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Spatial Proportional Reasoning Is Associated With Formal Knowledge About Fractions

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Proportional reasoning involves thinking about parts and wholes (i.e., about fractional quantities). Yet, research on proportional reasoning and fraction learning has proceeded separately. This study assessed proportional reasoning and formal fraction knowledge in 8- to 10-year-olds. Participants ($N = 52$) saw combinations of cherry juice and water in displays that highlighted either part–whole or part–part relations. Their task was to indicate on a continuous rating scale how much each mixture would taste of cherries. Ratings suggested the use of a proportional integration rule for both kinds of displays, although more robustly and accurately for part–whole displays. The findings indicate that children may be more likely to scale proportional components when being presented with part–whole as compared with part–part displays. Crucially, ratings for part–whole problems correlated with fraction knowledge, even after controlling for age, suggesting that a sense of spatial proportions is associated with an understanding of fractional quantities.

Reasoning about relative quantities is important for many science disciplines, as, for example, when one has to understand concentrations of liquids in chemistry or think about the density of objects in physics. However, thinking about relative quantities is also crucial for many problems that we encounter in everyday life: How much sugar is needed if I want to use a cake recipe calling for three eggs when I have only two eggs? Is buying three detergent packets for the price of two a better deal than getting one packet for half price? Answering these problems exactly requires formal calculation using fractions; even estimating the answers requires understanding the number system that goes beyond whole numbers. Unfortunately, students often exhibit difficulties when learning to understand and carry out calculations with fractions (e.g., Hecht & Vagi, 2010; Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004).

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Such findings documenting children's difficulties with fractions have led the National Mathematics Advisory Panel (2008, p. 18) to recommend that "the teaching of fractions must be acknowledged as critically important and improved." The importance of this goal is underlined by recent findings that sixth graders' fraction understanding is correlated with their mathematics achievement (Siegler, Thompson, & Schneider, 2011) and predicts mathematical proficiency up to 6 years later (Bailey, Hoard, Nugent, & Geary, 2012; Siegler et al., 2012). In particular, 10- to 14-year-old children's fraction understanding predicts their knowledge of algebra in high school (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014; Siegler et al., 2012). Thus, a well-developed understanding of fractions seems to be foundational for an understanding of higher mathematics.

Fractions can be defined as one part or several equal parts of a whole (or as a quotient p/q), and their components can be scaled without changing the value of the fraction (i.e., 1/5 = 2/10 = 3/15; cf. Boyer & Levine, 2012). To compare fractions or to create equivalent fractions, one has to understand "relations between relations" (Piaget & Inhelder, 1975) and thus be able to reason proportionally. Given the aforementioned findings that children often struggle with fractions, the question arises as to whether children's understanding of numeric fractions aligns with their sensitivity to proportions presented non-numerically.

The seminal studies of Piaget and Inhelder (1975) suggested that the answer may be "yes"; they argued that proportional reasoning emerges late, around the age of 11 years. In their studies, children were presented with two sets of red and white marbles that differed in absolute numbers and proportions. The children were then instructed to choose the set that was more likely to yield a red marble in a random draw. Children younger than 11 years predominantly selected the set with the higher number of red marbles, thus focusing on the absolute number instead of the relation between differently colored marbles. Because this task also required an understanding of "random draw" and probability, children's difficulties may not have arisen because of lack of proportional knowledge. However, similarly low performance in children younger than 11 years was reported in subsequent studies using different procedures that did not involve probability judgments—for example, tasks based on mixing juice and water (Fujimura, 2001; Noelting, 1980) or liquids of different temperature (Moore, Dixon, & Haines, 1991).

In sharp contrast to these studies, other research has suggested that proportional reasoning emerges much earlier (Sophian, 2000; Sophian & Wood, 1997; Spinillo & Bryant, 1991) and may even have its origins in infancy (McCrink & Wynn, 2007; Xu & Denison, 2009). For example, several studies have demonstrated that 5- to 6-year-olds showed successful proportional reasoning when presented with continuous amounts as opposed to discrete amounts (Boyer, Levine, & Huttenlocher, 2008; Jeong, Levine, & Huttenlocher, 2007; Spinillo & Bryant, 1999). Children also showed earlier competence at the age of 3 to 4 years when asked to produce equal proportions, possibly by tapping into their ability to reason by analogy (Goswami, 1989; Singer-Freeman & Goswami, 2001). Analogical reasoning may build on similar cognitive competencies as proportional reasoning, because it often also requires an understanding of relations between relations (e.g., bananas are related to fruits like cucumbers are related to vegetables; cf. Gentner, 1989). Furthermore, studies using functional measurement paradigms have shown that 5- to 7-year-olds made correct proportional judgments about the probability of events in complex situations (Acredolo, O'Connor, Banks, & Horobin, 1989; Anderson & Schlottmann, 1991; Schlottmann, 2001).

In functional measurement methodology, two variables are typically manipulated in a full factorial design, and the participants' task is to judge the combinations of these variables on a rating scale. Thus, a reason for earlier success in these tasks may be that children were asked to translate spatial proportions into spatial ratings, which might be more intuitively graspable than binary choice tasks (as used by Piaget & Inhelder, 1975). But does such intuitive sensitivity to proportions translate to explicit reasoning about proportions and numeric fractions?

Even though young children seem to possess some sense of proportional magnitudes, early instruction emphasizes whole numbers and counting instead. This experience with counting and whole-number calculation may initially interfere with the acquisition of fraction understanding (Mix, Levine, & Huttenlocher, 1999), which was underlined by findings showing that children who have greater proficiency with whole numbers have more trouble grasping the notion of fractional quantities (Paik & Mix, 2003; Thompson & Opfer, 2008). A possibility we explore in this study is that some children may be able to access continuous relative representations more than others, which in turn might help them in thinking and learning about formal fractions.

In initial support of this notion, two previous studies have shown a relation between children's understanding of non-numerical and numerical relative quantities (Ahl, Moore, & Dixon, 1992; Moore et al., 1991). However, these studies used a temperature-mixing task and thus involved a highly abstract physical property that is often challenging for children (Stavy & Berkovitz, 1980; see Wiser & Carey, 1983, for a history-of-science perspective). In fact, 8-year-olds showed poor understanding of the temperature task and even many 11- and 14-year-olds struggled with it (Moore et al., 1991). Furthermore, the same stimulus set was presented in the numerical and non-numerical conditions, with the only difference being that in the numerical condition, *additional* numeric information about the temperature was displayed. Thus, it is possible that performance scores in these conditions were related simply because children relied primarily on the visual cues in both and largely ignored the numeric information.

In the current study, we used two distinct tasks that differed in more than just additional numeric information and assessed whether 8- to 10-year olds' intuitive, non-numerical understanding of spatial proportions is related to their formal knowledge about numeric fractions. Children's understanding of spatial proportions was measured by a task that adopted a functional measurement approach (Anderson & Schlottmann, 1991; Schlottmann, 2001). As this methodology allows for assessing not only absolute but relative responses, we analyzed each child's information integration pattern and looked at children's absolute accuracies. In line with previous studies, we presented different combinations of continuous quantities of juice and water (Boyer & Levine, 2012; Boyer et al., 2008; Fujimura, 2001; Noelting, 1980), and children were asked to indicate on a rating scale how much these mixtures would taste of juice. By varying a concrete property (taste) that could be visually indicated by color, the task was expected to be easier as compared with previous studies that have used temperature-mixing tasks (Ahl et al., 1992; Moore et al., 1991).

Subsequently, participants were presented with a written test with formal fraction problems. This test measured school-taught fraction knowledge and covered several aspects of conceptual fraction knowledge (e.g., understanding fractional equivalence or comparing fractions; cf. Hallet, Nunes, & Bryant, 2010) and procedural fraction knowledge (e.g., performing mathematical algorithms with fractions; cf. Byrnes, 1992). We chose 8 years as the lower bound of the age range in this study, given that children do not receive much instruction about fractions prior to third grade. Results showing a relation between children's proportional reasoning and fraction knowledge would suggest that being able to think about proportions spatially may help to overcome the tendency to apply whole-number concepts to fraction problems. Such a relation could also signify that better understanding of formal fractions enhances reasoning about non-numerical proportions. Although a correlation would not allow for firm conclusions about the causal direction, finding a relation is a critical first step in supporting theorizing and developing viable interventions.

We also investigated whether the cognitive processes involved in spatial proportional reasoning differ for part–whole and part–part reasoning. Proportions can be represented as either part–whole relations (e.g., the amount of juice in relation to the total amount of liquid) or part–part relations (e.g., the amount of juice in relation to the amount of water). Some previous research has suggested that part–part encoding is easier for 6- to 8-year-old children (Spinillo & Bryant, 1991). Another study (Singer & Resnick, 1992) showed that 11- to 13-year-old children needed to have information about both parts to make decisions about proportional problems, whereas information about the whole was less crucial, indicating that children relied on part–part rather than part–whole relations. However, a study by Sophian and Wood (1997) yielded evidence that children performed better for problems involving part–whole reasoning than for problems involving part–part reasoning.

These results suggest that the framing of problems might influence young children's proportional reasoning and account for these differences. Therefore, in the present study, we varied the presentation such that half of the children saw proportions in which amounts of juice and water were presented on top of each other, making the parts as well as the part–whole relation easily accessible (stacked displays; see Figure 1). The other half saw proportions in which the amounts were presented aligned next to each other, thus providing easy access to the sizes of the parts but less obvious information about the part–whole relation (side-by-side displays). If encoding of part–whole relations is easier than encoding of part–part relations or vice versa, we expected to see differences in strategies and/or accuracies. Moreover, given that fractions are part–whole relations, it was reasonable to

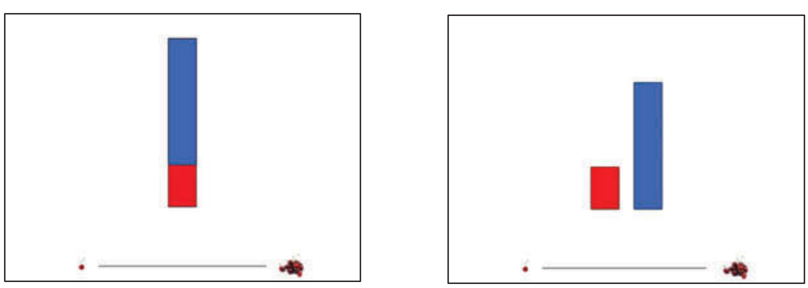

FIGURE 1 Examples of a stacked (left) and a side-by-side (right) presentation of cherry juice (e.g., 6 units) and water (e.g., 24 units) in the proportional reasoning task.

expect a more robust association between formal fraction knowledge and presentations that highlight part–whole relations.

To date, it also remains an open question as to *why* these different kinds of presentations might lead to different results. One possible reason may be that they promote a different understanding of how proportional components should be scaled. Scaling can be defined as a process of transforming absolute magnitudes while conserving relational properties, and it is therefore an important aspect of proportional reasoning (Barth, Baron, Spelke, & Carey, 2009; Boyer & Levine, 2012; McCrink & Spelke, 2010). The importance of scaling for proportional reasoning is evident in everyday life—for instance, when one wants to adjust the amounts of ingredients for a cake from 6 servings to 10 servings or prepare the same concentrations of syrup–water mixtures in different jugs. It is possible that during part–part reasoning, in which the focus lies on the parts themselves as well as on the relation of the parts to each other (e.g., Part A is bigger than Part B), it is harder to see how much the magnitudes have to be scaled, as compared with part–whole presentations, in which the focus lies on the total amount. To test this assumption, we took advantage of the fact that previous research has shown that error rates increased linearly with larger scaling factors (cf. Boyer & Levine, 2012; McCrink & Spelke, 2010; Möhring, Newcombe, & Frick, 2014), indicating that scaling entails cognitive costs. Thus, we presented proportions of different magnitudes, such that their sizes had to be transformed by four different scaling factors to match the size of the rating scale. If scaling was used predominantly in part–whole presentations, one could expect errors to increase as a linear function of scaling factor for part–whole displays but not for part–part displays.

METHOD

Participants

Fifty-two 8- to 10-year-old children participated in the present study. Half of the children were assigned to the stacked condition (n = 26, 14 girls; M_{age} = 9;3, range = 8;0–10;8), and the other half were assigned to the side-by-side condition (n = 26, 12 girls; M_{age} = 9;3, range = 8;1–10;8). Four additional children were tested but excluded from the final sample due to an unclear status in mathematics because of homeschooling (one 8-year-old), diagnoses of an attention-deficit disorder (one 9-year-old and one 10-year-old), or incomplete data on the proportional reasoning test (one 10-year-old). Children were recruited from a pool of families that had volunteered to take part in studies of child development and came from 28 different schools that were located in 15 different school districts near a large U.S. city. Children were predominantly Caucasian and from middle-class backgrounds.

Stimuli

The materials for the proportional reasoning task consisted of 16 pictures that were presented on white paper in a ring binder. The pictures showed a red and a blue rectangle, representing cherry juice and water, respectively. The rectangles were 2 cm wide; their length was varied systematically, according to a factorial design. Below the rectangles, a 12-cm-long horizontal line served as a rating scale. A single cherry was printed next to the left end of the scale, indicating a faint taste of cherries; a heap of many cherries was shown next to the right end of the scale, indicating a strong taste of cherries. In the stacked condition, the red and blue rectangles were presented stacked on top of each other; in the side-by-side condition, they were presented next to each other, aligned on the bottom with 1 cm between them (see Figure 1).

A test of fraction knowledge was developed based on the Common Core State Standards for Mathematics (for examples, see the Appendix). Several aspects of fraction understanding from Grade 3 to Grade 5 were included (e.g., Grade 3, using visual fraction models, understanding fraction equivalence by comparing fractions with equal denominators; Grade 4, understanding fraction equivalence by comparing fractions with unequal denominators, adding and subtracting fractions with equal denominators, multiplying fractions with whole numbers, understanding the decimal notation for fractions; Grade 5, adding and subtracting fractions with unequal denominators, multiplication and division of fractions, calculating with mixed numbers). The questions were presented numerically (i.e., no word problems were included) as fraction estimations or comparisons, missing value problems, or open-ended problems. All children worked on the same fractions test that consisted of problems addressing knowledge from Grade 3 to Grade 5. Children of every age group attempted all problems. There were a total of 25 problems that were scored with 1 point each if solved correctly, and the number of points was translated into a percentage score. Children were allowed to skip a problem if they did not know the answer, which was scored with 0 points.

Procedure and Design

Children were tested individually in a laboratory room. The experimenter first presented the proportional reasoning task by showing the child a picture of a bear and telling a short story about how the bear likes to drink cherry juice with water. The experimenter explained that cherry juice is made of cherries, very sweet and red. Then, the child was presented with different combinations of cherry juice and water and was asked to help the bear decide how much each combination would taste of cherry.

Children were randomly assigned to either the stacked or the side-by-side condition, and they received three instruction trials in the same format as the later test trials. The first two instruction trials served as end-anchor trials in which the experimenter explained the two end anchors of the scale and pointed out the two amounts of cherry juice and water using gestures by indicating their length between their index finger and thumb. For the first end-anchor trial (28 units of juice vs. 2 units of water, with 1 unit being equal to 0.5 cm), the experimenter placed a small rubber peg on the correct location on the 12-cm scale. In the second trial (2 units of juice vs. 28 units of water), the experimenter asked the child to guess how much this mixture would taste of cherry and to place the rubber peg accordingly. Children received corrective feedback on their responses. On the third instruction trial (22 units of juice vs. 8 units of water), children were asked to place the rubber peg at a point between the end anchors on the rating scale that would indicate the cherry taste of this mixture. This trial served to prevent children from only using the end positions of the scale and to further familiarize them with the rating scale and the placement of the rubber peg. The experimenter marked each child's response using a fine-tip wet-erase marker and flipped the page for the next trial.

Amounts of juice and water presented in instruction trials were different from those in subsequent test trials.

Test trials consisted of systematic combinations of cherry juice and water, such that the cherry juice part (3, 4, 5, and 6 units) as well as the total amount (6, 12, 18, and 24 units) varied on four levels. These 16 combinations were presented twice in two consecutive blocks, yielding a total of 32 trials that took about 10 min. Because the total amounts of 6, 12, 18, and 24 units had to be mapped onto a rating scale of 24 units (which equals 12 cm), children had to scale the total amount by a factor of 4, 2, 1.33, or not to scale (factor of 1), respectively. Thus, the design involved four scaling factors, in which the proportional components had to be either mapped directly (i.e., scaling factor 1:1) or scaled to fit the size of the rating scale (i.e., scaling factors 1:1.33, 1:2, or 1:4). Children did not receive any feedback. The combinations were presented in one of two different quasi-random orders, in which immediate repetitions of factor levels were avoided. Roughly half of the participants were randomly assigned to each order.

After the proportional reasoning task, children were presented with the paper-and-pencil fractions test involving numeric fractions. The experimenter read the questions aloud to each child and no feedback was given. The fraction test took about 15 min to 25 min.

RESULTS

In the first step, children's information integration strategies on the proportional reasoning task were classified by means of analyses of variance (ANOVAs) for children in the stacked and side-by-side conditions. Single main effects of either juice or total amount were taken as an indicator that participants focused on one of these two dimensions (centration). Main effects of both juice and total amount indicated that the two pieces of information were integrated. As can be seen in the normative response pattern in Figure 2, a correct proportional integration strategy would result in a fan-shaped pattern, which is statistically indicated by significant main effects as well as an interaction of total amount and juice. In contrast, a subtractive integration strategy would be evident in a parallel pattern and, statistically, in significant main effects only.[1] In a second step, we examined children's accuracy on an absolute level. That is, we focused on the question of how close their ratings were to the normative responses and tested how scaling factors influenced children's accuracy. Finally, the relation between children's accuracy in the proportional reasoning task and their fraction test scores was investigated.

[1] Strategies were also analyzed on an individual level (cf. Wilkening, 1979) to rule out averaging artifacts. The majority of children used a proportional integration rule in both conditions, but a slightly smaller percentage of children in the side-by-side condition (38.5%) than in the stacked condition (57.7%) did so. An equal number of children used a subtractive strategy in both conditions (30.8%). Fisher's exact tests showed no significant difference in strategy use between the two conditions ($p = .31$) nor between younger and older children ($p = .36$). Children who used a proportional or a subtractive integration rule applied this rule with very high consistency (i.e., Pearson correlations between measurement repetitions were $r = .93$ and $r = .76$, respectively).

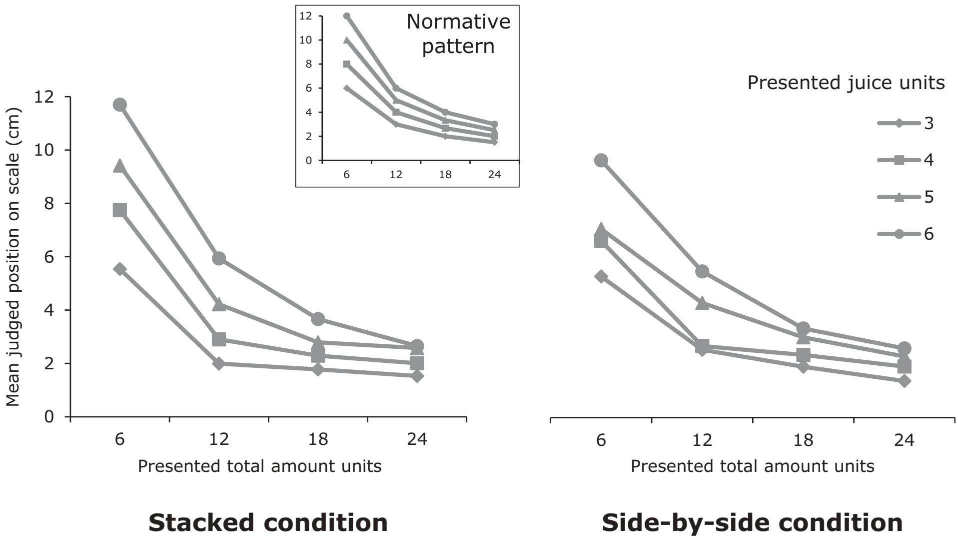

FIGURE 2 Children's ratings of the combined juice and total amount units (integration patterns) on the group level for children in the stacked and side-by-side conditions.

Information Integration Strategies on the Proportional Reasoning Task

A preliminary overall ANOVA of "cherriness" ratings (in cm) revealed a significant interaction of juice and sex, $F(3, 144) = 2.91$, $p < .05$, $\eta^2 = .06$, due to girls' higher ratings for the two largest juice amounts; however, Bonferroni-corrected post hoc tests revealed no significant differences (all *p*s > .05). As this interaction was unexpected and not easily interpretable and because there were no further significant effects of order and sex (all *F*s < 2.06, *p*s > .10), data were collapsed across these variables in subsequent analyses.

To investigate the effects of presentation type on children's responses, an ANOVA with this between-participants variable and the within-participant variables of total amount (4) and juice (4) was calculated. Given the relatively wide age range in the present study, children were divided into younger (M_{age} = 8;6, *SD* = 5 months) and older children (M_{age} = 10;1, *SD* = 6 months) using a median split, and age (younger vs. older children) was added to the analysis as a between-participants variable. This analysis revealed significant interactions of presentation type with total amount, $F(3, 144) = 6.70$, $p < .001$, $\eta^2 = .12$, and presentation type with juice, $F(3, 144) = 5.01$, $p < .01$, $\eta^2 = .10$, as well as a significant three-way interaction of presentation type, total amount, and juice, $F(9, 432) = 2.60$, $p < .01$, $\eta^2 = .05$. These effects indicate that children in the stacked and side-by-side conditions differed in their integration of the two components. In addition, the ANOVA revealed a significant interaction of age group and total amount, $F(3, 144) = 3.92$, $p < .05$, $\eta^2 = .08$. Older children's ratings differed more between the total amounts, whereas younger children's ratings were closer together. However, Bonferroni-corrected post hoc comparisons showed that only the two smallest total amounts of 6 units and 12 units (both *p*s < .05) differed significantly between younger and older children. There were no further significant effects of age group (all *F*s < 3.75, all *p*s > .059) or presentation type (all *F*s < 3.09, *p*s > .08).

To shed light on the three-way interaction of presentation type, total amount, and juice reported above, separate ANOVAs for the two conditions were carried out. In the stacked condition, the ANOVA yielded significant main effects of total amount, $F(3, 75) = 601.16$, $p < .001$, $\eta^2 = .96$, and of juice, $F(3, 75) = 242.06$, $p < .001$, $\eta^2 = .91$, and a significant interaction of total amount and juice, $F(9, 225) = 25.95$, $p < .001$, $\eta^2 = .51$. In the side-by-side condition, the same effects were found: a significant effect of total amount, $F(3, 75) = 80.77$, $p < .001$, $\eta^2 = .76$, and of juice, $F(3, 75) = 68.72$, $p < .001$, $\eta^2 = .73$, and a significant interaction of total amount and juice, $F(9, 225) = 9.20$, $p < .001$, $\eta^2 = .27$. These results indicate that on the group level, children integrated the information according to a proportional integration rule in both conditions. However, as Figure 2 indicates, the response pattern of children in the stacked condition looked almost identical to the normative pattern, but the pattern was somewhat less clear in the side-by-side condition. That is, even though children integrated both proportional components in both conditions, their integration pattern appeared less accurate on an absolute level in the side-by-side condition. Thus, in the next section, children's absolute accuracy was investigated further.

Children's Absolute Accuracy

To investigate children's absolute accuracy, it was necessary to transform the data and standardize scores across differences in using the rating scale. For example, one child might have used only a small part of the scale, whereas another child might have used the total length, by distributing the responses over the whole scale. Such individual usage of the rating scale does not affect analyses of the response strategies reported, because these are based on relative differences between single responses. However, it would be misleading when averaging across absolute accuracies. Furthermore, the slightly compressed response pattern in the side-by-side condition as compared with the stacked condition might have been a result of a restricted usage of the rating scale. To control for such restricted usage by different individuals or in different conditions, children's responses were standardized by dividing the raw responses by each child's individual standard deviation. This procedure, termed *ipsatization*, is one way of standardizing individual data and is typically used to address systematic response biases or tendencies to shift responses to one end of the rating scale (Fischer, 2004; Hicks, 1970). In the next step, a variable for children's overall performance in the proportional reasoning task was created. To this end, children's responses (ipsatized) were subtracted from the normative (ipsatized) responses. Then, the absolute values of these deviations from the norm were averaged across trials.

To find out whether children in the stacked and side-by-side conditions differed on an absolute level, an ANOVA was calculated with presentation type (stacked vs. side-by-side) and age group (younger vs. older) as between-participants variables and absolute deviation as a dependent variable. The analysis showed a significant main effect of presentation type, $F(1, 48) = 8.78$, $p < .01$, $\eta^2 = .16$, with children in the stacked condition ($M = 0.35$, $SE = 0.03$) showing smaller deviations from the correct response than children in the side-by-side condition ($M = 0.62$, $SE = 0.09$, all *p*s < .01). Age group also had a significant effect, $F(1, 48) = 7.19$, $p < .01$, $\eta^2 = .13$, with older children ($M = 0.37$, $SE = 0.04$) outperforming younger children ($M = 0.60$, $SE = 0.09$, all ps < .01). There were no further significant effects (all Fs < 2.49, all ps > .12).

The Influence of Scaling on Children's Accuracy

Children's absolute deviations were averaged across scaling factors and a repeated-measures ANOVA was calculated, with scaling factor (1:1, 1:1.33, 1:2, 1:4) as a within-participant variable and presentation type (stacked vs. side-by-side) and age group (younger vs. older) as between-participants variables. The ANOVA yielded a significant effect of scaling factor, $F(3, 144) = 4.22$, $p < .01$, $\eta^2 = .08$, which was qualified by a significant three-way interaction between scaling factor, presentation type, and age group, $F(3, 144) = 2.69$, $p < .05$, $\eta^2 = .05$. There were no further significant effects (all Fs < 2.50, all ps > .06). To shed light on this three-way interaction, separate ANOVAs with scaling factor and age group for the different presentation types were calculated. In the stacked condition, scaling factor had a significant effect, $F(3, 72) = 14.71$, $p < .001$, $\eta^2 = .38$, which was best explained by a linear function, $F(1, 24) = 26.60$, $p < .001$, $\eta^2 = .53$, indicating that deviations increased linearly with larger scaling factors (see Figure 3). There were no further significant effects (all Fs < 2.94, all ps > .09). By contrast, there was no effect of scaling factor in the side-by-side condition, $F(3, 72) = 0.68$, $p = .57$, $\eta^2 = .03$, and no interaction with age group, $F(3, 72) = 1.50$, $p = .22$, $\eta^2 = .06$. The ANOVA yielded a significant main effect of age group only, $F(1, 24) = 5.06$, $p < .05$, $\eta^2 = .17$, because older children ($M = 0.43$, $SE = 0.12$) outperformed younger children ($M = 0.81$, $SE = 0.12$). Thus, even though children's accuracy increased with age in the side-by-side condition, performance was not influenced by scaling factor, as it was in the stacked condition.

Test of Fraction Knowledge

An ANOVA with children's fraction test scores as the dependent variable and age group (younger vs. older) and presentation type (stacked vs. side-by-side) as between-participants variables yielded a significant main effect of age group, $F(2, 48) = 47.19$, $p < .001$, $\eta^2 = .50$, showing that older children ($M = 74.0\%$, $SE = 3.57$) performed better than younger children ($M = 44.9\%$, $SE = 2.13$; all ps < .001). The ANOVA yielded no other significant effects (all Fs < .16, ps > .69) and, therefore, no significant difference between the fraction test scores of children in the stacked ($M = 58.6\%$, $SE = 4.12$) and side-by-side conditions ($M = 60.3\%$, $SE = 4.13$). This difference was partly due to younger children (M 35.1 %, $SE = 2.88$) solving more problems incorrectly than older children ($M = 24.4\%$, $SE = 3.53$), $t(50) = 2.33$, $p < .05$, $d = 0.66$. In addition, younger children skipped more problems ($M = 20.0\%$, $SE = 3.76$) as compared with older children ($M = 1.1\%$, $SE = 0.47$), $t(50) = 5.00$, $p < .001$, $d = 1.41$.

Relation Between Proportional Reasoning and Fraction Knowledge

Pearson correlations between children's fraction knowledge (fraction test scores) and their proportional reasoning (mean absolute deviations) were calculated. If children's proportional reasoning is related to their fraction knowledge, a significant negative correlation

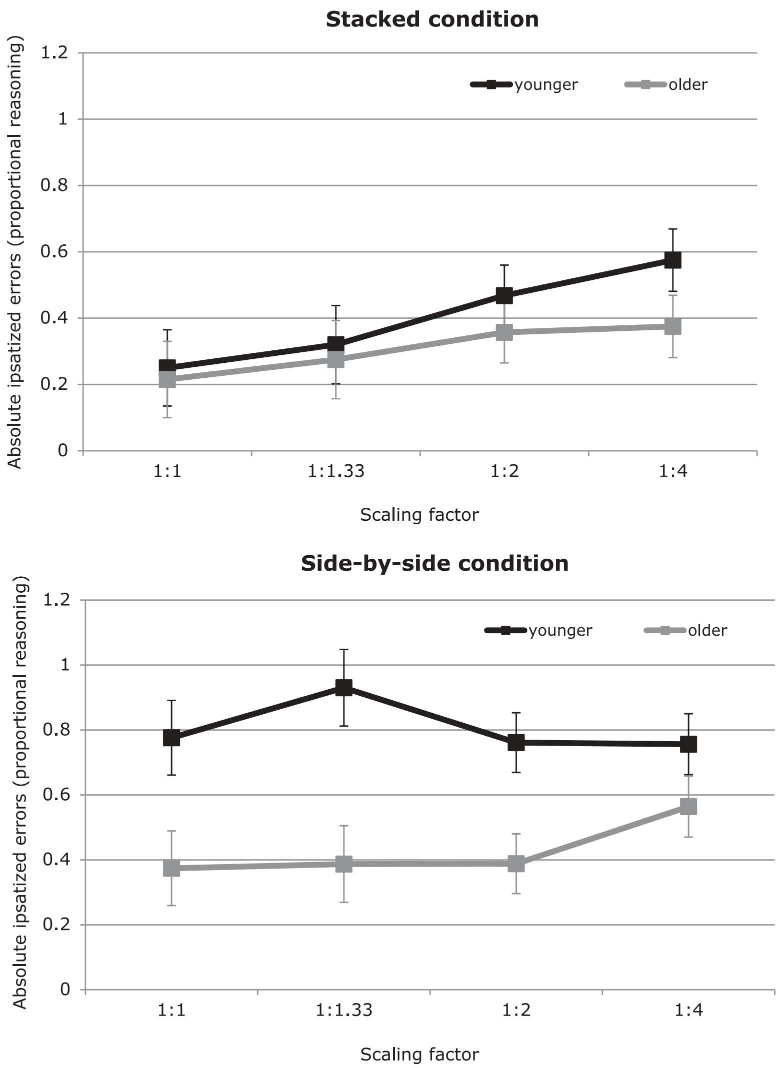

FIGURE 3 Absolute (ipsatized) errors averaged over scaling factors for younger and older children in the stacked and side-by-side conditions.

would be expected, with smaller deviations in the proportional reasoning task going along with a higher score in the fractions test. The correlation in the stacked condition was highly significant and negative, $r(24) = -.61$, $p < .001$, even after controlling for age, $r(23) = -.47$, $p < .05$. By contrast, the correlation in the side-by-side condition was not significant, $r(24) = -.28$, $p = .17$, and remained nonsignificant after controlling for age, $r(23) = .07$, $p = .75$. Using the Fisher's r-to-z transformation, the difference between these age-controlled correlations in the two conditions was found to be significant, $z = -1.97$, $p < .05$.

Linear regression analyses were carried out, with age entered in a first step, absolute deviation from the correct answer entered in a second step, and the fraction test score entered as the predicted variable. In the stacked condition, these two predictors accounted for a significant part of the variance, R^2 = .72, $F(2, 25)$ = 28.95, p < .001. As would be expected, age explained a significant part (63%) of the variance (β = .66, p < .001). However, adding proportional reasoning as a predictor significantly increased the explained variance of the model (ΔR^2 = .08, β = –.32, p < .05). In the side-by-side condition, the explained variance was R^2 = .40, $F(2, 25)$ = 7.60, p < .01. In this case, age explained all 40% of the variance (β = .66, p < .01), and proportional reasoning did not add any explained variance (β = .06, p = .75).

DISCUSSION

The present study investigated 8- to 10-year-olds' proportional reasoning in terms of their integration of proportional components, their absolute accuracy, and the relation between children's proportional reasoning and formal fraction understanding. Findings suggested that children as young as 8 years old were able to consider both components that constitute a proportion and integrate them in a normative proportional way.[2] These results stand in contrast to previous claims that proportional reasoning does not emerge before the age of 11 years (Moore et al., 1991; Noelting, 1980; Piaget & Inhelder, 1975), and they confirm other findings that even younger children are able to reason about proportions (Acredolo et al., 1989; Boyer & Levine, 2012; Boyer et al., 2008; Jeong et al., 2007; Schlottmann, 2001; Singer-Freeman & Goswami, 2001; Sophian, 2000; Spinillo & Bryant, 1991, 1999). In line with previous paradigms showing earlier success in children's proportional reasoning, it is possible that the presentation of continuous proportional quantities and the nature of the response mode (spatial ratings that are more intuitively graspable) led to children's success on our proportional reasoning task.

Part–Whole Versus Part–Part Encoding

Although on the group level, children in both presentation conditions integrated components proportionally, the compressed integration pattern in the side-by-side condition suggested that children differentiated the units of juice less than in the stacked condition. Even after controlling for idiosyncratic usage of the rating scale by normalizing the variance of the responses, children's deviations from the norm were significantly higher in the side-by-side condition than in the stacked condition. This finding indicates that the task was more difficult if the components were presented side by side as two separate objects. The finding that children in the two conditions did not differ in their average fraction test scores rules out the possibility that the present results are due to children in the stacked condition having a better overall understanding of rational numbers. In general, these results are in line with previous studies that have demonstrated better proportional reasoning performance in the context of part–whole rather

[2] A substantial number of children applied a proportional strategy on the individual level, suggesting that these group results were not due to averaging artifacts.

than part–part relations (Sophian & Wood, 1997). These results imply that the instruction of proportions in school may benefit from focusing on part–whole relations instead of comparing separate parts.

Analyses of how scaling influenced children's absolute errors revealed that children in the side-by-side condition showed large errors overall, but these deviations were not affected by scaling factor. Neither older nor younger children showed signs of scaling in the side-by-side condition, even though performance generally improved with age.[3] By contrast, children in the stacked condition showed smaller errors that increased with larger scaling factors, suggesting that they mentally expanded the proportional amounts to match them onto the rating scale (cf. Boyer & Levine, 2012; Möhring et al., 2014). Thus, it appears that children in the stacked condition were aware of the necessity to scale the magnitudes, whereas children in the side-by-side condition did not seem to transform the proportions accordingly. An understanding of scaling may have been more difficult in the side-by-side condition, because the separate parts were more prominent and a between-object relation had to be mapped onto a unitary rating scale, which may have included an additional processing step of mentally combining the two amounts. By contrast, in the stacked condition, the two amounts were presented already combined into one coherent Gestalt, which may have been easier to map onto the rating scale.

It is also conceivable that such between-object relations may have led children to focus on absolute amounts, which may have misled them to focus on *extensive* rather than *intensive* properties (Howe, Nunes, & Bryant, 2010; Jäger & Wilkening, 2001; Strauss & Stavy, 1982). Whereas intensive properties do not depend on the extent or absolute amount of the whole, extensive properties do. For example, if someone drank half of the cherry-water mixture in a glass, the remaining mixture would still taste the same (intensive property), whereas its volume would decrease (extensive property). Thus, in the side-by-side condition children may have focused on volume or absolute amount, whereas an understanding of proportion would require focusing on *intensive* quantities such as juice concentration (taste).

The Relation Between Proportional Reasoning and Fraction Understanding

Importantly, our results showed that children's proportional judgments were associated with their knowledge about fractions. However, this correlation was significant in the stacked condition only, which was the easier condition in that overall accuracy was significantly higher than in the side-by-side condition. This correlational finding is in line with previous findings that numerical magnitude estimations (i.e., ability to compare sets of dots or place whole numbers or fractions on a number line) are associated with mathematics achievement (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Halberda, Mazzocco, & Feigenson, 2008; Siegler & Booth, 2004; Siegler et al., 2011). Our findings extend these results by showing that children's estimations of *spatial, non-numerical proportions* are related to their

[3] The fact that older children outperformed younger children in the present proportional reasoning task could be explained by a general increase in cognitive abilities, but it could also be that older children benefitted more from feedback during the instruction trials (cf. Opfer & Thompson, 2014). Future studies may systematically investigate the importance of feedback for proportional reasoning at different ages.

formal, numerical fraction knowledge. This relation was found even after controlling for age, showing that individual differences in a spatial sense of proportions are associated with the ability to conceptualize formal fractions and perform mathematical operations on them above and beyond effects of age.

A possible explanation for why these abilities are related is that children who have a better understanding of the relative size of proportions are better able to visualize fractions in terms of spatial analogues, which in turn may help them to understand numerical fractions (perhaps because they can differentiate plausible and implausible answers). The importance of spatial analogues for students' understanding of fractional magnitudes was shown in a recent intervention study with at-risk children (Fuchs et al., 2013). This training mainly involved representing, comparing, ordering, and placing fractions on a number line from 0 to 1. Children in the training group showed considerable gains in their ability to carry out operations with fractions relative to a control group. Along the same lines, cultural differences in how fractional magnitudes are introduced in school have been shown to affect children's fraction understanding (Ma, 1999; Moseley, Okamoto, & Ishida, 2007). Whereas teachers in the United States explain fractions often with the concept of counting parts (e.g., 1/3 as one of three slices of a pizza), teachers in Japan or China explain fractions as distances on number lines. Even though in both cases, children may develop a representation of fractional magnitudes, imagining magnitudes by partitioning can be troublesome when it comes to very big fractions (e.g., 385/975), improper fractions (e.g., 5/4), and negative fractions (–1/4). However, the same examples of fractions can be imagined more easily on a number line, which might be one reason why Chinese and Japanese students show a better overall fraction understanding as compared with U.S. students (Ma, 1999; Moseley et al., 2007). In line with these observations, several researchers have suggested that teaching fractions in U.S. schools would profit from using multiple representations ranging from subdividing circles, to folding paper strips, and to using sets of discrete chips to represent a fraction (cf. *The Rational Number Project*, Cramer, Behr, Post, & Lesh, 1997/2009).

Another explanation for the correlation between proportional reasoning and fraction understanding in the stacked condition may be that children with better fraction knowledge performed better in the proportional reasoning task. That is, children's formal fraction knowledge may have helped them to encode spatial proportions and to reproduce them on the rating scale. Even though this possibility cannot be eliminated by our correlational results, it seems unlikely in light of many studies (Acredolo et al., 1989; Boyer & Levine, 2012; Boyer et al., 2008; Jeong et al., 2007; Schlottmann, 2001; Singer-Freeman & Goswami, 2001; Sophian, 2000; Spinillo & Bryant, 1991, 1999) showing signs of proportional reasoning at an age when understanding of formal fractions is not present (Hecht & Vagi, 2010; Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004). Nonetheless, future studies using longitudinal designs or training components are needed to pin down the causal direction of the relation we have identified. It should also be noted that children in our sample came from various schools and thus differed in how they learned about fractions. Even though we were able to control for general effects of fraction exposure in school by controlling for age, we were not able to investigate the specific effects that differences in fraction instruction had on children's proportional reasoning. Future studies may incorporate this aspect in their design and try to disentangle effects of differences in fraction instruction.

One testable implication of our findings is that experience and training with spatial proportions may facilitate children's understanding of fractions and thus their eventual success in mathematics. For example, fostering children's ability to visualize proportions may improve their understanding of fractional equivalence because they may realize that part–whole relations stay the same even though they involve different numbers of parts and different unit sizes (e.g., 1/5 and 2/10). Such training may also increase children's visual-spatial competencies, which have been found to be an important predictor for fraction concepts (Vukovic et al., 2014). In addition, children's fraction knowledge may benefit from experience localizing proportions on a (mental) line or scale. Overall, our finding of a significant relation between children's ability to rate proportional magnitudes and their ability to understand formal fractions adds to a growing body of research supporting the importance of *spatializing* the mathematics curriculum in the elementary school years (Mix & Cheng, 2012; Newcombe, 2013; Newcombe, Uttal, & Sauter, 2013).

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APPENDIX

Examples of Problems in the Fraction Test

Running head: SPATIAL PROPORTIONS AND FORMAL FRACTIONS

Appendix: Examples of problems in the fraction test.

Which fraction is smaller? Please circle your answer.

$\frac{4}{12}$ **or** $\frac{2}{12}$

$\frac{1}{20}$ **or** $\frac{3}{4}$

Please add/subtract/multiply/divide the following values:

$\frac{8}{12} - \frac{6}{12} =$ ________________ $\frac{1}{2} \times 6 =$ ________________

$\frac{5}{9} + \frac{2}{9} =$ ________________ $\frac{7}{10} + \frac{2}{100} =$ ________________

True or false? Please circle your answer.

$\frac{12}{13} = \frac{24}{25}$ **T / F** $\frac{6}{15} < \frac{36}{60}$ **T / F**

$\frac{3}{10} > \frac{40}{100}$ **T / F** $2\frac{1}{2} = \frac{2}{2} + \frac{1}{2}$ **T / F**