooks reported in Torbeyns et al. (2009). As indicated by the literature (Baroody, 1987, 1999; De Smedt et al., 2010; Torbeyns et al., 2009), when students do not possess meaningful understanding of procedures and strategies, mechanical practices may only result in inert learning. As such, Chinese textbooks' more robust contextual supports may have a greater potential to contribute to students' meaningful initial learning, laying a conceptual foundation for students' later numerical practices (Ding & Li, 2014). It is interesting to note that Chinese textbooks decreased the proportion of concrete worked examples from additive to

multiplicative inverses. In this study, we lack evidence to explain this observation. Future studies may explore possible reasons through interviews of textbook designers. For instance, is it possible that Chinese textbook designers might have expected students to transfer their learning from additive to multiplicative inverses, and thus provide more student self-practice problems?

Explicit Learning of Fundamental Mathematical Ideas: Structural Relations

Prior studies have also revealed that the current learning environments of inverse relations lacks support for students' explicit understanding, resulting in students' inability to spontaneously use this relation in new contexts (Baroody, 1999; De Smedt et al., 2010; Torbeyns et al., 2009). In this study, the U.S. textbooks demonstrated a procedural focus. The addition and subtraction diagrams and multiplication and division diagrams presented in EM textbooks literally listed quantities, but not the interaction among quantities, which may hinder students' understanding of the problem structures and underlying quantitative relationships (see elaboration in Murata, 2008). In addition, both EM and HM posed questions on inverse relations; yet, their expected student responses were mainly limited to surface similarities of numbers/quantities but not the interactions between them. This has been found to be a roadblock of deep initial learning (Chi & VanLehn, 2012).

In contrast, the Chinese textbooks demonstrated explicit attention to structural relations, which was first indicated by its deep use of worked examples. From the very beginning, worked examples of number composition and fact families aimed to teach students the part-whole relationships (part + part = whole; whole – part = part), which is consistent with prior findings (Zhou & Peverly, 2005). Chinese textbooks also systematically faded the concreteness into the abstract ideas (Goldstone & Son, 2005). For instance, when it came to checking, the Chinese

textbooks suggested going beyond the concrete situation to enable students' inverse understanding at a general level (e.g., difference + subtrahend = minuend). Such an approach will likely enable generalization from worked examples (Lewis, 1988). The second indicator of Chinese textbooks' explicit attention to structural relations lies in its consistent emphasis on inverse quantitative relationships. Through carefully matched quantities and situations of inverse word problems, students' attention may be drawn to the underlying structures (e.g., speed × time = distance; distance ÷ speed = time; distance ÷ time = speed). These results may partially explain findings from prior cross-cultural studies that have demonstrated Chinese students' superiority in understanding distance, time, and speed interrelations (e.g., Zhou, Peverly, Boehm, & Lin, 2000), a hard topic related to multiplicative reasoning (Thompson, 1994). Even with the problems that did not contain the matched quantities, the Chinese textbooks still attended to inverse quantitative relationships (e.g., sold + leftover = original, original - sold - leftover). Focusing on relational but not numerical calculation (Nunes et al., 2012; Thompson, 1994) likely facilitates inverse understanding at a structural level, which will transcend over contexts (e.g., whole numbers, decimals). Indeed, the more elaborated the underlying structures and principles are, the more effective subsequent learning will be (Ausubel, 1968).

Spaced Learning of Fundamental Mathematical Ideas: Task Design and a Dilemma

The learning of any fundamental ideas takes time (Pashler et al., 2007). Thus, it is important to provide spaced learning with frequent exposure of students to relevant practices. Consequently, textbooks, as part of the learning environments, should be carefully designed with coherence, especially vertical coherence (National Academy of Education, 2009). In this study, Chinese textbooks demonstrated purposeful task arrangement and connected structural relations

over grades. In contrast, both U.S. textbook series, although presenting inverse tasks with high frequency, were mainly computational and repetitive. This is different from the Chinese textbooks' task choices and focus shifts. The above cross-cultural differences in task design call for rethinking of learning opportunities when spacing learning over time. For example, what tasks should be chosen as worked examples to support deep, meaningful initial learning and how much practice is enough? When grade level increases, how may students' attention be shifted to new learning in order to strengthen rather than to simply repeat the learned concepts? Only with a clear task design goal, can one expect to overcome the common limitation – lack of depth – introduced by U.S. textbooks (Schmidt, Wang, & McKnight, 2005), thus offering more quality opportunities to learn fundamental mathematical ideas (CCSSI, 2010).

Related to spacing learning over time, an emerging dilemma occurs due to the observed weak connections between inverse relations and algebraic equation solving in all textbook series. According to the literature (e.g., Carpenter et al., 2003; Nunes et al., 2009), understanding the two-term inverse relations (e.g., if $a + b = c$ then $c - b = a$; if $a \times b = c$ then $c \div b = a$) would contribute to students' later solving of algebraic equations. However, findings in this study indicate that all textbooks approached algebraic equation solving using the property of equality (along with the three-term inversion principle). The Chinese teacher manual explained that the shift from using the two-term inverse relations to the property of equality is to bridge students' learning in middle school, where more complex algebraic solving will be taught. As such, it remains unclear in what ways the use of two-term inverse relations is inadequate for solving algebraic equations. Indeed, could students' mastery of the two-term inverse relations actually support students' algebraic equation solving? As instructed by the Chinese textbooks, students

who have mastered the detailed steps of equation solving may write a "shortcut" that involves the two-term inverse relations, which enhances the efficiency in equation solving. In fact, recent research asserts that there is a close relation between students' understanding of the two-term and three-term inverse relations (Baroody et al., 2009; Gilmore & Bryant, 2008). Future studies may explore how these two types of inverse relations may work together, along with property of equality, to support students' algebraic equation solving. Textbook designers may also consider how the advantage of two-term inverse relations obtained in previous years may be better taken to support the learning of algebraic equation solving. Such explorations will further understanding of the detailed paths to space learning of fundamental mathematical ideas in elementary grades.

Implementation and Conclusion

As Shimizu and Kaur (2013) emphasized, the purpose of cross-cultural comparison is to reflect upon one's practices and to learn from others. In this study, findings based on both U.S. and Chinese textbooks contribute insights to improve the learning environments of fundamental mathematical ideas. Chinese textbooks' skillful use of representations and stressing the underlying structural relations are consistent with prior findings on Chinese textbook presentations of the distributive property (Ding & Li, 2010, 2014), the associative property (Ding et al., 2012), and the equal sign (Li et al., 2008). These approaches in developing students' meaningful and explicit understanding may be learned by textbook designers in U.S. and others countries. Likewise, U.S. textbooks' unique problem tasks (e.g., fact triangle, frames and rows, part-whole mat, one problem solved with inverse operations, in-out activity or rate tables) may be learned by Chinese textbook designers and others. Given that examples with greater

variability can better facilitate the encoding of principles (Renkl et al., 2002), the new problem types may promote students' interests and effective learning. Indeed, findings about U.S. textbooks' preference in using rate tables are consistent with Cai et al.'s (2005) conclusion that U.S. textbooks seemed to stress more functional thinking.

The above findings together raise important questions of not only how to design effective textbooks, but also how to use them successfully in classrooms to support learning. Findings about Chinese textbooks in this study appear to be parallel with prior findings on Chinese teachers' knowledge structures and classroom teaching (e.g., Ding et al., 2013; Zhou, Peverly, & Xin, 2006). As such, future studies may include both types of inverse relations, across more countries and explore detailed connections between textbooks, classroom instruction, and student learning.

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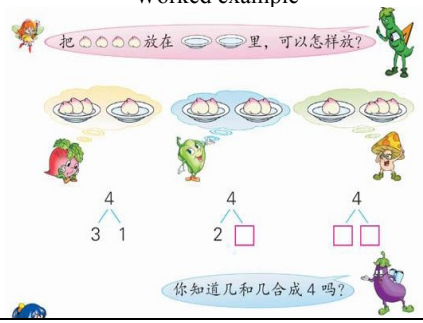
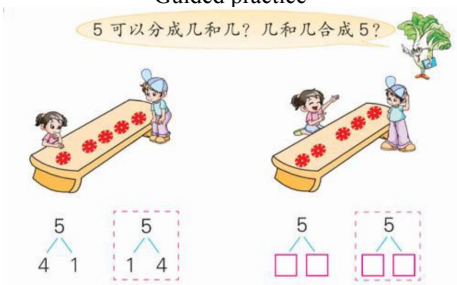
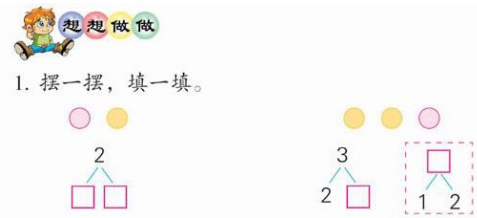
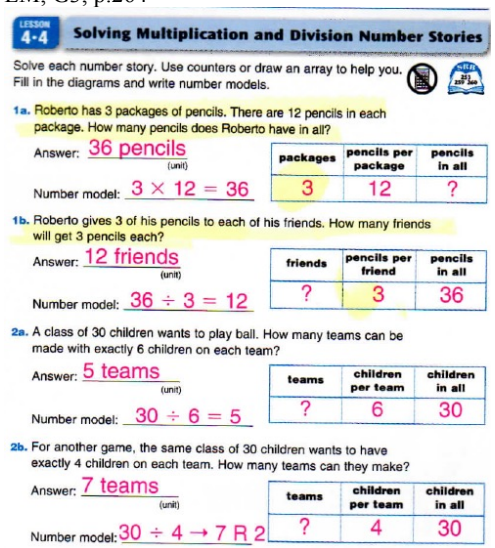
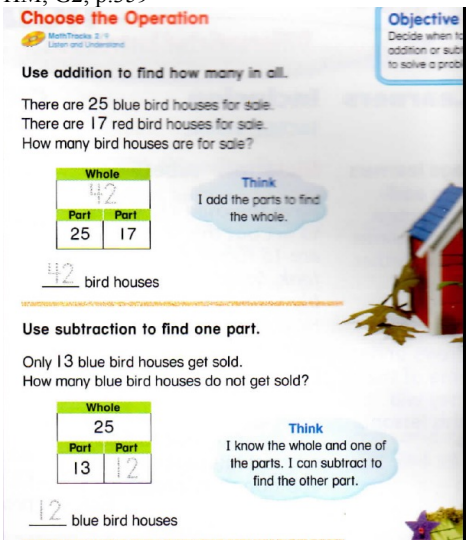
a. Number composing and decomposing	Chinse, G1, v1, p.30 Worked example  	Chinese, G1, v1, p.30 Guided practice  	Chinese, G1, v1, p.31 Independent practice 
b. Separate problems linked with similarity	Chinese, G1, v2, p.49 & p.55  	EM, G3, p.264  LESSON 4-4 **Solving Multiplication and Division Number Stories** Solve each number story. Use counters or draw an array to help you. Fill in the diagrams and write number models. 1a. Roberto has 3 packages of pencils. There are 12 pencils in each package. How many pencils does Roberto have in all? Answer: 36 pencils (unit) Number model: 3 × 12 = 36 packages: 3; pencils per package: 12; pencils in all: ? 1b. Roberto gives 3 of his pencils to each of his friends. How many friends will get 3 pencils each? Answer: 12 friends (unit) Number model: 36 ÷ 3 = 12 friends: ?; pencils per friend: 3; pencils in all: 36 2a. A class of 30 children wants to play ball. How many teams can be made with exactly 6 children on each team? Answer: 5 teams (unit) Number model: 30 ÷ 6 = 5 teams: ?; children per team: 6; children in all: 30 2b. For another game, the same class of 30 children wants to have exactly 4 children on each team. How many teams can they make? Answer: 7 teams (unit) Number model: 30 ÷ 4 → 7 R 2 teams: ?; children per team: 4; children in all: 30	HM, G2, p.359  **Choose the Operation** Objective Use addition to find how many in all. There are 25 blue bird houses for sale. There are 17 red bird houses for sale. How many bird houses are for sale? Whole: 42; Part: 25; Part: 17 Think: I add the parts to find the whole. 42 bird houses Use subtraction to find one part. Only 13 blue bird houses get sold. How many blue bird houses do not get sold? Whole: 25; Part: 13; Part: 12 Think: I know the whole and one of the parts. I can subtract to find the other part. 12 blue bird houses

Figure 1. Examples of coding difficulties across textbook series. G = Grade, V = Volume.

c. One problem involving inverse situations

Chinese examples:

G1:

	Refrigerator	Washer	TV
Original	56	( )	46
Sold	30	20	( )
Leftover	( )	18	5

G4

Speed	time	Distance
80	*t*	
V		*S*
	t	*S*

G6:

Material	Peanut	Bean	Rapeseed
Original	160	200	
Oil output	40		48
Output rate		16%	40%

EM examples:

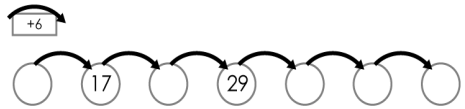
G2: In and out activity
Rule: 1 qt = 2 pt

qt	Pt
2	
3	
	10
8	

G4

Buses			36	180	…
Jets	1	2	3		

Frames and Arrows:



HM examples:

G5

M	*cm*
45	
	0.005
	862.3
A	

G6

s = €	*s* = 300mi/h	*s*=20mi/h
d=225mi	*d*=€	*d*=mi
t = 5h	*t*=6h	*t*=€

Figure 1. Examples of coding difficulties across textbook series (continued). The Chinese textbook series introduces the formula of "distance = speed × time" as $s = vt$ while the U.S. textbook series introduces $d = st$. All Figures from the Chinese textbooks are reproduced with permission from the textbook author, Lin Wang. The above EM material is authored by: University of Chicago School Mathematics Project (2007). *Everyday mathematics* (Teacher edition, Grade 3). Chicago: Wright Group/McGraw-Hill. This material is reproduced with permission of McGraw-Hill Education. The above HM material is from *Houghton Mifflin Mathematics*, Teachers' Edition (Grade 2). Copyright © by Houghton Mifflin Company. All rights reserved. Used by duplication is strictly prohibited unless written permission is obtained from Houghton Mifflin Harcourt Publishing Company.

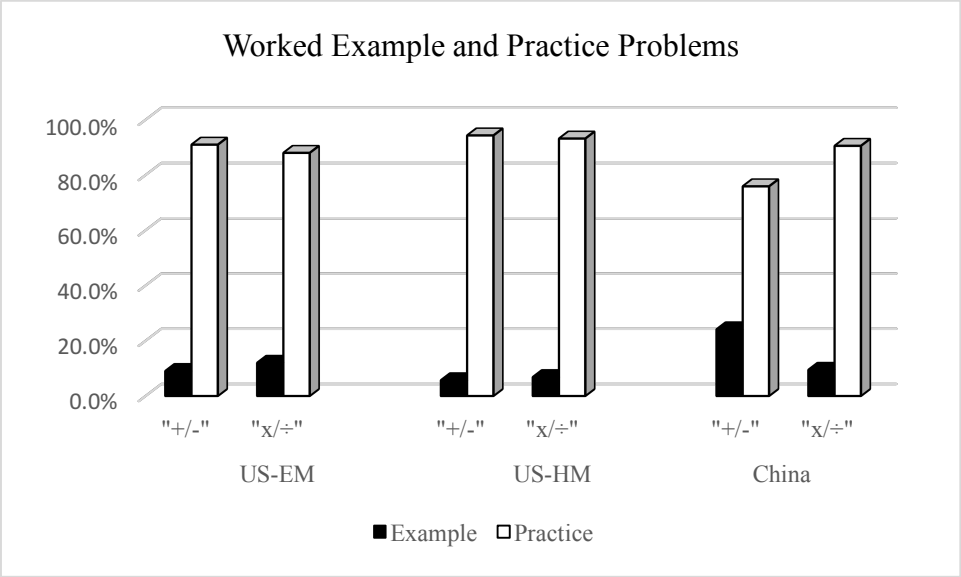


Figure 2. The percentages of worked examples and practice problems in each textbook series. "+/-" = additive inverses; "×/÷" = multiplicative inverses.

a. Checking subtraction using addition

(G2, v4, p.52)

b. Checking division using multiplication

(G3, v5, p.3)

Figure 3. Chinese textbooks' worked examples of checking using inverse relations. G = Grade; V = Volume. All Figures from the Chinese textbooks are reproduced with permission from the textbook author, Lin Wang.

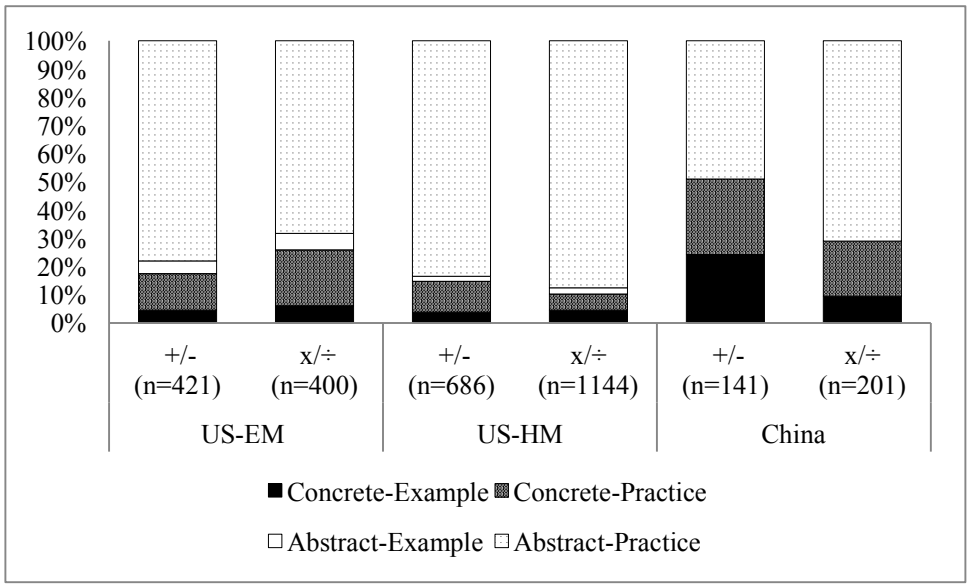


Figure 4. Representation uses in worked examples and practice problems. "+/-" = additive inverses; "×/÷" = multiplicative inverses. Concrete-Example = Worked examples that involve concrete representations. Abstract-Practice = Practice problems that involve abstract representations.

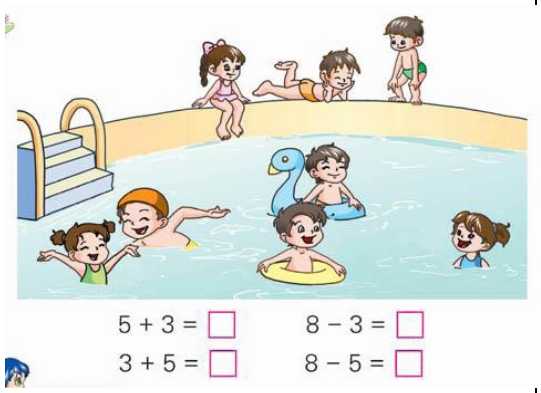
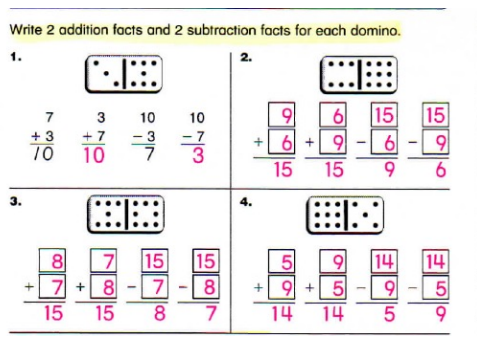
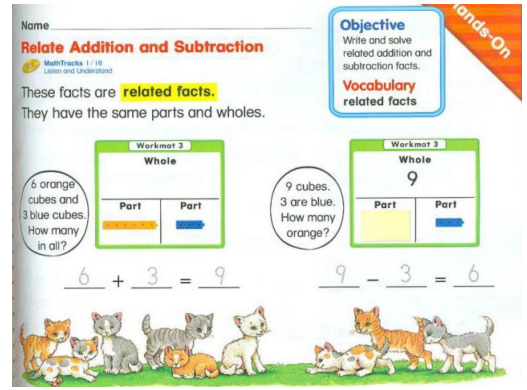
Chinese	US-EM	US-HM
Real-world situations	Fact family with dominos	Part-whole mode
		
(G1, v1, p.54)	(G2, p.124)	(G1, p.153)

Figure 5. Fact family in context in each textbook series. G = Grade; V = Volume. The Chinese textbook figure is reproduced with permission from the textbook author, Lin Wang. The above EM material is authored by: University of Chicago School Mathematics Project (2007). *Everyday mathematics* (Teacher edition, Grade 2). Chicago: Wright Group/McGraw-Hill. The text is reproduced with permission of McGraw-Hill Education. The above HM material is from *Houghton Mifflin Mathematics*, Teachers' Edition (Grade 1). Copyright © by Houghton Mifflin Company. All rights reserved. Used by duplication is strictly prohibited unless written permission is obtained from Houghton Mifflin Harcourt Publishing Company.

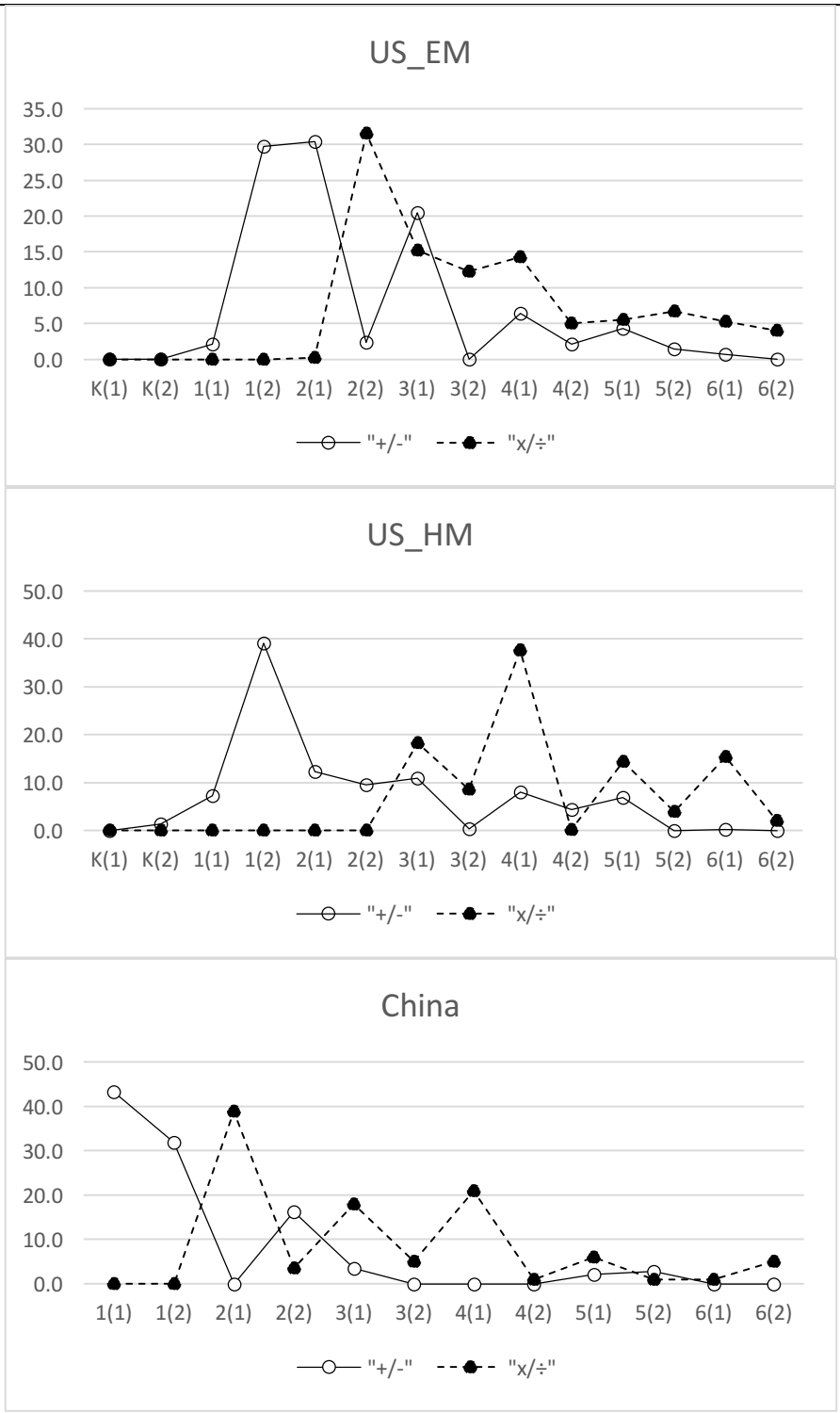


Figure 6. Percentages of additive and multiplicative inverses across grades. "+/-" = additive inverses; "×/÷" = multiplicative inverses.

Table 1. *Average Number of Instances Per Page on Each Textbook Series*

Textbook series	Grades	Pages coded[1]	# of instances		# of instances per page		% of instances	
			Additive	Multiplicative	Additive	Multiplicative	Additive	Multiplicative
EM	K	429	0	0	0.00	0.00	Na	Na
	1	918	134	0	0.15	0.00	100.0	0.0
	2	990	138	127	0.14	0.13	52.1	47.9
	3	982	86	110	0.09	0.11	43.9	56.1
	4	1018	36	77	0.04	0.08	31.9	68.1
	5	1044	24	49	0.02	0.05	32.9	67.1
	6	1004	3	37	0.00	0.04	7.5	92.5
HM	K	536	0	0	0.00	0.00	Na	na
	1	932	318	0	0.34	0.00	100.0	0.0
	2	932	149	0	0.16	0.00	100.0	0.0
	3	936	77	307	0.08	0.33	20.1	79.9
	4	936	85	431	0.09	0.46	16.5	83.5
	5	920	47	208	0.05	0.23	18.4	81.6
	6	920	1	198	0.00	0.22	0.5	99.5
China	1	209	106	0	0.51	0.00	100.0	0.0
	2	214	23	85	0.11	0.40	21.3	78.7
	3	228	5	46	0.02	0.20	9.8	90.2
	4	229	0	44	0.00	0.19	0.0	100.0
	5	254	7	14	0.03	0.06	33.3	66.7
	6	250	0	12	0.00	0.05	0.0	100.0

[1]The HM teacher edition does not provide the exact number of total pages. Rather, a teacher edition uses student textbook page (p.89) as a base for the corresponding explanation pages (p.89A, p.89B, p.89C). There are approximately an average of 200 explanation pages added to a student edition in each grade. The listing page numbers for HM in Table 1 were obtained by using the number of student textbook pages plus 200.

Table 2. *Types of Problems involving Concrete and Abstract Representations across Textbook Series*

Representation	Problem Type	US-EM_+/- (n = 421)		US-EM_×/÷ (n = 400)		US-HM_+/- (n = 686)		US-HM_×/÷ (n = 1144)		China_+/- (n = 141)		China_×/÷ (n = 201)	
		freq	%	freq	%	freq	%	freq	%	freq	%	freq	%
Concrete	Number composing and decomposing									27	19.1%		
	Fact family in context	32	7.6%			71	10.3%	50	4.4%	20	14.2%	6	3.0%
	Computing/checking in context	1	0.2%	10	2.5%	14	2%	32	2.8%	15	10.6%	28	13.9%
	Inverse word problems	2	0.5%	1	0.3%	7	1%	2	0.2%	1	0.7%	10	5.0%
	One problem solved with inverse operations	34	8.1%	30	7.5%	1	0.15%	1	0.1%				
	Separate problems linked with similarity	1	0.2%	14	3.5%	5	0.7%	8	0.7%	8	5.7%	2	1.0%
	One problem involving inverse situations	3	0.7%	48	12%	2	0.3%	22	1.9%	1	0.7%	12	6.0%
	Total	**73**	**17.3%**	**103**	**25.8%**	**100**	**14.6%**	**115**	**10.1%**	**72**	**51.1%**	**58**	**28.9%**
Abstract	Fact family/related facts	238	56.5%	216	54%	276	40.2%	146	12.8%	37	26.2%	59	29.4%
	Checking using inverse relations	5	1.2%	1	0.3%	217	31.6%	745	65%	18	12.8%	68	33.8%
	Computing using inverse relations	6	1.4%	9	2.3%	55	8.0%	80	7.0%	6	4.3%	10	5.0%
	In-out activity/frames and arrows	92	21.9%	64	16.1%	9	1.3%	33	2.9%				
	Filling in structural tables									4	2.8%	4	2.0%
	Finding the missing numbers	4	1%	2	0.5%	5	0.7%	2	0.2%				
	Solving algebraic equations	3	0.7%	5	1.3%	2	0.3%	12	1.0%	4	2.8%	2	1.0%
	Definition: explain the term/relation					22	3.2%	11	1.0%				
	Total	**348**	**82.7%**	**297**	**74.3%**	**586**	**85.4%**	**1029**	**89.9%**	**69**	**48.9%**	**143**	**71.1%**