LOCAL LAWS AND FLUCTUATIONS FOR SUPER-COULOMBIC RIESZ GASES

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Abstract. We study the local statistical behavior of the super-Coulombic Riesz gas of particles in Euclidean space of arbitrary dimension, with inverse power distance repulsion integrable near 0, and with a general confinement potential, in a certain regime of inverse temperature. Using a bootstrap procedure, we prove local laws on the next order energy and control on fluctuations of linear statistics that are valid down to the microscopic lengthscale, and provide controls for instance, on the number of particles in a (mesoscopic or microscopic) box, and the existence of a limit point process up to subsequences.

As a consequence of the local laws, we derive an almost additivity of the free energy that allows us to exhibit for the first time a CLT for Riesz gases corresponding to small enough inverse powers, at small mesoscopic length scales, which can be interpreted as the convergence of the associated potential to a fractional Gaussian field.

Compared to the Coulomb interaction case, the main new issues arise from the nonlocal aspect of the Riesz kernel. This manifests in (i) a novel technical difficulty in generalizing the transport approach of Leblé and the second author to the Riesz gas which now requires analyzing a degenerate and singular elliptic PDE, (ii) the fact that the transport map is not localized, which makes it more delicate to localize the estimates, (iii) the need for coupling the local laws and the fluctuations control inside the same bootstrap procedure.

1. Introduction

We are interested in proving local laws and studying the fluctuations of super-Coulombic Riesz gases. These are ensembles of point configurations $X_N = (x_1, \dots, x_N)$ with $x_i \in \mathbb{R}^{\mathsf{d}}$ whose law is given by

$$d\mathbb{P}_{N,\beta}(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N \tag{1.1}$$

with

$$\mathcal{H}_N(X_N) = \frac{1}{2}\sum_{i\neq j} \mathsf{g}(x_i - x_j) + N\sum_{i=1}^N V(x_i) \tag{1.2}$$

where

$$\mathsf{g}(x) = \begin{cases} \frac{1}{\mathsf{s}}|x|^{-\mathsf{s}} & \text{for } \mathsf{s} \neq 0 \\ -\log|x| & \text{for } \mathsf{s} = 0 \end{cases} \tag{1.3}$$

with the condition

$$\mathsf{d} - 2 < \mathsf{s} < \mathsf{d}. \tag{1.4}$$

The condition (1.4) that we will use throughout implies that the case $\mathsf{s} = 0$, or log gas case, is then only encountered in dimension $\mathsf{d} = 1$. The case $\mathsf{s} = \mathsf{d}-2$ corresponds to the Coulomb case in any dimension, this is why the condition (1.4) corresponds to a super-Coulombic Riesz gas. Note that as s becomes larger than d, the kernel becomes nonintegrable near 0 but integrable at infinity. The regime $\mathsf{s} > \mathsf{d}$, called the *hypersingular case* [BHS14], corresponds to a short-range interaction regime, and the behavior is quite different (see for instance [HLSS18]), hence the restriction to $\mathsf{s} < \mathsf{d}$. The sub-Coulomb case $\mathsf{s} < \mathsf{d} - 2$ on the other hand is longer range due to the slower decay of the interaction, bringing in new difficulties that are outside the scope of this paper.

The function V is an external confining potential, on which we shall place assumptions later.

The parameter $\beta > 0$ is an inverse temperature, and we have chosen to multiply the energy by $N^{-\frac{\mathsf{s}}{\mathsf{d}}}$ because $N^{\frac{\mathsf{s}}{\mathsf{d}}}$ is the typical energy per particle. Finally, the normalization factor

$$Z_{N,\beta} := \int_{(\mathbb{R}^{\mathsf{d}})^N} e^{-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}} \mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N \tag{1.5}$$

is called the partition function.

The Coulomb case is particularly natural and physical, because g is the fundamental solution to the Laplacian:

$$-\Delta \mathsf{g} = \mathsf{c}_{\mathsf{d}} \delta_0 \tag{1.6}$$

where c_{d} is a constant depending only on the dimension, and δ_0 is the Dirac mass. In the Riesz case with $\mathsf{s} \in (\mathsf{d}-2, \mathsf{d})$, and in that interval only, g is instead the fundamental solution to a *fractional Laplacian*

$$(-\Delta)^{\frac{\mathsf{d}-\mathsf{s}}{2}} \mathsf{g} = \mathsf{c}_{\mathsf{d},\mathsf{s}} \delta_0. \tag{1.7}$$

In the following we denote

$$\alpha = \frac{\mathsf{d}-\mathsf{s}}{2}. \tag{1.8}$$

The constant $\mathsf{c}_{\mathsf{d},\mathsf{s}}$ is given by (see [Kwa17])

$$\mathsf{c}_{\mathsf{d},\mathsf{s}} = \frac{\pi^{\frac{\mathsf{d}}{2}} 4^\alpha \Gamma(\alpha)}{\Gamma(\frac{\mathsf{d}}{2} - \alpha)}, \quad \alpha = \frac{\mathsf{d}-\mathsf{s}}{2}. \tag{1.9}$$

We recall that the fractional Laplacian is a nonlocal operator (contrarily to the Laplacian associated to the Coulomb case). It can be seen as an integral operator defined by (its various definitions can be found for instance in [Kwa17] and [LPG+20])

$$(-\Delta)^\alpha f(x) = \mathrm{P.V.} \int (f(x) - f(x+y)) \frac{c_{\mathsf{d},\alpha}}{|y|^{\mathsf{d}+2\alpha}} dy, \quad c_{\mathsf{d},\alpha} = \frac{4^\alpha \Gamma(\frac{\mathsf{d}}{2}+\alpha)}{\pi^{\frac{\mathsf{d}}{2}} |\Gamma(-\alpha)|} = \frac{\alpha 4^\alpha \Gamma(\frac{\mathsf{d}}{2}+\alpha)}{\pi^{\frac{\mathsf{d}}{2}} \Gamma(1-\alpha)}. \tag{1.10}$$

We will also use the homogeneous Sobolev norm defined by

$$\|f\|^2_{\dot{H}^{-\alpha}} := \iint \mathsf{g}(x-y) df(x) df(y) \tag{1.11}$$

which is equivalent with the usual H^s definition via Fourier transform

$$\|f\|^2_{\dot{H}^{-\alpha}} := \frac{2}{c_{\mathsf{d},\mathsf{s}}} \int |\xi|^{-2\alpha} |\hat{f}|^2(\xi) d\xi \tag{1.12}$$

as in [DNPV12, Proposition 3.4], where $c_{\mathsf{d},\mathsf{s}}$ is the constant in (1.10), since $\hat{\mathsf{g}}(\xi) = C_{\mathsf{d},\mathsf{s}} |\xi|^{\mathsf{s}-\mathsf{d}}$ for some constant $C_{\mathsf{d},\mathsf{s}}$ (cf. [Ser24, Proposition 2.14]).

The nonlocality of the operator creates much of the difficulties encountered in the Riesz case.

The Coulomb gas is an important model of statistical physics, in particular due to its connection to plasma physics, random matrix theory, quantum mechanics models, and conformal field theory. We refer to the introduction of [Ser24] for more detail. The Riesz gas is less understood but also physically interesting (in solid state physics, ferrofluids, elasticity), see [Maz11, BBDR05, CDR09, CDFR14, Tor18], and has attracted quite a bit of attention in the recent physics literature, see for instance [KKK+22, KKK+21, ADK+19].

1.1. **The equilibrium measure.** The first order or mean-field asymptotic behavior of the gas is well understood (see for instance [Ser24, Chap. 2]). From [Fro35], [Cho58], if the potential V is lower semicontinuous, bounded below, finite on a set of positive capacity, and satisfies the growth condition

$$\lim_{|x|\to+\infty} (V(x)+\mathsf{g}(x)) = +\infty, \tag{1.13}$$

then the continuous approximation of the Hamiltonian given by

$$\mathcal{E}(\mu) := \frac{1}{2}\iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)\,d\mu(x)d\mu(y) + \int_{\mathbb{R}^\mathsf{d}} V(x)\,d\mu(x) \tag{1.14}$$

is well-defined as long as $\mathsf{s} < \mathsf{d}$ (hence the restriction to that regime – this is called the *potential case*) and has a unique, compactly supported minimizer μ_V among the set of probability measures on $\mathbb{R}^\mathsf{d}$, called the *equilibrium measure* and characterized by the following Euler-Lagrange equation: there exists a constant c_V such that

$$\begin{cases} h^{\mu_V} + V = c_V & \text{quasi-everywhere on } \Sigma \\ h^{\mu_V} + V \geq c_V & \text{quasi-everywhere} \end{cases} \tag{1.15}$$

where $h^{\mu_V} = \mathsf{g} * \mu_V$ is the potential generated by μ_V, and Σ denotes the support of μ_V. In the following, we denote the corresponding *effective potential* by

$$\zeta_V := h^{\mu_V} + V - c_V. \tag{1.16}$$

It is worth noting that in the Coulomb case, since $-\Delta h^{\mu_V} = \mathsf{c_d}\mu_V$ in view of (1.6), taking the Laplacian of the first relation in (1.15) we find

$$\mu_V = \frac{1}{\mathsf{c_d}}\Delta V \quad \text{in } \mathring{\Sigma}, \tag{1.17}$$

where $\mathring{\Sigma}$ denotes the interior of Σ. In contrast, such a manipulation is no longer possible in the Riesz nonlocal case, and there is then no local or explicit expression for μ_V in terms of V. The interested reader can refer to the articles [CSW22, CSW23] for examples of Riesz equilibrium measures.

In the Riesz case $\mathsf{s} \in [\mathsf{d}-2, \mathsf{d})$, this equilibrium measure problem can be rephrased in terms of an obstacle problem / a fractional obstacle problem as was observed for instance in [Ser15, Chapter 2]; this will be very useful for us as it will allow us to use results in that area on the behavior of μ_V and additional results proved in the appendix by X. Ros-Oton. Let us recall more precisely the correspondence (the interested reader can also refer to [Ser24, Section 2.4], [CDM16, AS22]: the *fractional obstacle problem*

$$\min\{(-\Delta)^\alpha h, h-\varphi\} = 0 \tag{1.18}$$

with obstacle $\varphi = c_V - V$ and $\alpha = \frac{\mathsf{d}-\mathsf{s}}{2}$ has solution $h = h^{\mu_V}$. The much-studied classical obstacle problem corresponds to the (Coulomb) case $\alpha = 1$. The fractional obstacle problem, studied for instance in [CSS08a, CDSS17a, ROS17a], is another free boundary problem. There are two sets in the solution to (1.18): the set $\{h^{\mu_V} = \varphi\}$ where the solution touches the obstacle is known as the *contact set* or coincidence set, and its boundary is the *free boundary*. In the set where $h^{\mu_V} > \varphi$, then $\mu_V = (-\Delta)^\alpha h^{\mu_V}$ must vanish. This corresponds to the complement of Σ in (1.15). Note that in general the contact set $\{\zeta_V = h^{\mu_V} - \varphi = 0\}$ contains the *droplet* $\Sigma = \operatorname{supp} \mu_V$, but may be larger, although generically it is not. In the Coulomb case, imposing $\Delta V > 0$ in a neighborhood of Σ ensures that the two sets coincide by taking the Laplacian of $\zeta_V = 0$ (as in the computation for (1.17)), but in the fractional case this computation does not work. Since we want to appeal to the regularity of the free boundary, we take as an assumption that the droplet and free boundary coincide, this is accomplished by requiring that $\zeta_V > 0$ on Σ^c, which is guaranteed by assumption (**A1**).

In the fractional case (contrarily to the Coulomb case), the density of μ_V generically vanishes as one approaches $\partial\Sigma$ from the inside, near *regular points*. This is one of the main results of the appendix that we will need: if x_0 is a regular point of $\partial\Sigma$, then

$$\mu_V(x) \sim \operatorname{dist}(x, \partial\Sigma)^{1-\alpha} \quad \text{as } x \to x_0, x \in \Sigma. \tag{1.19}$$

Note that this behavior for instance matches the *semi-circle law* behavior of the equilibrium measure in the one-dimensional log case (for which $\mathsf{s} = 0$ and $\alpha = 1/2$).

The recent paper [CF25] also exploits the correspondence with the fractional obstacle problem to prove that this behavior near all boundary points is generic with respect to V in dimension $\mathsf{d} \le 3$, for any $\mathsf{s} \in [\mathsf{d}-2, \mathsf{d})$. For simplicity, we will thus assume that all boundary points are regular, which can be guaranteed by assumption (**A2**) and (**A3**) below.

Additionally, we will need some regularity on the quotient $\frac{\mu_V}{\operatorname{dist}(x,\partial\Sigma)^{1-\alpha}}$, this is assumption (**A4**).

Finally, the "lift-off" rate from the obstacle, i.e. the growth of ζ_V is known from the fractional obstacle problem literature: it is

$$\zeta_V(x) \ge c(x) \operatorname{dist}(x, \Sigma)^{1+\alpha} \tag{1.20}$$

with $c(x) > 0$ near regular points. This is in assumption (**A1**).

1.2. **Goal of the paper.** Proving local laws for the gas roughly corresponds to understanding how much the distribution of the points deviates from the mean-field distribution μ_V. We do so via local controls on the *next-order electric energy* (called modulated energy in the dynamics context) $\mathsf{F}_N(X_N, \mu_V)$, first introduced in [PS17] and defined via

$$\mathsf{F}_N(X_N, \mu) := \frac{1}{2} \iint_{\triangle^c} \mathsf{g}(x-y)\, d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu\Big)(x)\, d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu\Big)(y), \tag{1.21}$$

where $\triangle$ denotes the diagonal in $\mathbb{R}^\mathsf{d} \times \mathbb{R}^\mathsf{d}$. This quantity, whose properties are described in [Ser24, Chap. 4], is bounded below by $-CN^{1+\frac{\mathsf{s}}{\mathsf{d}}}$ where C depends only on $\|\mu\|_{L^\infty}$ and behaves effectively like the square of a distance between the empirical measure $\frac{1}{N}\sum_i \delta_{x_i}$ and the reference probability density μ, more precisely like

$$N^2 \left\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i} - \mu \right\|^2_{\dot{H}^{\frac{\mathsf{s}-\mathsf{d}}{2}}}$$

(where $\dot{H}$ denotes a homogeneous Sobolev norm as in (1.11)–(1.12)), but here defined in a *renormalized* manner that allows to admit Diracs thanks to the removal of the (infinite) diagonal self-interaction terms.

As shown in prior works [PS17, LS15], see [Ser24, Chap. 4], the electric energy F_N provides good controls on the difference $\frac{1}{N}\sum_i \delta_{x_i} - \mu$, such as rough bounds on linear statistics, bounds on minimal distances between points and bounds on charge discrepancies (i.e. integrals of the difference over balls or cubes). These controls, based on the *electric formulation* of the energy F_N, will be recalled in Section 3.1, particularly Proposition 3.5.

In parallel with the local laws, we wish to address the related question of understanding the asymptotic behavior of fluctuations of linear statistics, of the form

$$\mathrm{Fluct}_{\mu_V}(\varphi) := \sum_{i=1}^N \varphi(x_i) - N \int \varphi d\mu_V, \tag{1.22}$$

for regular enough test-functions φ.

This program has been in large part completed in the Coulomb case in prior works: [Leb17] for local laws in two dimensions (see [BBNY17] for related local laws), [AS22] for local laws in general dimension, [LS18, BBNY19] for fluctuations in the two-dimensional case, [Ser23] for fluctuations in general dimension. We also refer the reader to [Ser24] for a recap. These local laws are proved by a method of boostrap on scales first introduced in [Leb17], while the fluctuations analysis is based on a transport method, introduced in [LS18] and re-used in [Ser23].

The one-dimensional logarithmic case, which corresponds to the well-known situation of β-ensembles, and is closely related to random matrix theory, was intensively studied: fluctuations and questions similar to local laws were analyzed in [Joh98, Shc13, Shc14, BG13, BG24, BEY12, BEY14, BMP22, BL18, BLS18], closer to our analysis here is the paper [Pei24] which really studies local laws for (1.21) and serves partly as a blueprint for the present paper. Finally, the one-dimensional Riesz gas with general $\mathsf{s} < 1$ was extensively studied in [Bou23b, Bou23a], leveraging on the convexity of the interaction in dimension 1, which permits a very different treatment based on the Helffer-Sjöstrand representation that yields stronger results.

Our goal here is to carry out the program of [Leb17, LS18, AS22, Ser23, Pei24] for super-Coulombic Riesz gases of arbitrary dimension. Note that in contrast with [Ser23] and [Ser24], we will not be able to obtain estimates that are optimal in β as β gets small, because this would require using the "thermal equilibrium measure" (see [Ser24, Chap. 2]) instead of the equilibrium measure and, contrarily to the Coulomb case [AS22], its behavior in the nonlocal case is much less understood than the standard equilibrium measure for which we have detailed knowledge (describe above) thanks to the fractional obstacle problem correspondance.

The paper focuses in particular on the new difficulties brought by the nonlocality of the fractional Laplacian, which are the following:

- Generalizing the transport approach of [LS18] to the Riesz gas now requires analyzing a degenerate and singular elliptic PDE. This was not needed in [Pei24] because in the one-dimensional log case, and in that case only, explicit formulas are available for the "master operator inversion" or equivalently for the transport map.
- That transport map ends up having nonlocalized tails, even if the test function φ one considers is localized at a mesoscale.
- As in prior work, the local laws rely on a *screening procedure*, which itself relies on the electric formulation of the energy together with a dimension-extension procedure to handle the nonlocality. The screening in extended dimension requires subtle adaptations, in particular in this probabilistic setting, and controlling its errors requires

one to couple the local laws and the fluctuations control inside the same bootstrap procedure, a difficulty not present in the (local) Coulomb case.

1.3. **Assumptions.** We will use the notation $|f|_{C^\sigma}$ for the Hölder semi-norm of order σ for any $\sigma \geq 0$ (not necessarily integer). For instance $|f|_{C^0} = \|f\|_{L^\infty}$, $|f|_{C^k} = \|D^k f\|_{L^\infty}$ and if $\sigma \in (k, k+1)$ for some k integer, we let

$$|f|_{C^\sigma(\Omega)} = \sup_{x \neq y \in \Omega} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^{\sigma-k}}.$$

We emphasize that with this convention $f \in C^k$ does not mean that f is k times differentiable but rather that $D^{k-1}f$ is Lipschitz. For $k \geq 1$ integer, we will then use the notation

$$\|f\|_{C^k} = \|f\|_{L^\infty} + \sum_{m=1}^{k} |f|_{C^m}. \tag{1.23}$$

For our main results, we assume the following, for some integer $k \geq 3$ that will be specified and for some $\epsilon > 0$.

A1 (Nondegeneracy): $\zeta_V \geq c\,\mathrm{dist}(x, \partial\Sigma)^{1+\alpha}$ with $c > 0$ in some strict neighborhood of Σ.

A2 (Positive Laplacian): $\Delta V \geq 0$ on $\{c_V > V\}$, where c_V is as in (1.15).
A3 (Lipschitz Free Boundary): The free boundary $\partial\Sigma$ is a Lipschitz graph.
A4 (Regularity of Equilibrium Measure):

$$\mu_V(x) = s(x)\mathrm{dist}(x, \partial\Sigma)^{1-\alpha} \tag{1.24}$$

for some function $s(x) \in C^{k+\epsilon}(\Sigma)$ that is bounded from below.
A5 (Regularity and growth of ∇V): We assume $V \in C^{k+1}$ and there exist $r \geq 0$, $c > 0$ and $C > 0$ such that

$$|\nabla V| \geq c|x|^r, \quad |\nabla^{\otimes m} V| \geq C|x|^{r-m+1} \quad \text{for } |x| \geq C,\ 1 \leq m \leq k. \tag{1.25}$$

A6 (One-cut) $\Sigma = \mathrm{supp}\,\mu_V$ is a connected set.

Note that these assumptions are genericity assumptions that avoid irregular cases. (**A2**) allows us to apply Proposition A.3, which shows that all points of the free boundary are regular. The assumption (1.24) is for instance reasonable in view of the generic behavior (1.19), as shown in [CF25]. We also make the additional assumption (**A6**) that Σ is connected (what is commonly referred to the *one-cut* regime in the one-dimensional case) to avoid some of the additional difficulties that the nonlocality of the interaction creates in the multicut regime (cf. [BG24] for a treatment of the log-gas in the multicut regime). We refer to further discussion of these assumptions in Section 2.3.

1.4. **Main results.** We prove the following two theorems in conjunction, by a bootstrap on sales. We state both at the traditional scale (although the local law will be proven at blown up scale).

1.4.1. *Local laws and consequences.* The first theorem establishes local laws *down to the microscale* $N^{-1/\mathsf{d}}$. They are expressed in terms of a version of the energy localized in a cube $\square_\ell$ of size ℓ (centered at an unspecified point), denoted $\tilde{\mathsf{F}}_N^{\square_\ell}(X_N)$. The precise definition of $\tilde{\mathsf{F}}_N$ will be given in Section 3, it relies on the electric formulation of the energy (as first introduced in [PS17], see [Ser24, Chap. 4, Chap. 7]), in terms of the electric potential h_N associated to the couple (X_N, μ) via

$$h_N[X_N, \mu] = \mathsf{g} * \left(\sum_{i=1}^N \delta_{x_i} - N\mu \right) \tag{1.26}$$

and extended to be a function of $\mathbb{R}^{\mathsf{d}+1}$. Briefly,

$$\tilde{\mathsf{F}}_N^{\square_\ell}(X_N, \mu) = \int_{\square_\ell \times [-\ell, \ell]} |y|^\gamma |\nabla h_{N,\mathsf{r}}|^2$$

where $|y|^\gamma$ is a weight to be specified later, and $h_{N,\mathsf{r}}$ is a suitable truncation of $h_N[X_N, \mu]$. At this point what matters is to know that this quantity is positive, coercive, and controls charge discrepancies and fluctuations of linear statistics in the cube $\square_\ell$. The lengthscale ℓ will range from $\ell = 1$, which corresponds to the macroscale, to $\ell \propto N^{-1/\mathsf{d}}$ which corresponds to the local scale or microscale.

The local laws are only valid in the bulk, i.e. well in the interior of Σ, the support of μ_V, except if $\ell = 1$ (or ℓ is larger than a fixed constant) in which case they are easily seen to hold and are valid globally (we will not state this but refer to (3.6)). More precisely, they are valid in $\hat{\Sigma}$ defined as

$$\hat{\Sigma} := \{x \in \Sigma, \mathrm{dist}(x, \partial\Sigma) \geq \varepsilon\} \tag{1.27}$$

where $\varepsilon > 0$ is some small fixed number.

In the whole paper we will denote $\# I_\Omega$ the cardinality of $\{X_N\} \cap \Omega$, i.e. the number of points in the set Ω. The notation $A \lesssim B$ means that A/B is bounded by a constant independent of β, N, and other parameters of the problem, except possibly for $\mathsf{d}, \mathsf{s}, V, \varepsilon$. The notation C_β and $\mathcal{C}_\beta$ will denote constants which are independent of β when β is larger than a positive constant, say 1, and which may depend on β for $\beta \leq 1$. In the same way $A \lesssim_\beta B$ means $A \leq C_\beta B$ for such a C_β, and O_β is also defined in the same way.

Theorem 1 (Local laws)**.** *Assume* (**A1**)–(**A6**)*. There exists constants $C > 0, C_0 > 0, C_1 > 0, C_2 > 0$, depending only on $\mathsf{d}, \mathsf{s}, V, \varepsilon$, and $\mathcal{C}_\beta > 0$ depending on β only if $\beta < 1$, and a lengthscale $4 \leq \rho_\beta \lesssim_\beta 1$, such that the following holds. For any $\ell \geq \rho_\beta N^{-\frac{1}{\mathsf{d}}}$ and any cube $\square_\ell \subset \hat{\Sigma}$, there exists a good event $\mathcal{G}_\ell$ satisfying*

$$\mathbb{P}_{N,\beta}(\mathcal{G}_\ell^c) \leq C_1 e^{-C_2 \beta \ell^{\mathsf{d}} N} \tag{1.28}$$

such that, if $X_N \in \mathcal{G}_\ell$ then

$$\tilde{\mathsf{F}}_N^{\square_\ell}(X_N, \mu_V) \leq \mathcal{C}_\beta \ell^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \tag{1.29}$$

and

$$C_0 \# I_{\square_\ell} \leq \mathcal{C}_\beta \ell^{\mathsf{d}} N. \tag{1.30}$$

Here, in contrast with [AS22], the local laws are not obtained as exponential moments controls, but only with bounds on the event tails. This is due to the nonlocal nature of the problem and the need to couple local laws controls and fluctuations controls at each scale.

As in the Coulomb case in [AS22], the local laws are valid down to a *temperature-dependent minimal scale* ρ_β, which depends on β and is expected to blow up as $\beta \to 0$. Because we use a description in terms of the equilibrium measure rather than in terms of the more precise thermal equilibrium measure as in [Ser24], our β-dependence in the estimates is sharp for $\beta \geq 1$ but less good when $\beta \to 0$, and also our estimate of the minimal scale is not optimal.

Thanks to the coercivity of $\tilde{\mathsf{F}}_N^{\square_\ell}(X_N, \mu)$ and the controls provided in Proposition 3.5, imported from [Ser24, Chap. 4], we deduce the following corollaries.

Corollary 1.1. *Assume* (**A1**)–(**A6**). *There exist $C_1, C_2, C > 0$ depending only on $\mathsf{d}, \mathsf{s}, V, \varepsilon$ such that the following holds.*

(1) (Discrepancy control) Let B_ℓ be a ball of radius $\ell \geq \rho_\beta N^{-1/\mathsf{d}}$, and if $\ell < 1$ assume moreover that $B_\ell \subset \hat{\Sigma}$. Letting $D(B_\ell) := \int_{B_\ell} (\sum_{i=1}^N \delta_{x_i} - N\mu_V)$, we have either $|D(B_\ell)| \leq C N^{1-\frac{1}{\mathsf{d}}} \ell^{\mathsf{d}-1}$ or

$$\frac{D(B_\ell)^2}{\ell^{\mathsf{s}}} \left| \min\left(1, \frac{D(B_\ell)}{\ell^{\mathsf{d}}}\right)\right| \lesssim_\beta \ell^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \tag{1.31}$$

except with probability $\leq C_1 e^{-C_2 \beta N \ell^{\mathsf{d}}}$.

(2) (Fluctuations control) Assume $\square_\ell$ is a cube of sidelength $\ell \geq \rho_\beta N^{-1/\mathsf{d}}$ included in $\hat{\Sigma}$, and φ is a function such that $\square_\ell$ contains a $2N^{-1/\mathsf{d}}$-neighborhood of the support of φ. For any $\eta \geq N^{-1/\mathsf{d}}$, we have

$$\left| \int_{\mathbb{R}^{\mathsf{d}}} \varphi \left(\sum_{i=1}^N \delta_{x_i} - N d\mu \right) \right| \lesssim_\beta \left(\eta^{\gamma-1} \|\varphi\|^2_{L^2(\Omega)} + \eta^{\gamma+1} \|\nabla \varphi\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \ell^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}}, \tag{1.32}$$

except with probability $\leq C_1 e^{-C_2 \beta N \ell^{\mathsf{d}}}$.

(3) (Minimal distance control) For a configuration X_N, let

$$\mathsf{r}_i := \frac{1}{4} \min(\min_{j \neq i} |x_i - x_j|, N^{-1/\mathsf{d}}). \tag{1.33}$$

Assume $\square_\ell$ is a cube of sidelength ℓ included in $\hat{\Sigma}$. We have

$$\sum_{x_i \in \square_\ell} \mathsf{g}(\mathsf{r}_i) \lesssim_\beta \ell^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \qquad \text{if } \mathsf{s} \neq 0 \tag{1.34}$$

$$\sum_{x_i \in \square_\ell} \mathsf{g}(40 \mathsf{r}_i N^{-\frac{1}{\mathsf{d}}}) \lesssim_\beta \ell^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \qquad \text{if } \mathsf{s} = 0, \tag{1.35}$$

except with probability $\leq C_1 e^{-C_2 \beta N \ell^{\mathsf{d}}}$.

The control of linear statistics fluctuations of (1.32) is not optimal, we will provide a better one below under stronger regularity assumptions on φ.

Since (1.30) provides an N-independent control on the number of points in a ball of size $\rho_\beta N^{-1/\mathsf{d}} < \ell \lesssim N^{-1/\mathsf{d}}$, blowing up the configuration by the factor $N^{1/\mathsf{d}}$, taking large enough microscopic balls and using a Borel-Cantelli type argument, we obtain the existence of a limiting point process up to extraction (we will not provide details as they are almost identical to [AS22, proof of Corollary 1.1]).

Corollary 1.2. *Assume* (**A1**)–(**A6**). *Let $x \in \hat{\Sigma}$ and for every i, let $x_i' = N^{1/\mathsf{d}} x_i$. For fixed β, as $N \to \infty$, the law of the point configuration $\{x_1' - x, \dots, x_N' - x\}$ converges after extraction* *of a subsequence, to a limiting point process with simple points and finite first and second correlation functions.*

We thus provide the first evidence of the existence of a limiting point process, that one may call a Riesz-β point process, but only after subsequences. Note that the only situation outside of $\mathsf{s} = 0$ in which the existence of such a point process is known is the one-dimensional Riesz gas, thanks to the work [Bou23a]. For $\mathsf{s} = 0$, it is known in one dimension and is the sine-β point process, or in two dimensions in the special determinantal case of $\beta = 2$, with the Ginibre point process.

We note that with the ingredients developed for Theorem 1 we could prove a local version of the Large Deviations Principle on empirical fields obtained in [LS15] characterizing the local point process averaged at a scale $\gg N^{-1/\mathsf{d}}$ as the minimum of a rate function. In the particular case of energy minimizers, i.e. $\beta = \infty$, taking advantage of the β-dependence in the estimates, we could in particular derive precise equidistribution of the energy and number of points as in [PRN18] but without the extra decay assumption needed there. The details, both in the case of general β and $\beta = \infty$ would be completely similar to the proof in [AS21], so we omit these results.

1.4.2. *Fluctuations of linear statistics.* Our second theorem concerns fluctuations of linear statistics, as defined in (1.22). We will assume that $\varphi \in C^k_c(\mathbb{R}^{\mathsf{d}})$, that $\rho_\beta N^{-1/\mathsf{d}} < \ell < 1$ and $\operatorname{supp}\varphi \subset \square_\ell \subset \square_{2\ell} \subset \hat{\Sigma}$ for some cube $\square_\ell$ of sidelength ℓ. The case $\ell = 1$ corresponds to the macroscopic case, the case $\ell \ll 1$ to the mesoscopic case. We believe that in the macroscopic case $\ell = 1$, our results hold without the interior condition $\operatorname{supp}\varphi \subset \hat{\Sigma}$, however the PDE analysis of the transport map equation is a little trickier and we chose not to pursue this generality here.

Because of the nonlocal nature of the problem, the results always involve *α-harmonic extensions* of the test-functions. These are defined as follows: if φ is a function in $\mathbb{R}^{\mathsf{d}}$, we let φ^Σ denote its α-harmonic extension to Σ^c, that is the solution to

$$\begin{cases} \varphi^\Sigma = \varphi & \text{in } \Sigma \\ (-\Delta)^\alpha \varphi^\Sigma = 0 & \text{in } \Sigma^c. \end{cases} \tag{1.36}$$

This extension is possible for nice enough functions, and enjoys some regularity, that will be important to us. In particular it is shown in Lemma A.8 that

$$(-\Delta)^\alpha \varphi^\Sigma = w(x)\mathrm{dist}(x, \partial\Sigma)^{-\alpha} \quad \text{as } x \to \partial\Sigma, x \in \Sigma \tag{1.37}$$

for some bounded $w(x)$ and we will need to assume some regularity of w on $\partial\Sigma$.

We develop a Riesz transport method (counterpart of that of [LS18], described in [Ser24, Chap 9,10], in the Coulomb case), to obtain the following control of fluctuations of linear statistics, as defined in (1.22). Here $\square_\ell(z)$ denotes the cube of size ℓ centered at z.

Theorem 2 (Fluctuations control)**.** *Assume* (**A1**)–(**A6**) *with* $k = 5$ *and* (1.37) *with* $w \in C^{5+\epsilon}$.

Assume $\varphi \in C^5_c(\mathbb{R}^{\mathsf{d}})$*, that* $\ell \geq \rho_\beta N^{-1/\mathsf{d}}$*, and* $\operatorname{supp}\varphi \subset \square_\ell(z) \subset \square_{2\ell}(z) \subset \hat{\Sigma}$*, with the estimates*

$$\forall \sigma \leq 5, \quad |\varphi|_{C^\sigma} \leq \mathsf{M}\ell^{-\sigma} \tag{1.38}$$

for some constant $\mathsf{M} > 0$.

Then, there exist $C_1, C_2 > 0$ depending only on $\mathsf{d}, \mathsf{s}, \varepsilon, V, \mathsf{M}$ such that for N sufficiently large, there is an event $\mathcal{G}_\ell$ with

$$\mathbb{P}_{N,\beta}(\mathcal{G}_\ell^c) \le C_1 e^{-C_2 \beta N \ell^{\mathsf{d}}} \tag{1.39}$$

such that if $\tau(\ell N^{\frac{1}{\mathsf{d}}})^{\mathsf{s}-\mathsf{d}}$ is smaller than a constant depending only on $\mathsf{d}, \mathsf{s}, \varepsilon, V, \mathsf{M}$, [1] *we have*

$$\left| \log \mathbb{E}_{\mathbb{P}_{N,\beta}} \left[\exp\left(\tau \frac{\beta}{1+\beta} \mathrm{Fluct}_{\mu_V}(\varphi) \mathbf{1}_{\mathcal{G}_\ell} \right) \right] \right| \lesssim_\beta \left((|\tau| + |\tau|^2)(\ell N^{\frac{1}{\mathsf{d}}})^{\mathsf{s}} \right). \tag{1.40}$$

Note that if we did not make the assumption that Σ is connected but instead the union of n connected components, we would need to make the assumption

$$\int_{\Sigma_i} (-\Delta)^\alpha \varphi^\Sigma = 0 \text{ on each connected component } \Sigma_i \text{ of } \Sigma, \tag{1.41}$$

which suffices to build the needed transport map. Without the condition (1.41), the result may be false due to the possibility of integer fluctuations in the multi-component case (see [BG24] for the treatment of this situation in the one-dimensional log case). This will also be reflected in the fact that without this condition, the equation for the transport map may not be solvable. Since we couple the proofs of Theorem 2 and 1, and in particular make use of Theorem 2 for functions that may not solve (1.41), we restrict our attention throughout the paper to the case where $\mathrm{supp}\,(\mu_V) = \Sigma$ is connected.

In the Coulomb case, the transport method combined with a local free energy expansion with a good enough rate allows us in principle to derive a full Central Limit Theorem for fluctuations of linear statistics, as described in [Ser23] and [Ser24, Chap 9,10]. In dimension two (and one as well), any rate is sufficient to conclude, as seen in [LS18]. In dimension three and higher, the currently available rate, which corresponds to a surface error, is not enough to conclude. The situation in the Riesz case is similar: the method in principle allows to get convergence for any s and d such that $\mathsf{s} \in (\mathsf{d}-2, \mathsf{d})$, except that the rate we are able to obtain is only sufficient when s is small enough and ℓ is small enough.

Let us first discuss the question of free energy expansion for the Riesz gas. In [LS15], the following expansion was shown:

$$\begin{cases} \log Z_{N,\beta} = -\beta N^{2-\frac{\mathsf{s}}{\mathsf{d}}} \mathcal{E}(\mu_V) + \left(\frac{\beta}{2\mathsf{d}} N \log N \right) \mathbf{1}_{\mathsf{s}=0} + \log \mathsf{K}_{N,\beta}(\mu_V, \zeta_V) \\ \log \mathsf{K}_{N,\beta}(\mu_V, \zeta_V) = -N \mathrm{Ent}(\mu_V) + N \mathcal{Z}(\beta, \mu_V) + o(N). \end{cases} \tag{1.42}$$

Here $\mathrm{Ent}(\mu) = \int_{\mathbb{R}^{\mathsf{d}}} \mu \log \mu$, $\mathcal{E}$ is as in (1.14) and

$$\mathcal{Z}(\beta, \mu) := -\beta \int_{\mathbb{R}^{\mathsf{d}}} \mu^{1+\frac{\mathsf{s}}{\mathsf{d}}}(x) f_{\mathsf{d},\mathsf{s}}(\beta \mu^{\frac{\mathsf{s}}{\mathsf{d}}}(x)) dx + \frac{\beta}{2\mathsf{d}} \mathrm{Ent}(\mu) \mathbf{1}_{\mathsf{s}=0}. \tag{1.43}$$

The quantity $f_{\mathsf{d},\mathsf{s}}$ is a function of the inverse temperature (note that here an effective temperature $\beta \mu^{\mathsf{s}/\mathsf{d}}$ appears), and corresponds to the pressure at that temperature, meaning the large volume limit of the free energy per unit volume for a unit charge density. The expression (1.43) then simply corresponds to scaling that limit in terms of the effective density, in other words, to a local density approximation.

In [LS15], the function $f_{\mathsf{d},\mathsf{s}}$ was characterized via the minimum of a rate function governing a large deviations principle on point processes. Here we will characterize it directly as the pressure, by showing in Lemma 7.2 the existence of the large volume limit of the free energy

[1] Since $\mathsf{s} < \mathsf{d}$, this is satisfied for N large enough for instance as soon as $\ell \gg N^{-1/\mathsf{d}}$.

(per unit volume) of a Riesz gas confined to a box, *with a rate*, corresponding to surface additivity errors. This will be done by an almost additivity property of the free energy obtained as a consequence of the local laws, as done in [AS22] in the Coulomb case. The rate obtained in that limit allows in principle to improve the $o(N)$ error in (1.42). However, there are additional surface errors on $\partial\Sigma$ which damage this improvement. But in view of obtaining a CLT via the transport method, what matters is not the rate in the free energy expansion, but rather the rate in the *relative free energy expansion*

$$\log \mathsf{K}_{N,\beta}(\mu_t) - \log \mathsf{K}_{N,\beta}(\mu_0)$$

where μ_t is the *transported measure*, and $\mu_0 = \mu_V$ the original one, and for that we can obtain an improved rate.

To go further, we will thus assume a relative expansion of next-order partition functions (see Sections 5 and 7 for a formal definition)

$$\begin{aligned}(1.44)\quad \log \mathsf{K}^{\mathcal{G}_\ell}_{N,\beta}(\mu_t, \zeta\circ\Phi_t^{-1}) - \log \mathsf{K}_{N,\beta}(\mu_0,\zeta) + N(\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_0))\\ = N\left(\mathcal{Z}(\beta,\mu_t) - \mathcal{Z}(\beta,\mu_0)\right) + O((\beta+1)N\ell^{\mathsf{d}}\mathcal{R}_t)\end{aligned}$$

where $\mathcal{R}_t$ is the error rate and $\mathsf{K}^{\mathcal{G}_\ell}_{N,\beta}$ corresponds to the free energy restricted to the event $\mathcal{G}_\ell$ where local laws hold as in Theorem 1.

We will also need an additional assumption on $f_{\mathsf{d},\mathsf{s}}$: we assume that the function $y \mapsto f_{\mathsf{d},\mathsf{s}}(y)$ is p times differentiable and satisfies

$$(1.45)\qquad \forall n \le p,\ \text{for } y \in \left[\frac{1}{2}\beta \min_{\square_\ell} \mu^{\mathsf{s}/\mathsf{d}}, 2\beta \max_{\square_\ell}\mu^{\mathsf{s}/\mathsf{d}}\right],\ \text{we have } |(yf_{\mathsf{d},\mathsf{s}}(y))^{(n)}| \lesssim_\beta |y|^{1-n}.$$

This can be interpreted as a *no-phase transition* assumption at the effective temperature $\beta\mu_V(x)^{\frac{\mathsf{s}}{\mathsf{d}}}$. It is reasonable to expect that the CLT result may fail if there is a phase transition. We formulated an assumption that is uniform as $\beta \ge 1$ in order to be able to deduce a zero temperature result by taking a uniform limit as $\beta \to \infty$, but one may dispense with the uniformity in the assumption is one is not interested in that.

Our CLT result has two parts: the first is conditional and asserts that if $\mathcal{R}_t$ can be shown to be small enough (in terms of the scale $N^{1/\mathsf{d}}\ell$), then the result holds. The second part asserts that the rate can indeed be obtained small enough when $\mathsf{s} > 0$ is small enough and ℓ small enough, which effectively restricts the result to dimensions 1 and 2 (because of the constraint $\mathsf{s} > \mathsf{d} - 2$). This is the same limitation as encountered in the Coulomb case in [Ser23]. However, we expect that our error rate which saturates at surface errors is not sharp, and thus that the result holds much beyond this setting.

Theorem 3 (Central Limit Theorem)**.** *Assume* (**A1**)–(**A6**) *and* (1.37) *for some* $k \ge 5$. *Assume* $\varphi = \varphi_0(\frac{\cdot - z}{\ell})$ *for some* $\varphi_0 \in C^k_c(\mathbb{R}^{\mathsf{d}})$ *and that* $\rho_\beta N^{-1/\mathsf{d}} \le \ell \le 1$ *and* $\operatorname{supp}\varphi \subset \square_\ell(z) \subset \square_{2\varepsilon}(z) \subset \Sigma$ *for some cube* $\square_\ell(z)$ *of sidelength* ℓ. *If* $\mathsf{s} \neq 0$ *assume that* (1.45) *holds.*

Assume (1.44) *with a rate* $\mathcal{R}_t$ *such that*

$$(1.46)\qquad \left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{\mathsf{s}}{2}} \left(\max_{|s|\le \ell^{\mathsf{d}-\mathsf{s}}} \mathcal{R}_s\right)^{1-\frac{1}{p}} \to 0$$

as $N \to \infty$ *for some* $p \ge 2$ *such that* $k \ge 2p+3$. *Then,*

$$(1.47)\qquad \sqrt{2\beta}\frac{\mathrm{Fluct}_{\mu_V}(\varphi)}{\left(N^{\frac{1}{\mathsf{d}}}\ell\right)^{\frac{\mathsf{s}}{2}}} - \left(\ell N^{-\frac{1}{\mathsf{d}}}\right)^{-\frac{\mathsf{s}}{2}}\mathrm{Mean}(\varphi)$$

converges in distribution to a centered Gaussian with variance

$$\mathrm{Var}(\varphi) = \frac{c_{\mathsf{d},\frac{\mathsf{d}-\mathsf{s}}{2}}}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}} \begin{cases} \|\varphi_0^\Sigma\|^2_{\dot{H}^{\frac{\mathsf{d}-\mathsf{s}}{2}}} & \text{if } \ell = 1 \\ \|\varphi_0\|^2_{\dot{H}^{\frac{\mathsf{d}-\mathsf{s}}{2}}} & \text{if } \ell \to 0 \text{ as } N \to \infty, \end{cases} \tag{1.48}$$

where $\mathsf{c}_{\mathsf{d},\mathsf{s}}$ *and* $c_{\mathsf{d},\alpha}$ *are as in* (1.9) *and* (1.10)*, and*

$$\mathrm{Mean}(\varphi) = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{\beta}} \dfrac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \left(-1 + \dfrac{\beta}{2\mathsf{d}} \mathbf{1}_{\mathsf{s}=0}\right) \displaystyle\int_\Sigma (-\Delta)^\alpha \varphi^\Sigma (\log \mu_V) \\ -\dfrac{\sqrt{2\beta}}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \displaystyle\int_\Sigma (-\Delta)^\alpha \varphi^\Sigma \left((1 + \dfrac{\mathsf{s}}{\mathsf{d}}) f_{\mathsf{d},\mathsf{s}}(\beta \mu_V^{\frac{\mathsf{s}}{\mathsf{d}}}) \mu_V^{\frac{\mathsf{s}}{\mathsf{d}}} + \dfrac{\mathsf{s}}{\mathsf{d}} f'_{\mathsf{d},\mathsf{s}}(\beta \mu_V^{\frac{\mathsf{s}}{\mathsf{d}}}) \mu_V^{2\frac{\mathsf{s}}{\mathsf{d}}} \right) \mathbf{1}_{\mathsf{s}\geq 0} & \text{if } \ell = 1 \\ 0 & \text{if } \ell \to 0 \end{cases} \tag{1.49}$$

Moreover,

$$\sqrt{2} \frac{\mathrm{Fluct}_{\mu_V}(\varphi)}{\left(N^{\frac{1}{\mathsf{d}}} \ell\right)^{\frac{\mathsf{s}}{2}}} - \frac{1}{\sqrt{\beta}} \left(\ell N^{-\frac{1}{\mathsf{d}}}\right)^{-\frac{\mathsf{s}}{2}} \mathrm{Mean}(\varphi) \tag{1.50}$$

converges in distribution to 0 *as* $N \to \infty$*, uniformly as* $\beta \to \infty$*.*

The result holds without the additional assumption (1.46) *for* $\mathsf{d} = 1, 2$ *and* $\mathsf{s} \leq \mathsf{s}_0$*, where*

$$0 < \mathsf{s}_0 \approx \begin{cases} 0.03973 & \text{in } \mathsf{d} = 1, \\ 0.06059 & \text{in } \mathsf{d} = 2. \end{cases}$$

as long as

$$\ell \ll N^{-\frac{\mathsf{s}}{\mathsf{d}(\mathsf{s}+2)}}.$$

This theorem provides us with the precise (expected) order of the fluctuations after one removes a deterministic shift (the factor containing the mean) which is *divergent* as soon as $\mathsf{s} > 0$.

Notice that the additional assumption on ℓ small holds automatically in the case $\mathsf{s} \leq 0$ if $\ell = o(1)$, which allows to match the result of [Bou23b] in the one-dimensional case, which was the only prior results in the Riesz case (however much less regularity of the test function was needed in [Bou23b]). We could also have a result for larger $\mathsf{s} > 0$ but at the expense of being restricted to smaller ℓ's.

The second statement (1.50) in the theorem is meant to allow to take $\beta \to \infty$, which implies, under the same suitable assumptions that for minimizers of the energy $\mathcal{H}_N$, a convergence result for $\frac{\mathrm{Fluct}_{\mu_V}(\varphi)}{(N^{\frac{1}{\mathsf{d}}}\ell)^{\mathsf{s}/2}}$ after substracting off the appropriate shift.

Let us discuss the comparison of this result with the Coulomb case, which is solved only in dimension 1 (see [Ser24]) and more interestingly, in dimension 2 in [LS18, Ser23] (refer also to [Ser24, Ser23]). In the Coulomb case, when φ is supported in Σ, then its harmonic extension (in that case $\alpha = 1$) outside Σ is simply itself and the mean and variance expressions involve only φ itself. However, when the support of φ overlaps Σ^c, then, as first shown in [AHM15] in the particular (determinantal) setting of $\beta = 2$, the expressions involve φ^Σ instead of φ. In the Riesz case, even when φ is supported in Σ, it does not coincide with its α-harmonic extension due to the nonlocality of the fractional Laplacian, thus φ^Σ is always involved, and this counts among the difficulties in handling the Riesz case.

In the (two-dimensional) Coulomb case, the variance expression was $\int |\nabla\varphi|^2$ (or more generally $\int |\nabla\varphi^\Sigma|^2$), leading to interpreting the limit of $\Delta^{-1}(\sum_{i=1}^N \delta_{x_i} - N\mu_V)$ as the *Gaussian free field*. Here in the Riesz case the variance is a homogeneous fractional Sobolev norm of φ^Σ. Modulo the α-harmonic extension, this structure is characteristic of a *fractional Gaussian field* (for a review of the notion, we refer to [LSSW16]).

This leads to the following interpretation:

Corollary 1.3. *If convergence holds, the quantity $(-\Delta)^{-\alpha}(\sum_{i=1}^N \delta_{x_i} - N\mu_V)$ converges to a Gaussian field of Hurst parameter $-\mathsf{s}/2$ in the mesoscopic case $\ell \to 0$, or a variant*[2] *of it in the macroscopic case.*

To our knowledge, this is a new and natural occurence of such fractional Gaussian fields.

1.5. **The transport method.** Let us next outline the implementation of the transport method, emphasizing the differences with the local situation.

The starting point is the rewriting of the Laplace transform (or moment generation function) of fluctuations as a ratio of partition functions, as done in this context for instance since [Joh98]: a straightforward computation shows that $\mathrm{Fluct}_{\mu_V}(\varphi)$ being defined in (1.22), we have

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(-\beta t N^{1-\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{Fluct}_{\mu_V}(\varphi)\right)\right] = e^{t\beta N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\int \varphi\, d\mu_V}\frac{Z_{N,\beta}(V_t)}{Z_{N,\beta}(V)}, \tag{1.51}$$

where $V_t := V + t\varphi$ and $Z_{N,\beta}(V)$ denotes the partition function of the Riesz gas of potential V, as in (1.5). The interesting regime will be the regime of small t, thus we can think of the Laplace transform of fluctuations as a ratio of two partition functions, that of a Riesz gas with perturbed potential V_t to that of the original Riesz gas. It then appears that understanding the log Laplace transform of the fluctuations amounts to obtaining a fine expansion of the free energy $\log Z_{N,\beta}(V)$ in terms of V. Since we cannot get a precise enough expansion for this, we will bypass this by combining a cruder expansion of the free energy, with a second way of estimating the ratio of partition functions. That second way is via a transport, which is a good choice of change of variables. Computations reveal that a good choice is one that transports the original equilibrium measure μ_V into the equilibrium measure μ_{V_t} associated to the perturbed external potential. When working in the regime of small t, we do not need an exact transport map, it suffices to find a map which is a perturbation of identity and transports at leading order μ_V to μ_{V_t}, i.e. a map $\psi : \mathbb{R}^\mathsf{d} \to \mathbb{R}^\mathsf{d}$ such that

$$(I + t\psi)\#\mu_V = \mu_{V_t} + o(t), \tag{1.52}$$

which, by linearization of the Monge-Ampère equation for instance, amounts to solving

$$\mathrm{div}\,(\psi\mu_V) = \frac{\partial}{\partial t}\mu_{V_t}. \tag{1.53}$$

In the bulk Coulomb case where the perturbation φ is supported in the support of μ_V (call this the interior case), it is easy to check that the difference of the equilibrium measures is supported in Σ and is then explicit in view of (1.17) : it is

$$\mu_{V_t} - \mu_V = \frac{t}{\mathsf{c_d}}\Delta\varphi. \tag{1.54}$$

This makes solving (1.53) immediate : it suffices to take $\psi = \frac{\nabla\varphi}{\mathsf{c_d}\mu_V}$.

[2]due to the harmonic extension

Still in the Coulomb case, when φ is not supported in Σ, this is no longer possible, as the support of the equilibrium measure itself varies, as studied in [SS18], and the difference in equilibrium measures involves the harmonic extension of φ. The resolution of (1.53) becomes much more delicate. In the present Riesz case, we always encounter that difficulty because the support of the equilibrium measure always changes, even in the interior case. To carry the method through, we need not only to find a solution ψ, but to ensure it is sufficiently regular, and have sharp estimates for its derivatives in terms of those of φ, in particular as φ get supported on small scales. This is the core of the new analysis performed in Section 2, which is of analysis and PDE nature.

Note that the one-dimensional log case is also a nonlocal case. In that setting, finding ψ solving (1.53) turns out to be the same as the inversion of a so-called "master operator" (see for instance [BG13, BG24, BL18] and references therein). This is an integral operator in one dimension, for which an explicit inversion formula is available from classical analysis, see [Mus92], allowing to read off the needed estimates. This is the route that was taken in [Pei24]. In higher dimension, or for $\mathsf{s} \neq 0$, these formulae are not available, and the inversion of the master operator should rather be seen as the solution of the appropriate PDE (2.16) which is a rather nonstandard PDE, for which we need to get sharp estimates.

An important difficulty in the Riesz case is that, contrarily to the interior Coulomb case (where we could take $\psi = \frac{\nabla\varphi}{\mathsf{c_d}\mu_V}$), the transport map ψ is no longer supported in the support of φ. When we study fluctuations for test functions that live on a mesoscale, this prevents us from localizing the estimates to the support of φ. Instead, ψ has decaying tails, and we need to estimate their decay speed away from $\operatorname{supp}\varphi$, and to estimate the contributions of the tails of the transport to the errors, which need to remain small in terms of the chosen mesoscale.

Once the transport has been found, a change of variables reduces the analysis of the ratio of partition functions to one crucial term corresponding to the expectation of quantities of the form

$$\iint_{(\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d})\setminus\triangle}(\psi(x)-\psi(y))^{\otimes n} : \nabla^{\otimes n}\mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i}-N\mu\Big)(x)\Big(\sum_{i=1}^N \delta_{x_i}-N\mu\Big)(y), \tag{1.55}$$

which correspond to the n-th variation of the energy F_N along the transport $(I+t\psi)$, as described for instance in [Ser24, Chap. 6], and have a commutator structure. Such terms are the analogues of the so-called "loop equation terms" in the case $\mathsf{s}=0$. To control them, we will make crucial use of the sharp and localized commutator estimates recently proven in [RS25] and recalled in Proposition 3.8. These ensure that for all configurations, these terms are all controlled by a constant (which depends on norms of derivatives of ψ) times (the localized version of) $\mathsf{F}_N + CN^{1+\frac{\mathsf{s}}{\mathsf{d}}}$. The control of this quantity at first order ($n=1$), inserted into (2.1) after the change of variables already allows to obtain the fluctuation bounds of Theorem 2. To obtain Theorem 3, we need the commutator estimates up to order $n=p$, and the combination with another free energy expansion with a good enough rate.

1.6. **Rest of the proof and plan of the paper.** In addition to the transport problem, dealing with the nonlocal Riesz case involves other difficulties, already encountered in [Pei24]. First, as introduced in [PS17], we use the Caffarelli-Silvestre extension to represent the fractional Laplacian as a *local* divergence-form operator with singular weight. This makes the electric formulation possible and is described in Section 3. The next ingredient is the screening procedure which is performed in the extended space $\mathbb{R}^{\mathsf{d}+1}$, and is the crucial tool to show the local laws, almost additivity and perform the bootstrap procedure. The need to control the energy on good "heights" in the extended dimension in order to perform the screening requires to couple the bootstrap with the control of fluctuations.

The results are coupled and proved as follows:

- We assume local laws down to scale 2ℓ. We show that this implies the control of fluctuations for test functions varying on scales $\geq 2\ell$ stated in Theorem 2.
- Still assuming local laws hold down to scale 2ℓ, we use the fluctuations control to get a control on the "electric field" at heights comparable to ℓ. This allows to perform the screening, which then allows to show that local laws hold down to scale ℓ.
- One may then iterate down the scales to obtain the local laws at all scales $\ell \geq \rho_\beta N^{-1/\mathsf{d}}$, hence Theorem 1.
- This also implies that the fluctuations control of Theorem 2 holds at all scales $\ell \geq \rho_\beta N^{-1/\mathsf{d}}$.

Note that this coupling of the two proofs was not needed in the Coulomb case where no control of the electric field in the extended dimension is needed for the screening.

As mentioned above, the proof of the CLT relies on the free expansion with a rate of (1.44). This in turn is done as in [AS21, Ser23] by first obtaining a rate of convergence for the free energy in a cube with uniform background equilibrium measure, obtained by almost additivity of the free energy, then using the almost additivity again to split the domain into regions where μ_V is close to constant, and using the transport method to estimate the difference with the free energy with uniform background. This is done in Section 7.

The first part of the paper is devoted to proving the fluctuations bound stated in Theorem 2 at scale 2ℓ, assuming the local law Theorem 1 at scales $\geq 2\ell$. It starts with Section 2, devoted to solving the transport map problem and showing good estimates for it. Section 3 reviews the electric formulation and extends the transport method to the Riesz setting, and finally Section 4 concludes with the proof of the fluctuations bound stated in Theorem 2 at scale 2ℓ, assuming the local law Theorem 1 at scales $\geq 2\ell$. Much of the argument for Theorems 2 and 3 can be understood without a technical understanding of the local laws bootstrap, and so we present the proof of these theorems first, for readers who are interested primarily in fluctuations of linear statistics. The second part of the paper, in particular Section 6, is then devoted to proving that the local law, Theorem 1, holds at scale ℓ if it holds at scale $\geq 2\ell$. In Section 7, we obtain the almost additivity of the free energy as a consequence of the local law, and prove the free energy expansion with a rate via domain partitioning plus transport. Section 8 gathers some independent estimates on fractional Laplacians that are needed, in particular in the construction of the transport map. Section 9 presents the proof of the screening procedure, adapted from [PS17]. Finally the paper concludes with the appendix by Xavier Ros-Oton providing the fine behavior of solutions to the fractional obstacle problem.

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2. Solving the transport problem

2.1. **Preliminary: reexpressing the Laplace transform.** Let us start with the rewriting of the Laplace transform of linear statistics, which will make the correct choice of transport appear. We start by a generic change of variables Φ_t and will make a specific choice below.

Lemma 2.1 (Reexpressing the Laplace transform). *Let φ be a compactly supported measurable test function, and let $\mathrm{Fluct}_{\mu_V}(\varphi)$ be as in (1.22). Set $V_t := V + t\varphi$. Then, for any $t \in \mathbb{R}$ and any event $\mathcal{G} \subset \mathbb{R}^{\mathsf{d}N}$, we have*

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(-\beta t N^{1-\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{Fluct}_{\mu_V}(\varphi)\right)\mathbf{1}_{\mathcal{G}}\right] = \exp\left(t\beta N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\int_{\mathbb{R}^\mathsf{d}}\varphi\, d\mu_V\right)\frac{Z_{N,\beta}^{V_t,\mathcal{G}}}{Z_{N,\beta}}, \tag{2.1}$$

with

$$Z_{N,\beta}^{V_t,\mathcal{G}} := \int_{\mathcal{G}\subset(\mathbb{R}^\mathsf{d})^N}\exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\sum_{i\neq j}\frac{1}{2}\mathsf{g}(x_i-x_j) + N\sum_{i=1}^N V_t(x_i)\right)\right)\, dX_N.$$

Furthermore, we can expand

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(-\beta t N^{1-\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{Fluct}_{\mu_V}(\varphi)\right)\mathbf{1}_{\mathcal{G}}\right] = e^{T_0}\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[e^{T_1+T_2}\mathbf{1}_{\mathcal{G}}\right], \tag{2.2}$$

where

$$\begin{aligned} T_0 := &-\beta N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\left(\frac{1}{2}\iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}}(\mathsf{g}(\Phi_t(x)-\Phi_t(y))-\mathsf{g}(x-y))\,d\mu_V(x)d\mu_V(y) + \int(V_t\circ\Phi_t - V)\,d\mu_V\right)\\ &+ N\int_{\mathbb{R}^\mathsf{d}}(\log\det D\Phi_t)\,d\mu_V + \beta t N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\int_{\mathbb{R}^\mathsf{d}}\varphi\,d\mu_V, \end{aligned} \tag{2.3}$$

$$T_1 := -\beta N^{1-\frac{\mathsf{s}}{\mathsf{d}}}\int_{\mathbb{R}^\mathsf{d}}\left(\int_{\mathbb{R}^\mathsf{d}}\mathsf{g}(\Phi_t(x)-\Phi_t(y))-\mathsf{g}(x-y))\,d\mu_V(y) + (V_t\circ\Phi_t - V)(x)\right)d\mathrm{fluct}_{\mu_V}(x), \tag{2.4}$$

$$\begin{aligned} T_2 := &-\frac{\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}}{2}\iint_{\triangle^c}(\mathsf{g}(\Phi_t(x)-\Phi_t(y))-\mathsf{g}(x-y))\,d\mathrm{fluct}_{\mu_V}(y)d\mathrm{fluct}_{\mu_V}(x)\\ &+\int_{\mathbb{R}^\mathsf{d}}\log\det D\Phi_t\,d\mathrm{fluct}_{\mu_V}(x)\\ = &-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\mathsf{F}_N(\Phi_t(X_N),\Phi_t\#\mu_V)-\mathsf{F}_N(X_N,\mu_V)\right)+\mathrm{Fluct}_{\mu_V}(\log\det D\Phi_t) \end{aligned} \tag{2.5}$$

where we have let

$$\mathrm{fluct}_\mu := \sum_{i=1}^N \delta_{x_i} - N\mu. \tag{2.6}$$

Proof. The relation (2.1) is immediate from spelling out the definition of $\mathrm{Fluct}_{\mu_V}(\varphi)$. We then perform the change of variables $y_i = \Phi_t(x_i)$ in the integral defining $Z_{N,\beta}^{V_t,\mathcal{G}}$ to obtain

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(-\beta t N^{1-\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{Fluct}_{\mu_V}(\varphi)\right)\mathbf{1}_{\mathcal{G}}\right]\exp\left(-\beta t N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\int_{\mathbb{R}^\mathsf{d}}\varphi\,d\mu_V\right)\\ &= \frac{1}{Z_{N,\beta}}\int_{\mathcal{G}}\exp\Big(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\Big(\frac{1}{2}\sum_{i\neq j}\mathsf{g}(\Phi_t(x_i)-\Phi_t(x_j)) + N\sum_{i=1}^N V_t(\Phi_t(x_i))\Big) + \sum_{i=1}^N\log\det D\Phi_t(x_i)\Big)dX_N\\ &= \mathbb{E}_{\mathbb{P}_{N,\beta}}\Big[\exp\Big(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\Big(\frac{1}{2}\sum_{i\neq j}(\mathsf{g}(\Phi_t(x_i)-\Phi_t(x_j))-\mathsf{g}(x_i-x_j)) + N\sum_{i=1}^N(V_t\circ\Phi_t - V)(x_i)\Big)\\ &\qquad + \sum_{i=1}^N\log\det D\Phi_t(x_i)\Big)\mathbf{1}_{\mathcal{G}}\Big]. \end{aligned}$$

Expanding $\sum_{i=1}^N \delta_{x_i}$ as $N\mu_V + \mathrm{fluct}_{\mu_V}$ and collecting terms, we find the result. The relation (2.5) follows by the definition (1.21).

□

2.2. **Choice of transport and estimates.** The choice of transport is made so that Φ_t is a perturbation of identity of the form $\Phi_t = I + t\psi$, chosen so that, at leading order in $t \to 0$, the T_1 term in the expansion above vanishes, leaving only the constant term T_0 to compute, and the subleading order term T_2 to analyze.

Plugging in $\Phi_t = I + t\psi$ and linearizing the T_1 term in t, we find that ψ must be chosen so that

$$\int_{\mathbb{R}^\mathsf{d}} \left(\int_{\mathbb{R}^\mathsf{d}} \nabla \mathsf{g}(x-y) \cdot (\psi(x) - \psi(y)) d\mu_V(y) + \varphi(x) + \nabla V(x) \cdot \psi(x) \right) d\mathrm{fluct}_{\mu_V}(x). \tag{2.7}$$

We will choose ψ such that the term in factor of fluct_{μ_V} vanishes identically. The reader can recognize the condition $\Xi(\psi) = \varphi$ for a certain "master operator" Ξ that needs to be inverted (as in [BG13] and references therein for the one-dimensional log gas).

This can be rewritten as

$$(\nabla(\mathsf{g} * \mu_V) + \nabla V) \cdot \psi - \mathsf{g} * (\mathrm{div}\,(\psi\mu_V)) + \varphi = 0 \quad \text{in } \mathbb{R}^\mathsf{d}. \tag{2.8}$$

Using the notation h^f for the potential $\mathsf{g} * f$ generated by f, and recalling the definition of ζ_V in (1.16), this becomes

$$\psi \cdot \nabla \zeta_V - h^{\mathrm{div}\,(\psi\mu_V)} + \varphi = 0 \quad \text{in } \mathbb{R}^\mathsf{d}. \tag{2.9}$$

This is the generalization to arbitrary dimension and interaction potential of the master operator inversion.

Let us now explain how to solve (2.9). Since μ_V vanishes outside Σ, the function $h^{\mathrm{div}\,(\psi\mu_V)}$ is α-harmonic there, hence the only possibility to solve (2.9) is to make $\psi \cdot \nabla\zeta_V + \varphi$ be α-harmonic outside of Σ. Since ζ_V vanishes in Σ, that function equals φ in Σ. Thus, the only possibility to solve it continuously is to have $\psi \cdot \nabla\zeta_V = \varphi^\Sigma - \varphi$. In other words we need to find ψ such that, in $\mathbb{R}^\mathsf{d}$, there holds

$$\begin{cases} h^{\mathrm{div}\,(\psi\mu_V)} = \varphi^\Sigma \\ \psi \cdot \nabla\zeta_V = \varphi^\Sigma - \varphi \end{cases} \tag{2.10}$$

or equivalently

$$\begin{cases} \mathrm{div}\,(\psi\mu_V) = \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}(-\Delta)^\alpha \varphi^\Sigma \\ \psi \cdot \nabla\zeta_V = \varphi^\Sigma - \varphi. \end{cases} \tag{2.11}$$

It is sufficient to solve (2.11) in order to obtain (2.10) since convolution with g uniquely inverts $(-\Delta)^\alpha$ for L^1 functions, cf [Kwa19, Theorem 2.4]. By Lemma 8.1, $\varphi^\Sigma \in \dot{H}^\alpha$ which embeds into L^1 by [DNPV12, Theorem 6.5]. $\mathrm{div}\,(\psi\mu_V)$ is continuous in Σ, compactly supported and has an integrable singularity ($\sim (\mathrm{dist}(,x\partial\Sigma))^{-\alpha}$) at $\partial\Sigma$, so it is also in L^1.

Since $\alpha < 1$, by (1.24) μ_V vanishes at the boundary of Σ and, if ψ is regular enough, $\psi\mu_V$ is continuous across $\partial\Sigma$ and thus

$$\mathrm{div}\,(\psi\mu_V) = \mathrm{div}\,(\psi\mu_V)\mathbf{1}_\Sigma, \tag{2.12}$$

which means that the first equation needs to be solved in each connected component of Σ only. While we only prove Theorems 1-3 in the case where Σ is connected, the results we state in this section remain true in the case where Σ is a disjoint union of finitely many connected components. In that case, the solvability in each connected component is ensured by the condition (1.41).

Care is needed for the boundary condition because of the blow up of $(-\Delta)^\alpha \varphi^\Sigma$ as $w\,\mathrm{dist}(\cdot, \partial\Sigma)^{-\alpha}$ on the boundary as in (1.37). Because $\psi\mu_V$ behaves like $\psi s \mathrm{dist}(\cdot, \partial\Sigma)^{1-\alpha}$ as in (1.24), whose normal derivative along the boundary blows up like $\psi s(1-\alpha)\mathrm{dist}(\cdot, \partial\Sigma)^{-\alpha}$, equating the two blowing-up rates leads to imposing $\psi \cdot \vec{n} s(1-\alpha) = w$, thus we are led to solving

$$\begin{cases} \mathrm{div}\,(\psi\mu_V) = \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}(-\Delta)^\alpha \varphi^\Sigma & \text{in } \Sigma \\ \psi \cdot \vec{n} = \frac{w}{(1-\alpha)s} & \text{on } \partial\Sigma \\ \psi \cdot \nabla\zeta_V = \varphi^\Sigma - \varphi & \text{in } \Sigma^c. \end{cases} \tag{2.13}$$

We will next see that this is solvable, and that the last two relations are compatible at $\partial\Sigma$.

Before proving the solvability with estimates, let us mention a word about the Coulomb case in which $\alpha = 1$. In that case μ_V is generically discontinuous across the boundary of Σ and $\mathrm{div}\,(\psi\mu_V)$ has a singular part on the boundary. The equation to solve is then

$$\begin{cases} \mathrm{div}\,(\psi\mu_V) = -\frac{1}{\mathsf{c}_{\mathsf{d}}}\Delta\varphi & \text{in } \Sigma \\ \psi \cdot \vec{n} = \frac{1}{\mathsf{c}_{\mathsf{d}}\mu_V}[\nabla\varphi^\Sigma] \cdot \vec{n} & \text{on } \partial\Sigma \\ \psi \cdot \nabla\zeta_V = \varphi^\Sigma - \varphi & \text{in } \Sigma^c. \end{cases} \tag{2.14}$$

We refer to [LS18, Lemma 3.4] where this problem is formally derived and solved in any dimension.

Proposition 2.2 (Good transport map and estimates)**.** *Let U be an open neighborhood of Σ. Suppose that μ_V satisfies (**A4**) for $s(x) \in C^{k+\epsilon}(\Sigma)$ for some $\epsilon \in (0,1)$, and that $\partial\Sigma$ is $C^{k+1+\epsilon}$. Assume $V \in C^{k+1}(\mathbb{R}^{\mathsf{d}+1})$, $\varphi \in C_c^{k+1}(\mathbb{R}^{\mathsf{d}})$, that $\ell \le 1$ and $\mathrm{supp}\,\varphi \subset \square_\ell(z) \subset \square_{2\ell}(z) \subset \hat{\Sigma}$, with the estimates*

$$\forall \sigma \le k+1, \quad \|\varphi\|_{C^\sigma} \le \mathsf{M}\ell^{-\sigma} \tag{2.15}$$

for some constant $\mathsf{M} > 0$. Assume that φ satisfies (1.37) for some $w(x) \in C^{k+\epsilon}(U)$. Finally, assume (1.41).

Then, there exists a map ψ defined in $\mathbb{R}^{\mathsf{d}}$, continuous in $\mathbb{R}^{\mathsf{d}}$, and a map $\psi^\perp$ defined in $\mathbb{R}^{\mathsf{d}}\backslash\Sigma$, vanishing in $\mathbb{R}^{\mathsf{d}}\backslash U$ and continous in $\mathbb{R}^{\mathsf{d}}\backslash\Sigma$, perpendicular to $\nabla\zeta_V$, and such that

$$\begin{cases} \mathrm{div}\,(\psi\mu_V) = \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}(-\Delta)^\alpha \varphi^\Sigma & \text{in } \Sigma \\ \psi \cdot \vec{n} = \frac{w}{(1-\alpha)s} & \text{on } \partial\Sigma \\ \psi = (\varphi^\Sigma - \varphi)\frac{\nabla\zeta_V}{|\nabla\zeta_V|^2} + \psi^\perp & \text{in } \Sigma^c, \end{cases} \tag{2.16}$$

and ψ satisfies (2.9) in $\mathbb{R}^{\mathsf{d}}$. Moreover, $\psi \in C^k(\Sigma) \cap C^k(\Sigma^c) \cap C(U)$.

Furthermore, for any $m \le k-3$, we have for any $x \notin \partial\Sigma$,

$$|\nabla^{\otimes m}\psi(x)| \lesssim \mathsf{M} \begin{cases} \frac{1}{\ell^{\mathsf{d}-\mathsf{s}+m-1}} & \text{if } x \in \square_{2\ell}(z) \\ \frac{\ell^{\mathsf{d}}}{|x-z|^{2\mathsf{d}-\mathsf{s}+m-1}} & \text{if } x \in U\backslash\square_{2\ell}(z) \\ \frac{\ell^{\mathsf{d}}}{|x-z|^{\mathsf{s}+2+m}} & \text{if } x \in U^c. \end{cases} \tag{2.17}$$

Notice that in the one-dimensional log case $\mathsf{s} = 0$, this scaling agrees with that of the equivalent formulation of that transport in [Pei24, Lemma 4.10] obtained there by exact formulas for the inversion of the master operator. We have been careful to assume that $s, w \in C^{k+\epsilon}$ and $\partial\Sigma$ is $C^{k+1+\epsilon}$ since some of the fractional and degenerate regularity results we seek to apply, namely [TTV24, Theorem 1.1] and [ARO20a, Theorem 1.3], do not necessarily hold for integer Hölder classes.

Before moving on to the proof of Proposition 2.2, let us recall more about the solutions to the fractional obstacle problem.

2.3. **Preliminaries on the fractional obstacle problem.** Theorem 5 in the appendix recalls the result from the work of [CSS08a], [CDSS17a] and [ROS17a], which classifies the free boundary points as regular or degenerate (or singular) points.

In our notation, regular points are those for which

$$\zeta_V(x) = c(x_0)\mathrm{dist}(x, \partial\Sigma)^{1+\alpha}(x) + O(|x - x_0|^{1+\alpha+a}) \tag{2.18}$$

for some $c(x_0) > 0$ and $a > 0$ with $\alpha + a < 1$.

If we make an additional assumption on the obstacle, then we can guarantee that all points are regular. Namely, this holds if the free boundary is Lipschitz, $\psi \in C^{2+\gamma}$ for some $\gamma > \alpha$ and if $\Delta\psi < 0$ on $\{\psi = c_V - V > 0\}$; this is Proposition A.3 in the appendix. As we will see below, we actually need a slightly stronger condition on the free boundary; however, as long as (**A3**) and (**A2**) hold, we can conclude that all points of the free boundary are regular. With the additional regularity of V, X. Ros Oton proves in Proposition A.7 in the appendix a quantified decay rate of μ_V near the free boundary: as $x \to x_0 \in \partial\Sigma_V$ with x_0 a regular free boundary point then we have in our notation

$$\mu_V(x) \sim \mathrm{dist}(x, \partial\Sigma_V)^{1-\alpha}, \tag{2.19}$$

justifying the assumption (1.24).

Let $U \supset \Sigma$ be an open set such that (2.18) holds in $U \setminus \Sigma_V$. In order for us to define the transport map with C^σ regularity for a general $0 < \sigma < 1$, we will need the free boundary to have $C^{1+\sigma}$ regularity. As a result of [JN17, Theorem 1.1], we have $C^{2+\gamma}$ regularity at regular points for some $\gamma > 0$ dependent on σ (and thus, in particular, $C^{1+\sigma}$) once $V \in C^4$; hence, for standard existence of a general Hölder continuous transport, we will need to assume $V \in C^4$. Higher regularity can be obtained with higher regularity assumptions on V and in fact if V is smooth, the free boundary will be smooth as well. This result was established independently as well in [KRS19, Theorem 1.1]. The regularity assumptions (**A4**)-(**A5**) are the additional higher regularity assumptions that we will need to recover the C^k regularity estimates on ψ given in Proposition 2.2.

2.4. **Proof of Proposition 2.2.** We now give the proof of our main analytic result, on the existence of a good transport map with estimates on its derivatives. We will use some auxiliary results on fractional harmonic extensions proved in Section 8.

First, we observe that under our assumptions μ_V is an A_2-Muckenhoupt weight on Σ. Since μ_V is bounded below away from $\partial\Sigma$, we only need to verify this for small balls near the boundary. We use the assumption that all points of the free boundary are regular, and the quantitative form (**A4**) that we assume. Thus, for $\epsilon > 0$ small we find a uniform bound

$$\fint_{B_\epsilon} \mu_V \fint_{B_\epsilon} \mu_V^{-1} \lesssim \fint_{B_\epsilon} |x_1|^{1-\alpha} \fint_{B_\epsilon} |x_1|^{\alpha-1} \lesssim \epsilon^{1-\alpha}\epsilon^{\alpha-1} \lesssim 1,$$

establishing the required uniform bound, by definition an A_2 weight. We now seek to establish the existence of weak energy solutions in Σ of

$$\begin{cases} -\mathrm{div}\,(\mu_V \nabla u) = \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}(-\Delta)^\alpha \varphi^\Sigma & \text{in } \Sigma \\ \nabla u \cdot \vec{n} = \frac{w}{(1-\alpha)s} & \text{on } \Sigma \end{cases} \tag{2.20}$$

in the Hilbert space $H^1_{\mu_V}$. This is the Hilbert space (see [GU09, Theorem 1]) of functions u such that

$$\int_\Sigma u^2 \mu_V + \int_\Sigma |\nabla u|^2 \mu_V < \infty$$

equipped with inner product

$$\langle u, v\rangle = \int_\Sigma uv\mu_V + \int_\Sigma \nabla u \cdot \nabla v \mu_V. \tag{2.21}$$

These weak solutions are ones that satisfy

$$\int_\Sigma \nabla u \cdot \nabla \phi \mu_V = \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_\Sigma (-\Delta)^\alpha \varphi^\Sigma \phi \tag{2.22}$$

for all $\phi \in H^1_{\mu_V}$.

Step 1: Existence of weak energy solutions. Let us first restrict to the subspace H of all functions u with $\int u\mu_V = 0$, which is a closed subspace since $\int u\mu_V$ is a bounded linear functional on $H^1_{\mu_V}$ as

$$\left|\int u\mu_V\right| \le \int |u|\sqrt{\mu_V}\sqrt{\mu_V} \le \left(\int u^2\mu_V\right)^{1/2} \left(\int \mu_V\right)^{1/2} \le \|u\|_{H^1_{\mu_V}}.$$

Notice that on H, the inner product

$$\langle u, v\rangle_1 := \int \nabla u \cdot \nabla v \mu_V \tag{2.23}$$

induces a norm equivalent to the one defined in (2.21) because of the weighted Poincaré inequality for A_2-weights in [FKS82, Theorem 1.5]. Letting $\overline{u} = \int u\mu_V = \fint u\mu_V$

$$\langle u, u\rangle_1 \le \langle u, u\rangle = \langle u, u\rangle_1 + \int u^2\mu_V = \langle u, u\rangle_1 + \int (u - \overline{u})^2 \mu_V \le C\langle u, u\rangle_1$$

since $\overline{u} = 0$ in H. We will conclude as usual via Riesz representation theorem once we establish that $\phi \mapsto \int(-\Delta)^\alpha\varphi^\Sigma\phi$ is a bounded linear functional on H; this, however, presents some difficulties because $(-\Delta)^\alpha\varphi^\Sigma$ blows up like $\mathrm{dist}(x, \partial\Sigma)^{-\alpha}$ as $x \to \partial\Sigma$. To get around this, we follow the approach of [TTV24, Theorem 2.10] and show that $(-\Delta)^\alpha\varphi^\Sigma$ is at least locally the divergence of a regular function.

Straightening the Boundary. Let k be fixed as in the assumptions of Proposition 2.2. Choose $x_0 \in \partial\Sigma$, and without loss of generality set $x_0 = 0$. Locally then, in some open neighborhood $\mathcal{O}$ we can write $x_n = \phi(x_1, \dots, x_{n-1})$ with $\phi \in C^{1+k}(\mathcal{O} \cap \Sigma)$. Define the diffeomorphism

$$\Phi(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, x_n + \phi(x)) = (\overline{x_1}, \dots, \overline{x_n}) \tag{2.24}$$

which maps $B_R^+ = B_R \cap \{x_n > 0\}$ to $\mathcal{O} \cap \{\overline{x_n} > 0\}$ for some $R > 0$. We can assume $\Phi(0) = \Phi^{-1}(0) = 0$, and observe also that $\Phi(B_R \cap \{x_n = 0\}) \subset \mathcal{O} \cap \partial\Sigma$. The Jacobian of this map is then

$$(J_\Phi(x))_{ij} = \begin{cases} 1 & \text{if } i = j \\ \frac{\partial \phi}{\partial x_i} & \text{if } i = n,\ 1 \le j \le n-1 \\ 0 & \text{otherwise} \end{cases} \tag{2.25}$$

and satisfies $|\det(J_\Phi)| \equiv 1$. One can readily compute (albeit with some tedious multivariable calculus) that $-\mathrm{div}\,(\mu_V \nabla u) = \frac{1}{\mathsf{c_{d,s}}}(-\Delta)^\alpha \varphi^\Sigma$ in $\mathcal{O} \cap \Sigma$ if and only if

$$-\mathrm{div}\,(\overline{\mu_V}\overline{A}\nabla\overline{u}) = \overline{f} \tag{2.26}$$

in $B_R \cap \{x_n > 0\}$, with $\overline{\mu_V} = \mu_V \circ \Phi$, $\overline{u} = u \circ \Phi$, $\overline{A} = J_\Phi^{-1}(J_\Phi^{-1})^t$ and $\overline{f} = \frac{1}{\mathsf{c_{d,s}}}(-\Delta)^\alpha \varphi^\Sigma \circ \Phi$. Now, it follows (cf [TTV24, Lemma 2.4], as in the proof of [TTV24, Corollary 2.9]) from our regularity assumptions on the boundary that

$$\frac{\mathrm{dist}(x, \partial\Sigma) \circ \Phi}{x_n} \in C^k(B_R^+) \quad \text{and} \quad \frac{\mathrm{dist}(x, \partial\Sigma) \circ \Phi}{x_n} \ge c > 0$$

for some constant c. In particular, in view of (1.24), shrinking R if necessary, we have

$$-\mathrm{div}\,(x_n^{1-\alpha}\tilde{A}\nabla\overline{u}) = \overline{f} \tag{2.27}$$

with $\tilde{A} = (s \circ \Phi)\frac{\mathrm{dist}(\Phi(x), \partial\Sigma)^{1-\alpha}}{x_n^{1-\alpha}}\overline{A} \in C^k(B_R^+)$. Additionally, using (1.37), we can write

$$\overline{f} = (w \circ \Phi)\mathrm{dist}(\Phi(x), \partial\Sigma)^{1-\alpha} := \overline{w}x_n^{-\alpha},$$

with $\overline{w} \in C^k(B_R^+)$ and bounded from below. Then, if we define

$$F = \left(0, 0, \dots, 0, \frac{1}{x_n^{1-\alpha}}\int_0^{x_n} \overline{w}(x_1, \dots, x_{n-1}, t)t^{-\alpha}\, dt\right) \tag{2.28}$$

we have $F \in C^k(B_R^+)$ with $\|F\|_{C^k(B_R^+)} \lesssim \|\overline{w}\|_{C^k(B_R^+)} \lesssim \|w\|_{C^k(B_R^+)}$ as in [TTV24, Lemma 2.4] and the arguments therein. Furthermore, we can rewrite (2.27) as

$$-\mathrm{div}\,(x_n^{1-\alpha}\tilde{A}\nabla\overline{u}) = f = \mathrm{div}\,(x_n^{1-\alpha}F).$$

Reverting to our weight $\overline{\mu_V}$ and defining $\tilde{F} = \frac{x_n^{1-\alpha}}{\overline{\mu_V}}F \in C^k(B_R^+)$, we find

$$-\mathrm{div}\,(\overline{\mu_V}\overline{A}\nabla\overline{u}) = \mathrm{div}\,(\overline{\mu_V}\tilde{F}) \tag{2.29}$$

in B_R^+. Finally, if we define $G = J_\Phi(\tilde{F} \circ \Phi^{-1})$, we have $\|G\|_{C^k(\mathcal{O})} \lesssim \|w\|_{C^k}$ and $\tilde{F} = J_\Phi^{-1}(G \circ \Phi)$ and one can readily compute that

$$-\mathrm{div}\,(\overline{\mu_V}\overline{A}\nabla\overline{u}) = \mathrm{div}\,(\overline{\mu_V}J_\Phi^{-1}(G \circ \Phi))$$

in B_R^+ if and only if

$$-\mathrm{div}\,(\mu_V \nabla u) = \mathrm{div}\,(\mu_V G) \tag{2.30}$$

in $\mathcal{O} \cap \Sigma$.

Bounded Linear Functional. We use the divergence form above to show that $\int(-\Delta)^\alpha\varphi^\Sigma\phi$ is a bounded linear functional on H. Let $\{\mathcal{O}_i\}$ denote a finite collection of charts covering a small neighborhood E of $\partial\Sigma$, and let $\{\psi_i\}$ denote a partition of unity on E subject to this collection of charts. Using the previous substep, we can write

$$\frac{1}{\mathsf{c_{d,s}}}(-\Delta)^\alpha\varphi^\Sigma = \sum_i \psi_i \mathrm{div}\,(\mu_V G_i)$$

for some G_i with $\|G_i\|_{C^k(\mathcal{O}_i)} \lesssim \|w\|_{C^k}$, and so

$$\begin{aligned}\int_E \frac{1}{\mathsf{c_{d,s}}}\phi(-\Delta)^\alpha\varphi^\Sigma = \sum_i \int_{\mathcal{O}_i}\frac{1}{\mathsf{c_{d,s}}}\phi_i(-\Delta)^\alpha\varphi^\Sigma\psi_i &= \sum_i\int_{\mathcal{O}_i}\phi_i\mathrm{div}\,(\mu_V G_i)\psi_i \\ &= -\sum_i\int_{\mathcal{O}_i}\mu_V G_i\cdot\nabla(\psi_i\phi) \\ &= -\sum_i\int_{\mathcal{O}_i}\sqrt{\mu_V}G_i\cdot(\sqrt{\mu_V}\psi_i\nabla\phi+\sqrt{\mu_V}\phi\nabla\psi_i)\,.\end{aligned}$$

By Cauchy-Schwarz and the boundedness of ψ_i, we find

$$\left|\int_E \frac{1}{\mathsf{c_{d,s}}}\phi(-\Delta)^\alpha\varphi^\Sigma\right| \lesssim \sum_i\left(\int\mu_V|G_i|^2\right)^{1/2}\left(\int_{\mathcal{O}_i}\mu_V|\nabla\phi|^2+\mu_V\phi^2\right)^{1/2}\lesssim\|\phi\|_H. \tag{2.31}$$

We can similarly bound using Cauchy-Schwarz on $\Sigma\setminus E$ since $(-\Delta)^\alpha\varphi^\Sigma$ and μ_V are bounded above and below there to conclude that $\phi\mapsto\int\phi(-\Delta)^\alpha\varphi^\Sigma$ is a bounded linear functional on H.

Existence. The Riesz Representation theorem guarantees a unique $u\in H$ such that (2.22) holds for all $\phi\in H$. To extend (2.22) to all test functions $\phi\in H^1_{\mu_V}$, we need

$$\int_\Sigma(-\Delta)^\alpha\varphi^\Sigma = 0$$

for compatibility. This follows from the fact that $(-\Delta)^\alpha$ is a mean-zero operator and thus

$$0=\int(-\Delta)^\alpha\varphi^\Sigma=\int_\Sigma(-\Delta)^\alpha\varphi^\Sigma,$$

as desired.

Step 2: Boundary Condition. Next, we show that these weak energy solutions have the desired boundary condition. From [TTV24, Theorem 1.1], in the chart $\mathcal{O}_i$ the boundary condition satisfies

$$(\nabla u+G_i)\cdot\vec{n}=0,$$

where $\vec{n}$ is the outward pointing unit normal. Using the relation

$$\frac{1}{\mathsf{c_{d,s}}}(-\Delta)^\alpha\varphi^\Sigma = w\,\mathrm{dist}(x,\partial\Sigma)^{-\alpha}=\mathrm{div}\,(\mu_V G_i)=\mathrm{div}\,(s\,\mathrm{dist}(x,\partial\Sigma)^{1-\alpha}G_i)$$

in view of (1.24), we find

$$\begin{aligned}w\,\mathrm{dist}(x,\partial\Sigma)^{-\alpha} = \mathrm{dist}(x,\partial\Sigma)^{1-\alpha}G_i\nabla s + s(1-\alpha)\mathrm{dist}(x,\partial\Sigma)^{-\alpha}\nabla\mathrm{dist}(x,\partial\Sigma)\cdot G_i \\ + s\,\mathrm{dist}(x,\partial\Sigma)^{1-\alpha}\mathrm{div}\,G_i,\end{aligned}$$

which can be rewritten as

$$s(1-\alpha)\nabla \mathrm{dist}(x,\partial\Sigma)\cdot G_i = w - \mathrm{dist}(x,\partial\Sigma)\nabla s G_i - s\,\mathrm{dist}(x,\partial\Sigma)\mathrm{div}\, G_i.$$

Taking $x\to\partial\Sigma$ and using the regularity of s, w and G_i coupled with $\nabla\mathrm{dist}(x,\partial\Sigma)\to -\vec{n}$, we find

$$G_i\cdot\vec{n} = -\frac{w}{s(1-\alpha)}$$

and thus

$$\nabla u\cdot\vec{n} = \frac{w}{s(1-\alpha)}.$$

Setting $\psi=\nabla u$, we find a solution to (2.16) in Σ.

Step 3: Schauder Estimate up to the Boundary. First, let us examine the behavior of $\psi=\nabla u$ in the interior of Σ. Writing $E=\cup\mathcal{O}_i$ as above for the union of charts covering $\partial\Sigma$, observe that μ_V is bounded below in $\Sigma\setminus E$. So, we can appeal to standard elliptic regularity to control u inside E, thus finding

$$\|u\|_{C^{2+\sigma}(A)}\lesssim \|w\|_{C^\sigma(A)} \tag{2.32}$$

for any $A\subset E$, recalling that for integer σ we define C^σ as the Hölder space $C^{\sigma-1,1}$. Applying (2.32) to $A=\square_{2\ell}$ (we omit the z in the notation) and using (8.16), we obtain for any $1\le m\le k$,

$$\|\psi\|_{C^m(\square_{2\ell})}\lesssim \ell^{\mathsf{d}+2}\|(-\Delta)^\alpha\varphi\|_{L^\infty}+\ell^{2+\mathsf{s}-\mathsf{d}}\|\varphi\|_{C^{1+m}}\lesssim \mathsf{M}(\ell^{2+\mathsf{s}}+\ell^{\mathsf{s}-\mathsf{d}-m+1})$$

where we have also used (2.15) and Lemma 8.4. This proves (2.17) in $\square_{2\ell}$.

In $\hat{\Sigma}\backslash\square_{2\ell}$ we can obtain the decay as well by applying elliptic estimates in dyadic annuli centered around $\square_{2\ell}$ of radii $2^k\ell$ until we hit the edge of the bulk. More precisely, choose k_* such that $2^{k_*}\ell$ is at macroscopic scale and $\square_{2^{k_*}\ell}\subset\hat{\Sigma}$. Let $A_k=\square_{2^\ell}\setminus\square_{2^{k-2}\ell}$. We can use (2.32) and (8.16) to obtain that

$$\|\psi\|_{C^m(A_k)}\lesssim \left\|(-\Delta)^\alpha\varphi^\Sigma\right\|_{C^{m-1}(A_k)}\lesssim \ell^{\mathsf{d}+2}\|(-\Delta)^\alpha\varphi\|_{L^\infty}+\frac{\ell^{\mathsf{d}}\|\varphi\|_{L^\infty}}{(2^{k-2}\ell)^{2\mathsf{d}-\mathsf{s}+m-1}}.$$

The first term is dominant by Lemma 8.4, so

$$\|\psi\|_{C^m(A_k)}\lesssim\frac{\mathsf{M}\ell^{\mathsf{d}}}{(2^{k-2}\ell)^{2\mathsf{d}-\mathsf{s}+m-1}}$$

by (2.15). Adjusting constants and summing over A_k yields

$$|\nabla^{\otimes m}\psi(x)|\lesssim\frac{\mathsf{M}\ell^{\mathsf{d}}}{|x-z|^{2\mathsf{d}-\mathsf{s}+m-1}}$$

which is (2.17) in $\hat{\Sigma}\backslash\square_{2\ell}$.

Near the boundary of Σ, we need to be more careful due to the decay of μ_V and the blowup of $(-\Delta)^\alpha\xi^\Sigma$. Applying [TTV24, Theorem 1.1], we obtain the control

$$\|u\|_{C^{1+\sigma}(\mathcal{O}_i\cap\Sigma)}\lesssim\|u\|_{L^2(\mathcal{O}_i\cap\Sigma)}+\|G_i\|_{C^\sigma(\mathcal{O}_i\cap\Sigma)} \tag{2.33}$$

for any $\sigma\notin\mathbb{N}$. We can rewrite this estimate using the equation (2.16) and our divergence form for the right hand side. With $E=\cup\mathcal{O}_i$ as above and recalling that $\int|\nabla u|^2\mu_V$ is an equivalent norm on H, we write

$$\|u\|_H^2 = \int_\Sigma |\nabla u|^2 \mu_V = \int_\Sigma \frac{1}{\mathsf{c_{d,s}}} u(-\Delta)^\alpha \varphi^\Sigma = \int_E \frac{1}{\mathsf{c_{d,s}}} (-\Delta)^\alpha \varphi^\Sigma u + \int_{\Sigma\setminus E} \frac{1}{\mathsf{c_{d,s}}} (-\Delta)^\alpha \varphi^\Sigma u. \tag{2.34}$$

The integral on $\Sigma \setminus E$ can be controlled via Cauchy-Schwarz, writing

$$\int_{\Sigma\setminus E} u(-\Delta)^\alpha \varphi^\Sigma \lesssim \left(\int_{\Sigma\setminus E} \frac{\left|(-\Delta)^\alpha \varphi^\Sigma\right|^2}{\mu_V}\right)^{1/2} \left(\int_{\Sigma\setminus E} u^2 \mu_V\right)^{1/2}$$

and so since $\int \mu_V^{-1} < \infty$,

$$\left|\int_{\Sigma\setminus E} \frac{1}{\mathsf{c_{d,s}}} (-\Delta)^\alpha \varphi^\Sigma u\right| \lesssim \left\|(-\Delta)^\alpha \varphi^\Sigma\right\|_{L^\infty(\Sigma\setminus E)} \|u\|_H \lesssim \|w\|_{L^\infty(\Sigma)} \|u\|_H.$$

For the integral in E, we argue as in (2.31) that

$$\begin{aligned}\left|\int_E \frac{1}{\mathsf{c_{d,s}}} u(-\Delta)^\alpha \varphi^\Sigma\right| &\lesssim \sum_i \left(\int \mu_V G_i\right)^{1/2} \left(\int_{\mathcal{O}_i} \mu_V |\nabla u|^2 + \mu_V |u|^2\right)^{1/2} \\ &\lesssim \sum_i \|G_i\|_{L^\infty} \|u\|_H \lesssim \|w\|_{L^\infty(\Sigma)} \|u\|_H.\end{aligned}$$

Inserting these into (2.34) we find

$$\|u\|_H^2 \lesssim \|w\|_{L^\infty} \|u\|_H$$

and from (2.33) we conclude that

$$\|u\|_{C^{1+\sigma}(E)} \lesssim \|w\|_{C^\sigma(E)}.$$

for $\sigma \notin \mathbb{N}$. Recalling the definition $\psi = \nabla u$ in Σ, we can rephrase the global estimate as

$$\|\psi\|_{C^\sigma(\Sigma)} \lesssim \|w\|_{C^\sigma(\Sigma)} = \left\|\frac{(-\Delta)^\alpha \varphi^\Sigma}{\mathrm{dist}(x, \partial\Sigma)^{-\alpha}}\right\|_{C^\sigma(\Sigma)}. \tag{2.35}$$

In view of Hölder interpolation of(8.16), we have (applying (2.35) for $\sigma = k + \epsilon$)

$$\|\psi\|_{C^{k+\epsilon}(\Sigma\setminus\hat{\Sigma})} \lesssim \ell^{\mathsf{d}} \|\varphi\|_{L^\infty} + \ell^{\mathsf{d}+2} \|(-\Delta)^\alpha \varphi\|_{L^\infty}$$

which yields, in view of Lemma 8.4,

$$\|\psi\|_{C^k(\Sigma\setminus\hat{\Sigma})} \lesssim \mathsf{M}\ell^{\mathsf{d}} \tag{2.36}$$

using (2.15) and Lemma 8.4, which is (2.17) in $\Sigma \setminus \hat{\Sigma}$. Note that we needed to be careful to apply (2.35) for $\sigma \notin \mathbb{N}$ as [TTV24, Theorem 1.1] is stated only for noninteger σ.

Finally, the above estimates were only for $\nabla^{\otimes m}\psi$ with $m \geq 1$. Extending the above estimates on $\nabla^{\otimes m}\psi$ for $m \geq 1$ to an L^∞ ($m = 0$) bound on ψ then follows from using that $|\psi| = O(\ell^d)$ at $\partial\Sigma$ and integrating the derivative estimate.

Step 4: Continuity Across the Boundary. We now establish continuity of ψ across the boundary.

Rewriting the external transport at the boundary. Let $x_0 \in \partial\Sigma$, and let $c_1(x_0)$ be the unique constant such that

$$\nabla\zeta_V \cdot \vec{n} = c_1(x_0)\mathrm{dist}(x, \partial\Sigma)^\alpha + O\left(|x - x_0|^{\alpha+a}\right) \tag{2.37}$$

where $a > 0$ is such that $\alpha + a < 1$. Thus, in $U \setminus \Sigma$, as x approaches x_0 from the outside,

$$\lim_{x\to x_0} \psi \cdot \vec{n}(x) = \lim_{x\to x_0} \frac{\varphi^\Sigma - \varphi}{c_1(x_0)\mathrm{dist}(x, \partial\Sigma)^\alpha}$$

Furthermore, since $\varphi \in C^2$ we know that $(-\Delta)^\alpha\varphi \in L^\infty$; then, regularity for fractional elliptic problems (cf. [ROS14, Theorem 1.2]) gives us a unique constant $c_2(x_0)$ near x_0 such that

$$\varphi^\Sigma(x) - \varphi(x) = c_2(x_0)\mathrm{dist}(x, \partial\Sigma)^\alpha + o(\mathrm{dist}(x, \partial\Sigma)^\alpha) \quad \text{as} \quad x \to x_0 \tag{2.38}$$

and so

$$\lim_{x\to x_0} \psi \cdot \vec{n} = \frac{c_2(x_0)}{c_1(x_0)} \tag{2.39}$$

as $x \to x_0$ from $U \setminus \Sigma$.

Agreement of Boundary Conditions. We now claim that the Neumann boundary condition in (2.16) and (2.39) agree. First, recalling that for x_0 on the boundary

$$w(x_0) = \lim_{x\to x_0} \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}(-\Delta)^\alpha\varphi^\Sigma(x)\mathrm{dist}(x, \partial\Sigma)^\alpha$$

and $s(x_0) = \lim_{x\to x_0} \frac{\mu_V(x)}{\mathrm{dist}(x,\partial\Sigma)^{1-\alpha}}$, we can rewrite the Neumann condition in (2.16) as

$$\psi \cdot \vec{n}(x_0) = \lim_{x\to x_0} \frac{(-\Delta)^\alpha\varphi^\Sigma(x)\mathrm{dist}(x, \partial\Sigma)}{(1-\alpha)\mathsf{c}_{\mathsf{d},\mathsf{s}}\mu_V(x)}. \tag{2.40}$$

Lemma A.8 yields

$$(-\Delta)^\alpha\varphi^\Sigma = (-\Delta)^\alpha\varphi + \overline{c}_\alpha c_2(x_0)\mathrm{dist}(x, \partial\Sigma)^{-\alpha} + o(\mathrm{dist}(x, \partial\Sigma)^{-\alpha}) \tag{2.41}$$

where

$$\overline{c}_\alpha = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}. \tag{2.42}$$

Proposition A.7 also gives

$$(1-\alpha)\mathsf{c}_{\mathsf{d},\mathsf{s}}\mu_V(x) = \overline{c}_\alpha c_1(x_0)\mathrm{dist}(x, \partial\Sigma)^{1-\alpha} + o(\mathrm{dist}(x, \partial\Sigma)^{1-\alpha}) \quad \text{as} \quad x \to x_0 \tag{2.43}$$

where $c_1(x_0)$ is as in (2.37). We then compute

$$\begin{aligned}
\psi \cdot \vec{n}(x_0) &= \lim_{x\to x_0} \frac{(-\Delta)^\alpha\varphi^\Sigma(x)\mathrm{dist}(x, \partial\Sigma)}{(1-\alpha)\mathsf{c}_{\mathsf{d},\mathsf{s}}\mu_V(x)} \\
&= \lim_{x\to x_0} \frac{\left((-\Delta)^\alpha\varphi + \overline{c}_\alpha c_2(x_0)\mathrm{dist}(x, \partial\Sigma)^{-\alpha}\right)\mathrm{dist}(x, \partial\Sigma)}{\overline{c}_\alpha c_1(x_0)\mathrm{dist}(x, \partial\Sigma)^{1-\alpha}} \\
&= \lim_{x\to x_0} \frac{(-\Delta)^\alpha\varphi}{\overline{c}_\alpha c_1(x_0)}\mathrm{dist}(x, \partial\Sigma)^\alpha + \lim_{x\to x_0} \frac{\overline{c}_\alpha c_2(x_0)\mathrm{dist}(x, \partial\Sigma)^{1-\alpha}}{\overline{c}_\alpha c_1(x_0)\mathrm{dist}(x, \partial\Sigma)^{1-\alpha}} \\
&= \frac{c_2(x_0)}{c_1(x_0)},
\end{aligned} \tag{2.44}$$

which agrees with (2.39). Thus, ψ is continuous in its normal component across the boundary. We may then build ψ continuous exactly in the same way as in the proof of [LS18, Lemma 3.4] by compensating with an appropriate tangential vector field $\psi^\perp$: consider the trace $\psi - (\psi \cdot \vec{n})\vec{n}$ on $\partial\Sigma$ from the inside, and extend it to a regular vector field vanishing outside U, then subtract off the projection of that vector field onto $\nabla \zeta_V$ to obtain a vector field $\psi^\perp$ which remains perpendicular to $\nabla\zeta_V$ and vanishing in U^c. In view of (2.36), and the C^{k+1} regularity of $\partial\Sigma$, we can obtain $\psi^\perp$ such that, for $m \le k$,

$$\|\nabla^{\otimes m}\psi^\perp\|_{L^\infty(\Sigma^c)} \lesssim \|\varphi\|_{L^\infty}\ell^{\mathsf{d}}. \tag{2.45}$$

The vector field ψ is now defined in $\mathbb{R}^{\mathsf{d}}$.

Step 4: Exterior Schauder Estimate. We apply the boundary Harnack inequality for α-harmonic functions from [ARO20a, Theorem 1.3]. Up to a rotation, we may as well assume at a point $x_0 \in \partial\Sigma$ that $\vec{n} = \vec{e}_n$, the unit vector in the x_n direction. The regularity of $\psi^\perp$ is provided in (2.45), so it is sufficiently to consider

$$(\varphi^\Sigma - \varphi)\frac{\nabla\zeta_V}{|\nabla\zeta_V|^2}.$$

Notice that outside of Σ, we have $(-\Delta)^\alpha\zeta_V = \mu_V + (-\Delta)^\alpha(V - c_V) = (-\Delta)^\alpha(V - c_V)$. Since $\zeta_V \equiv 0$ in Σ, we then have

$$\begin{cases} (-\Delta)^\alpha(\partial_i\zeta_V) = (-\Delta)^\alpha(V - c_V) & \text{in } B_r(x_0)\cap\Sigma^c \\ \partial_i\zeta_V = 0 & \text{in } B_r(x_0)\cap\Sigma \end{cases}$$

for $x_0 \in \partial\Sigma$ and $r > 0$ sufficiently small. We also have

$$\begin{cases} (-\Delta)^\alpha(\varphi^\Sigma - \varphi) = -(-\Delta)^\alpha\varphi & \text{in } B_r(x_0)\cap\Sigma^c \\ \varphi^\Sigma - \varphi = 0 & \text{in } B_r(x_0)\cap\Sigma. \end{cases}$$

Notice also that it follows from (2.18) (see also [ROS17a]) that $\partial_n\zeta_V \gtrsim \mathrm{dist}(x,\partial\Sigma)^\alpha$. Now, it follows from [ARO20a, Theorem 1.3] that

$$\frac{\varphi^\Sigma - \varphi}{\partial_n\zeta_V}, \quad \frac{\partial_i\zeta_V}{\partial_n\zeta_V} \in C^\sigma$$

for any $\sigma \notin \mathbb{N}$ such that $\sigma + \alpha, \sigma - \alpha \notin \mathbb{N}$. As in the proof of [ARO20a, Theorem 1.1], we write

$$\frac{(\varphi^\Sigma - \varphi)\partial_i\zeta_V}{|\nabla\zeta_V|^2} = \frac{(\varphi^\Sigma - \varphi)}{\partial_n\zeta_V}\frac{\partial_i\zeta_V\partial_n\zeta_V}{|\nabla\zeta_V|^2}. \tag{2.46}$$

Now,

$$\frac{\partial_i\zeta_V\partial_n\zeta_V}{|\nabla\zeta_V|^2} = \frac{\frac{\partial_i\zeta_V}{\partial_n\zeta_V}}{1 + \sum_{i=1}^{n-1}\left(\frac{\partial_i\zeta_V}{\partial_n\zeta_V}\right)^2} \in C^\sigma(B_r(x_0))$$

by the boundary Harnack inequality, and

$$\begin{aligned} \left\|\frac{(\varphi^\Sigma - \varphi)}{\partial_n\zeta_V}\right\|_{C^\sigma(B_r(x_0))} &\lesssim \|(-\Delta)^\alpha\varphi\|_{C^{\sigma-\alpha}(B_r(x_0))} + \|\varphi^\Sigma - \varphi\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \|(-\Delta)^\alpha\varphi\|_{C^{\sigma-\alpha}(U\setminus\Sigma)}. \end{aligned}$$

where we have used [ROS16, Theorem 1.2] to control the L^∞ term. Inserting these into (2.46) and using [GT01, (4.7)], we find

$$\left\|(\varphi^\Sigma-\varphi)\frac{\nabla\zeta_V}{|\nabla\zeta_V|^2}\right\|_{C^{k+\epsilon}(B_r(x_0))}\lesssim\|(-\Delta)^\alpha\varphi\|_{C^{k+\epsilon-\alpha}(U\setminus\Sigma)},$$

which, using (8.7) and (2.15) coupled with Hölder interpolation completes the proof. We have been careful to apply [ARO20a, Theorem 1.3] for $\sigma=k+\epsilon$, since that result does not hold for integer k.

To control the decay of ψ in U^c, we note that by definition (1.16) we have $\nabla\zeta_V=\nabla h^{\mu_V}+\nabla V$. Since μ_V has compact support, ∇h^{μ_V} and its derivatives decay like those of g, i.e. faster than $|x|^{-\mathsf{s}-1}$. Since $\mathsf{s}>\mathsf{d}-2\geq-1$, this means that ∇h^{μ_V} and all its derivatives tend to 0 at infinity, thus $\nabla\zeta_V\sim\nabla V$ at infinity. It follows from the definition of ψ in (2.16) (and the fact that $\psi^\perp$ is compactly supported) that

$$\nabla^{\otimes m}\psi\sim\nabla^{\otimes m}\left(\varphi^\Sigma\frac{\nabla V}{|\nabla V|^2}\right)\quad\text{as }|x|\to\infty$$

Using (8.9), (2.15), Lemma 8.4, $\mathsf{s}>\mathsf{d}-2$ and (**A5**) we have that

$$|\psi|\lesssim\frac{\|(-\Delta)^\alpha\varphi\|_{L^\infty}\,\ell^{\mathsf{d}+2}+\|\varphi\|_{L^\infty}\ell^{\mathsf{d}}}{|x-z|^{\mathsf{s}+2}}\lesssim\frac{\mathsf{M}\ell^{\mathsf{d}}}{|x-z|^{\mathsf{s}+2}},$$

which is (2.17). For the derivatives, we have using (8.9), (**A5**), (2.15) and the Faa-di Bruno formula that

$$|\nabla^{\otimes m}\psi|\lesssim\sum_{j=0}^m\left\|D^j\varphi^\Sigma\right\|\left\|D^{m-j}\left(\frac{1}{|\nabla V|}\right)\right\|\lesssim\frac{\|(-\Delta)^\alpha\varphi\|_{L^\infty}\,\ell^{\mathsf{d}+2}+\|\varphi\|_{L^\infty}\ell^{\mathsf{d}}}{|x-z|^{\mathsf{s}+2+m}}\lesssim\frac{\mathsf{M}\ell^{\mathsf{d}}}{|x-z|^{\mathsf{s}+2+m}}$$

for $|x|$ large enough. This completes the proof of (2.17) and of Proposition 2.2.

We will often need the following set of consequences.

Lemma 2.3. *Assume* (2.15) *hold for $k=3$. For ψ as above, and any $n\geq1$ integer, for U as above, we have*

$$\int_U|\psi|^n\lesssim\mathsf{M}^n\ell^{(1-n)\mathsf{d}+n\mathsf{s}+n},\tag{2.47}$$

$$\int_U|D\psi|^n\lesssim\mathsf{M}^n\ell^{(1-n)\mathsf{d}+n\mathsf{s}},\tag{2.48}$$

$$\int_\Sigma\frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\,dy\lesssim\mathsf{M}^n\begin{cases}\ell^{n\mathsf{d}}\max(|x-z|,\ell)^{\mathsf{d}(1-2n)+\mathsf{s}(n-1)} & \text{if } x\in U\\ \ell^{n\mathsf{d}}|x-z|^{-n(\mathsf{s}+2)} & \text{if } x\in U^c\end{cases}\tag{2.49}$$

where z is the center of $\square_\ell$, and

$$\int_{\Sigma^2}\frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\,dxdy\lesssim\mathsf{M}^n\ell^{(2-n)\mathsf{d}+(n-1)\mathsf{s}}.\tag{2.50}$$

Proof. From the estimates (2.17), we have

$$\int_U|\psi|^n\lesssim\mathsf{M}^n\left(\frac{\ell^{\mathsf{d}}}{\ell^{n(\mathsf{d}-\mathsf{s}-1)}}+\ell^{n\mathsf{d}}\int_\ell^1\frac{r^{\mathsf{d}-1}}{r^{n(2\mathsf{d}-\mathsf{s}-1)}}\,dr\right)\lesssim\mathsf{M}^n\ell^{\mathsf{d}(1-n)+n(\mathsf{s}+1)}.\tag{2.51}$$

Next, we write

$$\int_U |D\psi|^n \lesssim \mathsf{M}^n \left(\frac{\ell^{\mathsf{d}}}{\ell^{n(\mathsf{d}-\mathsf{s})}} + \ell^{n\mathsf{d}} \int_\ell^1 \frac{r^{\mathsf{d}-1}}{r^{n(2\mathsf{d}-\mathsf{s})}}\, dr\right) \lesssim \mathsf{M}^n \ell^{(1-n)\mathsf{d}+n\mathsf{s}}. \tag{2.52}$$

For (2.49), we first consider $x \in \square_{2\ell}$. Then, using $\frac{|\psi(x)-\psi(y)|}{|x-y|} \lesssim \mathsf{M}\ell^{\mathsf{s}-\mathsf{d}}$ from (2.17), we find

$$\begin{aligned}\int_U \frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dy &\lesssim \int_{y\in U, |y-x|\le \ell} \frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, d + \int_{y\in U, |y-x|>\ell} \frac{|\psi(x)|^n + |\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dy \\ &\lesssim \frac{\mathsf{M}^n}{\ell^{n(\mathsf{d}-\mathsf{s})}} \int_0^\ell \frac{1}{r^{\mathsf{s}}} r^{\mathsf{d}-1}\, dr + \mathsf{M}^n \ell^{-n(\mathsf{d}-\mathsf{s}-1)} \int_{\ell\le|u|\le C} \frac{1}{|u|^{\mathsf{s}+n}}\, du \\ &\lesssim \mathsf{M}^n \ell^{(1-n)(\mathsf{d}-\mathsf{s})}.\end{aligned}$$

Next, for $x \in U\backslash\square_{2\ell}$, we obtain similarly using (2.17), z being the center of $\square_{2\ell}$,

$$\begin{aligned}&\int_U \frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dy \lesssim \int_{y\in U, |y-x|\le \frac12|x-z|} \frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dy + \int_{y\in U, |y-x|>\frac12|x-z|} \frac{|\psi(x)|^n}{|x-y|^{\mathsf{s}+n}}\, dy \\ &\qquad + \int_{y\in U, |y-x|>\frac12|x-z|} \frac{|\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dy \\ &\lesssim \frac{\mathsf{M}^n \ell^{n\mathsf{d}}}{|x-z|^{n(2\mathsf{d}-\mathsf{s})}} \int_0^{\frac12|x-z|} \frac{1}{r^{\mathsf{s}}} r^{\mathsf{d}-1}\, dr + \mathsf{M}^n \ell^{n\mathsf{d}} |x-z|^{-n(2\mathsf{d}-\mathsf{s}-1)} \int_{\frac12|x-z|\le|u|\le C} \frac{1}{|u|^{\mathsf{s}+n}}\, du \\ &\qquad + \mathsf{M}^n \int_{\frac12|x-z|\le|y-x|\le C} \frac{1}{|y-x|^{\mathsf{s}+n}} \frac{\ell^{n\mathsf{d}}}{\max(|y-z|,\ell)^{n(2\mathsf{d}-\mathsf{s}-1)}}\, dy \\ &\lesssim \mathsf{M}^n \ell^{n\mathsf{d}} |x-z|^{\mathsf{d}(1-2n)+\mathsf{s}(n-1)}.\end{aligned}$$

Finally, for $x \in U^c$, we obtain similarly

$$\begin{aligned}&\int_\Sigma \frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dy \lesssim \int_{y\in \Sigma, |y-x|\le \frac12|x-z|} \frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dy + \int_{y\in U, |y-x|>\frac12|x-z|} \frac{|\psi(x)|^n}{|x-y|^{\mathsf{s}+n}}\, dy \\ &\qquad + \int_{y\in \Sigma, |y-x|>\varepsilon} \frac{|\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dy \\ &\lesssim \mathsf{M}^n \ell^{n\mathsf{d}} |x-z|^{\mathsf{d}(1-2n)+\mathsf{s}(n-1)} + \mathsf{M}^n \frac{\ell^{n\mathsf{d}}}{|x-z|^{n(\mathsf{s}+2)}}.\end{aligned}$$

This proves (2.49).

For (2.50), we integrate (2.49) over U to find

$$\int_{\Sigma^2} \frac{|\psi(x)-\psi(y)|^n}{|x-y|^{\mathsf{s}+n}}\, dxdy \lesssim \mathsf{M}^n \left(\ell^{\mathsf{d}} \ell^{(\mathsf{d}-\mathsf{s})(1-n)} + \ell^{n\mathsf{d}} \ell^{\mathsf{d}+\mathsf{d}(1-2n)+\mathsf{s}(n-1)}\right),$$

which yields the result. □

3. Splitting, the electric formulation, and transport calculus

In order to complete the proof of Theorems 2-3, we need to recall the main ingredients of our electric-formulation based analysis, originating in [PS17]. Most of what follows is based on material that can be found in [Ser24].

3.1. **The next-order energy and partition functions.** As discussed in the introduction, under our assumptions on V the sequence of empirical measures $\frac{1}{N}\sum \delta_{x_i}$ converges in a large deviations sense [Ser24, Theorem 3.3] to the equilibrium measure μ_V. Splitting off the main term of the interaction leads to a definition of the next-order energy.

Lemma 3.1 (Splitting Formula). *Given any configuration $X_N \in (\mathbb{R}^{\mathsf{d}})^N$, the energy being as in (1.2), we have*

$$\mathcal{H}_N(X_N) = N^2\mathcal{E}(\mu_V) + N\sum_{i=1}^N \zeta_V(x_i) + \mathsf{F}_N(X_N,\mu_V) \tag{3.1}$$

where $\mathcal{E}$ is defined in (1.14) and the next-order energy $\mathsf{F}_N(X_N,\mu_V)$ is defined in (1.21).

We refer to [Ser24, Lemma 5.1] for the proof.

Associated to this next-order energy is the *next-order partition function*

$$\mathsf{K}_{N,\beta}(\mu,\zeta) := \int_{(\mathbb{R}^{\mathsf{d}})^N} \exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\mathsf{F}_N(X_N,\mu) + N\sum_{i=1}^N \zeta(x_i)\right)\right) dX_N. \tag{3.2}$$

The Gibbs measure can then be rewritten as

$$d\mathbb{P}_{N,\beta}(X_N) = \frac{1}{\mathsf{K}_{N,\beta}(\mu_V,\zeta_V)} \exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\mathsf{F}_N(X_N,\mu_V) + N\sum_{i=1}^N \zeta_V(x_i)\right)\right) dX_N. \tag{3.3}$$

We will also need in the proofs of Theorem 2 and 3 a next-order partition function restricted to a given event $\mathcal{G}$, which will be denoted as

$$\mathsf{K}^{\mathcal{G}}_{N,\beta}(\mu,\zeta) := \int_{(\mathbb{R}^{\mathsf{d}})^N} \exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\mathsf{F}_N(X_N,\mu,U) + N\sum_{i=1}^N \zeta(x_i)\right)\right) \mathbf{1}_{\mathcal{G}}\, dX_N. \tag{3.4}$$

In [Ser24, Corollary 5.23] an exponential moment control of the energy in the form

$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\frac{\beta}{2}N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\mathsf{F}_N(X_N,\mu_V) + \left(\frac{N}{2\mathsf{d}}\log N\right)\mathbf{1}_{\mathsf{s}=0} + N\sum_{i=1}^N \zeta_V(x_i)\right)\right)\right)\right| \le C(1+\beta)N, \tag{3.5}$$

where $C>0$ depends on $\mathsf{s},\mathsf{d},\zeta_V$ and $\|\mu_V\|_{L^\infty}$, are shown from upper and lower bounds on $\mathsf{K}_{N,\beta}(\mu,\zeta)$.

They can be seen as a "local law" at the macroscale. In particular it follows that, except with probability $e^{-C\beta N}$, we have

$$\mathsf{F}_N(X_N,\mu_V) + \left(\frac{N}{2\mathsf{d}}\log N\right)\mathbf{1}_{\mathsf{s}=0} + N\sum_{i=1}^N \zeta_V(x_i) \lesssim_\beta N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \tag{3.6}$$

where $C>0$ depends on $\mathsf{s},\mathsf{d},\zeta_V$ and $\|\mu_V\|_{L^\infty}$. We will thus be able to intersect all our good events with this large probability event.

3.1.1. *Electric formulation.* We will use the so-called *electrostatic* approach to studying F_N, introduced for the general Riesz gas in [PS17]. A key tool in this approach is the Caffarelli-Silvestre [CS07] extension procedure for reinterpreting fractional Laplace operators, as introduced in the Riesz context in [PS17]. The connection is based on the observation that (up to a constant) g is the solution kernel for the fractional Laplace operator $(-\Delta)^\alpha$, where $\alpha = \frac{\mathsf{d}-\mathsf{s}}{2}$.

Considering the extended function $\mathsf{g}(x,y)$ on $\mathbb{R}^{\mathsf{d}+1}$ (where $x\in\mathbb{R}^{\mathsf{d}}$ and $y\in\mathbb{R}$) one can observe that g is a fundamental solution of a degenerate elliptic equation, i.e. up to a constant solves

$$\begin{cases}\mathrm{div}\,(|y|^\gamma\nabla u)=0 & \text{in } \mathbb{R}^{\mathsf{d}}\times(\mathbb{R}\setminus\{0\})\\ -\lim_{|y|\downarrow 0}|y|^\gamma\partial_y u(\cdot,y)=\delta_0 & \text{on } \mathbb{R}^{\mathsf{d}}\times\{0\}\end{cases}\tag{3.7}$$

with γ satisfying

$$\mathsf{d}-1+\gamma=\mathsf{s}.\tag{3.8}$$

In particular, if we let μ denote a measure on $\mathbb{R}^{\mathsf{d}}$, then the extension of the potential $h^\mu=\mathsf{g}*\mu$ to $\mathbb{R}^{\mathsf{d}+1}$ given by

$$h^\mu(x,y)=\int_{\mathbb{R}^{\mathsf{d}+1}}\frac{1}{|(x,y)-(x',y')|^{\mathsf{s}}}\,d(\mu\delta)_{\mathbb{R}^{\mathsf{d}}}(x',y')=\int_{\mathbb{R}^{\mathsf{d}}}\frac{1}{|(x,y)-(x',0)|^{\mathsf{s}}}\,d\mu(x')\tag{3.9}$$

solves

$$-\mathrm{div}\,(|y|^\gamma\nabla h^\mu)=\mathsf{c}_{\mathsf{d},\mathsf{s}}\mu\delta_{\mathbb{R}^{\mathsf{d}}}\quad\text{in } \mathbb{R}^{\mathsf{d}+1},\tag{3.10}$$

where we denote by $\delta_{\mathbb{R}^{\mathsf{d}}}$ the uniform measure on $\mathbb{R}^{\mathsf{d}}\times\{0\}$ characterized by the fact that for any continuous function φ in $\mathbb{R}^{\mathsf{d}+1}$, $\int_{\mathbb{R}^{\mathsf{d}+1}}\varphi\,\delta_{\mathbb{R}^{\mathsf{d}}}=\int_{\mathbb{R}^{\mathsf{d}}}\varphi(x,0)dx$.

To formalize the electrostatic rewriting of the next-order energy, we will need a smearing procedure that regularizes the electrostatic potential generated by point charges. That procedure is only needed in the case $\mathsf{s}\geq 0$ where g is singular at the origin. While much of what we define can be found in [Ser24, Sec. 4.1.3], we restate much of it here for readability. If we define

$$\mathsf{f}_\eta(x)=(\mathsf{g}(x)-\mathsf{g}(\eta))_+\tag{3.11}$$

either for $x\in\mathbb{R}^{\mathsf{d}}$ or by extension for $x\in\mathbb{R}^{\mathsf{d}+1}$, this is a function supported in $B(0,\eta)$ and satisfying

$$-\mathrm{div}\,(|y|^\gamma\nabla\mathsf{f}_\eta)=\mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\delta_0-\delta_0^{(\eta)}\right)\tag{3.12}$$

where we define $\delta_{x_0}^{(\eta)}=-\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\mathrm{div}\,(|y|^\gamma\nabla\mathsf{g}_\eta(x-x_0))$. It is a weighted measure of mass 1 supported on $\partial B(0,\eta)$ satisfying

$$\int\varphi\delta_0^{(\eta)}=\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\partial B(0,\eta)}\varphi(x,y)|y|^\gamma\mathsf{g}'(\eta)$$

for smooth φ. If u solves

$$-\mathrm{div}\,(|y|^\gamma\nabla u)=\mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^N\delta_{(x_i,0)}-\mu\delta_{\mathbb{R}^{\mathsf{d}}}\right),\tag{3.13}$$

for any truncation vector $\vec\eta=(\eta_1,\dots,\eta_N)$, we define

$$u_{\vec\eta}(X):=u-\sum_{i=1}^N\mathsf{f}_{\eta_i}(x-(x_i,0))\tag{3.14}$$

and observe that

$$-\mathrm{div}\,(|y|^\gamma\nabla u_{\vec\eta})=\mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^N\delta_{(x_i,0)}^{(\eta_i)}-\mu\delta_{\mathbb{R}^{\mathsf{d}}}\right).\tag{3.15}$$

We will also commonly identify $x_i\in\mathbb{R}^{\mathsf{d}}$ with $(x_i,0)\in\mathbb{R}^{\mathsf{d}+1}$.

For our computations, we will often make use of the identification

$$\int_{U\times[-h,h]} \mathsf{f}_\eta(x,y)\, d\mu\delta_{\mathbb{R}^d}(x,y) = \int_U \mathsf{f}_\eta(x)\, d\mu(x). \tag{3.16}$$

We will frequently use that

$$\int_{\mathbb{R}^\mathsf{d}} |\nabla^{\otimes n}\mathsf{f}_\eta| \le C\eta^{\mathsf{d}-\mathsf{s}-n}. \tag{3.17}$$

A truncation parameter that we will often choose is the minimal distance (1.33). With this truncation defined, we can now integrate by parts. The following exact formula follows from an examination of the proof of [PS17, Proposition 1.6] using the decay of $h_{N,\vec{\eta}}$ at infinity; see [Ser24, Lemma 4.10].

Lemma 3.2 (Riesz electric formulation of the next order energy)**.** *Let X_N be a configuration of points in $\mathbb{R}^\mathsf{d}$, μ a probability density on $\mathbb{R}^\mathsf{d}$ such that*

$$\iint |\mathsf{g}(x-y)|\ d|\mu|(x)d|\mu|(y) < +\infty \tag{3.18}$$

and let $\vec{\eta}$ denote a truncation vector with $\eta_i \le \mathsf{r}_i$ for all $1 \le i \le N$. Let

$$h_N[X_N,\mu] := \mathsf{g} * \left(\sum_{i=1}^N \delta_{x_i} - N\mu\right) \tag{3.19}$$

(which will most often be abbreviated as h_N). Then, with the notation (3.14), we have

$$\mathsf{F}(X_N,\mu) = \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left(\int_{\mathbb{R}^{\mathsf{d}+1}} |y|^\gamma |\nabla h_{N,\vec{\eta}}|^2 - \mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{i=1}^N \mathsf{g}(\eta_i)\right) - N\sum_{i=1}^N \int_{\mathbb{R}^\mathsf{d}} \mathsf{f}_\eta(x-x_i)\, d\mu(x). \tag{3.20}$$

We will very often use the truncated potential at distances r_i, then simply denoted $h_{N,\mathsf{r}}$. We can then define the local energy as announced in the introduction.

Definition 3.3 (Local energy)**.** *If $\square_\ell$ is some cube of sidelength ℓ included in $\mathbb{R}^\mathsf{d}$, we let*

$$\tilde{\mathsf{F}}_N^{\square_\ell}(X_N,\mu) = \int_{\square_\ell\times[-\ell,\ell]} |y|^\gamma |\nabla h_{N,\mathsf{r}}|^2 \tag{3.21}$$

where h_N is defined as in (3.19).

The following provides minimal distance controls.

Proposition 3.4. *Assume $\mathsf{s} \in [(\mathsf{d}-2)_+,\mathsf{d})$. Let $\mu \in L^1(\mathbb{R}^\mathsf{d}) \cap L^\infty(\mathbb{R}^\mathsf{d})$ with $\int_{\mathbb{R}^\mathsf{d}} \mu = 1$, satisfying (3.18), and let $X_N \in (\mathbb{R}^\mathsf{d})^N$ be a pairwise distinct configuration.*

Let $\Omega \subset \mathbb{R}^\mathsf{d}$. For any $\vec{\eta}$ satisfying $\frac12\mathsf{r}_i \le \eta_i \le \mathsf{r}_i$ for every $1 \le i \le N$, it holds that

$$\frac{1}{\mathsf{c}_\mathsf{d}}\int_{\Omega\times[-N^{-1/\mathsf{d}},N^{-1/\mathsf{d}}]} |y|^\gamma |\nabla h_{N,\vec{\eta}}|^2 \ge \begin{cases} \dfrac{1}{C}\displaystyle\sum_{i:x_i\in\Omega,\mathrm{dist}(x_i,\partial\Omega)\ge\frac14 N^{-1/\mathsf{d}}} \mathsf{g}(\eta_i) & \text{if } \mathsf{s}\neq 0\\ \displaystyle\sum_{i:x_i\in\Omega,\mathrm{dist}(x_i,\partial\Omega)\ge\frac14 N^{-1/\mathsf{d}}} \mathsf{g}(40\eta_i N^{1/\mathsf{d}}) & \text{if } \mathsf{s}=0, \end{cases} \tag{3.22}$$

where $C>0$ depends only on d, s and $\|\mu\|_{L^\infty}$.

If $\Omega = \mathbb{R}^{\mathsf{d}}$, then we also have that

$$2\left(\mathsf{F}_N(X_N,\mu)+\frac{N\log N}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\right)+CN^{1+\frac{\mathsf{s}}{\mathsf{d}}}\ge\begin{cases}\frac{1}{C}\sum_{i=1}^N \mathsf{g}(\eta_i) & \text{if } \mathsf{s}\neq 0\\ \sum_{i=1}^N \mathsf{g}(\eta_i N^{1/\mathsf{d}}) & \text{if } \mathsf{s}=0,\end{cases}\tag{3.23}$$

and

$$\int_{\mathbb{R}^{\mathsf{d}+1}}|y|^\gamma|\nabla h_{N,\vec\eta}|^2\le C\left(\mathsf{F}_N(X_N,\mu)+\frac{N\log N}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\right)+CN^{1+\frac{\mathsf{s}}{\mathsf{d}}},\tag{3.24}$$

where $C>0$ depends only on d, s and $\|\mu\|_{L^\infty}$.

Note that it follows from (3.24) (if $\mathsf{s}<0$ there is nothing to prove) that

$$\mathsf{F}_N(X_N,\mu)+\frac{N\log N}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\ge -CN^{1+\frac{\mathsf{s}}{\mathsf{d}}},\tag{3.25}$$

where $C>0$ depends only on d,s and $\|\mu\|_{L^\infty}$, which proves that F_N is uniformly bounded below.

We note that (3.24) implies that the macroscopic law (3.6) can be expressed as a control of the form

$$\int_{\mathbb{R}^{\mathsf{d}+1}}|y|^\gamma|\nabla h_{N,\mathsf{r}}|^2+N\sum_{i=1}^N\zeta_V(x_i)\lesssim_\beta N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\tag{3.26}$$

except with probability $\le e^{-C\beta N}$, where $C>0$ depends only on $\mathsf{d},\mathsf{s},\zeta_V$ and $\|\mu_V\|_{L^\infty}$.

The following is obtained by combining Proposition 4.28, Lemma 4.20, Lemma 4.25 and Lemma 4.26 in [Ser24] (taking $N^{-1/\mathsf{d}}$ for the value of λ there since we do not track the $\|\mu\|_{L^\infty}$ dependence).

Proposition 3.5 (Control of fluctuations, discrepancies and minimal distances)**.** *Assume φ is a function such that $\Omega\subset\mathbb{R}^{\mathsf{d}}$ contains a $2N^{-1/\mathsf{d}}$-neighborhood of its support. For any configuration $X_N(\in\mathbb{R}^{\mathsf{d}})^N$, let I_Ω denote $\{i, x_i\in\Omega\}$ and $\#I_\Omega$ its cardinality. For any $\eta\ge N^{-1/\mathsf{d}}$, we have*

$$\begin{aligned}&\left|\int_{\mathbb{R}^{\mathsf{d}}}\varphi\left(\sum_{i=1}^N\delta_{x_i}-Nd\mu\right)\right|\\&\le C\left(\eta^{\gamma-1}\|\varphi\|^2_{L^2(\Omega)}+\eta^{\gamma+1}\|\nabla\varphi\|^2_{L^2(\Omega)}\right)^{\frac12}\left(\int_{\Omega\times[-2\eta,2\eta]}|y|^\gamma|\nabla h_{N,\mathsf{r}}|^2\right)^{\frac12}+C\#I_\Omega|\varphi|_{C^1}N^{-\frac{1}{\mathsf{d}}},\end{aligned}\tag{3.27}$$

where $C>0$ depends only on d and s and $\|\mu_V\|_{L^\infty}$.

If B_R is some ball of radius $R>2N^{-1/\mathsf{d}}$, letting $D(B_R)=\int_{B_R}d\left(\sum_{i=1}^N\delta_{x_i}-N\mu_V\right)$, we have either $|D(B_R)|\le CN^{1-\frac{1}{\mathsf{d}}}R^{\mathsf{d}-1}$ or

$$\frac{D(B_R)^2}{R^{\mathsf{s}}}\left|\min\left(1,\frac{D(B_R)}{R^{\mathsf{d}}}\right)\right|\le C\int_{B_{2R}\times[-2R,2R]}|y|^\gamma|\nabla h_{N,\mathsf{r}}|^2[X_N,\mu]\tag{3.28}$$

where $C>0$ depends on d,s and $\|\mu\|_{L^\infty}$.

If Ω is a general set of finite perimeter and Ω_δ its δ-neighborhood, if $D(\Omega) \geq 0$, for any $\|\mu\|_{L^\infty}^{-1/\mathsf{d}} < \delta \leq N^{1/\mathsf{d}}$,

$$\left(D(\Omega) - \|\mu\|_{L^\infty}|\Omega_\delta \backslash \Omega|\right)_+^2 \leq C\left(\frac{|\Omega_\delta|}{|\partial\Omega_\delta|}\right)^\gamma \frac{|\Omega_\delta|}{\delta}\int_{(\Omega_\delta\backslash\Omega)\times - \frac{|\Omega_\delta|}{|\partial\Omega_\delta|}-\delta, \frac{|\Omega_\delta|}{|\partial\Omega_\delta|}+\delta]} |y|^\gamma |\nabla h_{N,\mathsf{r}}|^2 \tag{3.29}$$

If $D(\Omega) \leq 0$, for any $-N^{1/\mathsf{d}} \leq \delta < -\|\mu\|_{L^\infty}^{-1/\mathsf{d}}$,

$$\left(D(\Omega) + \|\mu\|_{L^\infty}|\Omega \backslash \Omega_\delta|\right)_-^2 \leq C\left(\frac{|\Omega|}{|\partial\Omega|}\right)^\gamma \frac{|\Omega|}{\delta}\int_{(\Omega\backslash\Omega_\delta)\times [- \frac{|\Omega|}{|\partial\Omega|}-|\delta|, \frac{|\Omega|}{|\partial\Omega|}+|\delta|]} |y|^\gamma |\nabla h_{N,\mathsf{r}}|^2. \tag{3.30}$$

Thus, thanks to the control (1.29), which provides a control on $\#I_\Omega$, we easily deduce controls on the quantities in (3.27) and (3.28) except with small probability.

We will also need the following simple control on fluctuations.

Lemma 3.6. *Assume that $\varphi \in C^1(\mathbb{R}^\mathsf{d})$ is compactly supported in some cube $\square_r \subset \hat{\Sigma}$. Let X_N be a configuration such that the local laws of Theorem 1 hold on $\square_{2r} \supset \square_r$. Then,*

$$\left|\int_{\square_r} \varphi(x)\, d\mathrm{fluct}_{\mu_V}(x)\right| \lesssim_\beta N r^\mathsf{d} \|\varphi\|_{L^\infty(\square_r)}. \tag{3.31}$$

Proof. We may bound

$$\left|\int_{\square_r} \varphi(x)\, d\mathrm{fluct}_{\mu_V}(x)\right| \leq \|\varphi\|_{L^\infty(\square_r)}(\#I_{\square_R} + N|\square_r|\|\mu_V\|_{L^\infty}).$$

The local laws on $\square_{2r}$ provide the desired bound. □

3.2. **Transport calculus and commutator estimates.** We may now recast the change of variables made in the proof of Lemma 2.1 at this next-order level as follows. This is a "post-splitting" version of Lemma 2.1.

Lemma 3.7. *Let $\Phi_t = I + t\psi$ with ψ as in Proposition 2.2, and $\mu_t = \Phi_t \# \mu_V$. Let $\mathcal{G}$ be an event. We have*

(3.32)
$$\begin{aligned}
&\log \frac{\mathsf{K}^{\mathcal{G}}_{N,\beta}(\mu_t, \zeta_V \circ \Phi_t^{-1})}{\mathsf{K}_{N,\beta}(\mu_V, \zeta_V)} + N(\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_V)) \\
&= \log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\mathsf{F}_N(\Phi_t(X_N), \Phi_t\#\mu_V) - \mathsf{F}_N(X_N, \mu_V)\right) + \mathrm{Fluct}_{\mu_V}(\log\det D\Phi_t)\right)\mathbf{1}_{\mathcal{G}}\right) \\
&= \log \mathbb{E}_{\mathbb{P}_{N,\beta}}(e^{T_2}\mathbf{1}_{\mathcal{G}}),
\end{aligned}$$

where T_2 is as in (2.5).

Proof. By the change of variables $y_i = \Phi_t(x_i)$ we have

$$\frac{\mathsf{K}^{\mathcal{G}}_{N,\beta}(\mu_t, \zeta\circ\Phi_t^{-1})}{\mathsf{K}_{N,\beta}(\mu_V,\zeta_V)}$$
$$= \frac{1}{\mathsf{K}_{N,\beta}(\mu_V,\zeta_V)}\int_{(\mathbb{R}^{\mathsf{d}})^N} \exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\mathsf{F}_N(\Phi_t(X_N),\Phi_t\#\mu_V) + N\sum_{i=1}^N \zeta_V(x_i)\right)\prod_{i=1}^N \det D\Phi_t(x_i)\mathbf{1}_{\mathcal{G}}\, dX_N$$
$$= \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\mathsf{F}_N(\Phi_t(X_N),\Phi_t\#\mu_V) - \mathsf{F}_N(X_N,\mu_V)\right) + \sum_{i=1}^N \log\det D\Phi_t(x_i)\right)\right)\mathbf{1}_{\mathcal{G}}\, dX_N,$$

where we used (3.3). Since $\mu_t = \Phi_t\#\mu_0$ we have $\det D\Phi_t = \frac{\mu_0}{\mu_t\circ\Phi_t}$, and thus

$$\int \log\det D\Phi_t d\mu_0 = \int \log\mu_0 d\mu_0 - \int \log\mu_t(\Phi_t(x))d\mu_t = \mathrm{Ent}(\mu_0) - \mathrm{Ent}(\mu_t). \tag{3.33}$$

The result follows by (2.5). □

Wishing to linearize these expressions as $t\to 0$, we are led to considering successive derivatives of F_N along a general transport. More precisely, we denote for any ψ,

$$\mathsf{A}_1(X_N,\mu,\psi) := \frac12\int_{\triangle^c}\nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))d\left(\sum_{i=1}^N\delta_{x_i} - N\mu\right)(x)\left(\sum_{i=1}^N\delta_{x_i} - N\mu\right)(y) \tag{3.34}$$

and more generally

$$\mathsf{A}_n(X_N,\mu,\psi) := \frac12\int_{\triangle^c}\nabla^{\otimes n}\mathsf{g}(x-y) : (\psi(x)-\psi(y))^{\otimes n}d\left(\sum_{i=1}^N\delta_{x_i} - N\mu\right)(x)\left(\sum_{i=1}^N\delta_{x_i} - N\mu\right)(y).. \tag{3.35}$$

It is easy to check [Ser23, Lemma 4.1] that, if $\Phi_t = I + t\psi$, we have

$$\frac{d^n}{dt^n}\mathsf{F}_N(\Phi_t(X_N),\Phi_t\#\mu) = \mathsf{A}_n(\Phi_t(X_N),\Phi_t\#\mu,\psi\circ\Phi_t^{-1}). \tag{3.36}$$

To control such terms, we will need the following recent sharp and localized commutator estimates of all order from [RS25]. [3]

Proposition 3.8 ([RS25])**.** *Let $\mu\in L^1(\mathbb{R}^{\mathsf{d}})\cap L^\infty(\mathbb{R}^{\mathsf{d}})$ with $\int_{\mathbb{R}^{\mathsf{d}}}\mu\, dx = 1$ satisfying (3.18). There exists a constant $C>0$ depending only d, s and $\|\mu\|_{L^\infty}$ such that the following holds. Let ψ be a Lipschitz vector field $\psi:\mathbb{R}^{\mathsf{d}}\to\mathbb{R}^{\mathsf{d}}$ and Ω be a closed set containing a $2N^{-1/\mathsf{d}}$-neighborhood of* $\operatorname{supp}\psi$*. For any pairwise distinct configuration $X_N\in(\mathbb{R}^{\mathsf{d}})^N$, it holds that*

$$\left|\int_{(\mathbb{R}^{\mathsf{d}})^2\setminus\triangle}(\psi(x)-\psi(y))\cdot\nabla\mathsf{g}(x-y)d\Big(\sum_{i=1}^N\delta_{x_i}-N\mu\Big)^{\otimes 2}(x,y)\right| \le C\|\nabla\psi\|_{L^\infty}\left(\int_{\Omega\times[-\ell,\ell]}|y|^\gamma|\nabla h_{N,\mathsf{r}}|^2 + C\# I_\Omega N^{\frac{\mathsf{s}}{\mathsf{d}}}\right). \tag{3.37}$$

[3] They are slightly restated, because up to allowing constants that depend on $\|\mu\|_{L^\infty}$ we can work with $\lambda = N^{-1/\mathsf{d}}$ there.

Suppose in addition that $\nabla^{\otimes(n-1)}\psi$ *is Lipschitz and that* Ω *contains a* $(5\ell+N^{-1/\mathsf{d}})$*-neighborhood of* $\operatorname{supp}\psi$*, where* ℓ *satisfies* $\ell > 2N^{-1/\mathsf{d}}$*. For any* $n \geq 2$*, we have*

(3.38)
$$\left|\int_{(\mathbb{R}^\mathsf{d})^2\setminus\triangle} \nabla^{\otimes n}\mathsf{g}(x-y) : (\psi(x)-\psi(y))^{\otimes n} d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu\Big)^{\otimes 2}(x,y)\right|$$
$$\leq C\sum_{p=0}^n (\ell\|\nabla^{\otimes 2}\psi\|_{L^\infty})^p \sum_{\substack{1\leq c_1,\dots,c_{n-p}\\ n-p\leq c_1+\dots+c_{n-p}\leq 2n}} N^{\frac{1}{\mathsf{d}}((n-p)-\sum_{k=1}^p c_{n-k})}\|\nabla^{\otimes c_1}\psi\|_{L^\infty}\cdots\|\nabla^{\otimes c_{n-p}}\psi\|_{L^\infty}$$
$$\times\left(\int_{\Omega\times[-\ell,\ell]} |y|^\gamma|\nabla h_{N,\mathsf{r}}|^2 + C\# I_\Omega N^{\frac{\mathsf{s}}{\mathsf{d}}}\right).$$

3.3. **Variation of energy along a transport.** In the macroscopic case $\ell = 1$, taking $\Omega = \mathbb{R}^\mathsf{d}$, we can immediately control the energy of the transported configuration in terms of the initial configuration: let

$$\Xi(t) := \mathsf{F}_N(\Phi_t(X_N), \Phi_t\#\mu_0) + \frac{N\log N}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0} + CN^{1+\frac{\mathsf{s}}{\mathsf{d}}},$$

for C large enough; in view of (3.36), applying the first order commutator estimate (3.37) and combining it with (3.24), we find that

$$\Xi'(t) \leq C\|\nabla\psi\|_{L^\infty}\Xi(t), \tag{3.39}$$

and applying Gronwall's lemma yields

$$\Xi(t) \leq \exp(Ct\|\nabla\psi\|_{L^\infty})\Xi(0), \tag{3.40}$$

which gives the desired control.

The mesoscopic case $\ell < 1$ is much more delicate, and presents additional difficulties compared to the Coulomb case, due to the nonlocalized nature of the transport. We address this difficulty by considering the transport on increasing dyadic scales and leveraging the decay of the transport away from $\operatorname{supp}\varphi$.

Let as above $\square_\ell$ be a cube of size ℓ such that $\operatorname{supp}\varphi \subset \square_\ell \subset \square_{2\ell} \subset \hat{\Sigma}$ and assume without loss of generality, that it is centered at the origin. Let k^* be such that

$$U \subset \square_{2^{k^*}\ell}, \tag{3.41}$$

where U is a neighborhood of Σ as in Proposition 2.2. For k in $[0, k_*]$, let

$$D_k = \square_{2^k\ell}, \quad A_k = D_{k+4}\setminus D_{k-2}, \quad A_{k_*+1} = D_{k_*}^c. \tag{3.42}$$

Finally, denote

$$\rho = 2^k\ell$$

where the dependence in k is implicit.

First we prove a preliminary lemma, which is a weighted trace inequality.

Lemma 3.9. *Let* h *be a function in* $A_k\times[-\lambda\rho,\lambda\rho]$ *such that* $\int_{A_k\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h|^2 < \infty$*. Let* $\bar{h} = \fint_{A_k\times[-\lambda\rho,\lambda\rho]} h$*. Then*

$$\int_{\partial(D_k\times[-\lambda\rho,\lambda\rho])} |y|^\gamma|h-\bar{h}|^2 \leq C\rho\int_{A_k\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h|^2, \tag{3.43}$$

where $C > 0$ *depends only on* s*,* d *and* $\lambda > 0$*. The same holds with* A_k *replaced by* D_k*.*

Proof. Let us denote by $\tilde h_z = \fint_{A_k\times\{z\}} h$. Since $\bar h = \fint_{A_k\times[-\lambda\rho,\lambda\rho]} = \tilde h_{z_0}$ for some $z_0 \in [-\lambda\rho,\lambda\rho]$ by the mean value property, we may write, using Cauchy-Schwarz, assuming without loss of generality that $z_0 \le z$,

$$(3.44)\quad |\tilde h_z - \bar h|^2 \le \frac{C}{\rho^{2\mathsf{d}}}\left(\int_{z_0}^{z}\partial_y\left(\int_{A_k\times\{y\}} h(x,y)dx\right)dy\right)^2 \le \frac{C}{\rho^{2\mathsf{d}}}\left(\int_{A_k\times[z,z_0]}\partial_y h\right)^2$$
$$\le \frac{C}{\rho^{2\mathsf{d}}}\int_{A_k\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h|^2\int_{A_k\times[-\lambda\rho,\lambda\rho]}\frac{1}{|y|^\gamma} \le C\rho^{1-\gamma-\mathsf{d}}\int_{A_k\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h|^2,$$

where $C>0$ depends on d,s and λ. It follows that, letting $\bar h_+ = \fint_{A_k\times[\lambda\rho/2,\lambda\rho]} h$, $|\bar h_+ - \bar h|^2$ is controlled in the same way. Thus, using the standard trace theorem in $A_k\times[\lambda\rho/2,\lambda\rho]$ and the triangle inequality, we deduce that

$$(3.45)\quad \int_{A_k\times\{\lambda\rho\}}|y|^\gamma|h-\bar h|^2 \le 2\int_{A_k\times\{\lambda\rho\}}|y|^\gamma|h-\bar h_+|^2 + 2\int_{A_k\times\{\lambda\rho\}}|y|^\gamma|\bar h_+ - \bar h|^2$$
$$\le C\rho\int_{A_k\times[\lambda\rho/2,\lambda\rho]}|y|^\gamma|\nabla h|^2 + C\rho^{\gamma+\mathsf{d}+1-\gamma-\mathsf{d}}\int_{A_k\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h|^2$$
$$\le C\rho\int_{A_k\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h|^2.$$

The same relation holds on $A_k\times\{-\lambda\rho\}$. We next turn to the integral on $\partial A_k\times[-\lambda\rho,\lambda\rho]$. By standard trace theorem, we have

$$\int_{\partial A_k\times\{y\}}|y|^\gamma|h-\tilde h_y|^2 \le C\rho\int_{A_k\times\{y\}}|y|^\gamma|\nabla h|^2,$$

and using (3.44) and the triangle inequality, we deduce

$$\int_{\partial A_k\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|h-\bar h|^2 \le C\rho\int_{A_k\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h|^2.$$

The result is proven for A_k. The proof is the same over D_k. ☐

Lemma 3.10. *Assume μ is a bounded probability density satisfying (3.18). Let Ω be A_k for some $0\le k\le k_*$. Let $\tilde\psi$ be a map supported in Ω or in Ω^c and let $\tilde\Phi_t := I + t\tilde\psi$. Assume that $\partial\Omega$ is at distance $\ge \frac12\rho = 2^{k-1}\ell$ from $\operatorname{supp}\tilde\psi$, and $|\tau|\|\tilde\psi\|_{L^\infty} < \min(\frac14\rho,\frac12)$. Then letting η_i be the minimum over $t\in[0,\tau]$ of the r_i's of the $\tilde\Phi_t(X_N)$, we have*

(3.46)
$$\forall t\in[0,\tau],\quad \int_{\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h_{N,\vec\eta}[\tilde\Phi_t(X_N),\tilde\Phi_t\#\mu]|^2 \le C\exp\left(Ct(\rho^{-1}\|\tilde\psi\|_{L^\infty} + |\tilde\psi|_{C^1})\right)$$
$$\times\left(\int_{\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h_{N,\vec\eta}[X_N,\mu]|^2 + \#I_\Omega N^{\frac{\mathsf{s}}{\mathsf{d}}} + N^{-\frac{2}{\mathsf{d}}}(\#I_\Omega)^2\rho^{-\mathsf{s}-2}\right)$$

where C depends only on s, d, $\lambda>0$ and $\|\mu\|_{L^\infty}$.

Proof. For shortcut, let us drop the tildes for the proof, and denote $h_N^t = h_N[\tilde\Phi_t(X_N),\tilde\Phi_t\#\mu]$ as in (3.19). Let us start with the case where ψ is supported in $\Omega = A_k$. We note that since $\operatorname{supp}\psi$ is at distance $\ge\frac12\rho$ from $\partial\Omega$ and $|\tau|\|\psi\|_{L^\infty} < \frac14\rho$, Φ_t maps Ω to Ω, and coincides with the identity in Ω^c and in the part of Ω at distance $\le\frac14\rho$ from $\partial\Omega$, for any $t\in[0,\tau]$. Denoting

$\tilde\Omega = \{x \in \Omega, \mathrm{dist}(x, \partial\Omega) \ge \frac12\rho\}$, we have that $\operatorname{supp}\psi \subset \tilde\Omega$ and $\operatorname{supp}\psi$ is even at distance $\ge \frac14\rho \ge \frac14\ell \ge \frac12\rho_\beta N^{-1/\mathsf{d}} \ge 2N^{-1/\mathsf{d}}$ from $\partial\tilde\Omega$.

The commutator estimate provides us with an estimate on $\frac{d}{dt}\mathsf{F}_N(\Phi_t(X_N), \mu_t)$, so we need to estimate the difference between that and $\frac{d}{dt}\int_{\Omega\times[-\rho,\rho]} |y|^\gamma |\nabla h^t_{N,\mathsf{r}}|^2$.

Since Φ_t coincides with the identity map outside Ω for each $t \in [0,\tau]$, spelling out the definition (3.20) (the equality case with $\eta_i = \mathsf{r}_i$), we may write that

$$\begin{aligned}\mathsf{F}_N(\Phi_t(X_N), \mu_t) - \mathsf{F}_N(X_N, \mu_0) = &\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\Omega\times[-\lambda\rho,\lambda\rho]} |y|^\gamma|\nabla h^t_{N,\vec\eta}|^2 - |y|^\gamma |\nabla h^0_{N,\vec\eta}|^2\\ &+ \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\mathbb{R}^{\mathsf{d}+1}\setminus(\Omega\times[\lambda\rho,\lambda\rho])} |y|^\gamma|\nabla h^t_{N,\vec\eta}|^2 - |y|^\gamma|\nabla h^0_{N,\vec\eta}|^2,\\ &- N\sum_{i\in I_\Omega}\int_{\mathbb{R}^\mathsf{d}} \mathsf{f}_{\eta_i}(x - \Phi_t(x_i))d\mu_t(x) + N\sum_{i\in I_\Omega}\int_{\mathbb{R}^\mathsf{d}} \mathsf{f}_{\eta_i}(x - x_i)d\mu_0(x),\end{aligned} \tag{3.47}$$

where we note that r_i for X_N and η_i are within a factor in $[\frac12, 2]$ of each other.

Since points at distance $\le \frac14\rho$ from $\partial\Omega$ and points in Ω^c are fixed points of Φ_t, and since $\frac14\rho \ge \frac14\ell \ge \frac14\rho_\beta N^{-1,\mathsf{d}}$ and $\mathsf{r}_i \le \frac14 N^{-1/\mathsf{d}}$, since $\rho_\beta \ge 4$, the smeared charges $\delta_{x_i}^{(\eta_i)}$ coincide in Ω^c, and thus the function $h^t_{N,\vec\eta} - h^0_{N,\vec\eta} := u^t$ solves

$$-\mathrm{div}\,(|y|^\gamma\nabla u^t) = 0 \quad \text{in } \mathbb{R}^{\mathsf{d}+1}\setminus(\Omega\times[-\lambda\rho,\lambda\rho]),$$

and decays at infinity, and its gradient as well. Hence, writing $\nabla h^t_{N,\vec\eta} = \nabla h^0_{N,\vec\eta} + \nabla u^t$ and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{\mathsf{d}+1}\setminus(\Omega\times[-\lambda\rho,\lambda\rho])} |y|^\gamma|\nabla h^t_{N,\vec\eta}|^2 - |y|^\gamma|\nabla h^0_{N,\vec\eta}|^2 = \int_{\mathbb{R}^{\mathsf{d}+1}\setminus(\Omega\times[-\lambda\rho,\lambda\rho])} |y|^\gamma|\nabla u^t|^2\\ + 2\int_{\partial(\Omega\times[-\lambda\rho,\lambda\rho])} |y|^\gamma(h^0_{N,\vec\eta} - \bar h^0_{N,\vec\eta})\frac{\partial u^t}{\partial n}\end{aligned} \tag{3.48}$$

where $\vec n$ is the inner pointing unit normal to $\partial\Omega$ and $\bar h^0_{N,\vec\eta}$ is the weighted average of $h^0_{N,\vec\eta}$ on $\partial(\Omega\times[-\lambda\rho,\lambda\rho])$. By Cauchy-Schwarz and Lemma 3.9 we may write

$$\left|\int_{\partial(\Omega\times[-\lambda\rho,\lambda\rho])} |y|^\gamma(h^0_{N,\vec\eta} - \bar h^0_{N,\vec\eta})\frac{\partial u^t}{\partial n}\right| \le C\rho^{\frac12}\|\nabla h^0_{N,\vec\eta}\|_{L^2_{|y|^\gamma}(\Omega\times[-\lambda\rho,\lambda\rho])}\|\nabla u^t\|_{L^2_{|y|^\gamma}(\partial(\Omega\times[-\lambda\rho,\lambda\rho]))}. \tag{3.49}$$

We next estimate ∇u^t in the complement of $\Omega\times[-\lambda\rho,\lambda\rho]$. By definition,

$$u^t(x) = \int_{\mathbb{R}^\mathsf{d}} \mathsf{g}(x-x')d\left(\sum_{i=1}^N \delta^{(\eta_i)}_{\Phi_t(x_i)} - N\mu_t\right)(x') - \int_{\mathbb{R}^\mathsf{d}} \mathsf{g}(x-x')d\left(\sum_{i=1}^N \delta^{(\eta_i)}_{x_i} - N\mu_0\right)(x').$$

We compute that

$$\begin{aligned}\partial_t u^t(x) &= \partial_t\left(\sum_{i=1}^N \fint_{\partial B(0,\eta_i)} \mathsf{g}(x - \Phi_t(x_i) - \cdot) - N\int_{\mathbb{R}^\mathsf{d}} \mathsf{g}(x - \Phi_t(y))d\mu_0(y)\right)\\ &= \sum_{i=1}^N \int_{\mathbb{R}^\mathsf{d}} \nabla\mathsf{g}(x-y)\cdot\psi_t(\Phi_t(x_i))\delta^{(\eta_i)}_{\Phi_t(x_i)}(x') - N\int_{\mathbb{R}^\mathsf{d}}\nabla\mathsf{g}(x-x')\cdot\psi_t(x')d\mu_t(x')\end{aligned}$$

where we have denoted $\psi_t := \frac{d\Phi_t}{dt}$. Hence

$$u^t(x) = \int_0^t \int_{\mathbb{R}^{\mathsf{d}}} \nabla \mathsf{g}(x-x') \cdot \psi_s(x') d\left(\sum_{i=1}^N \delta_{\Phi_s(x_i)}^{(\eta_i)} - N\mu_s\right)(x')\, ds$$
$$+ \int_0^t \sum_{i=1}^N \int_{\mathbb{R}^{\mathsf{d}}} \nabla \mathsf{g}(x-x') \cdot (\psi_s(\Phi_s(x_i)) - \psi_s(x'))\, \delta_{\Phi_t(x_i)}^{(\eta_i)}(x')\, ds. \tag{3.50}$$

and

$$\nabla u^t(x) = -\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_0^t \int_{\mathbb{R}^{\mathsf{d}}} \nabla^{\otimes 2} \mathsf{g}(x-x') : \psi_s(x') \mathrm{div}\, (|y|^\gamma \nabla h_N^s)(x')\, ds$$
$$+ \int_0^t \sum_{i=1}^N \int_{\mathbb{R}^{\mathsf{d}} \times \mathbb{R}^1} \nabla^{\otimes 2} \mathsf{g}(x-y) : (\psi_s(\Phi_s(x_i)) - \psi_s(x'))\, \delta_{\Phi_t(x_i)}^{(\eta_i)}(x')\, ds. \tag{3.51}$$

The function $\chi_x(x') := \nabla^{\otimes 2} \mathsf{g}(x-x') : \psi_s(x')$ is compactly supported where ψ_s is, moreover, it can be checked to satisfy

$$|\nabla \chi_x(x')| \lesssim \left(\frac{\|\psi_s\|_{L^\infty(\Omega)}}{\mathrm{dist}(x, \mathrm{supp}\, \psi_s)^{\mathsf{s}+3}} + \frac{|\psi_s|_{C^1(\Omega)}}{\mathrm{dist}(x, \mathrm{supp}\, \psi_s)^{\mathsf{s}+2}}\right). \tag{3.52}$$

We extend the function χ_x to $\Omega \times [-\lambda\rho, \lambda\rho]$ by multiplying χ_x by a function $\varphi(y)$ supported in $[-\lambda\rho, \lambda\rho]$ with $\|\varphi\|_{L^\infty} \le 1$ and $\left\|\frac{d}{dy}\varphi\right\|_{L^\infty} \lesssim \frac{1}{\rho}$. Then, for $x \in \Omega^c$,

$$\int_{\Omega \times [-\lambda\rho, \lambda\rho]} |y|^\gamma |\nabla \chi_x|^2 \lesssim (\|\psi_s\|_{L^\infty(\Omega)} + \rho |\psi_s|_{C^1(\Omega)})^2 \rho^{-\mathsf{s}-4} \tag{3.53}$$

using $\mathsf{d} + \gamma = \mathsf{s} + 1$. Using Green's formula and the Cauchy-Schwarz inequality to control the first term in (3.51), and a rough bound for the second term, we are led to

$$|\nabla u^t(x)|^2 \lesssim$$
$$t \int_0^t (\|\psi_s\|_{L^\infty} + \rho |\psi_s|_{C^1})^2 \rho^{-\mathsf{s}-4} \int_{\Omega \times [-\lambda\rho, \lambda\rho]} |y|^\gamma |\nabla h_{N,\vec{\eta}}^s|^2 ds + t \int_0^t |\psi_s|_{C^1}^2 \left(\sum_{i \in I_\Omega} \eta_i \rho^{-\mathsf{s}-2}\right)^2 ds.$$

Since this is true for all $x \in (\Omega \times [-\lambda\rho, \lambda\rho])^c$, inserting $\eta_i \le N^{-1/\mathsf{d}}$, it follows that

$$\int_{\partial(\Omega \times [-\lambda\rho, \lambda\rho])} |y|^\gamma |\nabla u^t|^2 \lesssim \rho^{\mathsf{d}+\gamma}$$
$$\times t \int_0^t (\|\psi_s\|_{L^\infty} + \rho|\psi_s|_{C^1})^2 \left(\rho^{-\mathsf{s}-4} \int_{\Omega \times [-\lambda\rho, \lambda\rho]} |y|^\gamma |\nabla h_{N,\vec{\eta}}^s|^2 + N^{-\frac{2}{\mathsf{d}}} (\# I_\Omega)^2 \rho^{-2\mathsf{s}-6}\right) ds$$

Combining with (3.48) and (3.49), and noting that $\int_{\mathbb{R}^{\mathsf{d}+1} \setminus (\Omega \times [-\lambda\rho, \lambda\rho])} |y|^\gamma |\nabla u^t|^2$ is an order $O(t^2)$ term, it follows that

$$\left|\frac{d}{dt}_{|t=0} \int_{\mathbb{R}^{\mathsf{d}+1} \setminus (\Omega \times [-\lambda\rho, \lambda\rho])} |y|^\gamma |\nabla h_{N,\vec{\eta}}^t|^2\right| \lesssim \rho^{\frac{1}{2}} \|\nabla h_{N,\vec{\eta}}^0\|_{L^2_{|y|^\gamma}(\Omega \times [-\lambda\rho, \lambda\rho]))} \rho^{\frac{\mathsf{s}+1}{2}} (\|\psi_0\|_{L^\infty} + \rho |\psi_0|_{C^1})$$
$$\times \left(\rho^{-\frac{\mathsf{s}}{2}-2} \|\nabla h_{N,\vec{\eta}}^0\|_{L^2_{|y|^\gamma}(\Omega \times [-\lambda\rho, \lambda\rho])} + N^{-\frac{1}{\mathsf{d}}} (\# I_\Omega) \rho^{-\mathsf{s}-3}\right).$$

On the other hand, by definition (3.11), we have

$$\begin{aligned}&\left|\frac{d}{dt}_{|t=0}\sum_{i\in I_\Omega}\int_{\mathbb{R}^\mathsf{d}}\mathsf{f}_{\eta_i}(x-\Phi_t(x_i))d\mu_t(x)\right|=\left|\sum_{i\in I_\Omega}\int_{\mathbb{R}^\mathsf{d}}\nabla\mathsf{f}_{\eta_i}(x-x_i)\cdot(\psi_0(x)-\psi_0(x_i))d\mu_0(x)\right|\\&\le|\psi_0|_{C^1}\sum_{i\in I_\Omega}\int_{B(0,\eta_i)}|x|^{-\mathsf{s}}\le C\#I_\Omega|\psi_0|_{C^1}N^{-\frac{\mathsf{d}-\mathsf{s}}{\mathsf{d}}}.\end{aligned}$$

Combining the above with (3.47) and (3.48), and by definition (3.36) and the commutator estimate in the form (3.37), after using Young's inequality, we are led to

$$\begin{aligned}&\left|\frac{d}{dt}_{|t=0}\int_{\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h^t_{N,\vec\eta}|^2\right|\le C(\|\psi_0\|_{L^\infty}+\rho|\psi_0|_{C^1})\times\\&\left(\rho^{-1}\int_{\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h^0_{N,\vec\eta}|^2+\rho^{-1}\int_{\tilde\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h^0_{N,\mathsf{r}}|^2+N^{-\frac{2}{\mathsf{d}}}(\#I_\Omega)^2\rho^{-\mathsf{s}-3}+\#I_\Omega\rho^{-1}N^{\frac{\mathsf{s}}{\mathsf{d}}}\right).\end{aligned}$$

We next wish to estimate $\int|y|^\gamma|\nabla h_{N,\mathsf{r}}|^2$ in terms of $\int|y|^\gamma|\nabla h_{N,\vec\eta}|^2$. Using the definition (3.14), if α_i and κ_i are $\le\mathsf{r}_i$, we have

$$\nabla h_{N,\vec\alpha}=\nabla h_{N,\vec\kappa}+\sum_{i=1}^N\nabla(\mathsf{f}_{\kappa_i}-\mathsf{f}_{\alpha_i})(x-x_i)$$

hence, using that the $B(x_i,\mathsf{r}_i)$ are disjoint, we find

$$\begin{aligned}\int_{\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h_{N,\vec\alpha}|^2&\le2\int_{\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h_{N,\vec\kappa}|^2+\sum_{i\in I_\Omega}\int_{\mathbb{R}^{\mathsf{d}+1}}|y|^\gamma|\nabla(\mathsf{f}_{\kappa_i}-\mathsf{f}_{\alpha_i})|^2\\&\lesssim\int_{\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h_{N,\vec\kappa}|^2+\sum_{i\in\Omega}(\kappa_i^{-\mathsf{s}}+\alpha_i^{-\mathsf{s}}).\end{aligned}\tag{3.54}$$

Applying in $\tilde\Omega$ to $\alpha_i=\mathsf{r}_i$ and $\kappa_i=\eta_i\ge\frac12\mathsf{r}_i$, we obtain

$$\int_{\tilde\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h^0_{N,\mathsf{r}}|^2\lesssim\int_{\tilde\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h^0_{N,\vec\eta}|^2+\sum_{i\in I_{\tilde\Omega}}\mathsf{r}_i^{-\mathsf{s}},$$

and using (3.22) and the definition of $\tilde\Omega$, we may absorb these terms into the others.

The same reasoning yields that the relation is true at any t, so we find

$$\begin{aligned}&\left|\frac{d}{dt}\int_{\Omega\times[-\lambda\rho,\lambda\rho]}|y|^\gamma|\nabla h^t_{N,\vec\eta}|^2\right|\\&\le C(\rho^{-1}\|\psi_t\|_{L^\infty}+|\psi_t|_{C^1})\left(\int_{\Omega\times[-\lambda\rho,\lambda\rho]}|\nabla h^t_{N,\vec\eta}|^2+\#I_\Omega N^{\frac{\mathsf{s}}{\mathsf{d}}}+N^{-\frac{2}{\mathsf{d}}}(\#I_\Omega)^2\rho^{-\mathsf{s}-2}\right).\end{aligned}\tag{3.55}$$

Applying Gronwall's lemma, we deduce the result (3.10) in the case ψ supported in Ω.

Let us now turn to the case where the support of ψ is in Ω^c. With the same notation, we may write $h^t_{N,\vec{\eta}} - h^0_{N,\vec{\eta}} = u^t$ where u^t solves $-\mathrm{div}\,(|y|^\gamma \nabla u^t) = 0$ in $\Omega \times \mathbb{R}$, and we may write

$$\begin{aligned}(3.56)\quad &\int_{\Omega\times[-\lambda\rho,\lambda\rho]} |y|^\gamma |\nabla h^t_{N,\vec{\eta}}|^2 - |y|^\gamma |\nabla h^0_{N,\vec{\eta}}|^2 \\ &= \int_{\Omega\times[-\lambda\rho,\lambda\rho]} |y|^\gamma |\nabla u^t|^2 + 2\int_{\partial(\Omega\times[-\lambda\rho,\lambda\rho]} |y|^\gamma (h^0_{N,\vec{\eta}} - \bar{h}^0_{N,\vec{\eta}})\frac{\partial u^t}{\partial n} \\ &= O(t^2) + O\left(\left(\int_{\Omega\times[-\lambda\rho,\lambda\rho]} |y|^\gamma |\nabla h_{N,\vec{\eta}}|^2\right)^{\frac12} \left(\int_{\partial\Omega\times[-\lambda\rho,\lambda\rho]} |y|^\gamma \left|\frac{\partial u^t}{\partial n}\right|^2\right)^{\frac12}\right),\end{aligned}$$

where $\vec{n}$ is the outwards pointing unit normal to $\partial\Omega$ and $\bar{h}^0_{N,\vec{\eta}}$ is the weighted average of $h^0_{N,\vec{\eta}}$ on $\partial\Omega$. The rest of the computation is identical to the first step and yields (3.55), and we finish the proof in the same way. We can then check that all the cases considered in the statement lemma have been treated. □

Proposition 3.11. *Let ψ be as constructed in Proposition 2.2, and let $\Phi_t = I + t\psi$. Assume that $X_N \in \mathcal{G}_\ell$, a set of configurations with $\mathbb{P}_{N,\beta}(\mathcal{G}_\ell^c) \le C_1 e^{-\beta C_2 \ell^{\mathsf{d}} N}$ such that the local laws (1.29)–(1.30) hold in each D_k of (3.42) with $k \ge 1$ (in $\hat{\Sigma}$) and that (3.26) holds. Then if t is small enough that $|t|\ell^{\mathsf{s}-\mathsf{d}}\mathsf{M}$ is smaller than a constant depending only on the constant in (2.17), for any $k \in [1, k_+]$ integer and D_k as in (3.42), k_* as in (3.41), we have*

$$(3.57)\qquad \int_{(D_{k+\frac32}\setminus D_{k+\frac12})\times[-2^k\ell,2^k\ell]} |y|^\gamma |\nabla h_N[\Phi_t(X_N), \Phi_t\#\mu_V]|^2 \le C(2^k\ell)^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}},$$

and

$$(3.58)\qquad \int_{D_{3/2}\times[-\ell,\ell]} |y|^\gamma |\nabla h_N[\Phi_t(X_N), \Phi_t\#\mu_V]|^2 \le C\ell^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}},$$

where $C > 0$ depends only on $\mathsf{s}, \mathsf{d}, \|\mu_V\|_{L^\infty}$ and the prior constants.

Proof. Let, for k integer,

$$(3.59)\qquad B_k = D_{k+2}\setminus D_k \quad \text{for } k \le k_*, \quad B_{k_*+1} = D^c_{k_*+1}.$$

Let χ_k be a partition of unity associated to $\{B_k\}_{k=0}^{k_*+1}$, such that $\operatorname{supp}\chi_k \subset B_k$, $\sum_{k=0}^{k_*+1}\chi_k = 1$ and $|\nabla\chi_k| \lesssim 2^{-k}\ell^{-1}$.

Let $k \in [0, k_*]$ be a given integer. We are going to decompose $I + t\psi$ as a composition of transports $I + t\psi_j$ such that each ψ_j has support in B_j. For that, we let $m_0, \dots, m_{k_*+1}$ be the sequence given by

$$\begin{cases} m_0 = k-1 \\ m_1 = k \\ m_2 = k+1 \\ m_p = p-3 & \text{for } 3 \le p \le k+1 \\ m_p = p & \text{for } k+2 \le p \le k_*+1 \end{cases}$$

i.e. we take away the indices from $k-1$ to $k+1$ and put them at the beginning. If $k = 0$, we do the same but moving only two indices to the beginning.

We define inductively ψ_j by

$$I + t\psi_j = \Big(I + t(\sum_{p=0}^{j} \chi_{m_p})\psi\Big) \circ \Big(I + t(\sum_{p=0}^{j-1} \chi_{m_p})\psi\Big)^{-1}. \tag{3.60}$$

In other words, we have decomposed the transport into successive localized transports $I+t\psi_j$. One may check by induction that

$$I + t\psi = I + t\sum_{j=0}^{k_*+1} \chi_j\psi = (I + t\psi_{k_*+1}) \circ \cdots \circ (I + t\psi_0). \tag{3.61}$$

We decompose this as

$$I + t\psi = T_{\mathrm{out}} \circ T_{\mathrm{inn}} \tag{3.62}$$

where $T_{\mathrm{inn}} = (I + t\psi_2) \circ (I + t\psi_1) \circ (I + t\psi_0)$, and $T_{\mathrm{out}} = \prod_{j=3}^{k_*+1}(I + t\psi_j)$ (or the same with only two maps set aside if appropriate).

If $t\ell^{\mathsf{s}-\mathsf{d}}\|\varphi_0\|_{C^3}$ is small enough, then by (2.17), we can make $t\|\psi\|_{L^\infty} \lesssim t\|\varphi_0\|_{C^3}\frac{\ell}{\ell^{\mathsf{d}-\mathsf{s}}} < \frac{1}{4}\ell$. One may then check that in view of (3.60) and definition of the χ_j, for $j \ge 3$, ψ_j is supported in B_{m_j}, and letting $\rho_j = 2^{m_j}\ell$, and in view of (2.17) we have

$$|\psi_j|_{C^1} \lesssim \frac{\ell^{\mathsf{d}}}{\rho_j^{2\mathsf{d}-\mathsf{s}}}, \qquad \|\psi_j\|_{L^\infty} \lesssim \frac{\ell^{\mathsf{d}}}{\rho_j^{2\mathsf{d}-\mathsf{s}-1}}. \tag{3.63}$$

Thus, the transport T_{inn} leaves A_k invariant (where A_k is as in (3.42)), is supported at distance $\ge \frac{1}{4}2^k\ell$ from its boundary, and coincides with identity outside. Thus we may apply Lemma 3.10 in that set with $\tilde\psi = T_{\mathrm{inn}}$ (we have that $|t|\|T_{\mathrm{inn}}\|_{L^\infty} \lesssim \frac{\ell^{\mathsf{d}}}{(2^k\ell)^{2\mathsf{d}-\mathsf{s}}}$ hence if $|t|\ell^{\mathsf{s}-\mathsf{d}}\|\varphi_0\|_{C^1}$ is small depending on the constant in (2.17), the assumption $|t|\|T_{\mathrm{inn}}\|_{L^\infty} < \min(\frac{1}{4}(2^k\ell), \frac{1}{4})$ is verified) and obtain

$$\begin{aligned} &\int_{A_k\times[-2^k\ell,2^k\ell]} |\nabla h_{N,\mathsf{r}}[T_{\mathrm{inn}}(X_N), T_{\mathrm{inn}}\#\mu_V]|^2 \\ &\lesssim \exp\left(Ct\frac{\ell^{\mathsf{d}}}{(2^k\ell)^{2\mathsf{d}-\mathsf{s}}}\right)\int_{A_k\times[-2^k\ell,2^k\ell]} |\nabla h_{N,\mathsf{r}}[X_N,\mu_V]|^2 + \#I_{A_k}N^{\frac{\mathsf{s}}{\mathsf{d}}} + N^{-\frac{2}{\mathsf{d}}}(\#I_{A_k})^2(2^k\ell)^{-\mathsf{s}-2}. \end{aligned} \tag{3.64}$$

For $j \ge 3$, the transports $I + t\psi_j$ have support in $(D_{k+2}\backslash D_k)^c$, so we can say that their support is at distance $\ge 2^k\ell$ from $\partial(D_{k+\frac{3}{2}}\backslash D_{k+\frac{1}{2}})$ (resp. $\partial D_{3/2}$ if $k=0$). We may thus apply Lemma 3.10 iteratively in $D_{k+\frac{3}{2}}\backslash D_{k+\frac{1}{2}}$, resp. $D_{3/2}$ if $k=0$, (which is a set of similar nature as in the assumption) with $\tilde\psi = \psi_j$ (again we verify that the assumption $|t|\|\psi_j\|_{L^\infty} < \min(\frac{2^k\ell}{4}, \frac{1}{2})$

is satisfied if $|t|\ell^{\mathsf{s}-\mathsf{d}}\|\varphi_0\|_{C^1}$ is small enough), and find

$$\begin{aligned}(3.65)\quad &\int_{(D_{k+\frac32}\setminus D_{k+\frac12})\times[-2^k\ell,2^k\ell]} |\nabla h_{N,\mathsf{r}}[T_{\mathrm{out}}(T_{\mathrm{inn}}(X_N)), T_{\mathrm{out}}\#(T_{\mathrm{inn}}\#\mu_V)]|^2 \\ &\qquad\lesssim \exp\left(Ct\sum_{j=0}^{2^*+1}\frac{\ell^{\mathsf{d}}}{(2^j\ell)^{2\mathsf{d}-\mathsf{s}}}+\frac{\ell^{\mathsf{d}}}{2^k\ell(2^j\ell)^{2\mathsf{d}-\mathsf{s}-1}}\right) \\ &\qquad\times\Bigg(\int_{(D_{k+\frac32}\setminus D_{k+\frac12})\times[-2^k\ell,2^k\ell]} |\nabla h_{N,\mathsf{r}}[T_{\mathrm{inn}}(X_N), T_{\mathrm{inn}}(X_N)\#\mu_V]|^2 \\ &\qquad\qquad + \#I_{D_{k+\frac32}\setminus D_{k+\frac12}}N^{\frac{\mathsf{s}}{\mathsf{d}}} + N^{-\frac{2}{\mathsf{d}}}(\#I_{D_{k+\frac32}\setminus D_{k+\frac12}})^2(2^k\ell)^{-\mathsf{s}-2}\Bigg)\end{aligned}$$

Combining (3.64) and (3.65), we are led to

$$\begin{aligned}(3.66)\quad &\int_{(D_{k+\frac32}\setminus D_{k+\frac12})\times[-2^k\ell,2^k\ell]} |\nabla h_{N,\mathsf{r}}[(I+t\psi)(X_N), (I+t\psi)\#\mu_V]|^2 \\ &\lesssim e^{Ct\ell^{\mathsf{s}-\mathsf{d}}}\left(\int_{(A_k\times[-2^k\ell,2^k\ell]}|\nabla h_{N,\mathsf{r}}[X_N,\mu_V]|^2+\#I_{A_k}N^{\frac{\mathsf{s}}{\mathsf{d}}}+N^{-\frac{2}{\mathsf{d}}}(\#I_{A_k})^2(2^k\ell)^{-\mathsf{s}-2}\right).\end{aligned}$$

The result then follows from the local laws for X_N and (3.28) to control $(\#I_{A_k})^2$. We note that since $\square_\ell$ is at distance $\geq\varepsilon>0$ from $\partial\Sigma$, either A_k is included in the set where the local laws hold, or $2^k\ell$ is larger than a constant depending on ε, in which case we can use the macroscopic law (for $\ell=1$) valid up to the boundary.

□

These local controls on the transported energy cover all dyadic annuli and allow us to control the A_n terms, except with small probability.

Lemma 3.12. *Assume that $X_N\in\mathcal{G}_\ell$ with $\mathcal{G}_\ell$ as in Proposition 3.11. Let ψ be the transport constructed in Proposition 2.2 for φ satisfying (2.15) at order $k=5$. If $|t|\ell^{\mathsf{s}-\mathsf{d}}\mathsf{M}$ is smaller than a constant depending only on the constant in (2.17), we have*

$$(3.67)\qquad |\mathsf{A}_1(\Phi_t(X_N),\Phi_t\#\mu_V,\psi\circ\Phi_t^{-1})|\lesssim_\beta \mathsf{M}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\ell^{\mathsf{s}}$$

and more generally, if (2.15) holds at order $2n+3$, we have

$$(3.68)\qquad |\mathsf{A}_n(\Phi_t(X_N),\Phi_t\#\mu_V,\psi\circ\Phi_t^{-1})|\lesssim_\beta \mathsf{M}^nN^{1+\frac{\mathsf{s}}{\mathsf{d}}}\ell^{n\mathsf{s}-(n-1)\mathsf{d}}.$$

Proof. Let us first treat the easiest situation of $n=1$. Let k_* be as above. Let χ_k be the partition of unity relative to the A_k's this time defined as

$$(3.69)\qquad A_k=D_k\setminus D_{k-\frac32},\ \text{for } k\leq k_*+1,\quad A_{k*+2}=D^c_{k_*+\frac12}$$

with $|\nabla^{\otimes m}\chi_k|\lesssim(2^k\ell)^{-m}$. The A_k's are chosen to form a covering of $\mathbb{R}^{\mathsf{d}}$, and each A_k intersects only A_{k-1} and A_{k+1}, with A_{k-1} and A_{k+1} at positive distance $2^{k/2}\ell$ from each other. Let us decompose ψ into $\sum_{k=0}^{k_*+1}(\chi_k\psi)$ and use the linearity of A_1 with respect to ψ, the commutator estimate (3.37), the local laws (1.29), the law (3.26) in U^c, the estimate on ψ (2.17) and the result of the above proposition to obtain that for $X_N \in \mathcal{G}_\ell$, we have

$$
\begin{aligned}
|\mathsf{A}_1(\Phi_t(X_N), \Phi_t\#\mu_V, \psi)| &\lesssim \sum_{k=0}^{k_*+1} \|\nabla(\chi_k\psi)\|_{L^\infty} \int_{B_k\times[-2^k\ell,2^\ell]} |y|^\gamma |\nabla h_{N,\mathsf{r}}(\Phi_t(X_N), \Phi_t(\mu_V))|^2 \\
&\lesssim_\beta \left(\sum_{k=0}^{k_*} \|\nabla(\psi\chi_k)\|_{L^\infty} \left(2^k\ell\right)^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} + \|\nabla(\psi\chi_{k_*+1})\|_{L^\infty(U^c)} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \right) \\
&\lesssim_\beta \mathsf{M} \left(\sum_{k=0}^{k_*} \frac{\left(2^k\ell\right)^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \ell^{\mathsf{d}}}{(2^k\ell)^{2\mathsf{d}-\mathsf{s}}} + \ell^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \right) \\
&\lesssim_\beta \mathsf{M}\ell^{\mathsf{s}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \left(\sum_{k=0}^{k_*} \frac{1}{(2^{\mathsf{d}-\mathsf{s}})^k} + 1 \right).
\end{aligned}
$$

The proof for larger n relies similarly on a dyadic decomposition coupled with the higher order commutator estimates (3.38), although the combinatorics are a bit more complicated due to the tensor products in the integrand. We first describe the approach for $n = 2$ since it is instructive and easier to follow, before giving a detailed proof for all $n \geq 2$.

For notational ease, we will write X_N, μ and ψ instead of $\Phi_t(X_N)$, $\Phi_t\#\mu_V$ and $\psi \circ \Phi_t^{-1}$; this makes the notation below more readable, and the local law applies as desired by Proposition 3.11.

Description for $n = 2$.

As in the $n = 1$ case, we decompose ψ as $\sum_{k=0}^{k_*+1}(\chi_k\psi)$; denoting $\chi_k\psi$ by ψ^k for notational ease, we find

(3.70)
$$
\mathsf{A}_2(X_N, \mu, \psi) = \frac{1}{2} \sum_{k,m=0}^{k_*+1} \iint_{\triangle^c} \nabla^{\otimes 2}\mathsf{g}(x-y) : (\psi^k(x) - \psi^k(y)) \otimes (\psi^m(x) - \psi^m(y))\, d\mathrm{fluct}_N^{\otimes 2}(x,y).
$$

There are three kinds of terms in the summands:

(1) $k = m$
(2) $|m - k| = 1$
(3) $|m - k| \geq 2$.

The $k = m$ case is simplest, as the term is itself a higher order commutator

$$
\mathsf{A}_2(X_N, \mu, \psi) = \frac{1}{2} \iint_{\triangle^c} \nabla^{\otimes 2}\mathsf{g}(x-y) : (\psi^k(x) - \psi^k(y))^{\otimes 2}\, d\mathrm{fluct}_N^{\otimes 2}(x,y).
$$

which we will estimate using (3.38) and the local law (1.29) at the scale $2^k\ell$ at which ψ^k lives.

The case where $|m - k| = 1$ is a bit challenging, because although it is not literally a commutator it is very close to one as ψ^k and ψ^m live at similar scales. The idea is to use polarization; letting

$$
\varphi(k,m) := \iint_{\triangle^c} \nabla^{\otimes 2}\mathsf{g}(x-y) : (\psi^k(x) - \psi^k(y)) \otimes (\psi^m(x) - \psi^m(y))\, d\mathrm{fluct}_N^{\otimes 2}(x,y)
$$

we notice that $\varphi(k,m)$ is bilinear in k and m, where the sum $k+m$ corresponds to $\psi^k+\psi^m$. Polarizing, we find

$$\varphi(k,m)=\frac{\varphi(k+m,k+m)-\varphi(k,k)-\varphi(m,m)}{2}.$$

Each $\varphi(i,i)$ term is then a commutator, which we can control using (3.38) and the local law at the corresponding scale (1.29) as in Case 1.

In the third case, $|m-k|\geq 2$, the analysis is quite different since the supports of ψ^k and ψ^m are then disjoint. While this case will be a bit computationally intensive, it is conceptually easier because the summand can then be seen as the double fluctuation of a regular function. Notice that the domain of integration for

$$\begin{aligned}\iint_{\triangle^c}\nabla^{\otimes 2}\mathsf{g}(x-y):(\psi^k(x)-\psi^k(y))\otimes(\psi^m(x)-\psi^m(y))\;d\mathrm{fluct}_N^{\otimes 2}(x,y)\\ =\sum_{i,j=1}^{\mathsf{d}}\iint_{\triangle^c}\partial_{ij}\mathsf{g}(x-y)(\psi_i^k(x)-\psi_i^k(y))(\psi_j^m(x)-\psi_j^m(y))\;d\mathrm{fluct}_N^{\otimes 2}(x,y),\end{aligned}$$

where we have denoted the ith coordinate of ψ^k by ψ_i^k, is only on the support of $(\psi_i^k(x)-\psi_i^k(y))(\psi_j^m(x)-\psi_j^m(y))$. Both of these factors must be nonzero, and since the supports are disjoint then we need one of x and y to be in each. In particular, via the symmetry of g,

$$\begin{aligned}\sum_{i,j=1}^{\mathsf{d}}\iint_{\triangle^c}\partial_{ij}\mathsf{g}(x-y)(\psi_i^k(x)-\psi_i^k(y))(\psi_j^m(x)-\psi_j^m(y))\;d\mathrm{fluct}_N^{\otimes 2}(x,y)\\ =2\sum_{i,j=1}^{\mathsf{d}}\iint_{\triangle^c}\partial_{ij}\mathsf{g}(x-y)\psi_i^k(x)\psi_j^m(y)\;d\mathrm{fluct}_N^{\otimes 2}(x,y).\end{aligned}$$

We will apply the energy estimate Proposition 3.5 twice to obtain the requisite control, once on each fluctuation. With this background for $n=2$, we now explain how to work through the argument for generic n.

Detailed argument for generic $n\geq 2$.

Decomposing $\psi=\sum\psi^k$ as above, we write

$$\mathsf{A}_n(X_N,\mu,\psi)=\frac{1}{2}\sum_{\vec{k}\in[k_*+2]^n}\iint_{\triangle^c}\nabla^{\otimes n}\mathsf{g}(x-y):\bigotimes_{i=1}^n\left(\psi^{k_i}(x)-\psi^{k_i}(y)\right)\;d\mathrm{fluct}_N^{\otimes 2}(x,y)\tag{3.71}$$

where $[k]^n=\{0,1,\dots,k\}^n$. We have three kinds of terms in the summation.

Case 1: $\vec{k} = (k, k, \ldots, k)$. This corresponds to the first type discussed in the $n = 2$ case. It is a true commutator and we compute using (3.38), (1.29) and (2.17) at scale $2^k\ell$

$$\left|\iint_{\triangle^c} \nabla^{\otimes n}\mathsf{g}(x-y) : \left(\psi^k(x) - \psi^k(y)\right)^{\otimes n} \ d\mathrm{fluct}_N^{\otimes 2}(x,y)\right| \lesssim_\beta \left(2^k\ell\right)^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \sum_{p=0}^{n} \left(2^k\ell\|\nabla^{\otimes 2}\psi\|_{L^\infty}\right)^p$$

$$\times \sum_{\substack{1\le c_1,\ldots,c_{n-p}\\ n-p\le c_1+\cdots+c_{n-p}\le 2n}} N^{\frac{1}{\mathsf{d}}((n-p)-\sum_{k=1}^{n-p} c_k)}\|\nabla^{\otimes c_1}\psi\|_{L^\infty}\cdots\|\nabla^{\otimes c_{n-p}}\psi\|_{L^\infty}$$

$$\lesssim_\beta \left(2^k\ell\right)^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \sum_{p=0}^{n} \frac{\ell^{\mathsf{d}p}\mathsf{M}^p}{(2^k\ell)^{2\mathsf{d}-\mathsf{s}}} \sum_{\substack{1\le c_1,\ldots,c_{n-p}\\ n-p\le c_1+\cdots+c_{n-p}\le 2n}} N^{\frac{1}{\mathsf{d}}((n-p)-\sum_{k=1}^{n-p} c_k)} \prod_{q=1}^{n-p} \frac{\ell^{\mathsf{d}}\mathsf{M}}{(2^k\ell)^{2\mathsf{d}-\mathsf{s}+c_q-1}}$$

$$\lesssim_\beta \left(2^k\ell\right)^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \frac{\ell^{\mathsf{d}n}\mathsf{M}^n}{(2^k\ell)^{(2\mathsf{d}-\mathsf{s})n}} = \ell^{\mathsf{d}(1-n)+\mathsf{s}n} N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\mathsf{M}^n\left(2^{n(\mathsf{s}-\mathsf{d})+\mathsf{d}(1-n)}\right)^k$$

where we have used that the largest possible c_q is $2n$ and that

$$N^{\frac{1}{\mathsf{d}}((n-p)-\sum_{k=1}^{n-p} c_k)}\left(2^k\ell\right)^{(n-p)+\sum_{q=1}^{n-p} c_q} \lesssim 1$$

since $N^{-1/\mathsf{d}} \le 2^k\ell$ and $(n-p) - \sum_{k=1}^{n-p} c_k \ge 0$. Summing over k yields

$$\sum_{\vec{k}=(k,k,\ldots,k),k\in[k_*+1]} \left|\iint_{\triangle^c} \nabla^{\otimes n}\mathsf{g}(x-y) : \bigotimes_{i=1}^{n}\left(\psi^{k_i}(x) - \psi^{k_i}(y)\right) \ d\mathrm{fluct}_N^{\otimes 2}(x,y)\right|$$
$$\lesssim_\beta \ell^{(1-n)\mathsf{d}+\mathsf{s}n} N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\mathsf{M}^n$$

In the case $k = k_*+2$, we use the bound (2.17) and (3.26) to obtain that this term is controlled by $\ell^{\mathsf{d}}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}$, which can be incorporated into the previous estimate.

Case 2: $\bigcup_i \mathrm{supp}\,(\psi^{k_i})$ is connected. This corresponds to the second type discussed in the $n = 2$ case. Since the supports of ψ^k and ψ^m only overlap for $|m-k| \le 1$, this is only possible if the index vector $\vec{k}$ takes the form (up to reordering)

$$\vec{k} = (k, k+a_1, k+a_1+a_2, \ldots, k+a_1+\cdots+a_{n-1}), \qquad a_i \in \{0,1\}$$

Note that for each k, there are only a constant dependent on n number of such vectors. For these indices, we make use of multilinear polarization. For notational ease, we again let

$$\varphi(k_1,\ldots,k_n) := \iint_{\triangle^c} \nabla^{\otimes n}\mathsf{g}(x-y) : \bigotimes_{i=1}^{n}\left(\psi^{k_i}(x) - \psi^{k_i}(y)\right) \ d\mathrm{fluct}_N^{\otimes 2}(x,y).$$

Notice that $\varphi(k_1,\ldots,k_n)$ is symmetric and multilinear in $\vec{k}$, where the sum $k+m$ again corresponds to $\psi^k + \psi^m$. Using the polarization formula for symmetric, multilinear forms we have

$$\varphi(k_1,\ldots,k_n) = \frac{1}{n!}\sum p = 1^n \sum_{1\le j_1<\cdots<j_p\le n} (-1)^{n-p}\varphi(k_{j_1}+\cdots+k_{j_p})$$

where we have simplified $\varphi(m) := \varphi(m,m,\ldots,m)$. Now, for a vector of the form $\vec{k} = (k, k+a_1, k+a_1+a_2, \ldots, k+a_1+\cdots+a_{n-1})$ with a_i as above, all of the k_i live at the same scale (up to an n-dependent constant). Hence, we can apply (1.29) and (2.17) at scale $2^k\ell$, up to adjusting constants, which coupled with (3.38) as in Case 1 yields

$$|\varphi(k_1,\dots,k_n)| \lesssim_\beta \ell^{\mathsf{d}(1-n)+\mathsf{s}n} N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \mathsf{M}^n \left(2^{n(\mathsf{s}-\mathsf{d})+(1-n)\mathsf{d}}\right)^k$$

In the case $k = k_* + 2$, we again use the bound (2.17) and (3.26) to obtain that this term is controlled by $\ell^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}}$, which can be incorporated into the previous estimate. Summing over k again yields

$$\sum_{\vec{k}\text{ is Case 2}} \left| \iint_{\triangle^c} \nabla^{\otimes n}\mathsf{g}(x-y) : \bigotimes_{i=1}^n \left(\psi^{k_i}(x) - \psi^{k_i}(y)\right)\, d\mathrm{fluct}_N^{\otimes 2}(x,y)\right| \\ \lesssim_\beta \mathsf{M}^n \ell^{n\mathsf{s}-(n-1)\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}}$$

using that there are an $O(n)$ number of $\vec{k}$ associated to each k.

Case 3: $\bigcup_i \mathrm{supp}\,(\psi^{k_i})$ *is not connected.* This corresponds to the third type discussed in the $n = 2$ case. The first observation to make here is that we only need to consider the situation where $\bigcup_i \mathrm{supp}\,(\psi^{k_i})$ is a disjoint union of two connected sets. Indeed, suppose that it could be written as a disjoint union of three connected sets. Then, since the support of distinct ψ^k and ψ^m only overlap for $|m-k| \le 1$, we can write the $\vec{k}$ (up to reordering) as

$$\vec{k} = (k, k+a_0, \dots, k+a_1+\cdots+a_{p-1}, m, m+b_0, \dots, m+b_1+\cdots+b_{q-1}, \\ r, r+c_0, \dots, r+c_1+\cdots+c_{n-p-q-1})$$

where all of the a_i, b_i and c_i belong to $\{0,1\}$ and the sets

$$A = \bigcup_{i=1}^p \mathrm{supp}\,(\psi^{k+a_1+\cdots+a_{i-1}}), \quad B = \bigcup_{i=1}^q \mathrm{supp}\,(\psi^{m+b_1+\cdots+b_{i-1}}), \quad C = \bigcup_{i=1}^{n-p-q} \mathrm{supp}\,(\psi^{r+c_1+\cdots+c_{i-1}})$$

are pairwise disjoint. We have set $a_0 = b_0 = c_0$ for notational ease. Now, let us examine the corresponding terms of A_n, which look like

$$\sum_{i_1,\dots,i_n} \iint_{\triangle^c} \partial_{i_1,\dots,i_n}\mathsf{g}(x-y) \prod_{j=1}^p \left(\psi_{i_j}^{k+a_0+\cdots+a_{i-1}}(x) - \psi_{i_j}^{k+a_0+\cdots+a_{i-1}}(y)\right) \times \\ \prod_{j=1}^q \left(\psi_{i_{p+j}}^{m+b_0+\cdots+b_{i-1}}(x) - \psi_{i_{p+j}}^{m+b_0+\cdots+b_{i-1}}(y)\right) \times \\ \prod_{j=1}^{n-p-1} \left(\psi_{i_{p+q+j}}^{r+c_0+\cdots+c_{i-1}}(x) - \psi_{i_{p+q+j}}^{r+c_0+\cdots+c_{i-1}}(y)\right)\, d\mathrm{fluct}_N^{\otimes 2}(x,y)$$

In order for the integral to be nonzero, all three factors need to not vanish. So, at least one of x or y needs to belong to A, B and C. However, since A, B and C are pairwise disjoint this is impossible! The same argument works for more than three disjoint sets. Hence, $\bigcup_i \mathrm{supp}\,(\psi^{k_i})$ is a disjoint union of two connected sets and $\vec{k}$ takes the form (up to reordering)

$$\vec{k} = (k, k+a_1, \dots, k+a_1+\cdots+a_{p-1}, m, m+b_1, \dots, m+b_1+\cdots+b_{n-p-1})$$

where all of the a_i, b_i and c_i belong to $\{0,1\}$ and the sets A and B as above are disjoint. There are a constant bounded by a number dependent only on n number of vectors associated to each k and m. The corresponding terms of A_n take the form

$$\iint_{\triangle^c} \partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\prod_{j=1}^{p}\left(\psi_{i_j}^{k+a_0+\cdots+a_{i-1}}(x)-\psi_{i_j}^{k+a_0+\cdots+a_{i-1}}(y)\right)\times \\ \prod_{j=1}^{n-p}\left(\psi_{i_{p+j}}^{m+b_0+\cdots+b_{i-1}}(x)-\psi_{i_{p+j}}^{m+b_0+\cdots+b_{i-1}}(y)\right)\, d\mathrm{fluct}_N^{\otimes 2}(x,y)$$

and the integral is only nonzero if $x\in A$ and $y\in B$ (or vice versa). Without loss of generality, assume $x\in A$. Then, the terms above become

$$\iint_{\triangle^c} \partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\prod_{j=1}^{p}\psi_{i_j}^{k+a_0+\cdots+a_{i-1}}(x)\prod_{j=1}^{n-p}\psi_{i_{p+j}}^{m+b_0+\cdots+b_{i-1}}(y)\, d\mathrm{fluct}_N^{\otimes 2}(x,y) \\ := \iint_{\triangle^c}\partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\varphi_1(x)\varphi_2(y)\, d\mathrm{fluct}_N^{\otimes 2}(x,y)$$

with $\partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\sim|x-y|^{-\mathsf{s}-n}$. We bound as described in the third type discussed in the $n=2$ case. We set

$$f(x)=\int\partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\varphi_2(y)\, d\mathrm{fluct}_N(y)$$

and first bound f and its derivatives using Proposition 3.5 and (1.29) and (2.17) at scale $2^m\ell$ (if $m=k_*+2$ then the same estimates hold with $2^m\ell\sim 1$). Using $|x-y|\gtrsim 2^m\ell$, we have

$$\begin{aligned}\|\partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\varphi_2(y)\|_{L^2}^2 &\lesssim_\beta \int_{\bigcup_{k=m}^{m+b_0+\cdots+b_{n-p-1}}A_k}\frac{1}{|x-y|^{2\mathsf{s}+2n}}\left(\frac{\ell^{2\mathsf{d}}\mathsf{M}^2}{|y|^{4\mathsf{d}-2\mathsf{s}-2}}\right)^{n-p} dy\\ &\lesssim_\beta \ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}\int_{O\left(\left(2^{m+b_0+\cdots+b_{n-p-1}}\right)^{\mathsf{d}}\right)}\frac{r^{\mathsf{d}-1}}{r^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)}}\, dr\\ &\lesssim_\beta \frac{\ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}}{(2^m\ell)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-\mathsf{d}}}.\end{aligned}$$

An analogous computation yields

$$\begin{aligned}\|\nabla\left(\partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\varphi_2(y)\right)\|_{L^2}^2 &\lesssim_\beta \frac{\ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}}{(2^m\ell)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-\mathsf{d}+2}}\\ \|\nabla^{\otimes 2}\left(\partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\varphi_2(y)\right)\|_{L^2}^2 &\lesssim_\beta \frac{\ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}}{(2^m\ell)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-\mathsf{d}+4}}\end{aligned}$$

We first use Proposition 3.5 to get L^∞ bounds on h and its gradient in $\operatorname{supp}\psi_i^k$. As a simplification, notice that it is sufficient to control

$$\left(\eta^{\gamma-1}\|\varphi\|_{L^2(\Omega)}^2+\eta^{\gamma+1}\|\nabla\varphi\|_{L^2(\Omega)}^2\right)^{1/2}\left((2^m\ell)^{\mathsf{d}}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\right)^{1/2}$$

for η to be chosen below, with $\varphi(y)=\partial_{i_1,\dots,i_n}\mathsf{g}(x-y)\varphi_2(y)$ using (1.29) at scale $2^m\ell$, since the error term is strictly smaller; one can bound $\#I_\Omega(N^{-\frac{1}{\mathsf{d}}})^{\mathsf{d}-\mathsf{s}}$ by the energy at scale $2^m\ell$. A computation shows that the terms above are balanced by choosing $\eta = 2^m\ell$, and we conclude by Proposition 3.5 that

$$
\begin{aligned}
|f(x)|^2 &\lesssim_\beta \frac{\ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}\left(2^m\ell\right)^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\ell\right)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-\mathsf{d}+1-\gamma}}\\
|\nabla f(x)|^2 &\lesssim_\beta \frac{\ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}\left(2^m\ell\right)^{\mathsf{d}} N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\ell\right)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-\mathsf{d}+3-\gamma}}.
\end{aligned}
$$

We now use these estimates to control

$$
\int \varphi_1(x)f(x)\, d\mathrm{fluct}_N(x)
$$

using Proposition 3.5. We find using (2.17) (again, if $k = k_*+2$ then the same estimates hold with $2^k\ell \sim 1$)

$$
\begin{aligned}
\|\varphi_1(x)f(x)\|_{L^2}^2 &\lesssim_\beta \frac{\ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\ell\right)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-2\mathsf{d}+1-\gamma}}\int_{\cup_{m=k}^{k+a_0+\cdots+a_{p-1}}A_k}\left(\frac{\ell^{2\mathsf{d}}\mathsf{M}^2}{|y|^{4\mathsf{d}-2\mathsf{s}-2}}\right)^p dy\\
&\lesssim_\beta \frac{\ell^{2\mathsf{d}n}\mathsf{M}^{2n}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\ell\right)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-2\mathsf{d}+1-\gamma}\left(2^k\ell\right)^{(4\mathsf{d}-2\mathsf{s}-2)p-\mathsf{d}}}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\|\nabla(\varphi_1(x)f(x))\|_{L^2}^2 &\lesssim_\beta \frac{\ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\ell\right)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-2\mathsf{d}+3-\gamma}}\int_{\cup_{m=k}^{k+a_0+\cdots+a_{p-1}}A_k}\left(\frac{\ell^{2\mathsf{d}}\mathsf{M}^2}{|y|^{4\mathsf{d}-2\mathsf{s}-2}}\right)^p dy\\
&+\frac{\ell^{2\mathsf{d}(n-p)}\mathsf{M}^{2(n-p)}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\ell\right)^{2\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-2\mathsf{d}+1-\gamma}}\int_{\cup_{m=k}^{k+a_0+\cdots+a_{p-1}}A_k}\left(\frac{\ell^{2\mathsf{d}}\mathsf{M}^2}{|y|^{4\mathsf{d}-2\mathsf{s}-2}}\right)^{p-1}\left(\frac{\ell^{2\mathsf{d}}\mathsf{M}^2}{|y|^{4\mathsf{d}-2\mathsf{s}}}\right) dy
\end{aligned}
$$

which yields the bound

$$
\|\nabla(\varphi_1(x)f(x))\|_{L^2}^2 \lesssim_\beta \frac{\ell^{2\mathsf{d}n}\mathsf{M}^{2n}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\ell\right)^{\mathsf{s}+2n+(4\mathsf{d}-2\mathsf{s}-2)(n-p)-\mathsf{d}}\left(2^k\ell\right)^{(4\mathsf{d}-2\mathsf{s}-2)p-\mathsf{d}}}\left(\frac{1}{\left(2^m\ell\right)^2}+\frac{1}{\left(2^k\ell\right)^2}\right).
$$

using $\gamma+\mathsf{d}-1=\mathsf{s}$. Applying Proposition 3.5 with $\eta = \left(\frac{1}{\left(2^m\ell\right)^2}+\frac{1}{\left(2^k\ell\right)^2}\right)^{-1/2}$ and (1.29) at scale $2^k\ell$ to find

$$
\left|\int \varphi_1(x)f(x)\; d\mathrm{fluct}_N(x)\right| \lesssim_\beta \frac{\ell^{\mathsf{d}n}\mathsf{M}^n\left(2^k\ell\right)^{\frac{\mathsf{d}}{2}}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\ell\right)^{\frac{\mathsf{s}}{2}+n+(2\mathsf{d}-\mathsf{s}-1)(n-p)-\frac{\mathsf{d}}{2}}\left(2^k\ell\right)^{(2\mathsf{d}-\mathsf{s}-1)p-\frac{\mathsf{d}}{2}}}\left(\frac{1}{\left(2^m\ell\right)^2}+\frac{1}{\left(2^k\ell\right)^2}\right)^{\frac{1-\gamma}{4}}.
$$

which, simplifying, yields

$$
\begin{aligned}
&\iint_{\triangle^c}\partial_{i_1,\ldots,i_n}\mathsf{g}(x-y)\prod_{j=1}^p\psi_{i_j}^{k+a_0+\cdots+a_{i-1}}(x)\prod_{j=1}^{n-p}\psi_{i_{p+j}}^{m+b_0+\cdots+b_{i-1}}(y)\; d\mathrm{fluct}_N^{\otimes 2}(x,y)\\
&\qquad\lesssim_\beta \frac{\mathsf{M}^n\ell^{n\mathsf{s}-(n-1)\mathsf{d}}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}}{\left(2^m\right)^{\frac{\mathsf{s}}{2}+n+(2\mathsf{d}-\mathsf{s}-1)(n-p)-\frac{\mathsf{d}}{2}}+\left(2^k\right)^{(2\mathsf{d}-\mathsf{s}-1)p-\frac{\mathsf{d}}{2}}}\left(\frac{1}{2^{2m}}+\frac{1}{2^{2k}}\right)^{\frac{1-\gamma}{4}}.
\end{aligned}
$$

Summing over all choices of k and m, we find

$$\sum_{\vec{k}\text{ is Case 3}}\left|\iint_{\triangle^c}\nabla^{\otimes n}\mathsf{g}(x-y):\bigotimes_{i=1}^{n}\left(\psi^{k_i}(x)-\psi^{k_i}(y)\right)\, d\mathrm{fluct}_N^{\otimes 2}(x,y)\right|\lesssim_\beta \mathsf{M}^n\ell^{n\mathsf{s}-(n-1)\mathsf{d}}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}.$$

Combining Cases $1-3$ yields the result. □

As a corollary, we obtain a first bound on the term T_2 defined in (2.5).

Corollary 3.13 (A first bound on T_2)**.** *Under the same assumptions, supposing that* (2.15) *holds at order* 4*, if $t\ell^{\mathsf{s}-\mathsf{d}}\mathsf{M}$ is small enough, for every N sufficiently large, we have*

$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}(e^{T_2}\mathbf{1}_{\mathcal{G}_\ell})\right|\lesssim_\beta \beta\mathsf{M}|t|N\ell^{\mathsf{s}} \tag{3.72}$$

where $\mathcal{G}_\ell$ is as in Proposition 3.11.

Proof. Let $X_N\in\mathcal{G}_\ell$. From (3.36), we have

$$\mathsf{F}_N(\Phi_t(X_N),\Phi_t\#\mu_0)-\mathsf{F}_N(X_N,\mu_0)=\int_0^t \mathsf{A}_1(\Phi_s(X_N),\mu_s,\psi\circ\Phi_s^{-1})ds.$$

In view of (3.67) this is $\lesssim_\beta \mathsf{M}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\ell^{\mathsf{s}}$. Similarly, using (3.31) on dyadic scales can bound

$$|\mathrm{Fluct}_{\mu_V}(\log\det D\Phi_t)|\lesssim_\beta |tN|\mathsf{M}\sum\left(2^k\ell\right)^{\mathsf{d}}\frac{\ell^{\mathsf{d}}}{(2^k\ell)^{2\mathsf{d}-\mathsf{s}}}\lesssim_\beta |t|N\|\varphi_0\|_{C^4}\ell^{\mathsf{s}}$$

using the transport bounds (2.17). Combining all of the above with (2.5) yields the result. □

4. Proof of Theorems 2 and 3

We now have all of the tools needed to examine fluctuations of linear statistics, and complete the proofs of Theorems 2 and 3. Let us first recall the expansion in Lemma 2.1, which allows us to expand the Laplace transform of $\mathrm{Fluct}_{\mu_V}(\varphi)$ on a good event $\mathcal{G}$ by

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(-\beta tN^{1-\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{Fluct}_{\mu_V}(\varphi)\right)\mathbf{1}_{\mathcal{G}}\right]=e^{T_0}\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[e^{T_1+T_2}\mathbf{1}_{\mathcal{G}}\right].$$

We now examine each of these terms individually.

4.1. Computation of T_0.

Using the explicit choice of ψ in Proposition 2.2, we can compute T_0, which is a completely deterministic computation. We start with the $\log\det D\Phi_t$ term.

Lemma 4.1. *Let $\Phi_t=I+t\psi$ with ψ as in Proposition 2.2. Denoting $\mu_t=\Phi_t\#\mu_V$, we have*

$$\int_{\mathbb{R}^{\mathsf{d}}}\log\det D\Phi_t\, d\mu_V=\mathrm{Ent}(\mu_V)-\mathrm{Ent}(\mu_t)=t\,M(\varphi)+O\left(t^2\mathsf{M}^2\ell^{2\mathsf{s}-\mathsf{d}}\right) \tag{4.1}$$

with

$$M(\varphi)=\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_\Sigma(-\Delta)^\alpha\varphi^\Sigma(\log\mu_V). \tag{4.2}$$

Proof. Since $\Phi_t=I+t\psi$, and $\mu_t=\Phi_t\#\mu_V$ we have $\det D\Phi_t=\frac{\mu_V}{\mu_t\circ\Phi_t}$, and thus

$$\int\log\det D\Phi_t d\mu_V=\int\log\mu_V d\mu_V-\int\log\mu_t(\Phi_t(x))d\mu_V=\mathrm{Ent}(\mu_V)-\mathrm{Ent}(\mu_t). \tag{4.3}$$

We can also write explicitly

$$\log\det D\Phi_t=t\,\mathrm{div}\,\psi+O\left(t^2|D\psi|^2\right) \tag{4.4}$$

and have

$$\int_{\mathbb{R}^\mathsf{d}} (\mathrm{div}\,\psi)\mu_V = \int_{\mathbb{R}^\mathsf{d}} \mathrm{div}\,(\psi\mu_V) - \int_{\mathbb{R}^\mathsf{d}} \psi\mu_V \cdot \nabla \log \mu_V = \int_{\mathbb{R}^\mathsf{d}} \mathrm{div}\,(\psi\mu_V)(1+\log\mu_V). \tag{4.5}$$

Thus, using (2.12) and (2.16)

$$\begin{aligned}\int_{\mathbb{R}^\mathsf{d}} (\mathrm{div}\,\psi)\mu_V &= \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_\Sigma (-\Delta)^\alpha \varphi^\Sigma (1+\log\mu_V)\\ &= \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\mathbb{R}^\mathsf{d}} (-\Delta)^\alpha \varphi^\Sigma (1+\log\mu_V)\\ &= \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_\Sigma (-\Delta)^\alpha \varphi^\Sigma \log\mu_V,\end{aligned}$$

where we have used that $(-\Delta)^\alpha\varphi^\Sigma$ is mean zero.

Integrating the error term and using (2.48), we obtain the result. □

Let us turn to the remaining terms.

Lemma 4.2. *Let $\Phi_t = I + t\psi$ with ψ as in Proposition 2.2 with (2.15) at order 4. Let T_0 be as in (2.3). Then, we have*

$$T_0 = \frac{\beta N^{2-\frac{\mathsf{s}}{\mathsf{d}}} t^2}{2}\mathrm{Var}(\varphi) + tNM(\varphi) + O\left(t^3 N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\beta \mathsf{M}^3 \ell^{2\mathsf{s}-\mathsf{d}} + t^2\mathsf{M}^2 N\ell^{2\mathsf{s}-\mathsf{d}}\right) \tag{4.6}$$

where

$$\mathrm{Var}(\varphi) = \frac{c_{\mathsf{d},\alpha}}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left\|\varphi^\Sigma\right\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} \tag{4.7}$$

with $M(\varphi)$ as in (4.2) and $c_{\mathsf{d},\alpha}$ as in (1.10).

Proof. The proof is based on a second order Taylor expansion in t. First, a careful computation yields

$$\begin{aligned}\frac12\left(\mathsf{g}(\Phi_t(x)-\Phi_t(y)) - \mathsf{g}(x-y)\right) &= \frac{t}{2}\nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))\\ &\quad + \frac{t^2}{4}\sum_{i,j}\partial_{i,j}\mathsf{g}(x-y)(\psi_i(x)-\psi_i(y))(\psi_j(x)-\psi_j(y)) + O\left(t^3\frac{|\psi(x)-\psi(y)|^3}{|x-y|^{3+\mathsf{s}}}\right)\end{aligned}$$

where the O is uniform despite the singularity in the derivative of g; this again can be obtained by factoring out $|x-y|$ so that we Taylor expand instead about $\frac{x-y}{|x-y|}$, where g is bounded. Similarly,

$$V(x+t\psi(x)) - V(x) = t\nabla V\cdot\psi(x) + \frac{t^2}{2}\sum_{i,j}\partial_{i,j}V\psi_i(x)\psi_j(x) + O\left(t^3|\psi(x)|^3\right)$$

and

$$t\varphi(x+t\psi(x)) - t\varphi(x) = t^2\nabla\varphi\cdot\psi(x) + O\left(t^3|D^2\varphi||\psi|^2\right).$$

After integration, the first order terms give by symmetry

$$\begin{aligned}\frac{t}{2}\iint \nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))\,d\mu_V(x)d\mu_V(y) + t\int\nabla V\cdot\psi(x)\,d\mu_V(x)\\ = t\int\nabla\left(h^{\mu_V}+V\right)\cdot\psi(x)\,d\mu_V(x) = 0,\end{aligned}$$

where the vanishing is thanks to (1.16) and the fact that ζ_V vanishes in the support of μ_V. Now, expanding $(\psi_i(x)-\psi_i(y))(\psi_j(x)-\psi_j(y))$ and using symmetry we find

$$\begin{aligned}&\iint \frac{t^2}{4}\sum_{i,j}\partial_{i,j}\mathsf{g}(x-y)(\psi_i(x)-\psi_i(y))(\psi_j(x)-\psi_j(y))\,d\mu_V(x)d\mu_V(y)\\&=\frac{t^2}{2}\sum_{i,j}\iint \partial_{i,j}\mathsf{g}(x-y)\psi_i(x)\psi_j(x)\,d\mu_V(x)d\mu_V(y)-\frac{t^2}{2}\sum_{i,j}\iint \partial_{i,j}\mathsf{g}(x-y)\psi_i(x)\psi_j(y)\,d\mu_V(x)d\mu_V(y).\end{aligned}$$

Notice that

$$\begin{aligned}\int\sum_{i,j}\partial_{ij}V\psi_i(x)\psi_j(x)\ d\mu_V(x)+\sum_{i,j}\iint \partial_{i,j}\mathsf{g}(x-y)\psi_i(x)\psi_j(x)\,d\mu_V(x)d\mu_V(y)&\\=\int \partial_{i,j}(V+\mathsf{g}*\mu_V)(x)\psi_i(x)\psi_j(x)\,d\mu_V(x)=0&\end{aligned}$$

by the same argument as above using (1.16). Thus, the order t^2 terms that remain are just

$$t^2\int\nabla\varphi\cdot\psi\,d\mu_V-\frac{t^2}{2}\sum_{i,j}\iint \partial_{i,j}\mathsf{g}(x-y)\psi_i(x)\psi_j(y)\,d\mu_V(x)d\mu_V(y).$$

For the first term, integrating by parts and using (2.12) and (2.16) yields

$$t^2\int_{\mathbb{R}^\mathsf{d}}\nabla\varphi\cdot\psi\,d\mu_V=-t^2\int_{\mathbb{R}^\mathsf{d}}\varphi\,\mathrm{div}\,(\psi\mu_V)=-\frac{t^2}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int\varphi^\Sigma(-\Delta)^\alpha\varphi^\Sigma=-\frac{t^2}{2}\frac{c_{\mathsf{d},\alpha}}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left\|\varphi^\Sigma\right\|^2_{H^\alpha}$$

since we have via fractional integration by parts and [DNPV12, Proposition 3.6] that

$$\int f(-\Delta)^\alpha f=\int|(-\Delta)^{\alpha/2}f|^2=\frac{c_{\mathsf{d},\alpha}}{2}\|f\|^2_{\dot H^\alpha}$$

with $c_{\mathsf{d},\alpha}$ the constant in (1.10), in view of the definitions (1.10) and (1.12). Integrating the second term by parts in x and y and using (2.16) yields

$$\begin{aligned}&-\frac{t^2}{2}\sum_{i,j}\iint \partial_{i,j}\mathsf{g}(x-y)\psi_i(x)\psi_j(y)\,d\mu_V(x)d\mu_V(y)=-\frac{t^2}{2}\iint \mathsf{g}(x-y)\mathrm{div}\,(\psi\mu_V)(x)\mathrm{div}\,(\psi\mu_V)(y)\\&=\frac{-t^2}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}^2}\iint \mathsf{g}(x-y)(-\Delta)^\alpha\varphi^\Sigma(y)(-\Delta)^\alpha\varphi^\Sigma(x)\\&=\frac{-t^2}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int\varphi^\Sigma(x)(-\Delta)^\alpha\varphi^\Sigma(x)=\frac{-t^2}{4}\frac{c_{\mathsf{d},\alpha}}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left\|\varphi^\Sigma\right\|^2_{H^\alpha}.\end{aligned}$$

Finally, we need to integrate the error terms. This is done using (2.47), (2.48) and (2.50), to write

$$\int|D^2\varphi||\psi|^2\lesssim\frac{1}{\ell^2}\ell^{-\mathsf{d}+2\mathsf{s}+2}\mathsf{M}^3,$$

and similarly for the other terms. Combining with (4.1) we have thus established (4.6). □

4.2. **Control of T_1.** We next turn our attention to T_1 as in (2.4); the choice of transport ψ made in Section 4 was so that T_1 would vanish at leading order in t. We verify that here, and estimate the next order in t.

Lemma 4.3. *Let $\Phi_t = I + t\psi$ with ψ as in Proposition 2.2 with (2.15) at order 5. If t is small enough that $\|t\psi\|_{C^1} < \frac{1}{2}$ and $|t|\ell^{\mathsf{d}-\mathsf{s}}\mathsf{M}$ is smaller than a constant, then*

$$T_1 = \beta N^{1-\frac{\mathsf{s}}{\mathsf{d}}} t^2 \int_{\mathbb{R}^\mathsf{d}} u(x) d\mathrm{fluct}_{\mu_V}(x)$$

where u satisfies

$$|u(x)| \lesssim \mathsf{M}^2 \begin{cases} \ell^{2\mathsf{d}} \max(|x-z|,\ell)^{-3\mathsf{d}+\mathsf{s}} & \text{if } x \in U \\ \ell^{2\mathsf{d}} |x-z|^{-2(\mathsf{s}+2)} & \text{if } x \in U^c. \end{cases} \tag{4.8}$$

and

$$\|Du\|_{L^\infty} \lesssim \frac{\mathsf{M}^2}{\ell^{\mathsf{d}-\mathsf{s}+1}}. \tag{4.9}$$

Proof. The approach is similar to [Pei24, Lemma 4.14, Lemma 5.5].

We use Taylor's formula to write (m denoting multi-indices of length 2)

$$\begin{aligned}
&\int_{\mathbb{R}^\mathsf{d}} (\mathsf{g}(\Phi_t(x) - \Phi_t(y)) - \mathsf{g}(x-y)) d\mu_V(y) + (V_t \circ \Phi_t - V)(x)) \\
&= t\left(\int \nabla \mathsf{g}(x-y) \cdot (\psi(x) - \psi(y)) d\mu_V(y) + (\nabla V \cdot \psi + \varphi)(x)\right) \\
&+ 2t^2 \int_{\mathbb{R}^\mathsf{d}} \sum_{|m|=2} \frac{1}{m!} (\psi(x) - \psi(y))^m \int_0^1 (1-a) D^m \mathsf{g}(x - y + at(\psi(x) - \psi(y)) \, da \, \mu_V(y) \\
&+ 2t^2 \int \sum_{|m|=2} \frac{1}{m!} \psi(x)^m \int_0^1 (1-a) D^m V(x + at\psi(x)) \, da + t^2 \psi(x) \cdot \int D\varphi(x + at\psi(x)) \, da,
\end{aligned}$$

and denote $t^2 u(x)$ the sum of the last two lines.

As in (2.8)–(2.9), we then may rewrite

$$\int \nabla \mathsf{g}(x-y) \cdot (\psi(x) - \psi(y)) d\mu_V(y) + \nabla V \cdot \psi + \varphi = \psi \cdot \nabla \zeta_V - h^{\mathrm{div}\,(\psi\mu_V)} + \varphi$$

which vanishes by our definition of ψ (see Proposition 2.2). It follows that

$$T_1 = \beta N^{1-\frac{\mathsf{s}}{\mathsf{d}}} t^2 \int_{\mathbb{R}^\mathsf{d}} u(x) d\mathrm{fluct}_{\mu_V}(x).$$

Step 1: Pointwise control on u. Let

$$u_1(x) := 2 \sum_{|m|=2} \frac{1}{m!} \int (\psi(x) - \psi(y))^m \int_0^1 (1-a) D^m \mathsf{g}(x - y + at(\psi(x) - \psi(y))) \, da \, d\mu_V(y). \tag{4.10}$$

A computation shows that

$$D^{ij}\mathsf{g}(z) = \frac{-\mathsf{s}}{|z|^{\mathsf{s}+2}} \mathbf{1}_{i=j} + \frac{\mathsf{s}(\mathsf{s}+2) z_i z_j}{|z|^{\mathsf{s}+4}} \quad \text{for } z \neq 0$$

and so the da-integrand in (4.10) can be reexpressed as

$$\int_0^1 (1-a)\Big(\frac{-\mathsf{s}}{|x-y+at(\psi(x)-\psi(y))|^{\mathsf{s}+2}} + \frac{\mathsf{s}(\mathsf{s}+2)(x-y+at(\psi(x)-\psi(y)))_i(x-y+at(\psi(x)-\psi(y)))_j}{|x-y+at(\psi(x)-\psi(y))|^{\mathsf{s}+4}}\Big)\, da.$$

Now, using $\|t\psi\|_{C^1} \le \frac{1}{2}$ we can write $|x-y+at(\psi(x)-\psi(y))| \gtrsim |x-y|$ and control (4.10), using (2.49), by

$$|u_1(x)| \lesssim \int \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{\mathsf{s}+2}}\, d\mu_V(y) \lesssim \mathsf{M}^2 \begin{cases} \ell^{2\mathsf{d}} \max(|x-z|,\ell)^{-3\mathsf{d}+\mathsf{s}} & \text{if } x \in U \\ \ell^{2\mathsf{d}} |x|^{-2(\mathsf{s}+2)} & \text{if } x \in U^c. \end{cases} \tag{4.11}$$

We can give a similar control, much more easily, for

$$u_2(x) = 2\sum_{|m|=2} \frac{1}{m!}\psi(x)^m \int_0^1 (1-a) D^m V(x+at\psi(x))\, da + \psi(x)\cdot \int_0^1 D\varphi(x+at\psi(x))\, da. \tag{4.12}$$

For $x \in \square_{2\ell}$, using (2.17), we can immediately bound

$$|u_2(x)| \lesssim \mathsf{M}^2 \ell^{2(\mathsf{s}-\mathsf{d}+1)} + \frac{1}{\ell}\mathsf{M}^2 \ell^{-\mathsf{d}+\mathsf{s}+1} \lesssim \mathsf{M}^2(\ell^{\mathsf{s}-\mathsf{d}+(\mathsf{s}-\mathsf{d}+2)} + \ell^{\mathsf{s}-\mathsf{d}}) \lesssim \mathsf{M}^2 \ell^{\mathsf{s}-\mathsf{d}}.$$

For $x \notin \square_{2\ell}$, if $\|t\psi\|_{C^1}$ is small enough, $x + t\psi(x) \notin \operatorname{supp}\varphi$, hence $\int_0^1 D\varphi(x+at\psi(x))\, da$ vanishes. Thus, we can immediately apply the decay of $\psi(x)$ to the remaining term to control

$$|u_2(x)| \lesssim |\psi(x)|^2$$

finally giving the decay bound

$$|u_2(x)| \lesssim \mathsf{M}^2 \begin{cases} \ell^{2\mathsf{d}} \max(|x-z|,\ell)^{-4\mathsf{d}+2\mathsf{s}+2} & \text{if } x \in U \\ \frac{\ell^{2\mathsf{d}}}{|x|^{2\mathsf{s}+4}} & \text{if } x \in U^c. \end{cases} \tag{4.13}$$

Taking the dominant terms in (4.11) and (4.13), and recalling that $u = u_1 + u_2$, yields the decay estimate in (4.8).

Step 2: Derivative control on u. We will only need an L^∞ control on Du. A computation allows us to write

$$\int (\mathsf{g}(\Phi_t(x)-\Phi_t(y)) - \mathsf{g}(x-y))\, d\mu_V(y) - t\int \nabla\mathsf{g}(x-y)\cdot(\psi(x)-\psi(y))\, d\mu_V(y) \tag{4.14}$$
$$= \int \frac{1}{|x-y|^{\mathsf{s}}}\left(\mathsf{g}\left(\frac{x-y}{|x-y|} + t\frac{\psi(x)-\psi(y)}{|x-y|}\right) - 1 - t\nabla\mathsf{g}\left(\frac{x-y}{|x-y|}\right)\cdot\frac{\psi(x)-\psi(y)}{|x-y|}\right)\, d\mu_V(y);$$

the utility of this computation is that the derivatives of g are uniformly bounded in a neighborhood of $\frac{x-y}{|x-y|}$. Differentiating with respect to any x variable, we see that the derivative either falls on $\frac{1}{|x-y|^{\mathsf{s}}}$ or the function in parentheses.

Let us do the latter first. We can write the function in parentheses as

$$2t^2 \sum_{|m|=2} \frac{1}{m!}\left(\frac{\psi(x)-\psi(y)}{|x-y|}\right)^m \int_0^1 (1-a) D^m\mathsf{g}\left(\frac{x-y}{|x-y|} + at\left(\frac{\psi(x)-\psi(y)}{|x-y|}\right)\right)\, da. \tag{4.15}$$

A computation, using that the derivatives of g are uniformly bounded near $\frac{x-y}{|x-y|}$ and a mean value bound, yields that for $|x-y| \le \ell$,

$$\left| D\left(2t^2 \sum_{|m|=2} \frac{1}{m!} \left(\frac{\psi(x)-\psi(y)}{|x-y|} \right)^m \int_0^1 (1-a) D^m \mathsf{g}\left(\frac{x-y}{|x-y|} + at\left(\frac{\psi(x)-\psi(y)}{|x-y|} \right) \right) da \right) \right| \lesssim t^2 \|\psi\|_{C^1} \|\psi\|_{C^2} + t^2 \|\psi\|_{C^1}^2 + t^3 \|\psi\|_{C^1}^2 \|\psi\|_{C^2} \lesssim t^2 \|\psi\|_{C^1} \|\psi\|_{C^2},$$

where we have used $\|t\psi\|_{C^1} \lesssim 1$. For $|x-y| \ge \ell$, we do not apply a mean value control to quotients of the form $\frac{\psi(x)-\psi(y)}{x-y}$ (except for those coming from the chain rule on $D^m \mathsf{g}$) and instead bound

$$\left| D\left(2t^2 \sum_{|m|=2} \frac{1}{m!} \left(\frac{\psi(x)-\psi(y)}{|x-y|} \right)^m \int_0^1 (1-a) D^m \mathsf{g}\left(\frac{x-y}{|x-y|} + at\left(\frac{\psi(x)-\psi(y)}{|x-y|} \right) \right) da \right) \right| \lesssim t^2 \frac{\|\psi\|_{L^\infty}^2}{|x-y|^3} + t^2 \frac{\|\psi\|_{L^\infty}^2}{|x-y|^2} + t^3 \frac{\|\psi\|_{L^\infty}^2 \|\psi\|_{C^1}}{|x-y|^3} \lesssim t^2 \frac{\|\psi\|_{L^\infty}^2}{|x-y|^3}$$

where we have again used $\|t\psi\|_{C^1} \lesssim 1$ and $|x-y| \lesssim 1$. An explicit analysis of this computation in one dimension can be found in [Pei24, Appendix C]. Integrating these bounds, we find for any i,

$$\begin{aligned}
&\left| \int \frac{1}{|x-y|^{\mathsf{s}}} \partial_i \left(\mathsf{g}\left(\frac{x-y}{|x-y|} + t\frac{\psi(x)-\psi(y)}{|x-y|} \right) - 1 - t\nabla \mathsf{g}\left(\frac{x-y}{|x-y|} \right) \cdot \frac{\psi(x)-\psi(y)}{|x-y|} \right) d\mu_V(y) \right| \\
&\lesssim \int_{|x-y|\le \ell} \frac{t^2 \|\psi\|_{C^1} \|\psi\|_{C^2}}{|x-y|^{\mathsf{s}}} \, d\mu_V(y) + \int_{|x-y|\ge \ell} t^2 \frac{\|\psi\|_{L^\infty}^2}{|x-y|^{3+\mathsf{s}}} \, d\mu_V(y) \\
&\lesssim \frac{t^2 \mathsf{M}^2}{\ell^{2\mathsf{d}-2\mathsf{s}+1}} \int_0^\ell \frac{1}{r^{\mathsf{s}}} r^{\mathsf{d}-1} \, dr + \frac{t^2 \mathsf{M}^2}{\ell^{2\mathsf{d}-2\mathsf{s}-2}} \int_\ell^\infty \frac{1}{r^{3+\mathsf{s}}} r^{\mathsf{d}-1} \, dr \\
&\lesssim \frac{t^2 \mathsf{M}^2}{\ell^{\mathsf{d}-\mathsf{s}+1}}.
\end{aligned}$$

Now, returning to (4.14), we also need to deal with terms of the form

$$\int \partial_i \left(\frac{1}{|x-y|^{\mathsf{s}}} \right) \left(\mathsf{g}\left(\frac{x-y}{|x-y|} + t\frac{\psi(x)-\psi(y)}{|x-y|} \right) - 1 - t\nabla \mathsf{g}\left(\frac{x-y}{|x-y|} \right) \cdot \frac{\psi(x)-\psi(y)}{|x-y|} \right) d\mu_V(y)$$

where ∂_i indicates differentiation with respect to the ith component of the x variable. This can be dealt with via integration by parts, using that the integrand is symmetric in x and y and that $\mu_V(y)$ vanishes on $\partial\Sigma$; in the second and third lines of the following environment, ∂_i denotes differentiation in y:

$$\begin{aligned}
&\int \partial_i \left(\frac{1}{|x-y|^{\mathsf{s}}} \right) \left(\mathsf{g}\left(\frac{x-y}{|x-y|} + t\frac{\psi(x)-\psi(y)}{|x-y|} \right) - 1 - t\nabla \mathsf{g}\left(\frac{x-y}{|x-y|} \right) \cdot \frac{\psi(x)-\psi(y)}{|x-y|} \right) d\mu_V(y) \\
&= \int \left(\frac{1}{|x-y|^{\mathsf{s}}} \right) \partial_i \left(\mathsf{g}\left(\frac{x-y}{|x-y|} + t\frac{\psi(x)-\psi(y)}{|x-y|} \right) - 1 - t\nabla \mathsf{g}\left(\frac{x-y}{|x-y|} \right) \cdot \frac{\psi(x)-\psi(y)}{|x-y|} \right) d\mu_V(y) \\
&+ \int \left(\frac{1}{|x-y|^{\mathsf{s}}} \right) \left(\mathsf{g}\left(\frac{x-y}{|x-y|} + t\frac{\psi(x)-\psi(y)}{|x-y|} \right) - 1 - t\nabla \mathsf{g}\left(\frac{x-y}{|x-y|} \right) \cdot \frac{\psi(x)-\psi(y)}{|x-y|} \right) \partial_i \mu_V(y) \, dy.
\end{aligned}$$

The second line is what we have just controlled. Furthermore, since $\mu_V(y) \sim s(y) \mathrm{dist}(y, \partial\Sigma)^{1-\alpha}$ by (1.24), $\partial_i \mu_V(y) \sim s(y) \mathrm{dist}(y, \partial\Sigma)^{-\alpha}$ is integrable at the boundary. Hence, we can repeat the proof of (4.11) (which never used the explicit density μ_V, just that it was integrable at $\partial\Sigma$) and retrieve the same estimate. In particular,

(4.16)
$$\left\| D \int (\mathsf{g}(\Phi_t(x) - \Phi_t(y)) - \mathsf{g}(x-y))\, d\mu_V(y) - t \int \nabla \mathsf{g}(x-y) \cdot (\psi(x) - \psi(y))\, d\mu_V(y) \right\|_{L^\infty} \lesssim \frac{t^2 \mathsf{M}^2}{\ell^{\mathsf{d}-\mathsf{s}+1}}.$$

The remaining terms are much easier to control. These terms are

$$V_t \circ \Phi_t(x) - V(x) - t\nabla V \cdot \psi(x) - \varphi(x).$$

Differentiating the expansion (4.12) yields control

$$\begin{aligned}
&\|D\left(V_t \circ \Phi_t(x) - V(x) - t\nabla V \cdot \psi(x) - \varphi(x)\right)\|_{L^\infty} \\
&\quad \lesssim t^2 \|\psi\|_{L^\infty} \|D\psi\|_{L^\infty} + t^2 \|\psi\|_{L^\infty}^2 + t^3 \|\psi\|_{L^\infty}^2 \|D\psi\|_{L^\infty} + t^2 \|D\psi\|_{L^\infty} \|\varphi\|_{C^1} \\
&\qquad + t^2 \|\psi\|_{L^\infty} \|\varphi\|_{C^2} + t^3 \|\psi\|_{L^\infty} \|\varphi\|_{C^2} \|\psi\|_{C^1}
\end{aligned}$$

which controlling $\|t\psi\|_{C^1} \lesssim 1$ allows us to simplify, using (2.17) and (2.15), as

$$\begin{aligned}
&\|D\left(V_t \circ \Phi_t(x) - V(x) - t\nabla V \cdot \psi(x) - \varphi(x)\right)\|_{L^\infty} \\
&\quad \lesssim t^2 \|\psi\|_{L^\infty} \|D\psi\|_{L^\infty} + t^2 \|\psi\|_{L^\infty}^2 + \frac{t^2}{\ell} \|D\psi\|_{L^\infty} \mathsf{M} + \frac{t^2}{\ell^2} \|\psi\|_{L^\infty} \mathsf{M} \\
&\quad \lesssim \frac{t^2 \mathsf{M}^2}{\ell^{2\mathsf{d}-2\mathsf{s}-2}} + \frac{t^2 \mathsf{M}^2}{\ell^{\mathsf{d}-\mathsf{s}+1}} \lesssim \frac{t^2 \mathsf{M}^2}{\ell^{\mathsf{d}-\mathsf{s}+1}},
\end{aligned}$$

where we have used that $2\mathsf{d} - 2\mathsf{s} - 1 \le \mathsf{d} - \mathsf{s} + 1$. Coupling this with (4.16) yields the $\|Du\|_{L^\infty}$ bound in (4.9).

□

Notice that in the mesoscopic case, $\|t\psi\|_{C^1}$ is small for $\ell N^{\frac{1}{\mathsf{d}}} \to +\infty$, since by (2.17)

$$t\|\psi\|_{C^1} \lesssim N^{-1+\frac{\mathsf{s}}{\mathsf{d}}} \ell^{-\mathsf{s}+\mathsf{d}} = \left(N^{\frac{1}{\mathsf{d}}} \ell\right)^{-(\mathsf{d}-\mathsf{s})} \to 0.$$

Once we have these scaling estimates, we can estimate T_1.

Lemma 4.4. *Let $\Phi_t = I + t\psi$ with ψ as in Proposition 2.2 with (2.15) satisfied at order 5. Assume that $X_N \in \mathcal{G}_\ell$, with $\mathcal{G}_\ell$ as in Proposition 3.11 and Lemma 3.12, a set of configurations such that the local laws (1.29)–(1.30) hold for each D_k in (3.42) (in $\hat{\Sigma}$) and (3.26) holds. Then, for N large enough, we have*

$$(4.17) \qquad |T_1| \lesssim_\beta \beta t^2 N^{2-\frac{\mathsf{s}}{\mathsf{d}}} \mathsf{M}^2 \times \begin{cases} \ell^{\mathsf{s}} \left(\left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{(\mathsf{d}-\mathsf{s})(\mathsf{s}-2\mathsf{d})}{2(3\mathsf{d}-\mathsf{s})}} + \ell^{2\mathsf{d}-\mathsf{s}} \right) & \text{if } \ell \lesssim N^{\frac{\mathsf{s}-\mathsf{d}}{\mathsf{d}(7\mathsf{d}-3\mathsf{s})}} \\ \ell^{2\mathsf{d}} \left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{\mathsf{d}-\mathsf{s}}{2}} \ell^{3\mathsf{d}-\mathsf{s}} \right)^{\frac{-2\mathsf{s}-4}{\frac{\mathsf{d}}{2}+2\mathsf{s}+4}} & \text{otherwise.} \end{cases}$$

In either case, we can write the (suboptimal) bound

$$(4.18) \qquad |T_1| \lesssim_\beta t^2 N^{2-\frac{\mathsf{s}}{\mathsf{d}}} \mathsf{M}^2 \ell^{\mathsf{s}} (1 + (N^{\frac{1}{\mathsf{d}}}\ell)^{-\sigma})$$

for some $\sigma > 0$.

Proof. The idea is to use Proposition 3.5 on the cube $\square_R(z)$ with $R > 2\ell$ to be determined below, and estimate more easily with Lemma 3.6 outside. Notice that, for u as in Lemma 4.3,

$$\|u\|^2_{L^2(\square_R)} \lesssim \frac{\mathsf{M}^4 R^{\mathsf{d}}}{\ell^{2(\mathsf{d}-\mathsf{s})}}$$

and

$$\|\nabla u\|^2_{L^2(\square_R)} \lesssim \frac{\mathsf{M}^4 R^{\mathsf{d}}}{\ell^{2(\mathsf{d}-\mathsf{s})}\ell^2}.$$

If $\ell < 1$ and $R < \varepsilon$ (as in the definition of $\hat{\Sigma}$) then the local laws holds in $\square_R$, if not we use the global law (3.26). Using Proposition 3.5 (applied with $\eta = \ell$) coupled with the local / macroscopic law depending on the case, we obtain

$$\left|\int_{\square_R} u(x)\, d\mathrm{fluct}_{\mu_V}(x)\right|^2 \lesssim_\beta \left(\ell^{\gamma-1}\frac{\mathsf{M}^4 R^{\mathsf{d}}}{\ell^{2(\mathsf{d}-\mathsf{s})}} + \ell^{\gamma+1}\frac{\mathsf{M}^4 R^{\mathsf{d}}}{\ell^{2(\mathsf{d}-\mathsf{s})}\ell^2}\right)\min(1, R^{\mathsf{d}}) N^{1+\frac{\mathsf{s}}{\mathsf{d}}}$$

where we have also used (3.28) (or its consequence (1.30)) to show that the additive error term in (3.27) is strictly smaller. We find

$$\left|\int_{\square_R} u(x)\, d\mathrm{fluct}_{\mu_V}(x)\right| \lesssim_\beta \left(\ell^{\gamma-1+2\mathsf{s}-2\mathsf{d}}\mathsf{M}^4 \min(R^{\mathsf{d}}, R^{2\mathsf{d}}) N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\right)^{\frac12}$$
$$\lesssim_\beta \mathsf{M}^2 \ell^{\frac32(\mathsf{s}-\mathsf{d})}\min(R^{\mathsf{d}/2}, R^{\mathsf{d}}) N^{\frac12+\frac{\mathsf{s}}{2\mathsf{d}}} \tag{4.19}$$

using $\mathsf{d} - 1 + \gamma = \mathsf{s}$. On the other hand, using Lemma 4.3 coupled with the local laws on dyadic scales $\geq 2\ell$, as long as $2^k\ell < \varepsilon$, or (3.26) otherwise, and Lemma 3.6, we obtain

$$\left|\int_{\square_R^c \cap U} u(x)\, d\mathrm{fluct}_{\mu_V}(x)\right| \lesssim_\beta \sum_{k \geq \log_2 \frac{R}{\ell}} N\left(\min(1, 2^k\ell)\right)^{\mathsf{d}} \frac{\ell^{2\mathsf{d}}\mathsf{M}^2}{(2^k\ell)^{3\mathsf{d}-\mathsf{s}}}$$
$$\lesssim_\beta N\ell^{\mathsf{s}}\mathsf{M}^2 \sum_{k \geq \log_2 \frac{R}{\ell}} \min\left(\ell^{-\mathsf{d}}(2^{\mathsf{s}-3\mathsf{d}})^k, (2^{\mathsf{s}-2\mathsf{d}})^k\right) \lesssim_\beta N\ell^{\mathsf{s}}\mathsf{M}^2\left(\frac{\ell}{R}\right)^{2\mathsf{d}-\mathsf{s}}\min(1, R^{-\mathsf{d}})$$
$$\lesssim_\beta N\mathsf{M}^2\ell^{2\mathsf{d}}\min(R^{\mathsf{s}-2\mathsf{d}}, R^{\mathsf{s}-3\mathsf{d}}).$$

Note that if R exceeds a large enough constant, then $\square_R^c \cap U = \varnothing$, so the integral vanishes, and we can thus replace the result by

$$\left|\int_{\square_R^c \cap U} u(x)\, d\mathrm{fluct}_{\mu_V}(x)\right| \lesssim_\beta N\mathsf{M}^2\ell^{2\mathsf{d}}R^{\mathsf{s}-2\mathsf{d}}. \tag{4.20}$$

Finally, using a very crude bound and (4.8), we have

$$\left|\int_{Q_R^c \cap U^c} u(x)\, d\mathrm{fluct}_{\mu_V}(x)\right| \lesssim_\beta N\|u\|_{L^\infty(U^c)} \lesssim N\ell^{2\mathsf{d}}\min(1, R^{-2(\mathsf{s}+2)})\mathsf{M}^2. \tag{4.21}$$

We then choose

$$R := \ell\left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{\mathsf{d}-\mathsf{s}}{2(3\mathsf{d}-\mathsf{s})}} \tag{4.22}$$

if this is such that R is smaller than a constant, which happens when $\ell \lesssim N^{\frac{\mathsf{s}-\mathsf{d}}{\mathsf{d}(7\mathsf{d}-3\mathsf{s})}}$. Then (4.19) and (4.20) balance and $R \gg \ell$. Indeed,

$$\ell^{\frac{3}{2}(\mathsf{s}-\mathsf{d})} R^{\mathsf{d}} N^{\frac{1}{2}+\frac{\mathsf{s}}{2\mathsf{d}}} = N\ell^{2\mathsf{d}} R^{\mathsf{s}-2\mathsf{d}} \iff R^{3\mathsf{d}-\mathsf{s}} = \ell^{\frac{7}{2}\mathsf{d}-\frac{3}{2}\mathsf{s}} N^{\frac{1}{2}-\frac{\mathsf{s}}{2\mathsf{d}}} = \ell^{3\mathsf{d}-\mathsf{s}} \left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{\mathsf{d}-\mathsf{s}}{2}}.$$

We arrive at the desired result in that case.

Otherwise, we optimize the sum of (4.19) and (4.21) and take

$$R = \left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{\mathsf{d}-\mathsf{s}}{2}} \ell^{3\mathsf{d}-\mathsf{s}}\right)^{\frac{1}{\frac{\mathsf{d}}{2}+2\mathsf{s}+4}}; \tag{4.23}$$

notice that this is $\gtrsim 1$ precisely when $\ell \gtrsim N^{\frac{\mathsf{s}-\mathsf{d}}{\mathsf{d}(7\mathsf{d}-3\mathsf{s})}}$, which is the regime under consideration. Then, for N large enough we have $\square_R^c \cap U = \varnothing$. In that case we also obtain the result by substituting (4.23) into (4.19) and (4.21).

□

4.3. **Proof of Theorem 2.** As in the Coulomb case, once we have control of the T_0 term via explicit controls of the transport, of the T_1 term via bounds on fluctuations, and of the T_2 term from the commutator estimate, we have a first bound on the fluctuations, which is Theorem 2. We take $\mathcal{G}_\ell$ as in Proposition 3.11 and Lemma 3.12, a set of configurations such that the local laws (1.29)–(1.30) hold for each D_k in (3.42) (in $\hat{\Sigma}$) and (3.26) holds, which we can assume satisfies $\mathbb{P}_{N,\beta}(\mathcal{G}_\ell^c) \le C_1 e^{-C_2\beta\ell^{\mathsf{d}}N}$, up to adjusting the definitions of C_1 and C_2.

Combining (2.2), (4.6), (4.18) and (3.72), we are led to

$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(e^{-\beta t N^{1-\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{Fluct}_{\mu_V}(\varphi)}\mathbf{1}_{\mathcal{G}_\ell}\right) - \frac{\beta N^{2-\frac{\mathsf{s}}{\mathsf{d}}} t^2}{2}\frac{c_{\mathsf{d},\frac{\mathsf{d}-\mathsf{s}}{2}}}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left\|\varphi^\Sigma\right\|^2_{\dot{H}^{\frac{\mathsf{d}-\mathsf{s}}{2}}} - tNM(\varphi)\right|$$
$$\lesssim_\beta \left(t^3 N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\beta \mathsf{M}^3\ell^{2\mathsf{s}-\mathsf{d}} + t^2(1+\beta)\mathsf{M}^2 N\ell^{2\mathsf{s}-\mathsf{d}}\right) + \beta t^2 N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\mathsf{M}^2\ell^{\mathsf{s}}(1+(N^{\frac{1}{\mathsf{d}}}\ell)^{-\sigma}) + \beta\mathsf{M}|t|N\ell^{\mathsf{s}},$$

for some $\sigma > 0$, with $M(\varphi)$ is as in (4.2). We then let $t = -\frac{1}{1+\beta}N^{-1+\frac{\mathsf{s}}{\mathsf{d}}}\tau$, and note that the condition $t\ell^{\mathsf{s}-\mathsf{d}}\mathsf{M}$ small enough that was needed for our proofs amounts to $\tau(\ell N^{\frac{1}{\mathsf{d}}})^{\mathsf{s}-\mathsf{d}}\mathsf{M}$ small enough. In view of the definition (4.2) and Lemma 8.6, we obtain the result under this assumption.

4.4. **Hölder trick for the CLT.** We can now turn to the proof of the Central Limit Theorem for fluctuations. We now assume that $\varphi = \varphi_0(\frac{\cdot - z}{\ell})$, which implies (2.15) with $\mathsf{M} = \|\varphi_0\|_{C^{k+1}}$. The goal of this section is to improve the estimate on T_2, assuming that we have an expansion of the relative free energy with a good enough rate. The method of proof, introduced in [LS18] consists in comparing this expansion of the difference of free energies with the relative expansion obtained by transport in (3.32).

We next assume that we have an expansion of the form (1.44), that is

$$\log \mathsf{K}_{N,\beta}^{\mathcal{G}_\ell}(\mu_t, \zeta_V\circ\Phi_t^{-1}) - \log \mathsf{K}_{N,\beta}(\mu_0,\zeta_V) + N(\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_0))$$
$$= N\left(\mathcal{Z}(\beta,\mu_t) - \mathcal{Z}(\beta,\mu_0)\right) + O((1+\beta)N\ell^{\mathsf{d}}\mathcal{R}_t) \tag{4.24}$$

Here, $\mathcal{Z}$ is as in (1.43), $\mathsf{K}_{N,\beta}^{\mathcal{G}}$ is as in (3.4), $\mathcal{R}_t$ is the error rate, and $f_{\mathsf{d},\mathsf{s}}$ is the pressure for the unit density system defined in Lemma 7.2. Indeed, this is precisely the expansion that we will prove in Proposition 7.5 leveraging the local laws of Theorem 1 down to microscopic scales. We will analyze when the rate $\mathcal{R}_t$ we obtain is sufficient at the end of this section.

Let us record some information about the function $\mathcal{Z}$.

Lemma 4.5. *Denote $\Phi_t = I + t\psi$, $\mu_t := \Phi_t \# \mu_V$ where ψ is the transport map constructed in Proposition 2.2, with $\varphi_0 \in C^5$. Assume that the function $y \mapsto f_{\mathsf{d},\mathsf{s}}(y)$ is p times differentiable and satisfies (1.45). Letting then $B_k(\beta, \mu, \psi)$ be the k-th derivative at $t = 0$ of the function $\phi(t) := \mathcal{Z}(\beta, \mu_t)$, if $|t|\ell^{\mathsf{s}-\mathsf{d}}\|\varphi_0\|_{C^5}$ is small enough, we have*

$$\mathcal{Z}(\beta,\mu_t) - \mathcal{Z}(\beta,\mu_0) = \sum_{k=1}^{p-1} \frac{t^k}{k!} B_k(\beta,\mu_0,\psi) + O\left(t^p \beta \|\varphi_0\|_{C^4}^p \ell^{(1-p)\mathsf{d}+p\mathsf{s}}\right). \tag{4.25}$$

and

$$|B_k(\beta,\mu_0,\psi)| \lesssim_\beta \beta \|\varphi_0\|_{C^4}^k \ell^{(1-k)\mathsf{d}+k\mathsf{s}}. \tag{4.26}$$

We also have the explicit expression

$$\begin{aligned} B_1(\beta,\mu_V,\psi) = &-\frac{\beta}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}(1+\frac{\mathsf{s}}{\mathsf{d}}) \int (-\Delta)^\alpha \varphi^\Sigma f_{\mathsf{d},\mathsf{s}}(\beta \mu_V^{\frac{\mathsf{s}}{\mathsf{d}}}) \mu_V^{\frac{\mathsf{s}}{\mathsf{d}}} - \frac{\beta}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \frac{\mathsf{s}}{\mathsf{d}} \int f'_{\mathsf{d},\mathsf{s}}(\beta \mu_V^{\frac{\mathsf{s}}{\mathsf{d}}}) \mu_V^{2\frac{\mathsf{s}}{\mathsf{d}}} (-\Delta)^\alpha \varphi^\Sigma \\ &+ \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \frac{\beta}{2\mathsf{d}} \mathbf{1}_{\mathsf{s}=0} \int (-\Delta)^\alpha \varphi^\Sigma \log \mu_V. \end{aligned} \tag{4.27}$$

Proof. We may write

$$\int \beta \mu_t^{1+\frac{\mathsf{s}}{\mathsf{d}}} f_{\mathsf{d},\mathsf{s}}(\beta \mu_t^{\frac{\mathsf{s}}{\mathsf{d}}}) = \int \beta \mu_t^{\frac{\mathsf{s}}{\mathsf{d}}} f_{\mathsf{d},\mathsf{s}}(\beta \mu_t^{\frac{\mathsf{s}}{\mathsf{d}}}) \Phi_t \# \mu_0 = \int \beta \left(\mu_t \circ \Phi_t\right)^{\frac{\mathsf{s}}{\mathsf{d}}} f_{\mathsf{d},\mathsf{s}} \left(\beta (\mu_t \circ \Phi_t)^{\frac{\mathsf{s}}{\mathsf{d}}}\right) d\mu_0.$$

Next we recall that by definition of the push forward we have

$$\mu_t \circ \Phi_t = \frac{\mu_V}{\det(I + tD\psi)}, \tag{4.28}$$

hence if $t|D\psi| < \frac{1}{2}$, which in view of (2.17) is implied by $|t|\ell^{\mathsf{s}-\mathsf{d}}\|\varphi_0\|_{C^5}$ small enough, we may bound

$$\left| \frac{d^j}{dt^j} \mu_t \circ \Phi_t \right| \le |D\psi|^j. \tag{4.29}$$

We also set $g(x) = \beta x^{\frac{\mathsf{s}}{\mathsf{d}}} f_{\mathsf{d},\mathsf{s}}(\beta x^{\frac{\mathsf{s}}{\mathsf{d}}})$ and check that by the assumption (1.45), we have $|g^{(n)}(y)| \le C\beta$ for $n \le p$ when y takes values $\beta \mu_V(x)^{\frac{\mathsf{s}}{\mathsf{d}}}$. Using the Faa di Bruno formula, we now have

$$\frac{d^k}{dt^k} g(\mu_t \circ \Phi_t) = \sum_{j_1 + 2j_2 + \cdots + kj_k = k} C_{\mathbf{j}} g^{(j_1 + \cdots + j_k)}(\mu_t \circ \Phi_t) \prod_{l=1}^k \left(\frac{d^l}{dt^l} \mu_t \circ \Phi_t \right)^{j_l}$$

where $C_{\mathbf{j}}$ is some combinatorial factor, and inserting the above estimates we deduce that

$$\left| \frac{d^k}{dt^k} g(\mu_t \circ \Phi_t) \right| \lesssim \beta |D\psi|^k,$$

with a uniform constant over $\beta \ge 1$. In the same way

$$\int \mu_t \log \mu_t = \int \log(\mu_t \circ \Phi_t) d\mu_V$$

and the derivatives of $\log(\mu_t \circ \Phi_t)$ are bounded by $C|D\psi|^k$. Integrating against $d\mu_V$ on the support of ψ, and using (2.48), we deduce that

$$|\phi^{(k)}(t)| \le C\beta \int_\Sigma |D\psi|^k \lesssim \beta \|\varphi_0\|_{C^5}^k \ell^{(1-k)\mathsf{d}+k\mathsf{s}}, \tag{4.30}$$

with a constant that is uniform for $\beta \geq 1$. The result (4.26) follows by Taylor expansion. For (4.27), a direct calculation using that $\partial_t \mu_t|_{t=0} = -\mathrm{div}\,(\psi\mu_0)$ yields

$$B_1(\beta,\mu_V,\psi) = \beta(1+\frac{\mathsf{s}}{\mathsf{d}})\int \mathrm{div}\,(\psi\mu_V) f_{\mathsf{d},\mathsf{s}}(\beta\mu_V^{\frac{\mathsf{s}}{\mathsf{d}}})\mu_V^{\frac{\mathsf{s}}{\mathsf{d}}} + \beta\frac{\mathsf{s}}{\mathsf{d}}\int f'_{\mathsf{d},\mathsf{s}}(\beta\mu_V^{\frac{\mathsf{s}}{\mathsf{d}}})\mu_V^{2\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{div}\,(\psi\mu_V) - \frac{\beta}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\int \mathrm{div}\,(\psi\mu_V)\log\mu_V.$$

Inserting (2.12) and (2.16), we obtain the result. □

We can now obtain our main result on the expansion of partition functions relevant to T_2.

Proposition 4.6. *Assume the same hypotheses as in the previous lemma. Let $\mu_t = (I + t\psi)\#\mu_V$ as in the previous lemma. Let $\mathcal{G}_\ell$ be as in Proposition 3.11 and Lemma 3.12, a set of configurations such that the local laws (1.29)–(1.30) hold for each $\square_{2^k\ell}$ with $k \geq 1$ (in $\hat\Sigma$) and (3.26) holds. Assume we know that for each $r \leq \ell^{\mathsf{d}-\mathsf{s}}$, we have*

(4.31)
$$\log\frac{\mathsf{K}^{\mathcal{G}_\ell}_{N,\beta}(\mu_r,\zeta_V\circ\Phi_r^{-1})}{\mathsf{K}_{N,\beta}(\mu)} + N(\mathrm{Ent}(\mu_r)-\mathrm{Ent}(\mu_0)) = N\left(\mathcal{Z}(\beta,\mu_r)-\mathcal{Z}(\beta,\mu_0)\right) + O((\beta+1)N\ell^{\mathsf{d}}\mathcal{R}_r)$$

with $\max_{|r|\leq \ell^{\mathsf{d}-\mathsf{s}}}\mathcal{R}_r \leq C$ and $\mathcal{R}_r$ continuous in r for $|r| \leq \ell^{\mathsf{d}-\mathsf{s}}$. For any integer $p \geq 1$ such that $\varphi_0 \in C^{2p+3}$, for every t such that $|t|\ell^{\mathsf{d}-\mathsf{s}}\|\varphi_0\|_{C^{2p+3}}\left(\max_{|r|\leq\ell^{\mathsf{d}-\mathsf{s}}}\mathcal{R}_r\right)^{-1/p}$ is smaller than a small enough constant (depending only on $\mathsf{d},\mathsf{s},\mu_V,p$), a being the largest number $\leq \ell^{\mathsf{d}-\mathsf{s}}$ such that

$$a = \frac{c}{\|\varphi_0\|_{C^{2p+3}}}\left(\max_{[-a,a]}\mathcal{R}_r\right)^{\frac{1}{p}}\ell^{\mathsf{d}-\mathsf{s}}, \tag{4.32}$$

for some $c > 0$ small enough, we have

$$\left|\log\mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\Big(\sum_{k=1}^p\gamma_k t^k\Big)\mathbf{1}_{\mathcal{G}_\ell}\right)\right| \leq C|t|(\beta+1)N\ell^{\mathsf{s}}\|\varphi_0\|_{C^{2p+3}}\left(\max_{|r|\leq a}\mathcal{R}_r\right)^{1-\frac{1}{p}} \tag{4.33}$$

where $\gamma_k = \beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\mathsf{A}_k(X_N,\mu_0,\psi) - \frac{N}{k!}B_k(\beta,\mu_0,\psi)$. Moreoever,

$$\left|\log\mathbb{E}_{\mathbb{P}_{N,\beta}}\left(e^{T_2}\mathbf{1}_{\mathcal{G}_\ell}\right) - N\sum_{k=1}^{p-1}\frac{t^k}{k!}B_k(\beta,\mu_V,\psi)\right| \lesssim_\beta t(\beta+1)N\ell^{\mathsf{s}}\Big(\max_{|r|\leq a}\mathcal{R}_r\Big)^{1-\frac{1}{p}} + t^p\beta\|\varphi_0\|^p_{C^{2p+3}}N\ell^{(1-p)\mathsf{d}+p\mathsf{s}}, \tag{4.34}$$

where the bound depends on p and the above bounds.

Proof. Expanding the next-order energy to order p via (3.36), we have

$$\mathsf{F}_N(\Phi_t(X_N),\Phi_t\#\mu_0) - \mathsf{F}_N(X_N,\mu_0) = \sum_{k=1}^{p-1}\frac{t^k}{k!}\mathsf{A}_k(X_N,\mu_0,\psi) + \frac{1}{p!}\int_0^t (t-s)^{p-1}\mathsf{A}_p(\Phi_s(X_N),\mu_s,\psi\circ\Phi_s^{-1})ds. \tag{4.35}$$

In the same way

$$\log\det D\Phi_t = \sum_{k=1}^{p-1} t^k c_k(\psi) + O(t^p |D\psi|^p), \tag{4.36}$$

for certain expressions $c_k(\psi)$. Inserting into (3.32) and using (3.68) to bound A_p, we obtain

$$\begin{aligned} e^{T_2}\mathbf{1}_{\mathcal{G}_\ell} = \exp\Bigg(&-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\sum_{k=1}^{p-1} t^k \mathsf{A}_k(X_N,\mu_0,\psi) + O_\beta\left(|t|^p \|\varphi_0\|^p_{C^{2p+3}}\ell^{(1-p)\mathsf{d}+p\mathsf{s}}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\right)\right) \\ &+\sum_{k=1}^{p-1} t^k \mathrm{Fluct}_{\mu_V}(c_k(\psi)) + O\left(\mathrm{Fluct}_{\mu_V}\left(t^p|D\psi|^p\right)\right)\Bigg)\mathbf{1}_{\mathcal{G}_\ell}. \end{aligned} \tag{4.37}$$

Let us first bound $\mathrm{Fluct}_{\mu_V}(|D\psi|^p)$. For that we rewrite ψ as $\sum_k(\chi_k\psi)$ where χ_k is a partition of unity relative to the A_k's as in the proof of Lemma 3.12. We use the rough bound of Lemma 3.6 (and a rough bound by $N\|\psi\|_{L^\infty}$ in A_{k_*+2}), and obtain using (2.17) that

$$\mathrm{Fluct}_{\mu_V}\left(|D\psi|^p\right)\mathbf{1}_{\mathcal{G}_\ell} \lesssim \sum_{k=0}^{\infty}\mathrm{Fluct}_{\mu_V}\left(|D(\chi_k\psi)|^p\right)\mathbf{1}_{\mathcal{G}_\ell} \lesssim N\|\varphi_0\|^p_{C^5}\left(\sum_{k=0}^{\infty}\left(\frac{\ell^{\mathsf{d}}}{(2^k\ell)^{2\mathsf{d}-\mathsf{s}}}\right)^p (2^k\ell)^{\mathsf{d}} + \ell^{\mathsf{d}}\right) \tag{4.38}$$

$$\lesssim \|\varphi_0\|^p_{C^5} N\left(\ell^{\mathsf{d}(1-p)+\mathsf{s}p} + \ell^{\mathsf{d}}\right) \lesssim \|\varphi_0\|^p_{C^5} N \ell^{\mathsf{d}(1-p)+\mathsf{s}p}. \tag{4.39}$$

Inserting this into (4.37) and inserting (4.37) into (3.32), we obtain an expansion that we may equate with the expansion (4.31) and (4.25). Setting

$$\gamma_k = -\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\mathsf{A}_k(X_N,\mu_0,\psi) + \mathrm{Fluct}(c_k(\psi)) - \frac{N}{k!}B_k(\beta,\mu_0,\psi) \tag{4.40}$$

this yields that

$$\log\mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\Big(\sum_{k=1}^{p-1} t^k\gamma_k\Big)\mathbf{1}_{\mathcal{G}_\ell}\right) = O_\beta\left(|t|^p(1+\beta)N\|\varphi_0\|^p_{C^{2p+3}}\ell^{(1-p)\mathsf{d}+p\mathsf{s}}\right) + O\left((1+\beta)N\ell^{\mathsf{d}}(\mathcal{R}_t+\mathcal{R}_0)\right).$$

We next wish to choose $a \le \ell^{\mathsf{d}-\mathsf{s}}$ such that

$$a^p\|\varphi_0\|^p_{C^{2p+3}}\ell^{(1-p)\mathsf{d}+p\mathsf{s}} \le C\ell^{\mathsf{d}}(\mathcal{R}_0+\mathcal{R}_a).$$

For that we choose

$$a = \sup\left\{ b\le \ell^{\mathsf{d}-\mathsf{s}},\ b \le \frac{c}{\|\varphi_0\|_{C^{2p+3}}}\left(\max_{r\in[-b,b]}\mathcal{R}_r\right)^{\frac1p}\ell^{\mathsf{d}-\mathsf{s}}\right\}.$$

We note that $a < \ell^{\mathsf{d}-\mathsf{s}}$ if $\max_{r\in[-\ell^{\mathsf{d}-\mathsf{s}},\ell^{\mathsf{d}-\mathsf{s}}]}\mathcal{R}_r$ is bounded and $c>0$ is chosen small enough. Thus by continuity, we must have

$$a = \frac{c}{\|\varphi_0\|_{C^{2p+3}}}\left(\max_{[-a,a]}\mathcal{R}_r\right)^{\frac1p}\ell^{\mathsf{d}-\mathsf{s}}. \tag{4.41}$$

With this choice we then have

$$\log\mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\Big(\sum_{k=1}^{p-1} a^k\gamma_k\Big)\mathbf{1}_{\mathcal{G}_\ell}\right) = O_\beta\left((\beta+1)N\ell^{\mathsf{d}}\max_{|r|\le a}\mathcal{R}_r\right). \tag{4.42}$$

We note by Hölder's inequality we have $\mathbb{E}(e^L)\mathbb{E}(e^{-L}) \geq 1$ thus we can transform (4.43) into

$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\Big(\pm\sum_{k=1}^{p-1} a^k\gamma_k\Big)\mathbf{1}_{\mathcal{G}_\ell}\right)\right| = O_\beta\left((\beta+1)N\ell^{\mathsf{d}}\max_{|r|\leq a}\mathcal{R}_r\right). \tag{4.43}$$

Let now $X_1,\dots X_p$ be p equally-spaced points in $[1/2,1]$, and let P_i's be the Lagrange interpolation polynomials of degree $p-1$ associated to $X_1,\dots,X_p$, i.e. such that $P_i(X_j) = \delta_{ij}$. We may expand each P_i as $\sum_{n=0}^{p-1} c_{i,n}X^n$, where the coefficients $c_{i,n}$ depend only on p. Expressing the polynomial $\sum_{n=1}^{p-1}\gamma_n a^n X^n$ along the P_i's we obtain that

$$\sum_{n=1}^{p-1}\gamma_n a^n X^n = \sum_{i=1}^{p}(\sum_{k=1}^{p-1}\gamma_k a^k X_i^k)P_i = \sum_{n=0}^{p-1}\sum_{i=1}^{p}(\sum_{k=1}^{p-1}\gamma_k a^k X_i^k)c_{i,n}X^n.$$

Equating the coefficients, it follows that for $1\leq n\leq p-1$,

$$\gamma_n a^n = \sum_{i=1}^{p} c_{i,n}\left(\sum_{k=1}^{p}\gamma_k a^k X_i^k\right). \tag{4.44}$$

Note that (4.43) is also true with a replaced by aX_i (for $X_i\in[\frac12,1]$). Choosing C a constant large enough (depending only on p) and using the generalized Hölder's inequality, we may write that for each $1\leq n\leq p-1$, in view of (4.43),

$$\begin{aligned}\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\frac{\gamma_n a^n}{C}\right)\mathbf{1}_{\mathcal{G}_\ell}\right) &\leq \sum_{i=0}^{p}|c_{i,n}|\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\sum_{k=1}^{p-1}\gamma_k a^k X_i^k\right)\mathbf{1}_{\mathcal{G}_\ell}\right)\right| \\ &\lesssim_\beta (\beta+1)N\ell^{\mathsf{d}}\max_{|r|\leq a}\mathcal{R}_r,\end{aligned} \tag{4.45}$$

where C depends on p. The same result holds starting from the opposite of (4.44), that is we can also obtain

$$\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(-\frac{\gamma_n a^n}{C}\right)\mathbf{1}_{\mathcal{G}_\ell}\right) \lesssim_\beta (\beta+1)N\ell^{\mathsf{d}}\max_{|r|\leq a}\mathcal{R}_r. \tag{4.46}$$

Using again $\mathbb{E}(e^L)\mathbb{E}(e^{-L})\geq 1$, we obtain a two-sided bound

$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\pm\frac{\gamma_n a^n}{C}\right)\mathbf{1}_{\mathcal{G}_\ell}\right)\right| \lesssim_\beta (\beta+1)N\ell^{\mathsf{d}}\max_{|r|\leq a}\mathcal{R}_r, \tag{4.47}$$

Using Hölder's inequality again we deduce that if $|t|/a$ is small enough (in particular <1),

$$\begin{aligned}\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\sum_{k=1}^{p-1}\gamma_k t^k\right)\mathbf{1}_{\mathcal{G}_\ell}\right) &\leq \sum_{k=1}^{p-1}\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\mathrm{sgn}(t^k)\frac{\gamma_k a^k}{C}\mathbf{1}_{\mathcal{G}_\ell}\right)\right)\right|^{C\frac{|t|^k}{a^k}} \\ &\leq C\frac{|t|}{a}\sum_{k=1}^{p-1}\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\mathrm{sgn}(t^k)\frac{\gamma_k a^k}{C}\mathbf{1}_{\mathcal{G}_\ell}\right)\right)\right| \lesssim_\beta \frac{|t|}{a}(\beta+1)N\ell^{\mathsf{d}}\max_{|r|\leq a}\mathcal{R}_r.\end{aligned} \tag{4.48}$$

Inserting (4.40) and (4.48) into (4.37) and using the definition of a, we obtain (4.33). Inserting (4.33) into (4.37), we obtain (4.34) after another application of Hölder's inequality. $\square$

4.5. **Proof of Theorem 3.** The Hölder trick of the previous subsection gives us improved control on T_2, which will allow us to obtain the CLT.

Lemma 4.7. *Under the same assumptions, suppose there is some $p \geq 2$ such that*

$$\left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{\mathsf{s}}{2}} \left(\max_{|r|\leq a} \mathcal{R}_r\right)^{1-\frac{1}{p}} \to 0 \quad \text{as } N \to \infty. \tag{4.49}$$

If $t = -\frac{\tau\sqrt{2}}{\sqrt{\beta}} N^{-1+\frac{\mathsf{s}}{2\mathsf{d}}} \ell^{-\frac{\mathsf{s}}{2}}$ where τ is fixed, then, as $N \to \infty$, we have

$$\log \mathbb{E}_{\mathbb{P}_{N,\beta}} \left(e^{T_2} \mathbf{1}_{\mathcal{G}_\ell}\right) = -\frac{\tau\sqrt{2}}{\sqrt{\beta}} N^{-1+\frac{\mathsf{s}}{2\mathsf{d}}} \ell^{-\frac{\mathsf{s}}{2}} N B_1(\beta, \mu_V, \psi) \mathbf{1}_{\mathsf{s}\geq 0} + o(1), \tag{4.50}$$

where $o(1)$ may depend on β.

If $t = -\frac{\tau\sqrt{2}}{\beta} N^{\frac{\mathsf{s}}{2\mathsf{d}}-1} \ell^{-\frac{\mathsf{s}}{2}}$ where τ is fixed, then, as $N \to \infty$, we have

$$\log \mathbb{E}_{\mathbb{P}_{N,\beta}} \left(e^{T_2} \mathbf{1}_{\mathcal{G}_\ell}\right) = -\frac{\tau\sqrt{2}}{\beta} N^{\frac{\mathsf{s}}{2\mathsf{d}}-1} \ell^{-\frac{\mathsf{s}}{2}} N B_1(\beta, \mu_V, \psi) \mathbf{1}_{\mathsf{s}\geq 0} + o(1), \tag{4.51}$$

where $o(1)$ is uniform as $\beta \to \infty$.

Moreover, (4.49) holds for p large enough if

$$\ell \ll N^{-\frac{\mathsf{s}}{\mathsf{d}(\mathsf{s}+2)}} \tag{4.52}$$

and $\mathsf{s} < \mathsf{s}_0$, where s_0 is approximately

$$\begin{cases} 0.03973 & \text{in } \mathsf{d} = 1, \\ 0.06059 & \text{in } \mathsf{d} = 2. \end{cases} \tag{4.53}$$

Notice that (4.52) holds automatically in the case $\mathsf{s} < 0$ and in the case $\mathsf{s} = 0$ as soon as $\ell = o(1)$.

Proof. The first item is an immediate consequence of (4.34): with the choice of t, if (4.49) holds we have

$$\left| t N \ell^{\mathsf{s}} \left(\max_{|r|\leq a} \mathcal{R}_r\right)^{1-\frac{1}{p}} \right| \lesssim \left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{\mathsf{s}}{2}} \left(\max_{|r|\leq a} \mathcal{R}_r\right)^{1-\frac{1}{p}} \to 0.$$

Moreover,

$$\left| t^p N \ell^{(1-p)\mathsf{d}+p\mathsf{s}} \right| \lesssim N^{-p+\frac{p\mathsf{s}}{2\mathsf{d}}} \ell^{-\frac{\mathsf{s}p}{2}} N \ell^{(1-p)\mathsf{d}+p\mathsf{s}} = \left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{p}{2}(\mathsf{s}-\mathsf{d})+(1-\frac{p}{2})\mathsf{d}} \to 0$$

since $\mathsf{s} < \mathsf{d}$ and $p \geq 2$, establishing the desired $o(1)$ limiting behavior for $\log \mathbb{E}_{\mathbb{P}_{N,\beta}} \left(e^{T_2} \mathbf{1}_{\mathcal{G}}\right) - \sum_{k=1}^{p-1} t^k \frac{N}{k!} B_k(\beta, \mu_0, \psi)$. Finally, using (4.26) and (2.17) we obtain

$$\left| t^k \frac{N}{k!} B_k(\beta, \mu_0, \psi) \right| \lesssim N N^{\frac{\mathsf{s}k}{2\mathsf{d}}-k} \ell^{-\frac{\mathsf{s}k}{2}} \ell^{(1-k)\mathsf{d}+\mathsf{s}k} = \left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{(1-k)\mathsf{d}+\frac{\mathsf{s}k}{2}}.$$

For any $k \geq 2$, the exponent is negative as $\mathsf{d} - \mathsf{d}k + \frac{\mathsf{s}k}{2} \leq (k-1)(\mathsf{s}-\mathsf{d}) < 0$. For $k = 1$, the exponent is only negative for $\mathsf{s} < 0$. In the $\mathsf{s} \geq 0$ regime, we possibly have some limit $\lim_{N\to\infty} (N^{-\frac{1}{\mathsf{d}}}\ell)^{-\frac{\mathsf{s}}{2}} B_1(\beta, \mu_V, \psi)$. This establishes (4.50). The proof of (4.51) is analogous and left to the reader.

We now examine the condition under which (4.49) holds. From Proposition 7.5, for all $|t|\ell^{\mathsf{s}-\mathsf{d}} \leq c$ small enough, we have

$$\mathcal{R}_t \le \max\left((|t|\ell^{\mathsf{s}-\mathsf{d}})^{\frac{\mathsf{d}}{2\mathsf{d}-\mathsf{s}}} \left(\mathscr{E}\left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}}\right)\right)^{\frac{\mathsf{d}-\mathsf{s}}{2\mathsf{d}-\mathsf{s}}}, |t|\varepsilon^{\mathsf{s}-\mathsf{d}} \right)$$

with

$$\kappa := \begin{cases} 1 - \frac{4}{4+\mathsf{d}-\mathsf{s}_+} & \text{if } \mathsf{d} \le 5 \text{ or } \mathsf{s} < \mathsf{d}-1 \\ 1 - \frac{\mathsf{d}-1}{(\mathsf{d}-1)+(\mathsf{d}-\mathsf{s})} & \text{otherwise} \end{cases}$$

and

$$\mathscr{E}(R) := R^{-\kappa}\log R$$

The definition (4.32) together with $a \le \ell^{\mathsf{d}-\mathsf{s}}$ thus implies that

$$\max_{|r|\le a} \mathcal{R}_r \le \max\left(\left(\mathscr{E}\left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}}\right)\right)^{\frac{\mathsf{d}-\mathsf{s}}{2\mathsf{d}-\mathsf{s}}}, a\varepsilon^{\mathsf{s}-\mathsf{d}} \right). \tag{4.54}$$

Moreover, abbreviating $\mathscr{E}\left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}}\right)$ into $\mathscr{E}$, in view of (4.32), we have

$$a \lesssim \max(\mathscr{E}^{\frac{\mathsf{d}-\mathsf{s}}{p(2\mathsf{d}-\mathsf{s})}}, (a\varepsilon^{\mathsf{s}-\mathsf{d}})^{\frac{1}{p}})\ell^{\mathsf{d}-\mathsf{s}}, \tag{4.55}$$

which together with (4.54) yields

$$\max_{|r|\le a} \mathcal{R}_r \lesssim \max\left(\left(\mathscr{E}\left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}}\right)\right)^{\frac{\mathsf{d}-\mathsf{s}}{2\mathsf{d}-\mathsf{s}}}, (\frac{\ell}{\varepsilon})^{\frac{p}{p-1}}, \left(\mathscr{E}((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}}\right)^{\frac{\mathsf{d}-\mathsf{s}}{p(2\mathsf{d}-\mathsf{s})}} (\frac{\ell}{\varepsilon})^{\mathsf{d}-\mathsf{s}} \right). \tag{4.56}$$

Taking for instance $p = 2$, in order to guarantee (4.49), it is thus sufficient to guarantee that

$$\frac{\mathsf{s}}{2} - \frac{1}{4}\frac{\kappa}{\kappa+1}\left(\frac{\mathsf{d}-\mathsf{s}}{2\mathsf{d}-\mathsf{s}}\right) < 0, \qquad \left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{\mathsf{s}}{2}}\ell = o(1) \text{ for some } p \ge 1. \tag{4.57}$$

This condition will only be possibly satisfied for small enough s, which requires $\mathsf{d} = 1, 2$. The second condition is satisfied as soon as $\ell \ll N^{-\frac{\mathsf{s}}{\mathsf{d}(\mathsf{s}+2)}}$, which yields (4.52). For the first condition in (4.57), we examine each dimension separately. In $\mathsf{d} = 2$ we only have $\mathsf{s} > 0$, and the condition becomes

$$\frac{\mathsf{s}}{2} - \frac{1}{4}\left(\frac{2-\mathsf{s}}{8-2\mathsf{s}}\right)\left(\frac{2-\mathsf{s}}{4-\mathsf{s}}\right) < 0 \iff 4\mathsf{s}^3 - 33\mathsf{s}^2 + 68\mathsf{s} - 4 < 0,$$

which is true for all $\mathsf{s} < \mathsf{s}_0 :\approx 0.06059$. In $\mathsf{d} = 1$ we always have (4.49) for $\mathsf{s} < 0$, so we only need to look at $\mathsf{s} \ge 0$; there, we need to check

$$\frac{\mathsf{s}}{2} - \frac{1}{4}\left(\frac{1-\mathsf{s}}{6-2\mathsf{s}}\right)\left(\frac{1-\mathsf{s}}{2-\mathsf{s}}\right) < 0 \iff 4\mathsf{s}^3 - 21\mathsf{s}^2 + 26\mathsf{s} - 1 < 0,$$

which is true for all $\mathsf{s} < \mathsf{s}_0 :\approx 0.03973$. □

We can now prove Theorem 3.

Proof of Theorem 3. Letting $t = -\frac{\tau\sqrt{2}}{\sqrt{\beta}}N^{\frac{\mathsf{s}}{2\mathsf{d}}-1}\ell^{-\frac{\mathsf{s}}{2}}$ and assembling the results of Lemma 2.1, (4.6), Lemma 4.4 and (4.50), under the assumptions (1.45) and (4.49) we obtain

$$\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\tau\sqrt{2\beta}(N^{\frac{1}{\mathsf{d}}}\ell)^{-\frac{\mathsf{s}}{2}}\mathrm{Fluct}_{\mu_V}(\varphi)\right)\mathbf{1}_{\mathcal{G}_\ell}\right) =$$

$$-\tau\frac{\sqrt{2}}{\sqrt{\beta}}\left(\ell N^{-\frac{1}{\mathsf{d}}}\right)^{-\frac{\mathsf{s}}{2}}\left(\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_\Sigma (-\Delta)^\alpha\varphi^\Sigma(\log\mu_V) + B_1(\beta,\mu_V,\psi)\right) + \frac{\tau^2}{2}\ell^{-\mathsf{s}}\frac{c_{\mathsf{d},\frac{\mathsf{d}-\mathsf{s}}{2}}}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left\|\varphi^\Sigma\right\|^2_{\dot{H}^{\frac{\mathsf{d}-\mathsf{s}}{2}}} + o_N(1).$$

Furthermore, since

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\tau\sqrt{2\beta}(N^{\frac{1}{\mathsf{d}}}\ell)^{-\frac{\mathsf{s}}{2}}\mathrm{Fluct}_{\mu_V}(\varphi)\right)\mathbf{1}_{\mathcal{G}_\ell}\right)=\mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\tau\sqrt{2\beta}(N^{\frac{1}{\mathsf{d}}}\ell)^{-\frac{\mathsf{s}}{2}}\mathrm{Fluct}_{\mu_V}(\varphi)\mathbf{1}_{\mathcal{G}_\ell}\right)\right)+\mathbb{P}_{N,\beta}(\mathcal{G}_\ell^c)$$

and $\mathbb{P}_{N,\beta}(\mathcal{G}_\ell^c)\sim C_1e^{-C_2\beta\ell^{\mathsf{d}}N}\to 0$, we have obtained that the Laplace transform of

$$\sqrt{2\beta}(N^{\frac{1}{\mathsf{d}}}\ell)^{-\frac{\mathsf{s}}{2}}\mathrm{Fluct}_{\mu_V}(\varphi)\mathbf{1}_{\mathcal{G}_\ell}$$

converges in distribution to that of a Gaussian. Inserting into the above (8.19), we thus have that

$$\sqrt{2\beta}\frac{\mathrm{Fluct}_{\mu_V}(\varphi)\mathbf{1}_{\mathcal{G}_\ell}}{\left(\ell N^{\frac{1}{\mathsf{d}}}\right)^{\frac{\mathsf{s}}{2}}}+\frac{\sqrt{2}}{\sqrt{\beta}}\left(\ell N^{-\frac{1}{\mathsf{d}}}\right)^{-\frac{\mathsf{s}}{2}}\left(\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_\Sigma(-\Delta)^\alpha\varphi^\Sigma(\log\mu_V)+B_1(\beta,\mu_V,\psi)\right)$$

converges in distribution to a centered Gaussian of variance equal to

$$\begin{cases}\|\varphi_0^\Sigma\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} & \text{if } \ell=1\\ \|\varphi_0\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} & \text{if } \ell\to 0.\end{cases}$$

In view of (8.20), (4.27) and $\mathbb{P}_{N,\beta}(\mathcal{G}_\ell^c)\to 0$, this establishes the first claim of Theorem 3.

If we take $t=-\frac{\tau\sqrt{2}}{\beta}N^{\frac{\mathsf{s}}{2\mathsf{d}}-1}\ell^{-\frac{\mathsf{s}}{2}}$, we instead use (4.51), and obtain the result in the same way. □

5. Local Energies and Screening

We turn now to the proof of Theorem 1. The argument here is inspired by the bootstrap on scales carried out in [Leb17], [AS21] and [Pei24], and relies heavily on the electric energy formulation introduced in [PS17]. The main notions for this approach were introduced in Section 3, where the next-order energy $\mathsf{F}_N(X_N,\mu_V)$ was introduced. The typical order of this quantity is well understood, from [Ser24, Corollary 5.23], see (3.5) and the consequence in (3.26) and serves as the base case for our bootstrap on scales. This section introduces notation and terminology for examining the system at the blown-up scale and for the localization to length scales $\ell\ll 1$.

5.1. **Blowup and Subadditive Approximation.** First, it is convenient to change coordinates so that the typical interparticle distance is order 1. In the blown-up scale the length scale $\rho_\beta N^{-1/\mathsf{d}}\le\ell\le 1$ will be replaced by $\rho_\beta\le L\le N^{\frac{1}{\mathsf{d}}}$.

Setting $X_N'=N^{\frac{1}{\mathsf{d}}}X_N$ and $\mu_V'(x)=\mu_V(xN^{-\frac{1}{\mathsf{d}}})$, we can compute that

$$\mathsf{F}_N(X_N,\mu_V)=N^{\frac{\mathsf{s}}{\mathsf{d}}}\mathsf{F}(X_N',\mu_V')-\left(\frac{N}{2\mathsf{d}}\log N\right)\mathbf{1}_{\mathsf{s}=0}\tag{5.1}$$

where, for any nonnegative density μ with $\mu(\mathbb{R}^{\mathsf{d}})=N$, we let F be defined by

$$\mathsf{F}(X_N,\mu):=\frac{1}{2}\iint_{\triangle^c}\mathsf{g}(x-y)d\left(\sum_{i=1}^N\delta_{x_i}-\mu\right)(x)d\left(\sum_{i=1}^N\delta_{x_i}-\mu\right)(y).\tag{5.2}$$

Note that (5.1) and (3.25) yield that

$$\mathsf{F}(X_N,\mu)\ge -CN\tag{5.3}$$

where $C>0$ depends only on $\mathsf{d},\mathsf{s},\|\mu\|_{L^\infty}$.

A similar blow up computation yields

$$N\zeta_V(x_i) = N^{\frac{\mathsf{s}}{\mathsf{d}}}\zeta_V'(x_i')$$

where

$$\zeta_V'(x) = h^{\mu_V'} + N^{1-\frac{\mathsf{s}}{\mathsf{d}}} V\left(\frac{x}{N^{\frac{1}{\mathsf{d}}}}\right) - N^{1-\frac{\mathsf{s}}{\mathsf{d}}} c_V.$$

We can then expand as before

$$Z_{N,\beta} = \exp\left(-\beta N^{2-\frac{\mathsf{s}}{\mathsf{d}}}\mathcal{E}(\mu_V)\right)\mathsf{K}_\beta(\mu_V', \zeta_V')$$

where we define more generally

$$\mathsf{K}_\beta(\mu,\zeta) := N^{-N}\int_{(\mathbb{R}^{\mathsf{d}})^N} \exp\left(-\beta\left(\mathsf{F}(X_N,\mu) + \sum_{i=1}^N \zeta(x_i)\right)\right) dX_N. \tag{5.4}$$

In the subsequent analysis, we will need subadditive and superadditive minimal approximations of our local energy that are purely local quantities, unlike the above energies which depend on the global configuration. We proceed to define the subadditive approximation now, again in analogy with the Coulomb gas [AS21, Section 2] and the 1-d log gas [Pei24, Section 2]. The superadditive approximation is easier to present following a discussion of the screening procedure, and so we postpone that discussion to later in this section.

5.1.1. *Subadditive approximation.* Let us start with our subadditive approximation, which is a Neumann energy, following the steps of [Ser24, Chap. 7]. It is now better to work with a generic nonnegative density μ in a generic domain U. Let $U \subset \mathbb{R}^{\mathsf{d}}$ be a domain with piecewise C^1 boundary and such that $\int_U \mu = N$ is an integer. When U is unbounded, we will need an additional decay assumption in the case $\mathsf{s} \le 0$: there exists $m > 0$ and a set $\check{U}$ such that $\mu \ge m > 0$ in $\check{U}$, such that

$$\frac{1}{\mu((\check{U})^c)} \iint_{(\check{U})^c\times(\check{U})^c} \mathsf{g}(x-y)d\mu(x)d\mu(y) \ge -CN, \tag{5.5}$$

which is easily satisfied by the blown-up equilibrium measure in the generic case we are studying.

We need a new version of the minimal distance to make the energy subadditive: we set

$$\hat{\mathsf{r}}_i := \frac{1}{4}\min\left(\min_{x_j\in U, j\neq i}|x_i - x_j|, \mathrm{dist}(x_i,\partial U), 1\right). \tag{5.6}$$

This will shrink the radii of the balls when they approach ∂U, ensuring that all $B(x_i, \hat{\mathsf{r}}_i)$ remain included in U if $x_i \in U$. Let now Λ be a set of the form $U \times [-H, H]$ for some $H \in (0, +\infty]$ (interior case), or of the form $(U^c \times [-H,H])^c$ (exterior case).

If $\mu(U) = N$, for a configuration X_N of points in $U \subset \mathbb{R}^{\mathsf{d}}$ and $\Lambda \subset \mathbb{R}^{\mathsf{d}+1}$ of the form above, we let u solve

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma \nabla u) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^N \delta_{x_i} - \mu\delta_{\mathbb{R}^{\mathsf{d}}}\right) & \text{in } \Lambda \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Lambda \\ \nabla u \to 0 & \text{at } \infty. \end{cases} \tag{5.7}$$

If U and Λ are bounded, the condition at ∞ is dropped. The associated *subadditive approximation* is the Neumann energy

$$\mathsf{F}(X_N,\mu,\Lambda) := \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left(\int_\Lambda |y|^\gamma |\nabla u_{\hat{\mathsf{r}}}|^2 - \mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{i=1}^N \mathsf{g}\left(\hat{\mathsf{r}}_i\right)\right) - \sum_{i=1}^N \int_U \mathsf{f}_{\hat{\mathsf{r}}_i}(x-x_i)\, d\mu(x), \tag{5.8}$$

where f_η is defined in (3.11). Note that $\mathsf{F}(X_N,\mu,\mathbb{R}^{\mathsf{d}+1}) = \mathsf{F}(X_N,\mu)$, the quantity defined in (5.2).

Notice also that for any $\Lambda_1 \subset \Lambda_2$, we have

$$\mathsf{F}\left(X_N,\mu,\Lambda_2\right) \le \mathsf{F}(X_N,\mu,\Lambda_1) + \mathsf{F}\left(X_N,\mu,\Lambda_2\setminus\Lambda_1\right). \tag{5.9}$$

The proof is as in [Ser24, Corollary 7.6] and relies on the following projection lemma, which tells us that gradients minimize energy, cf. [PS17].

Lemma 5.1 (Projection lemma)**.** *Assume that U is an open subset of $\mathbb{R}^{\mathsf{d}}$ with piecewise C^1 boundary, and let $X_N \subset U\times\{0\} \subset \Lambda \subset \mathbb{R}^{\mathsf{d}+1}$, where Λ is an open subset of $\mathbb{R}^{\mathsf{d}+1}$ with piecewise C^1 boundary. Assume E is a vector-field satisfying a relation of the form*

$$\begin{cases} -\mathrm{div}\left(|y|^\gamma E\right) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^N \delta_{x_i} - \mu\delta_{\mathbb{R}^{\mathsf{d}}}\right) & \text{in } \Lambda \\ E\cdot n = 0 & \text{on } \partial\Lambda, \end{cases} \tag{5.10}$$

and u solves

$$\begin{cases} -\mathrm{div}\left(|y|^\gamma \nabla u\right) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^N \delta_{x_i} - \mu\delta_{\mathbb{R}^{\mathsf{d}}}\right) & \text{in } \Lambda \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Lambda, \end{cases}$$

and $u(\frac{\partial u}{\partial n} - E\cdot n) \to 0$ as $|x|\to\infty, x\in\Lambda$ if Λ is unbounded. Then

$$\int_\Lambda |y|^\gamma |\nabla u_{\hat{\mathsf{r}}}|^2 \le \int_\Lambda |y|^\gamma |E_{\hat{\mathsf{r}}}|^2,$$

where $E_{\hat{\mathsf{r}}} := E - \sum_{i=1}^N \nabla \mathsf{f}_{\hat{\mathsf{r}}_i}(x-x_i)$.

Extending $\nabla u_{\hat{\mathsf{r}}}$ by a zero vector field and using the projection lemma as in [Ser24, Corollary 7.7], we have that

$$\mathsf{F}(X_N,\mu,\Lambda) \ge \mathsf{F}(X_N,\mu\mathbf{1}_\Lambda,\mathbb{R}^{\mathsf{d}}) = \mathsf{F}(X_N,\mu\mathbf{1}_\Lambda), \tag{5.11}$$

which implies in view of (5.3) the lower bound

$$\mathsf{F}(X_N,\mu,\Lambda) \ge -CN \tag{5.12}$$

with C depending only on $\mathsf{d},\mathsf{s},\|\mu\|_{L^\infty}$.

5.1.2. *Local versions.* We next turn to local versions of these energies. First we define a new minimal distance relative to $\partial\Omega$ where $\Omega\subset\mathbb{R}^{\mathsf{d}}$:

$$\tilde{\mathsf{r}}_i := \frac{1}{4}\begin{cases} \min\left(\min_{x_j\in\Omega, j\neq i}|x_i-x_j|, \mathrm{dist}(x_i,\partial U\cap\Omega), 1\right) & \text{if } \mathrm{dist}(x_i,\partial\Omega)\ge\frac12 \\ \min(1,\mathrm{dist}(x_i,\partial U\cap\Omega)) & \text{if } \mathrm{dist}(x_i,\partial\Omega\backslash\partial U)\le\frac14 \\ t\min\left(\min_{x_j\in\Omega,j\neq i}|x_i-x_j|, \mathrm{dist}(x_i,\partial U\cap\Omega\backslash\partial U),1\right) & \\ +(1-t)\min(1,\mathrm{dist}(x_i,\partial U\cap\Omega)) & \text{if } \mathrm{dist}(x_i,\partial\Omega\backslash\partial U) = \frac{1+t}{4}, t\in[0,1]. \end{cases} \tag{5.13}$$

We note that this minimal distance coincides with (5.6) when taking $\Omega = \mathbb{R}^{\mathsf{d}}$. Here, the balls are enlarged to their largest possible values for points that approach the boundary of Ω (except for the part included in ∂U). This way, balls can potentially overlap the boundary and not be disjoint. This also ensures that for $x_i \in \Omega$, $\tilde{\mathsf{r}}_i$ does not depend on the points of the configuration that lie outside of Ω. In addition the definition is made so that the radii are continuous with respect to the position of the points.

Given a set $\tilde{\Omega} \subset \mathbb{R}^{\mathsf{d}+1}$ of the form $\Omega \times [-h, h]$ or its complement, we then let

$$\mathsf{F}^{\tilde{\Omega}}(X_N, \mu, \Lambda) := \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}} \left(\int_{\tilde{\Omega} \cap \Lambda} |y|^{\gamma} |\nabla u_{\tilde{\mathsf{r}}}|^2 - \mathsf{c}_{\mathsf{d},\mathsf{s}} \sum_{i \in I_{\tilde{\Omega}}} \mathsf{g}\left(\tilde{\mathsf{r}}_i\right) \right) - \sum_{i \in I_{\tilde{\Omega}}} \int_U \mathsf{f}_{\tilde{\mathsf{r}}_i}(x - x_i)\, d\mu(x). \tag{5.14}$$

This definition provides the following important superadditivity property (see [Ser24, Section 4.5, Lemma 7.8])

$$\mathsf{F}(X_N, \mu, \Lambda) \geq \mathsf{F}^{\tilde{\Omega}}(X_N, \mu, \Lambda) + \mathsf{F}^{(\tilde{\Omega})^c}(X_N, \mu, \Lambda). \tag{5.15}$$

We have the following control, as in [Ser24, Proposition 4.28, Lemma 7.8]: there exists $C_0 > 0$ depending only on d, s and $\|\mu\|_{L^\infty}$ such that

$$\int_{\tilde{\Omega}} |y|^{\gamma} |\nabla h_{\tilde{\mathsf{r}}}|^2 \leq 4\mathsf{c}_{\mathsf{d},\mathsf{s}} \left(\mathsf{F}^{\tilde{\Omega}}(X_N, \mu, \Lambda) + C_0 \# I_{\tilde{\Omega}} \right). \tag{5.16}$$

5.1.3. *Local partition function.* Associated to $U \subset \mathbb{R}^{\mathsf{d}}$ we also define a *local partition function* respect to a height H in the extended dimension. For $\zeta \geq 0$ such that $\int e^{-\zeta(x)} dx$ is convergent, ζ vanishing in the support of μ, and u associated to U via (5.7), let

$$\mathsf{K}_{\beta,H}(U, \mu, \zeta) := N^{-N} \int_{U^N} \exp\left(-\beta \left(\mathsf{F}\left(X_N, \mu, U \times [-H, H]\right) + \sum_{i=1}^N \zeta(x_i) \right) \right) dX_N. \tag{5.17}$$

We associate a measure to this partition function by

$$d\mathbb{Q}_{\beta,H}(U, \mu, \zeta) := \frac{1}{N^N \mathsf{K}_{\beta,H}(U, \mu, \zeta)} \exp\left(-\beta \left(\mathsf{F}\left(X_N, \mu, U \times [-H, H]\right) + \sum_{i=1}^N \zeta(x_i) \right) \right) dX_N, \tag{5.18}$$

and we also introduce external partition functions

$$\mathsf{K}^{\mathrm{ext}}_{\beta,H}(U^c, \mu, \zeta) := (N')^{-N'} \int_{(U^c)^{N'}} \exp\left(-\beta \left(\mathsf{F}\left(X_{N'}, \mu, (U \times [-H, H])^c\right) + \sum_{i=1}^{N'} \zeta(x_i) \right) \right) dX_{N'}, \tag{5.19}$$

where $N' = \mu(U^c)$, assumed to be integer.

Coupled with (5.9), we have the following superadditivity of partition functions.

Lemma 5.2. *Let U be as above, and suppose U is partitioned into p disjoint sets Q_i, $1 \leq i \leq p$ with $\mu(Q_i) = N_i \in \mathbb{Z}$. Let $H, h_1, \dots, h_p \in (0, +\infty]$ and suppose $h_i \leq H$ for all i. Then,*

$$\mathsf{K}_{\beta,H}(U, \mu, \zeta) \geq \frac{N! N^{-N}}{N_1! \cdots N_p! N_1^{-N_1} \cdots N_p^{-N_p}} \prod_{i=1}^p \mathsf{K}_{\beta,h_i}(Q_i, \mu, \zeta). \tag{5.20}$$

We also have, if $\mu(\mathbb{R}^{\mathsf{d}}) = N$ and $\mu(U) = \bar{\mathrm{n}}$,

$$\mathsf{K}_{\beta,H}(\mathbb{R}^{\mathsf{d}}, \mu, \zeta) \geq \frac{N! N^{-N}}{\bar{\mathrm{n}}! \bar{\mathrm{n}}^{-\bar{\mathrm{n}}} (N - \bar{\mathrm{n}})! (N - \bar{\mathrm{n}})^{N - \bar{\mathrm{n}}}} \mathsf{K}_{\beta,H}(U, \mu, \zeta) \mathsf{K}^{\mathrm{ext}}_{\beta,H}(U^c, \mu, \zeta). \tag{5.21}$$

Proof. The argument is exactly as in [Ser24, Lemma 7.3], with the added observation in (5.20) that we can decrease H by appending a zero electric field for $|y| > h_i$ in the extended dimension to the vector field defining $\mathsf{F}(X_N|_{Q_i}, \mu, Q_i \times [-h_i, h_i])$ and applying Lemma 5.1 above. □

5.2. **Preliminary free energy controls.** To obtain free energy controls, we use, as in [Ser24], the following rewriting.

Lemma 5.3. *Let U be an open subset of $\mathbb{R}^{\mathsf{d}}$ with bounded and piecewise C^1 boundary and μ a bounded nonnegative density such that $\mu(U) = N$ is an integer. Let $h \in (0, +\infty]$, and let G_U solve*

$$\begin{cases} -\mathrm{div}_x(|y|^\gamma \nabla G_U(x, x_0)) = \mathsf{c}_{\mathsf{d},\mathsf{s}} \left(\delta_{x_0}(x) - \frac{1}{\mu(U)}\mu(x)\delta_{\mathbb{R}^{\mathsf{d}}}\right) & \text{in } U \times [-h, h] \\ \frac{\partial G_U}{\partial n} = 0 & \text{on } \partial (U \times [-h, h]) \\ \nabla G_U \to 0 & \text{at } \infty. \end{cases} \tag{5.22}$$

Let

$$H_U(x, x') = G_U(x, x') - \mathsf{g}(x - x'),$$

where g is naturally extended to $\mathbb{R}^{\mathsf{d}+1}$. Then for any configuration X_N of points in U, we have

(5.23)
$$\mathsf{F}(X_N, \mu, U \times [-H, H]) = \frac{1}{2} \iint_{\mathbb{R}^{\mathsf{d}} \setminus \triangle} \mathsf{g}(x - y) d\left(\sum_{i=1}^N \delta_{x_i} - \mu \mathbf{1}_U\right)(x) d\left(\sum_{i=1}^N \delta_{x_i} - \mu \mathbf{1}_U\right)(y)$$
$$+ \frac{1}{2} \iint_{U \times U} H_U(x, y) d\left(\sum_{i=1}^N \delta_{x_i} - \mu\right)(x) d\left(\sum_{i=1}^N \delta_{x_i} - \mu\right)(y).$$

The proof is analogous to [Ser24, Lemma 7.9] except working in extended space with natural extension of g and G_U.

We can now obtain the a priori bound on the Neumann free energies.

Proposition 5.4 (Neumann free energy bound)**.** *Let U be an open subset of $\mathbb{R}^{\mathsf{d}}$ with bounded and piecewise C^1 boundary and μ a bounded nonnegative density in U such that $\mu(U) = N$ is an integer. Let $H \in (0, +\infty]$. Under the assumption (5.5), we have*

$$\log \mathsf{K}_{\beta,H}(U, \mu, \zeta) + \mathrm{Ent}(\mu) \geq -C(1 + \beta)N \tag{5.24}$$

where $C > 0$ depends only on $\mathsf{d}, \mathsf{s}, \|\mu\|_{L^\infty}$ and the constants in the assumptions.

Under the assumption that U is bounded, we have

$$\log \mathsf{K}_{\beta,H}(U, \mu, 0) \leq C\beta N + N \log \frac{|U|}{N} \tag{5.25}$$

Proof. Let us start with the lower bound. The difference with the proof of [Ser24, Prop 7.10] is that the reference measure in the definition of $\mathsf{K}_{\beta,H}$ is $e^{-\zeta(x)}dx$ and not μ. By definition and using Jensen's inequality, we may write

(5.26)
$$\log \mathsf{K}_{\beta,H}(U,\mu,\zeta) = \log\left(N^{-N}\int_{U^N}\exp\left(-\beta\mathsf{F}(X_N,\mu,U)+\sum_{i=1}^N\zeta(x_i)-\sum_{i=1}^N\log\mu(x_i)\right)d\mu^{\otimes N}(X_N)\right)$$
$$\geq \frac{1}{N^N}\int_{U^N}\left(-\beta\mathsf{F}(X_N,\mu,U)+\sum_{i=1}^N\zeta(x_i)-\sum_{i=1}^N\log\mu(x_i)\right)d\mu^{\otimes N}(X_N)$$

Inserting the rewriting (5.23), expanding all the sums and observing that the terms involving H_U cancel after integration against $d\mu^{\otimes N}$, we are led to

$$\log \mathsf{K}_{\beta,H}(U,\mu,\zeta) \geq \int_U \zeta d\mu - \int_U \mu\log\mu + \frac{\beta}{2N}\iint \mathsf{g}(x,y)d\mu(x)d\mu(y).$$

If $\mathsf{s}>0$ then $\mathsf{g}\geq 0$ and we deduce the lower bound. If $\mathsf{s}\leq 0$, we argue by splitting the region into cells of size R and optimizing over R. It is an adaptation of the proof of Propositions 7.10 and 5.14 in [Ser24], left to the reader, and using (5.5).

For the upper bound, it suffices to insert the lower bound (5.12) for F into (5.17). □

5.3. **Riesz Screening and Superadditive Approximation.** The main technical tool that we use in the proof of Theorem 1 is a screening procedure for Riesz gases, adapted from the procedure for Coulomb gases described in [Ser24, Section 7.2]. This kind of procedure is based on ideas from [ACO09] and [SS12], and adapted to Riesz gases in [PS17], then also used extensively in [LS15]. The difficulty in the adaptation is in having to work in the extended space. The version below is optimized from [PS17] as was done for the Coulomb gas in [AS21], introducing a new approach in dealing with the extended dimension for the so-called "outer screening" (which was up to [AS21] called inner screening).

A detailed description of the ideas and motivations involved in the screening can be found in [Ser24, Section 7.2], some of which we summarize here. Consider a hyperrectangle Ω in which the background measure μ is bounded from below, the localization superadditivity as in (5.15) yields

$$\mathsf{F}(X_N,\mu,\mathbb{R}^{\mathsf{d}}) \geq \mathsf{F}^{\Omega\times[-h,h]}(X_N,\mu,\mathbb{R}^{\mathsf{d}}) + \mathsf{F}^{(\Omega\times[-h,h])^c}(X_N,\mu,\mathbb{R}^{\mathsf{d}}). \tag{5.27}$$

A matching upper bound is of course not true, but if the energy on Ω is (reasonably) well controlled we can screen the configuration X_N, which means modify it near $\partial\Omega$ and produce new configurations $Y_{\bar{\mathrm{n}}}$ and $Y_{N-\bar{\mathrm{n}}}$ with corresponding electric fields that have a zero Neumann boundary condition and energy smaller than the original ones, up to small errors, i.e.

$$\mathsf{F}\left(Y_{\bar{\mathrm{n}}},\tilde{\mu},\Omega\times[-h,h]\right) \leq \mathsf{F}^{\Omega\times[-h,h]}(X_N,\mu,\mathbb{R}^{\mathsf{d}}) + \text{ screening errors}$$

and

$$\mathsf{F}\left(Y_{N-\bar{\mathrm{n}}},\tilde{\mu},(\Omega\times[-h,h])^c\right) \leq \mathsf{F}^{(\Omega\times[-h,h])^c}(X_N,\mu,\mathbb{R}^{\mathsf{d}}) + \text{ screening errors}.$$

Here the configurations $Y_{\bar{\mathrm{n}}}$ and $Y_{N-\bar{\mathrm{n}}}$ coincide with X_N except in a boundary layer near $\partial\Omega$, the same for $\tilde{\mu}$ with μ. By subadditivity of the Neumann energy (5.9), gluing together $Y_{\bar{\mathrm{n}}}$ and $Y_{N-\bar{\mathrm{n}}}$ into a new configuration Y_N on $\mathbb{R}^{\mathsf{d}}$, we then have

$$\mathsf{F}(Y_N,\mu,\mathbb{R}^{\mathsf{d}}) \leq \mathsf{F}\left(Y_{\bar{\mathrm{n}}},\tilde{\mu},\Omega\times[-h,h]\right) + \mathsf{F}\left(Y_{N-\bar{\mathrm{n}}},\tilde{\mu},(\Omega\times[-h,h])^c\right)$$
$$\leq \mathsf{F}^{(\Omega\times[-h,h])^c}(X_N,\mu,\mathbb{R}^{\mathsf{d}}) + \mathsf{F}^{\Omega\times[-h,h]}(X_N,\mu,\mathbb{R}^{\mathsf{d}}) + \text{ screening errors}.$$

The configuration Y_N being considered as an approximation of X_N, this constitutes a sort of converse (with error) to the superadditivity of (5.27), providing an estimate of additivity error. We will be able to do this for most point configurations and corresponding electric fields, this will allow the above relations to be integrated over configurations and turned into a free energy almost additivity result. The crucial point here is that the screening errors hence addivity errors are negligible with respect to the volume of the box Ω, even when Ω is small, which is what will allow to obtain the local laws down to the microscale of Theorem 1. An important fact is that screening can only be performed in regions where the density μ is bounded below by some constant $m > 0$.

5.3.1. *Riesz screenability.* Before delving into a more detailed description of the procedure, we first give a definition of screenable electric fields. We let $\mathcal{Q}_R$ be the set of closed hyperrectangles of the form $Q_R \times [-h, h]$ in $\mathbb{R}^{\mathsf{d}+1}$ with the sidelengths of Q_R in $[\frac{1}{2}R, 2R]$ and which are such that $\int_{Q_R} \mu$ is an integer.

We will screen fairly generic electric fields satisfying relations of the form (5.28), respecting Neumann boundary data constraints if there are any, and for such vector fields we define their truncations as in (3.14), with the truncation radii $\hat{\mathsf{r}}$, which we notice coincides with $\tilde{\mathsf{r}}$ for points at distance larger than 1 from all considered boundaries (hence the truncated fields coincide as well once at distance ≥ 1 from the boundary). The main difference with the Coulomb case is in the need to control the energy on horizontal slices parallel to $\mathbb{R}^{\mathsf{d}}$ in the extended space. In what follows $m > 0$ is a positive constant: screening can only be performed in regions where the density μ is bounded away from 0.

Definition 5.5. *Let μ be a nonnegative bounded density. Assume Λ is either $\mathbb{R}^{\mathsf{d}+1}$ or a the cartesian product of a hyperrectangle with an interval $[-H, H]$, or the complement of such a set. Let $h \leq R/2$ and let $\tilde{\Omega} = (Q_R \times [-h, h]) \cap \Lambda$ (inner case), resp. $\tilde{\Omega} = \Lambda \backslash (Q_R \times [-h, h])$ (outer case) where Q_R is a hyperrectangle of sidelengths in $[R, 2R]$ with sides parallel to those of Λ, and such that $\mu(\tilde{\Omega}) = \bar{\mathrm{n}}$, an integer. Let ℓ and $\tilde{\ell}$ be such that $R \geq \tilde{\ell} \geq \ell \geq C$, where C is some constant dependent on m. In the inner case, assume that $\mu \geq m > 0$ in $((Q_R \backslash Q_{R-2\tilde{\ell}}) \times \{0\}) \cap \Lambda$ and in the outer case, assume that that $\mu \geq m > 0$ in $((Q_{R+2\tilde{\ell}} \backslash Q_R) \times \{0\}) \cap \Lambda$. In the outer case, also assume that the faces of ∂Q_R are at distance $\geq 2\tilde{\ell}$ from their respective parallel faces of $\partial \Lambda$.*

In the inner screening, let X_n be a configuration of points in $\tilde{\Omega}$ and let w solve

$$\begin{cases} -\mathrm{div}\left(|y|^\gamma \nabla w\right) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^n \delta_{x_i} - \mu \delta_{\mathbb{R}^d}\right) & \text{in } \tilde{\Omega} \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial \Lambda \cap \tilde{\Omega}. \end{cases} \tag{5.28}$$

In the outer screening, let X_n be a configuration of points in $\tilde{\Omega}$ and let w solve

$$\begin{cases} -\mathrm{div}\left(|y|^\gamma \nabla w\right) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^n \delta_{x_i} - \mu \delta_{\mathbb{R}^{\mathsf{d}}}\right) & \text{in } \tilde{\Omega} \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial \Lambda \cap \tilde{\Omega}. \end{cases} \tag{5.29}$$

In the inner screening, denote

$$S(X_n,w,h)=\int_{((Q_{R-\tilde\ell}\backslash Q_{R-2\tilde\ell})\times[-h,h])\cap\Lambda}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \tag{5.30}$$

$$S'(X_n,w,h)=\sup_x\int_{(((Q_{R-\tilde\ell}\backslash Q_{R-2\tilde\ell})\cap\square_\ell(x))\times[-h,h])\cap\Lambda}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \tag{5.31}$$

$$e(X_n,w,h)=\int_{Q_R\times\{-h,h\}}|y|^\gamma|\nabla w|^2. \tag{5.32}$$

In the outer screening, denote

$$S(X_n,w,h)=\int_{((Q_{R+2\tilde\ell}\backslash Q_{R+\tilde\ell})\times[-h,h])\cap\Lambda}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \tag{5.33}$$

$$S'(X_n,w,h)=\sup_x\int_{(((Q_{R+2\tilde\ell}\backslash Q_{R+\tilde\ell})\cap\square_\ell(x))\times[-h,h])\cap\Lambda}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \tag{5.34}$$

$$e(X_n,w,h)=\int_{Q_R\times\{(-(R+h),(R+h)\}}|y|^\gamma|\nabla w|^2. \tag{5.35}$$

We say that a configuration X_n and potential w are screenable at height h if

$$\max\left(\frac{1}{R^{\mathsf{d}-2}\tilde\ell^2}h^\gamma e(X_n,w,h),\frac{h^{1+\gamma}}{\ell^{\mathsf{d}+1}}\min\left(S'(X_n,w,h),\frac{S(X_n,w,h)}{\tilde\ell}\right)\right)\le\mathsf{c} \tag{5.36}$$

where c is a constant dependent only on upper and lower bounds for μ (and defined in (9.13)).

With a notion of screenability in hand, we can define the minimal energy approximation that we will need for the screening statement. It is analogous to a Dirichlet energy.

Definition 5.6 (Best screenable potential and energy)**.** *With the same notation as above, we let*

(5.37)

$$\mathsf{G}^{\mathrm{inn/ext}}_{a,h}(X_n,\tilde\Omega)=\min\Big\{\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left(\int_{\tilde\Omega}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2-\mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{i=1}^n\mathsf{g}(\hat{\mathsf{r}}_i)\right)-\sum_{i=1}^n\int_{\tilde\Omega}\mathsf{f}_{\hat{\mathsf{r}}_i}(x-x_i)d\mu(x)\delta_{\mathbb{R}^{\mathsf{d}}},$$
$$w\text{ inner/outer screenable satisfying a relation of the form}$$
$$\begin{cases}-\mathrm{div}\,(|y|^\gamma\nabla w)=\mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^n\delta_{x_i}-\mu\delta_{\mathbb{R}^d}+\sum_j\delta_{x_j}^{(\eta_j)}\right) & \text{in } \tilde\Omega\\ \frac{\partial w}{\partial n}=0 & \text{on } \partial\Lambda\cap\tilde\Omega\end{cases}$$
$$\text{with } x_j\notin\tilde\Omega,\eta_j\le\frac14\min(1,\mathrm{dist}(x_j,\partial\Lambda))$$
$$\text{and satisfying (5.36) at level } h \text{ and } e(X_n,w,h)\le a\Big\},$$

(with the min *understood as $+\infty$ if the set is empty), with $e(X_n,w,h)$ as in (5.32) and (5.35), respectively. By the direct method in the calculus of variations, one may check that the minima are achieved. Note that G depends on Λ and μ but for the sake of lightness we do not retain it in the notation.*

We also define

(5.38)

$$\bar{S}_{a,h}(X_n) = \inf\{S(X_n, w, h), w \text{ achieving the min in } \mathsf{G}^{\mathrm{inn}}_{a,h}(X_n, \tilde{\Omega}), \text{ resp. } \mathsf{G}^{\mathrm{ext}}_{a,h}(X_n, \tilde{\Omega})\}$$

(5.39)

$$\bar{S}'_{a,h}(X_n) = \inf\{S'(X_n, w, h), w \text{ achieving the min in } \mathsf{G}^{\mathrm{inn}}_{a,h}(X_n, \tilde{\Omega}), \text{ resp. } \mathsf{G}^{\mathrm{ext}}_{a,h}(X_n, \tilde{\Omega})\}.$$

When u is inner/outer screenable at level h with an a-bound on e, it is a competitor in the definition of $\mathsf{G}^{\mathrm{inn/out}}$, thus we have

$$\mathsf{F}^{\tilde{\Omega}}(X_N, \mu, \Lambda) \ge \mathsf{G}^{\mathrm{inn/out}}(X_N|_{\tilde{\Omega}}, \tilde{\Omega}). \tag{5.40}$$

5.3.2. *Riesz screening.* Before we state the screening procedure formally let us give a heuristic description (see Figure 1). For a screenable field and configuration, we select by a mean-value argument a "good boundary" enclosing a set $\mathcal{O}$ (like old), in which we keep the configuration and field unchanged. We let $\mathcal{N}$ (like new) be the complement layer to $\mathcal{O}$, and $\mathcal{N}_\eta$ be $\mathcal{N}$ with a buffer layer of size $\eta < 1$ removed. We place new points in $\mathcal{N}_\eta$ in a way that neutralizes the background measure, and define a new field in $\mathcal{N} \times [-h, h]$ with a zero Neumann boundary condition on $\partial\mathcal{O} \times [-h, h]$. A novel component of the screening in the Riesz case is that we need to complete the field away from the subspace $\mathbb{R}^{\mathsf{d}}$; this is done in $\Omega \times ([-R, R] \setminus [-h, h])$ by matching the current field at level h and setting a Neumann zero boundary condition elsewhere.

The inner screening is analogous, except we are now working with the field in $(\Omega \times [-R, R])^c$; see diagram (B) in Figure 1 below, where E stands for the electric field ∇w.

We now state the screening procedure formally.

Proposition 5.7 (Riesz Screening)**.** *Let us take the same assumptions as in Definition 5.5, and suppose $0 \le \eta < 1$. Then, there exists a $C > 5$ dependent only on $\mathsf{d}, \mathsf{s}, m$ and $\|\mu\|_{L^\infty}$ such that the following holds. Suppose that X_n and w satisfy (5.28) (respectively, (5.29)) and are screenable at height h in the sense of Definition 5.5. Then, there exists a set $\mathcal{O} \subset \mathbb{R}^{\mathsf{d}}$ such that*

$$(Q_{R-2\tilde{\ell}} \times \{0\}) \cap \Lambda \subset \mathcal{O} \times \{0\} \subset (Q_{R-2\tilde{\ell}} \times \{0\}) \cap \Lambda \quad \textit{(inner screening)},$$

resp.

$$(Q^c_{R+2\tilde{\ell}} \times \{0\}) \cap \Lambda \subset \mathcal{O} \times \{0\} \subset (Q^c_{R+\tilde{\ell}} \times \{0\}) \cap \Lambda \quad \textit{(outer screening)},$$

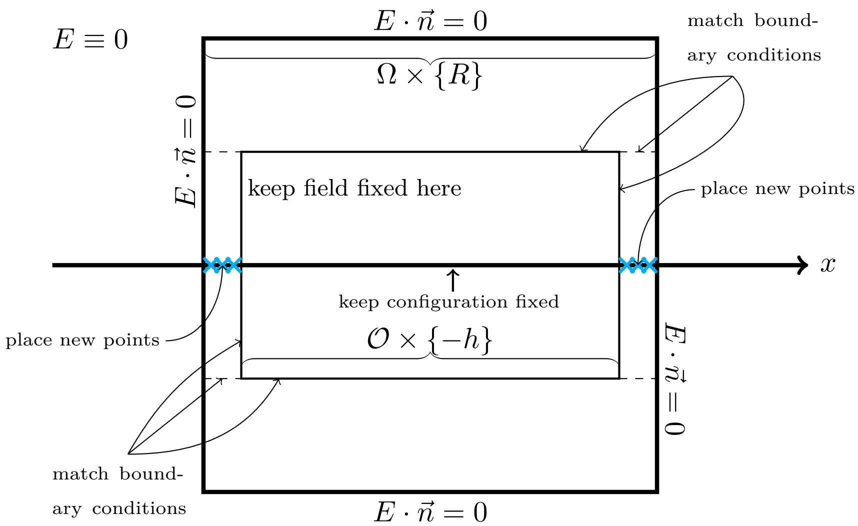
a subset $I_\partial \subset \{1, \dots, n\}$, and a nonnegative density $\tilde{\mu}$ supported in $\mathcal{N}_\eta = \{x \in \mathcal{N} : \mathrm{dist}(x, \mathcal{O}) \ge \eta\}$ with $\mathcal{N} \subset \mathbb{R}^{\mathsf{d}}$ such that $\mathcal{N} \times \{0\} = Q_R \times \{0\} \cap \Lambda \setminus (\mathcal{O} \times \{0\})$ in the inner screening and $\mathcal{N} \times \{0\} = ((Q^c_R \times \{0\}) \cap \Lambda) \setminus (\mathcal{O} \times \{0\})$ in the outer screening, such that the following holds:

(1) $n_{\mathcal{O}}$ being the number of points of X_n such that $B(x_i, \hat{\mathsf{r}}_i)$ intersects $\mathcal{O}$, we have

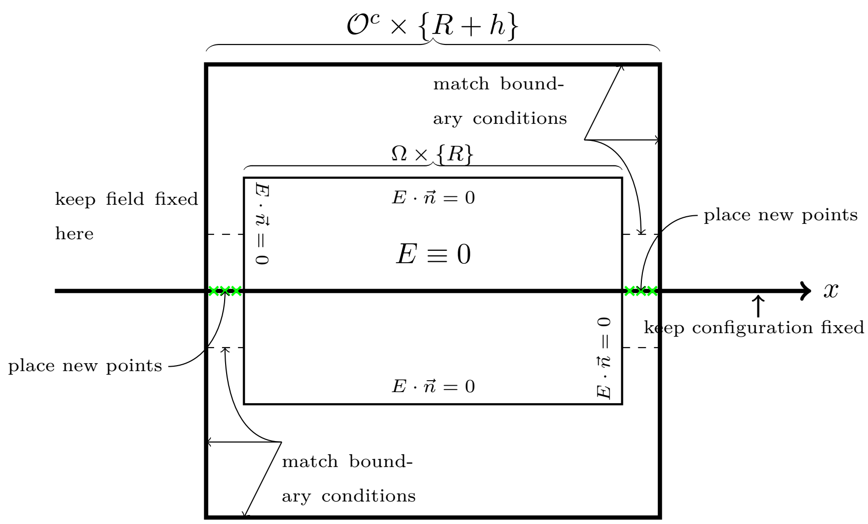
$$\tilde{\mu}(\mathcal{N}) = \tilde{\mu}(\mathcal{N}_\eta) = \bar{\mathrm{n}} - n_{\mathcal{O}}, \qquad |\mu(\mathcal{N}) - \tilde{\mu}(\mathcal{N})| \le C\left(R^{\mathsf{d}-1} + \frac{S(X_n, w, h)}{\tilde{\ell}}\right) \tag{5.41}$$

$$\|\mu - \tilde{\mu}\|_{L^\infty(\mathcal{N}_\eta)} \le \frac{m}{2}, \qquad \int_{\mathcal{N}_\eta} (\tilde{\mu} - \mu)^2 \le C\frac{S(X_n, w, h)}{\ell\tilde{\ell}} + C\frac{\eta^2}{\ell} R^{\mathsf{d}-1} \tag{5.42}$$

(2) $\# I_\partial \le C\frac{S(X_n,w,h)}{\tilde{\ell}}$.



(A) inner screening



(B) outer screening

FIGURE 1. The screening procedure

(3) The Neumann approximation is comparable to the original energy, i.e.

(5.43)

$$\begin{aligned}
&\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, Q_R \times [-R, R]) - \left(\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}} \left(\int_{Q_R \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 - \mathsf{c}_{\mathsf{d}} \sum_{i=1}^n \mathsf{g}(\hat{\mathsf{r}}_i) \right) + \sum_{i=1}^n \int \mathsf{f}_{\hat{\mathsf{r}}_i}(x - x_i) d\mu(x) \right) \\
&\leq C \frac{h S(X_n, w)}{\tilde{\ell}} + \sum_{(i,j) \in J} \mathsf{g}(x_i - z_j) + |n - \bar{\mathrm{n}}| + \left(\frac{R^2}{\tilde{\ell}} + R \right) e(X_n, w, h) \\
&\quad + \left(\tilde{\ell} + h^{-\gamma} \right) R^{\mathsf{d}-1} + \mathsf{F}(Z_{\bar{\mathrm{n}} - n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h, h])
\end{aligned}$$

where

$$J = \{(i,j) \in I_\partial \times \{1, \dots, \bar{\mathrm{n}} - n_{\mathcal{O}}\} : |x_i - z_j| \le \hat{\mathsf{r}}_i\}$$

in the inner case, and

(5.44)

$$\begin{aligned}
&\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, (Q_R \times [-R,R])^c) - \left(\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left(\int_{(Q_R\times[-R,R])^c} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 - \mathsf{c}_{\mathsf{d}} \sum_{i=1}^n \mathsf{g}(\hat{\mathsf{r}}_i)\right) + \sum_{i=1}^n \int \mathsf{f}_{\hat{\mathsf{r}}_i}(x - x_i) d\mu(x)\right) \\
&\le C \frac{hS(X_n, w)}{\tilde{\ell}} + \sum_{(i,j)\in J} \mathsf{g}(x_i - z_j) + |n - \bar{\mathrm{n}}| + \frac{R}{\min(h, \tilde{\ell})}\left(\frac{R^2}{\tilde{\ell}} + R\right) e(X_n, w, h) \\
&\quad + \left(\tilde{\ell} + h^{-\gamma}\right) R^{\mathsf{d}-1} + \mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h,h])
\end{aligned}$$

in the outer case.

The method of proof essentially adapts and optimizes the approach of [PS17, Section 6] in an analogous way to the optimization of [AS21, Appendix C] for Coulomb gases. A thorough discussion can be found in [Ser24, Chapter 7]. The approach to outer screening is a novel adaptation for the Riesz gas. We will present the proof in Section 9.

Remark 5.8. *It will be important to screen fields defined at heights $R' > R$ in the proof of almost additivity in Section 7. We can start with a field ∇w defined in $Q_R \times [-R', R']$ for any $R' > R$ and apply the screening procedure to its restriction to $Q_R \times [-R,R]$ to obtain a screened field in $Q_R \times [-R,R]$. Notice then that by appending an electric field that is uniformly zero for $|y| > R$ and using Lemma 5.1 above, we can replace $\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, Q_R \times [-R,R])$ in (5.43) with $\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, Q_R \times [-R', R'])$ for any $R' > R$. This yields*

$$\begin{aligned}
\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, Q_R \times [-R', R']) - \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{Q_R\times[-R',R']} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 \\
\le \mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, Q_R \times [-R,R]) - \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{Q_R\times[-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2,
\end{aligned}$$

from which it follows that the screening result above allows us to screen in $Q_R \times [-R', R']$ for any $R' > R$ with the same errors as in (5.43)-(5.44).

6. Main Bootstrap

In this section we prove the main probabilistic control on local energies, following a bootstrap on scales. The argument is a generalization of [Pei24] to the higher dimensional Riesz case, which was in turn a generalization of [AS21] to the 1d-log case. See also [Ser24, Chapters 7-8] for a thorough description of the method.

We are going to prove local laws in blown-up scale. In order to prove Theorem 1, we need to prove them for $\Lambda = \mathbb{R}^{\mathsf{d}+1}$ and μ equal to the blown-up of the equilibrium measure μ_V. For the proof of the almost additivity of the energy in the next section, we will also need to have proven the local laws in cubes. This is why we continue to work with a generic density μ and set Λ in the setup of Section 5.1.1, and a general probability law $\mathbb{Q}_{\beta,H}$ as in (5.18). In the case of a cube, the local laws will be valid up to the boundary. In the case of the whole space, the local laws are valid only in the bulk, i.e. on cubes separated from the set where

μ is small by a fixed positive distance in original scale. For that reason, we let, in the case where $\Lambda = \mathbb{R}^{\mathsf{d}+1}$,

$$\hat{\Sigma}' := \{x \in \mathbb{R}^{\mathsf{d}}, \mathrm{dist}(x, \{\mu = 0\}) > \varepsilon N^{1/\mathsf{d}}\}, \tag{6.1}$$

and assume that $\mu \geq m > 0$ on $\hat{\Sigma}'$. Note that by assumption (1.24) we know that μ_V is bounded below at distance ε from $\partial\Sigma$, so μ_V' satisfies this condition. In the case where Λ is a hypercube of projection onto $\mathbb{R}^{\mathsf{d}}$ equal to U, we just let $\hat{\Sigma}' = U$. All the constants in the local laws will depend on m, hence on ε.

We wish to prove that there exists constants C_1, C_2 independent of the scale and of β, $\mathcal{C}_\beta$ independent of β when $\beta \geq 1$, such that, u being defined in (5.7), for all $L > \rho_\beta$ and any cube $\square_L \subset \hat{\Sigma}'$, there exists an event $\mathcal{G}_L$ such that

$$\forall X_N \in \mathcal{G}_L, \quad \mathsf{F}^{\square_L \times [-L,L]}(X_N, \mu, \Lambda) + 2C_0 \# I_{\square_L} \leq \mathcal{C}_\beta L^{\mathsf{d}} \tag{6.2}$$

where C_0 is the constant in (5.16), with the event $\mathcal{G}_L$ satisfying $\mathbb{Q}_{\beta,H}(\mathcal{G}_L^c) \leq C_1 e^{-C_2 \beta L^{\mathsf{d}}}$.

As in [Leb17], [AS21] and [Pei24, Theorem 1], this is achieved by an induction on the scale. To do so, we consider a cube $\square_{2\varepsilon}(z) \subset \Sigma'$ (later we will drop the z from the notation). Then $\square_\varepsilon(z) \subset \hat{\Sigma}'$. We then consider $2^{-k_*}\varepsilon = L$, and assume that there exists an event $\mathcal{G}_{2L}$ with $\mathbb{Q}_{\beta,H}(\mathcal{G}_{2L}^c) \leq C_1 e^{-C_2\beta(2L)^{\mathsf{d}}}$ such that for every $1 \leq k \leq k_*$

$$\forall X_N \in \mathcal{G}_{2L}, \quad \mathsf{F}^{\square_{2^k L} \times [-2^k L, 2^k L]}(X_N, \mu, \Lambda) + 2C_0 \# I_{\square_{2^k L}} \leq \mathcal{C}_\beta (2^k L)^{\mathsf{d}}. \tag{6.3}$$

Note that for $k \geq k_*$ this is also automatically satisfied thanks to (3.26) (for a constant depending on ε).

In view of (5.16), this implies in particular

$$\forall X_N \in \mathcal{G}_{2L}, \forall 1 \leq k, \quad \int_{\square_{2^k L} \times [-2^k L, 2^k L]} |y|^\gamma |\nabla u_{\mathsf{r}}|^2 \leq 4\mathsf{c}_{\mathsf{d},\mathsf{s}} \mathcal{C}_\beta \left(2^k L\right)^{\mathsf{d}} \quad \text{and} \quad C_0 \# I_{\square_{2^k L}} \leq \mathcal{C}_\beta \left(2^k L\right)^{\mathsf{d}}. \tag{6.4}$$

We then wish to prove that there exists an event $\mathcal{G}_L$ such that $\mathbb{Q}_{\beta,H}(\mathcal{G}_L^c) \leq C_1 e^{-C_2 \beta L^{\mathsf{d}}}$ and such that for all $X_N \in \mathcal{G}_L$, (6.3) holds for $k = 0$ as long as $L \geq \rho_\beta$, with the same constants $C_1, C_2, \mathcal{C}$. This easily suffices to imply that (6.2) holds for any cube $\square_L \subset \hat{\Sigma}'$ as long as $L \geq \rho_\beta$. Note that in order to prove that (6.3) holds for $k = 0$, it suffices to show it over a hyperrectangle Q_L such that $\square_L \subset Q_L \subset \square_{\frac{3}{2}L}$, which will allow us to choose Q_L such that $\mu(Q_L)$ is an integer.

Proposition 6.1. *Let the setup be as in Section 5.1.1 with Λ equal to $\mathbb{R}^{\mathsf{d}+1}$ or a hypercube of height H, $\mathbb{Q}_{\beta,H}(U, \mu, \zeta)$ as in (5.18) and u as in (5.7). Suppose that there exists an event $\mathcal{G}_{2L}$ such that (6.4) holds. Then, there is a scale $\rho_\beta > 0$ (depending only on β), $C > 0$, with $4 \leq \rho_\beta \lesssim_\beta 1$ such that if $L \geq \rho_\beta$, the following holds.*

$$\forall X_N \in \mathcal{G}_L, \quad \mathsf{F}^{\square_L \times [-L,L]}(X_N, \mu, \Lambda) + 2C_0 \# I_{\square_L} \leq \mathcal{C}_\beta L^{\mathsf{d}} \tag{6.5}$$

with

$$\mathbb{Q}_{\beta,h}(U, \mu, \zeta)(\mathcal{G}_{2L} \setminus \mathcal{G}_L) \leq e^{-C\beta L^{\mathsf{d}}} \tag{6.6}$$

with $C > 0$ depending only on $\mathsf{d}, \mathsf{s}, m, \varepsilon, \|\mu\|_{L^\infty}$ and $\mathcal{C}_\beta \geq 1$ also possibly depending on β when $\beta \leq 1$.

Once this proposition is proved, Theorem 1 follows by a bootstrap on the scale starting from (3.6) and by using (5.16) in the same way as the proof of [Pei24, Theorem 3.1] from [Pei24, Proposition 3.2]. One starts at the macroscopic scale in $\hat{\Sigma}'$, where one has the local law outside of an exponentially small event by (3.6), and applies Proposition 6.1 iteratively down to scale L. At each application at scale $2^k L$, we lose an event of probability no more than $e^{-C\beta 2^k L^{\mathsf{d}}}$, so the local law at scale L holds off of an event of size at most

$$\sum_{k=0}^{\infty} e^{-C\beta 2^k L^{\mathsf{d}}} \le C_1 e^{-C_2 \beta L^{\mathsf{d}}},$$

for some constants C_1 and C_2 independent of scale L.

The key technical tool that we use in the proof of Proposition 6.1 is the screening procedure Proposition 5.7, which allows us to localize our rather nonlocal next order energy and exhibit an almost additivity on scales. In order to successfully employ this procedure, we will need to guarantee that the errors generated when we screen are sufficiently small.

6.1. **Control of Screening Errors.** The first result we will need is on the rate of decay of the electric field away from the subspace $\mathbb{R}^{\mathsf{d}}$, which allows to control the e terms (as in (5.32)) in the screening estimates, a question which is absent in the Coulomb case. This is done by viewing the electric field as a fluctuation and using as an input the local law at scales $2^k L$ (6.4). In the sequel we let $\mathsf{s}_+ = \max(\mathsf{s}, 0)$.

Proposition 6.2 (Decay estimate - control of the e term)**.** *Let Λ be $\mathbb{R}^{\mathsf{d}+1}$ or a hyperrectangle as above. Let L be such that $L = 2^{-k_*}\varepsilon$, with $\square_\varepsilon \subset \hat{\Sigma}'$ in the case $\Lambda = \mathbb{R}^{\mathsf{d}+1}$.*

(1) Let $\mathcal{G}_{2L}$ be the event that (6.4) holds. Then, for any $\epsilon > 0$ and $K \ge 1$ large enough so that

$$\frac{L}{K} \le \epsilon^{\frac{2}{4+\mathsf{s}-\mathsf{d}}} N^{\frac{1}{\mathsf{d}}},$$

there exists $4 \le \rho_\beta \lesssim_\beta 1$ and a constant $C > 0$ dependent only on K and ϵ such that if $L > C(\beta^{-1}\log\frac{1}{\epsilon})^{\frac{1}{\mathsf{d}}}$ and $h = L/K$, there is an event $\mathcal{G} \subset \mathcal{G}_{2L}$ such that

$$\int_{\square_{2L}\times\{\pm h\}} |y|^\gamma |\nabla u|^2 \lesssim_\beta \epsilon L^{\mathsf{d}-1} \quad \text{on } \mathcal{G}, \tag{6.7}$$

with

$$\mathbb{Q}_{\beta,H}(U,\mu,\zeta)(\mathcal{G}_{2L}\setminus\mathcal{G}) \le e^{-C\beta L^{\mathsf{d}}}. \tag{6.8}$$

(2) Let $\mathcal{G}_h$ be the event that (6.4) holds for $2L = \rho_\beta$. Then, there exists a constant $C > 0$ such that given $M \ge 1$, if $L > h \ge \rho_\beta$, there is an event $\mathcal{G} \subset \mathcal{G}_h$ such that

$$\int_{\square_{2L}\times\{\pm h\}} |y|^\gamma |\nabla u|^2 \lesssim_\beta M L^{\mathsf{d}} h^{\mathsf{s}_+-\mathsf{d}-1} \quad \text{on } \mathcal{G}, \tag{6.9}$$

with

$$\mathbb{Q}_{\beta,H}(U,\mu,\zeta)(\mathcal{G}_h\setminus\mathcal{G}) \le M^{-\frac{\mathsf{d}}{2}} L^{\mathsf{d}} h^{\frac{\mathsf{d}(\mathsf{d}-\mathsf{s}_+-2)}{2}} e^{-CM}. \tag{6.10}$$

Proof. Let us first consider the case $\Lambda = \mathbb{R}^{\mathsf{d}+1}$. In that case, u given by (5.7) is, up to an additive constant, equal to $\mathsf{g} * \left(\sum_{j=1}^N \delta_{x_j'} - \mu\right)$. By symmetry in y, it suffices to prove the result at height $+h$. For notational ease we will let E be a shorthand for ∇u. The main idea rests on the observation that the components of E_i, $1 \le i \le \mathsf{d}+1$, at a point $(\vec{a}, h) = (a_1, a_2, \ldots, a_{\mathsf{d}}, h)$

with $h > 0$, are fluctuations of smooth linear statistics. Namely, denoting x'_j for the points of the configuration (at the blown-up scale) and spelling out $\nabla \mathsf{g}$, we have

$$E_i(\vec{a}, h) = \int_{\mathbb{R}^{\mathsf{d}}} \kappa^k_{(\vec{a},h)}(x) d\left(\sum_{j=1}^N \delta_{x'_j} - \mu\right)(x) = \mathrm{Fluct}_\mu\left(\kappa^k_{(\vec{a},h)}\left(N^{\frac{1}{\mathsf{d}}} x\right)\right) \tag{6.11}$$

where

$$\kappa^i_{(\vec{a},h)}(x) = \begin{cases} \frac{-(a-x)_i}{(|x-a|^2+h^2)^{\frac{\mathsf{s}+2}{2}}} & \text{if } 1 \le i \le \mathsf{d} \\ \frac{-h}{(|x-a|^2+h^2)^{\frac{\mathsf{s}+2}{2}}} & \text{if } i = \mathsf{d}+1. \end{cases} \tag{6.12}$$

With this observation, the main idea is as follows:

- Place P points on $\square_{2L} \times \{h\}$; estimate $E(z) = E(z_p) + (E(z) - E(z_p))$
- Estimate $E(z_p)$ in probability using Theorem 2 and the local laws.
- Use elliptic regularity and the local laws to control $\nabla_{\mathbb{R}^{\mathsf{d}}} E$ and thus $E(z) - E(z_p)$.

Step 1: Setup. We split $\square_{2L}$ into P equally sized subrectangles I_p of sidelength comparable to $LP^{-1/\mathsf{d}}$, and let $z_p = (\vec{a}_p, h)$ denote the center of the subcube I_p. On each subcube I_p, we estimate

$$\begin{aligned} \int_{I_p \times \{h\}} |E|^2 &\le 2 \sum_{i=1}^{\mathsf{d}+1} \left(\int_{I_p \times \{h\}} |E_i(z) - E_i(z_p)|^2 + \int_{I_p \times \{h\}} |E_i(z_p)|^2 \right) \\ &\lesssim \|\nabla_{\mathbb{R}^{\mathsf{d}}} E\|^2_{L^\infty(I_p \times \{h\})} \int_{I_p \times \{h\}} |z - z_p|^2 + \frac{L^{\mathsf{d}}}{P} \sum_{i=1}^{\mathsf{d}+1} |E_i(z_p)|^2 \\ &\lesssim \frac{L^{\mathsf{d}+2}}{P^{1+\frac{2}{\mathsf{d}}}} \|\nabla_{\mathbb{R}^{\mathsf{d}}} E\|^2_{L^\infty(I_p \times \{h\})} + \frac{L^{\mathsf{d}}}{P} |E(z_p)|^2. \end{aligned} \tag{6.13}$$

where $\nabla_{\mathbb{R}^{\mathsf{d}}}$ denotes gradient in $\mathbb{R}^{\mathsf{d}}$. We now estimate each term separately.

Step 2. Control of $|E(z_p)|^2$. This is done via estimates on fluctuations of linear statistics associated to the functions $\kappa^i_{(\vec{a},h)}$. The main difference between our approach here and that of [Pei24, Proposition 3.3] is that we use Theorem 2 for a rescaled test function directly, instead of running transport estimates for the functions κ^i. Even though these test functions are not literally rescaled versions of a compactly supported test function, we can treat them as such after a dyadic splitting.

Without loss of generality, let $\vec{a} = \vec{0}$; we also focus on $i = \mathsf{d}+1$, since the computation for $i \le \mathsf{d}$ is analogous and produces the same result. Let x be the space variable at the non-blown-up scale. Notice that

$$h^{1+\mathsf{s}} \kappa^{\mathsf{d}+1}_{\vec{0},h}\left(N^{\frac{1}{\mathsf{d}}} x\right) = \frac{-h^{2+\mathsf{s}}}{\left(|N^{\frac{1}{\mathsf{d}}} x|^2 + h^2\right)^{\frac{\mathsf{s}+2}{2}}} = -\frac{1}{\left(|\frac{N^{\frac{1}{\mathsf{d}}} x}{h}|^2 + 1\right)^{\frac{\mathsf{s}+2}{2}}} := \varphi_0\left(\frac{x N^{1/\mathsf{d}}}{h}\right),$$

where φ_0 is a smooth scale-independent function. Let χ_k be a partition of unity associated to dyadic annuli $B(0, 2^{k+2}) \backslash B(0, 2^{k-2})$ (as in Section 3). We may choose k_* such that $2^{k_*}\ell$ is bounded below by $\varepsilon > 0$ and $B(0, 2^{k_*}\ell)$ does not intersect any other connected component of $\operatorname{supp} \mu$ (if there is more than one) than that of 0.

We may then write $\varphi_0 = \sum_{k=0}^{k_*}(\chi_k\varphi_0) + \varphi_1$, with $\chi_k\varphi_0$ a function supported in a dyadic annulus and $\|\varphi_1\|_{L^\infty}$ is bounded by $O((hN^{-1/\mathsf{d}})^{\mathsf{s}+2})$. Computing directly shows as well that

$$\left\|\nabla\left(\varphi_1\left(\frac{\cdot N^{1/\mathsf{d}}}{h}\right)\right)\right\|_{L^\infty} = O\left(\left(hN^{-1/\mathsf{d}}\right)^{\mathsf{s}+2}\right).$$

The function $(2^k)^{\mathsf{s}+2}\chi_k\varphi_0$ satisfies (1.38) at scale $2^k\ell$ for some constant $\mathsf{M} > 0$. We may thus apply Theorem 2 to it. For the proof of item (1) of the proposition, we use the local laws at scales $2^k L$ to have a control at scale $2^k hN^{-1/\mathsf{d}} = 2^k\frac{L}{K}N^{-1/\mathsf{d}}$, we thus obtain the bound stated in Theorem 2 : for $\tau_k(2^k h)^{\mathsf{s}-\mathsf{d}}$ small enough,

(6.14)
$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(\tau_k\frac{\beta}{1+\beta}\mathrm{Fluct}_\mu\left((2^k)^{\mathsf{s}+2}(\chi_k\varphi_0)\left(\frac{\cdot N^{1/\mathsf{d}}}{h}\right)\right)\mathbf{1}_{\mathcal{G}_{2L}}\right)\right]\right| \lesssim_\beta (|\tau_k| + |\tau_k|^2)(2^k h)^{\mathsf{s}}$$

where the constant depends on K. Applying to $\tau_k = \tilde\tau_k(2^k h)^{\mathsf{d}-\mathsf{s}}$, we find that if $\tilde\tau_k$ is small enough,

$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}}\left[\exp\left(\frac{\beta}{1+\beta}\tilde\tau_k(2^k h)^{\mathsf{d}-\mathsf{s}}(2^k)^{\mathsf{s}+2}\mathrm{Fluct}_\mu\left((\chi_k\varphi_0)\left(\frac{\cdot N^{1/\mathsf{d}}}{h}\right)\right)\mathbf{1}_{\mathcal{G}'_{2L}}\right)\right]\right| \lesssim_\beta (\tilde\tau_k(2^k h)^{\mathsf{d}} + \tilde\tau_k^2(2^k h)^{2\mathsf{d}-\mathsf{s}}).$$

For any λ, Markov's inequality yields

$$\mathbb{P}_{N,\beta}\left(\left\{\mathrm{Fluct}_\mu\left((\chi_k\varphi_0)\left(\frac{\cdot N^{1/\mathsf{d}}}{h}\right)\right) \geq \lambda\right\}\cap\mathcal{G}_{2L}\right) \leq \exp\left(C_\beta(\tilde\tau_k(2^k h)^{\mathsf{d}} + \tilde\tau_k^2(2^k h)^{2\mathsf{d}-\mathsf{s}}) - \tilde\tau_k\frac{\beta}{1+\beta}(2^k h)^{\mathsf{d}-\mathsf{s}}(2^k)^{\mathsf{s}+2}\lambda\right) \tag{6.15}$$

hence, choosing $\tilde\tau_k = \epsilon\frac{\beta}{C_\beta}(2^k h)^{\frac{\mathsf{s}-\mathsf{d}}{2}}$ and $\lambda = 2\epsilon\frac{1+\beta}{\beta}(2^k h)^{\frac{\mathsf{d}+\mathsf{s}}{2}}(2^k)^{-\mathsf{s}-2}$ we find that as soon as ϵ is small enough and h is large enough (depending on K and the other constants), in $\mathcal{G}_{2L}$, it holds that

$$\mathrm{Fluct}_\mu\left((\chi_k\varphi_0)\left(\frac{\cdot N^{1/\mathsf{d}}}{h}\right)\right) \leq 2\epsilon\frac{1+\beta}{\beta}(2^k h)^{\frac{\mathsf{d}+\mathsf{s}}{2}}(2^k)^{-\mathsf{s}-2} \tag{6.16}$$

except on an event of probability $\leq \exp(-\beta\epsilon^2(2^k h)^{\mathsf{d}})$. Using Proposition 3.5 at macroscopic scale and with $\eta = 1$, we also have

$$\mathrm{Fluct}_\mu\left(\varphi_1\left(\frac{\cdot N^{1/\mathsf{d}}}{h}\right)\right) \lesssim_\beta \left(hN^{-1/\mathsf{d}}\right)^{\mathsf{s}+2} N^{\frac12+\frac{\mathsf{s}}{2\mathsf{d}}}. \tag{6.17}$$

Taking a union bound on all these events, summing and using that $\mathsf{s} > \mathsf{d}-2$, we obtain that except for an event of probability $\leq \exp(-C'\beta\epsilon^2 h^{\mathsf{d}})$, we have

$$\mathrm{Fluct}_\mu\left(\kappa^i_{0,h}\right) \lesssim_\beta h^{-1-\mathsf{s}}\left(\epsilon h^{\frac{\mathsf{d}+\mathsf{s}}{2}}\sum_{k=0}^{k_*}(2^k)^{\frac{\mathsf{d}-\mathsf{s}}{2}-2} + \left(hN^{-1/\mathsf{d}}\right)^{\mathsf{s}+2}N^{\frac12+\frac{\mathsf{s}}{2\mathsf{d}}}\right) \lesssim_\beta \epsilon h^{-1+\frac{\mathsf{d}-\mathsf{s}}{2}} + hN^{\frac{\mathsf{d}-\mathsf{s}-4}{2\mathsf{d}}}. \tag{6.18}$$

The second term can be absorbed into the first, since $h \le \epsilon^{\frac{2}{4+\mathsf{s}-\mathsf{d}}} N^{1/\mathsf{d}}$ and $\mathsf{d}-\mathsf{s}-4<0$, hence

$$hN^{\frac{\mathsf{d}-\mathsf{s}-4}{2\mathsf{d}}} \le \epsilon h^{1+\frac{\mathsf{d}-\mathsf{s}-4}{2}} = \epsilon h^{-1+\frac{\mathsf{d}-\mathsf{s}}{2}}.$$

Hence, we conclude that

$$|E(z_p)|^2 \lesssim_\beta \epsilon^2 h^{\mathsf{d}-\mathsf{s}-2}, \tag{6.19}$$

except on an event of probability $\le \exp(-C\beta\epsilon^2 h^{\mathsf{d}})$. For the proof of item (2) of the proposition where h and L are no longer comparable, we run the same argument, using that local laws hold on scale h directly, except that after (6.15), we choose

$$\begin{cases} \tilde{\tau}_k = \frac{1}{\sqrt{C_\beta}}(2^k)^\varepsilon \sqrt{M}(2^k h)^{-\mathsf{d}+\frac{\mathsf{s}}{2}}, & \lambda = \frac{2}{\sqrt{C_\beta}}\frac{1+\beta}{\beta}(2^k)^\varepsilon\sqrt{M}(2^k h)^{\frac{\mathsf{s}}{2}}(2^k)^{-\mathsf{s}-2} \quad \text{if } \mathsf{s}<0 \\ \tilde{\tau}_k = \frac{1}{C_\beta}(2^k)^\varepsilon M(2^k h)^{-\mathsf{d}}, & \lambda = 2\frac{1+\beta}{\beta}(2^k)^\varepsilon (2^k h)^{\mathsf{s}}(2^k)^{-\mathsf{s}-2} \quad \text{if } \mathsf{s}\ge 0. \end{cases}$$

This yields that in $\mathcal{G}_h$, except with probability $\exp(-(2^k)^\varepsilon M)$, we have

$$\mathrm{Fluct}_\mu\left((\chi_k\varphi_0)\left(\frac{\cdot N^{1/\mathsf{d}}}{h}\right)\right) \lesssim_\beta \begin{cases} \sqrt{M}(2^k h)^{\frac{\mathsf{s}}{2}}(2^k)^{-\mathsf{s}-2+\varepsilon} & \text{if } \mathsf{s}<0 \\ (2^k h)^{\mathsf{s}}(2^k)^{-\mathsf{s}-2+\varepsilon} & \text{if } \mathsf{s}\ge 0. \end{cases}$$

Taking a union bound over these events and summing over k in the same manner as above, we conclude that, choosing $\varepsilon>0$ small enough,

$$|E(z_p)| \lesssim_\beta \begin{cases} h^{-1-\frac{\mathsf{s}}{2}}\sqrt{M} & \text{if } \mathsf{s}<0 \\ h^{-1} & \text{if } \mathsf{s}\ge 0 \end{cases} \lesssim_\beta \sqrt{M}h^{-1+(-\frac{1}{2}\mathsf{s})_+}. \tag{6.20}$$

where $(\cdot)_+$ denotes the positive part.

Step 3. Bound on $|\nabla_{\mathbb{R}^{\mathsf{d}}} E|$. Let us start with the case of item (1). We observe that, if $K\ge 2$, $2h\le L$, hence in $\square_L\times[h/2,2h]$, if $\square_L\subset\hat{\Sigma}'$, the bound

$$\int_{\square_L\times[h/2,2h]} |y|^\gamma |E|^2 \lesssim_\beta h^{\mathsf{d}}$$

is verified from (6.2) in $\mathcal{G}_{2L}$, with a constant depending on K. In addition, $\nabla_{\mathbb{R}^{\mathsf{d}}} E$ satisfies

$$-\mathrm{div}\,(|y|^\gamma \nabla_{\mathbb{R}^{\mathsf{d}}} E) = 0 \quad \text{in } \square_L\times[h/2,2h].$$

Elliptic regularity estimates then yield that x being the center of $\square_L$,

$$|\nabla_{\mathbb{R}^{\mathsf{d}}} E(x,h)|^2 \lesssim_\beta \frac{1}{h^{2+\mathsf{d}+1+\gamma}}\int_{\square_h\times[h/2,2h]} |y|^\gamma|E|^2 \lesssim_\beta \frac{h^{\mathsf{d}}}{h^{3+\mathsf{d}+\gamma}} \lesssim_\beta h^{-4+\mathsf{d}-\mathsf{s}} \tag{6.21}$$

using $\mathsf{d}-1+\gamma=\mathsf{s}$. For the proof of item (2), we use that the local laws hold down to scale h hence all the estimates above hold directly on $\square_h\times[-h/2,2h]$ and obtain the same result.

Step 4: Conclusion in the case of the whole space. For item (1), inserting (6.21) and (6.19) into (6.13), and recalling that $\mathsf{d}+\gamma=\mathsf{s}+1$, taking a union bound over the bad events, we obtain that except with probability $\le Pe^{-C\beta L^{\mathsf{d}}}$, we have, using $\gamma+\mathsf{d}-\mathsf{s}=1$,

$$\int_{\square_L\times\{h\}} |y|^\gamma|E|^2 \lesssim_\beta Ph^\gamma\left(\frac{L^{\mathsf{d}+2}}{P^{1+\frac{2}{\mathsf{d}}}}h^{\mathsf{d}-\mathsf{s}-4} + \frac{L^{\mathsf{d}}}{P}\epsilon^2 h^{\mathsf{d}-\mathsf{s}-2}\right) \lesssim_\beta \frac{L^{\mathsf{d}-1}}{P^{\frac{2}{\mathsf{d}}}} + \epsilon^2 L^{\mathsf{d}-1}.$$

Choosing $P=\epsilon^{-\frac{\mathsf{d}}{2}}$, we obtain the desired result since $\log P$ can be absorbed into $O(\beta L^{\mathsf{d}})$ when L is larger than a constant times $(\beta^{-1}\log\frac{1}{\epsilon})^{\frac{1}{\mathsf{d}}}$.

For item (2), we obtain instead, inserting (6.21) and (6.20) into (6.13),

$$\begin{aligned}\int_{\square_L\times\{h\}}|y|^\gamma|E|^2 &\lesssim_\beta Ph^\gamma\left(\frac{L^{\mathsf{d}+2}}{P^{1+\frac{2}{\mathsf{d}}}}h^{\mathsf{d}-\mathsf{s}-4}+M\frac{L^{\mathsf{d}}}{P}h^{\max(-2\mathsf{s},0)-2}\right)\\ &\lesssim_\beta\left(\frac{L^{\mathsf{d}+2}}{P^{\frac{2}{\mathsf{d}}}h^3}+ML^{\mathsf{d}}h^{\mathsf{s}-\mathsf{d}-1+(-\mathsf{s})_+}\right).\end{aligned}\tag{6.22}$$

Taking $P=M^{-\frac{\mathsf{d}}{2}}L^{\mathsf{d}}h^{\frac{\mathsf{d}(\mathsf{d}-\mathsf{s}_+-2)}{2}}$ to equate the last two terms yields the result. We have thus concluded the proof in the case where $\Lambda=\mathbb{R}^{\mathsf{d}+1}$.

Step 5. The case where Λ is a hypercube. First, let us justify that Theorem 2 holds as well for the Neumann energy setup as follows: let U be a hyperrectangle in $\mathbb{R}^{\mathsf{d}}$ of sidelengths in $[\ell/2,2\ell]$ at the original scale, μ a density bounded below in U with $N\mu(U)=\bar{\mathrm{n}}$ integer, F_N defined as in (5.8), except at the original scale, and the associated Gibbs measure $\mathbb{Q}_{N,\beta,H}(U,\mu)$ and partition function $\mathsf{K}_{N,\beta,H}(U,\mu)$ (we can assume that $\zeta=0$) as in (5.18) and (5.17) but in original scale. Let φ be a test function in U satisfying $\frac{\partial\varphi}{\partial n}=0$. While our analysis in Sections 2-4 is based on an analysis via transport before applying the splitting formula Lemma 3.1, we can conduct a similar analysis post splitting as in [LS18, Sections 3-4], finding that the expansion of next-order partition functions is governed by terms T_1 and T_2. More precisely, we can write analogously to Lemma 2.1 that

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_{N,\beta,H}(U,\mu)}\left[\exp\left(-\beta tN^{1-\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{Fluct}_\mu(\varphi)\right)\mathbf{1}_{\mathcal{G}}\right]\\ &\quad=\frac{1}{\mathsf{K}_{N,\beta,H}(U,\mu)}\int_{\mathcal{G}}\exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}(tN\mathrm{Fluct}_\mu(\varphi)+\mathsf{F}_N(X_N,\mu,U\times[-H,H]))\right)dX_N\end{aligned}$$

We can use (5.23) to observe, by "completing the square" that if ν solves

$$\int_U G_U(x,y)d\nu(y)=\varphi(x)\tag{6.23}$$

then

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_{N,\beta,H}(U,\mu)}\left[\exp\left(-\beta tN^{1-\frac{\mathsf{s}}{\mathsf{d}}}\mathrm{Fluct}_\mu(\varphi)\right)\mathbf{1}_{\mathcal{G}}\right]\\ &\quad=\exp\left(-\frac{\beta}{2}N^{2-\frac{\mathsf{s}}{\mathsf{d}}}t^2\iint_{U^2}G_U(x,y)d\nu(x)d\nu(y)\right)\frac{\mathsf{K}_{N,\beta,H}(U,\mu+t\nu)}{\mathsf{K}_{N,\beta,H}(U,\mu)}.\end{aligned}\tag{6.24}$$

To solve (6.23), we may reflect and periodize φ across the faces of the hyperrectangle U, in such a way that φ remains continuous and $\nabla\varphi$ as well (thanks to the assumption $\nabla\varphi\cdot\vec{n}=0$ on ∂U). Call φ^{per} the reflected and periodized function, then we can check that $\nu=\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}(-\Delta)^\alpha(\varphi^{\mathrm{per}})$ computed over $\mathbb{R}^{\mathsf{d}}$ solves (6.23) in U, where the fractional Laplacian of a periodic function is defined via the Fourier series representation as in [RS15]. We can then analyze (6.24) as in the case of the full space by using the transport map

$$\begin{cases}\mathrm{div}\,(\psi\mu)=\nu & \text{in } U\\ \psi\cdot\vec{n}=0 & \text{on } \partial U.\end{cases}$$

for which estimates as a function of φ are easy to obtain, in fact more easily than in the full space case treated in Section 2. Starting from (6.24), we may then obtain the analogue of Theorem 2 for φ. Spelling out ∇G_U we obtain an expression for the electric field in the

Neumann case analogous to (6.11), which expresses it as the fluctuation of a function in the class of the φ's just analyzed and we can complete the proof in the same way.

$\square$

This estimate allows us to better understand the screening errors in Proposition 5.7, and show that the initial and screened fields are comparable.

Corollary 6.3. *Let Λ be $\mathbb{R}^{\mathsf{d}+1}$ or a hyperrectangle as above. Let $\square_L(z)$ be as above, i.e. such that $\square_{2^{k_*}L} = \square_\varepsilon \subset \hat{\Sigma}'$ in the case $\Lambda = \mathbb{R}^{\mathsf{d}+1}$. Let $\Omega = Q_L \cap \Lambda$ with Q_L a hyperrectangle such that $\square_L \cap \Lambda \subset Q_L \cap \Lambda \subset \square_{\frac{3}{2}L} \cap \Lambda$, and $\int_\Omega \mu$ is an integer. Let $X_n = X_N|_\Omega$ and let $Y_{\bar{\mathrm{n}}}$ be as in Proposition 5.7, and recall the definitions of $\mathsf{G}^{\mathrm{inn}}_{a,h}$ and $\mathsf{G}^{\mathrm{ext}}_{a,h}$ from (5.37). We next make the choice*

$$a = \begin{cases} C_\beta \epsilon L^{\mathsf{d}-1} & \text{in case (1) below} \\ C_\beta M L^{\mathsf{d}} h^{\mathsf{s}_+ - \mathsf{d}-1} & \text{in case (2) below,} \end{cases} \tag{6.25}$$

for the C_β implicitly appearing in (6.7), resp. (6.9).

(1) Let $\mathcal{G}_{2L}$ be the event in (6.4). Let $\epsilon > 0$. There exists $K, K_1 \geq 1$ such that for $h = L/K$, $h \geq \rho_\beta$, $\tilde{\ell} = \frac{L}{K_1}$, and for $\mathcal{G} \subset \mathcal{G}_{2L}$ as in part (1) of Proposition 6.2, we have for all configurations X_N in $\mathcal{G}$,

$$\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, \Omega \times [-L, L]) - \mathsf{G}^{\mathrm{inn}}_{a,h}(X_n, \Omega \times [-L, L]) \lesssim \sum_{(i,j)\in J} \mathsf{g}(x_i - z_j) + \mathsf{F}\left(Z_{\bar{\mathrm{n}} - n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h, h]\right) + |n - \bar{\mathrm{n}}| + C_\beta \epsilon L^{\mathsf{d}}, \tag{6.26}$$

and

$$\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, (\Omega \times [-L, L])^c) - \mathsf{G}^{\mathrm{ext}}_{a,h}(X_{N-n}, (\Omega \times [-L, L])^c) \lesssim \sum_{(i,j)\in J} \mathsf{g}(x_i - z_j) + \mathsf{F}(Z_{\bar{\mathrm{n}} - n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-L, L]) + |n - \bar{\mathrm{n}}| + C_\beta \epsilon L^{\mathsf{d}}. \tag{6.27}$$

(2) Let $\mathcal{G}_h$ be as in item (2) of Proposition 6.2. Let $M \geq 1$, $L > h \geq \rho_\beta$, and $\mathcal{G} \subset \mathcal{G}_h$ as in item (2) of Proposition 6.2. Suppose that

$$C_\beta \frac{M L^2 h^{\mathsf{s}+\mathsf{s}_+ - 2\mathsf{d}}}{\tilde{\ell}^2} \text{ sufficiently small}, \qquad C_\beta h L^{\frac{1-\mathsf{d}}{\mathsf{d}-\gamma}} \text{ sufficiently large } \textit{if } \mathsf{s} \geq \mathsf{d} - 1. \tag{6.28}$$

Then, for all configurations in $\mathcal{G}$,

$$\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, \Omega \times [-L, L]) - \mathsf{G}^{\mathrm{inn}}_{a,h}(X_n, \Omega \times [-L, L]) \lesssim \sum_{(i,j)\in J} \mathsf{g}(x_i - z_j) + |n - \bar{\mathrm{n}}| + \mathsf{F}(Z_{\bar{\mathrm{n}} - n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h, h]) + \left(\frac{h^2}{\tilde{\ell}} + \tilde{\ell} + h^{-\gamma}\right) L^{\mathsf{d}-1} + C_\beta \left(\frac{L^2}{\tilde{\ell}} + L\right) M L^{\mathsf{d}} h^{\mathsf{s}_+ - \mathsf{d}-1} \tag{6.29}$$

and

$$\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, (\Omega \times [-L, L])^c) - \mathsf{G}^{\mathrm{ext}}_{a,h}(X_{N-n}, (\Omega \times [-L, L])^c) \lesssim \sum_{(i,j)\in J} \mathsf{g}(x_i - z_j) + |n - \bar{\mathrm{n}}| + \mathsf{F}(Z_{\bar{\mathrm{n}} - n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h, h]) + \left(\frac{h^2}{\tilde{\ell}} + \tilde{\ell} + h^{-\gamma}\right) L^{\mathsf{d}-1} + C_\beta \frac{L}{\min(h, \tilde{\ell})} \left(\frac{L^2}{\tilde{\ell}} + L\right) M L^{\mathsf{d}} h^{\mathsf{s}_+ - \mathsf{d}-1}. \tag{6.30}$$

Proof. We will apply Proposition 5.7, and control the error terms using Proposition 6.2. Let us focus on the proof for the inner screening, since the outer case is analogous. Let w be any potential associated to Ω as in (5.28) with

$$\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\Omega\times[-L,L]}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2\le\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\Omega\times[-L,L]}|y|^\gamma|\nabla u_{\hat{\mathsf{r}}}|^2$$
$$\int_{\Omega\times\{\pm h\}}|\nabla w|^2\le a.$$

At least one such potential exists, namely the true potential u given in (5.7).

Step 1: Screenability. Let us comment on screenability in the general case (2) first. For $\mathsf{s}\le\mathsf{d}-1$ we verify screenability on $S'(X_n,w,h)$, obtaining, for h large enough

$$\frac{h^{1+\gamma}}{h^{\mathsf{d}+1}}S'(X_n,w,h)\le\frac{\mathcal{C}_\beta h^{\mathsf{d}+1+\gamma}}{h^{\mathsf{d}+1}}\le h^\gamma\le\mathsf{c}$$

on $\mathcal{G}_h$, since $\mathsf{s}<\mathsf{d}-1$ implies that $\gamma<0$ and $h\gg1$. If $\mathsf{s}\ge\mathsf{d}-1$, we instead need to verify screenability with $S(X_n,w,h)$. By a covering argument we have

$$S(X_n,w,h)\le\left(\frac{L}{h}\right)^{\mathsf{d}-1}\mathcal{C}_\beta h^{\mathsf{d}}\le\mathcal{C}_\beta hL^{\mathsf{d}-1}\tag{6.31}$$

and obtain

$$\frac{h^{1+\gamma}}{h^{\mathsf{d}+2}}\mathcal{C}_\beta hL^{\mathsf{d}-1}=\mathcal{C}_\beta\frac{L^{\mathsf{d}-1}}{h^{\mathsf{d}-\gamma}}$$

which can be made $\le\mathsf{c}$ so long as $h\gg L^{\frac{\mathsf{d}-1}{\mathsf{d}-\gamma}}$, yielding the additional condition on h. Coupling these estimates with Proposition 6.2 yields the second item in (6.28).

For item (1), we specialize to $h=\frac{L}{K}$ and the announced a. We also set $\ell=\frac{L}{K}$ and $\tilde{\ell}=\frac{L}{K_1}$, with $K>K_1$ to be determined. Using the bound (6.7), we see that the first item of (5.36) becomes

$$C_\beta\frac{\epsilon L^{\mathsf{d}-1}h^\gamma}{L^{\mathsf{d}-2}\tilde{\ell}^2}=C_\beta\frac{\epsilon K_1^2L^{1+\gamma}}{K^\gamma L^2}\le\mathsf{c}$$

which holds on the event $\mathcal{G}$ of Proposition 6.2 for appropriate choice of constants since $\gamma<1$.

For item (2), we instead find using (6.9) that the first item in (5.36) becomes

$$C_\beta\frac{h^\gamma ML^{\mathsf{d}}h^{\mathsf{s}_+-\mathsf{d}-1}}{L^{\mathsf{d}-2}\tilde{\ell}^2}=C_\beta\frac{ML^2h^{\mathsf{s}+\mathsf{s}_+-2\mathsf{d}}}{\tilde{\ell}^2}\le\mathsf{c}$$

using $\gamma=\mathsf{s}+1-\mathsf{d}$, which is the first item in (6.28).

Step 2: Screening. We now apply and control the errors in Proposition 5.7. Let us start with the general case (2). First, we have

$$\frac{h}{\tilde{\ell}}S(X_n,w,h)\le\mathcal{C}\frac{h^2}{\tilde{\ell}}L^{\mathsf{d}-1}$$

using the covering argument from the screenability computation in Step 1. Substituting in the bound from Proposition 6.2 yields

$$\begin{aligned}\mathsf{F}(Y_{\bar{\mathrm{n}}},\tilde{\mu},\Omega\times[-L,L]) - \left(\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\left(\int_{Q_R\times[-R,R]}|y|^\gamma|\nabla w_{\hat{\mathrm{r}}}|^2 - \mathsf{c}_\mathsf{d}\sum_{i=1}^n \mathsf{g}(\hat{\mathrm{r}}_i)\right) + \sum_{i=1}^n\int \mathsf{f}_{\hat{\mathrm{r}}_i}(x-x_i)d\mu(x)\right) \\ \lesssim \sum_{(i,j)\in J}\mathsf{g}(x_i-z_j) + \mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}},\tilde{\mu},\mathcal{N}_\eta\times[-h,h]) + |n-\bar{\mathrm{n}}| \\ + \left(\frac{h^2}{\tilde{\ell}} + \tilde{\ell} + h^{-\gamma}\right)L^{\mathsf{d}-1} + C_\beta\left(\frac{L^2}{\tilde{\ell}}+L\right)ML^\mathsf{d}h^{\mathsf{s}_+-\mathsf{d}-1}.\end{aligned}$$

Since the right hand side is uniform for any w associated to Ω as in (5.28) with the requisite decay, we obtain the result by inserting the definition of $\mathsf{G}^{\mathrm{inn}}_{a,h}(X_n,\tilde{\Omega})$.

In case (1), the screening errors in the previous computation then take the form

$$\begin{aligned}\mathcal{C}_\beta\left(\frac{h^2}{\tilde{\ell}} + \tilde{\ell} + h^{-\gamma}\right)L^{\mathsf{d}-1} + \left(\frac{L^2}{\tilde{\ell}}+L\right)e(X_n,w,h) \\ \le \mathcal{C}_\beta\left(\frac{K_1}{K^2}+\frac{1}{K_1}\right)L^\mathsf{d} + K^\gamma L^{\mathsf{d}-1-\gamma} + C_\beta\epsilon\left(K_1^2+1\right)L^\mathsf{d}\end{aligned}$$

where we have inserted the control on $e(X_n,w,h)$ with the local laws parameters from Proposition 6.2. By choosing K and K_1 appropriately and redefining ϵ from Proposition 6.2, we have that this is $\lesssim_\beta \epsilon L^\mathsf{d}$ as desired, since $|\gamma|<1$. A similar computation holds for the error terms in the outer screening. □

6.2. **Analysis of Exponential Moments.** To get control on the level of exponential moments, we need a sufficient volume of configurations for which we can screen. This is given by the following.

Proposition 6.4. *Keep the same assumptions and cases as in Corollary 6.3. Let $\mathcal{G}$ denote the good event of Corollary 6.3, and let a be as in (6.25). Let $n\le N$, and define*

$$\mathcal{G}_\Omega = \{X_n\in\Omega^n : X_n = X_N|_\Omega \text{ for some } X_N\in\mathcal{G}\} \tag{6.32}$$

$$\mathcal{G}_{\Omega^c} = \{X_{N-n}\in(\Omega^c)^{N-n} : X_{N-n} = X_N|_{\Omega^c} \text{ for some } X_N\in\mathcal{G}\} \tag{6.33}$$

Then, we have

$$n^{-n}\int_{\mathcal{G}_\Omega}\exp\left\{-\beta\mathsf{G}^{\mathrm{inn}}_{a,h}(X_n,\mu,\Omega\times[-L,L])\right\}\,d\rho^{\otimes n}(X_n) \le \mathsf{K}_{\beta,L}(\Omega,\mu,\zeta)\exp(\beta\varepsilon_e+\varepsilon_v)$$

and

$$\begin{aligned}(N-n)^{n-N}\int_{\mathcal{G}_{\Omega^c}}\exp\left\{-\beta\mathsf{G}^{\mathrm{ext}}_{a,h}(X_{N-n},\mu,(\Omega\times[-L,L])^c)\right\}\,d\rho^{\otimes N-n}(X_{N-n}) \\ \le \mathsf{K}^{\mathrm{ext}}_{\beta,L}(\Omega,\mu,\zeta)\exp(\beta\varepsilon_e+\varepsilon_v),\end{aligned}$$

where ρ denotes the confinement measure $d\rho(x)=e^{-\zeta(x)}\,dx$, ε_e denotes the energy error

$$\varepsilon_e = \begin{cases}|\bar{\mathrm{n}}-n| + C_\beta\epsilon L^\mathsf{d} + \tilde{\ell}^\mathsf{d} + \frac{\mathcal{C}L^\mathsf{d}}{\tilde{\ell}} & \text{in case (1)}\\ |\bar{\mathrm{n}}-n| + \tilde{\ell}^\mathsf{d} + \frac{\mathcal{C}hL^{\mathsf{d}-1}}{\tilde{\ell}} + \left(\frac{h^2}{\tilde{\ell}}+\tilde{\ell}+h^{-\gamma}\right)L^{\mathsf{d}-1} + C_\beta\left(\frac{L^2}{\tilde{\ell}}+L\right)ML^\mathsf{d}h^{\mathsf{s}_+-\mathsf{d}-1} & \text{in case (2)}\end{cases}$$

in the inner screening, and

$$\varepsilon_e = \begin{cases} |\bar{\mathrm{n}} - n| + C_\beta \epsilon L^{\mathsf{d}} + \tilde{\ell}^{\mathsf{d}} + \frac{\mathcal{C} L^{\mathsf{d}}}{\tilde{\ell}} & \textit{in case } (1) \\ |\bar{\mathrm{n}} - n| + \tilde{\ell}^{\mathsf{d}} + \frac{\mathcal{C} h L^{\mathsf{d}-1}}{\tilde{\ell}} + \left(\frac{h^2}{\tilde{\ell}} + \tilde{\ell} + h^{-\gamma}\right) L^{\mathsf{d}-1} + & \\ C_\beta \frac{L}{\min(h, \tilde{\ell})} \left(\frac{L^2}{\tilde{\ell}} + L\right) M L^{\mathsf{d}} h^{\mathsf{s}_+ - \mathsf{d} - 1} & \textit{in case } (2) \end{cases}$$

in the outer screening, and ε_v denotes the volume error, with estimate

$$\varepsilon_v \leq C \tilde{\ell}^d + \log \frac{\mathcal{C}_\beta \tilde{\ell}}{\eta} + (\bar{\mathrm{n}} - n - \alpha) \log \frac{\alpha}{\alpha'} + \left(\bar{\mathrm{n}} - n - \alpha - \frac{1}{2}\right) \log \left(1 + \frac{n - \bar{\mathrm{n}}}{\alpha}\right) + \frac{1}{2} \log \frac{n}{\bar{\mathrm{n}}},$$

where α, α' are integers satisfying $\alpha, \alpha' \lesssim \tilde{\ell}^{\mathsf{d}}$ and $|\alpha - \alpha'| \lesssim L^{\mathsf{d}-1} h^{1+\gamma} + \frac{L^{\mathsf{d}}}{\tilde{\ell}}$.

Proof. We focus on the integral over $\mathcal{G}_\Omega$ (inner screening), since the argument for the integral over $\mathcal{G}_{\Omega^c}$ is an analogous application of Corollary 6.3. Much of the proof is as in [Pei24, Proposition 3.5] and [AS21, Proposition 4.2], with the screening and combinatorial accounting updated as in [Ser24, Proposition 8.2].

Each configuration $X_n \in \mathcal{G}_\Omega$ is screenable by assumption, and screening produces a set $\mathcal{O}(X_n)$ of the form $Q_t \cap \Lambda$. We partition $\mathcal{G}_\Omega$ into a disjoint union $\cup_k \mathcal{E}_k$ based on what t is in terms of the η thickness of the point-free layer, i.e.

$$\mathcal{E}_k = \{X_n \in \mathcal{G}_\Omega : \mathcal{O}(X_n) = Q_t,\ t \in [R - 2\tilde{\ell} + k\eta, R - 2\tilde{\ell} + (k+1)\eta]\}$$

All but $O\left(\frac{\tilde{\ell}}{\eta}\right)$ of the sets are empty. For each configuration in one of the nonempty $\mathcal{E}_k$, X_n yields a number of points $n - n_{\mathcal{O}}$ of points that are removed. There are $\binom{n}{n_{\mathcal{O}}}$ ways of doing this, and each deletion corresponds to a volume of configurations no larger than $\rho(\mathcal{N})^{n - n_{\mathcal{O}}}$. Then we insert $\bar{\mathrm{n}} - n_{\mathcal{O}}$ new points, but the resulting configurations are equivalent up to a permutation of indices; this leads to an overcounting of factor $\binom{\bar{\mathrm{n}}}{n_{\mathcal{O}}}$. Furthermore, there is no issue with a lack of injectivity here. Any two configurations in $\mathcal{E}_k$ must have $\partial\mathcal{O}$ at distance no more than η from each other, and so it is impossible to place new points within the $\mathcal{O}$ of the other configuration. Hence, the same new configuration $Y_{\bar{\mathrm{n}}}$ cannot be produced from two separate configurations in the same $\mathcal{E}_k$.

Coupling this with the error estimate of Corollary 6.3, we find

$$\bar{\mathrm{n}}^{\bar{\mathrm{n}}} \mathsf{K}_{\beta, L}(\Omega, \mu, \zeta) \geq \frac{\binom{\bar{\mathrm{n}}}{n_{\mathcal{O}}}}{\binom{n}{n_{\mathcal{O}}} |\mathcal{N}|^{n - n_{\mathcal{O}}}} \frac{\eta}{C \tilde{\ell}} \int_{\mathcal{G}_\Omega} \exp\left(-\beta \left(\mathsf{G}^{\mathrm{inn}}_{a,h}(X_n, \Omega \times [-L, L]) + \varepsilon_e\right)\right)$$
$$\int_{\mathcal{N}_\eta^{\bar{\mathrm{n}} - n_{\mathcal{O}}}} \exp\left(-\beta C \left(\sum_{(i,j) \in J} \mathsf{g}(x_i - z_j) + \mathsf{F}(Z_{\bar{\mathrm{n}} - n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h, h])\right)\right) dZ_{\bar{\mathrm{n}} - n_{\mathcal{O}}} d\rho^{\otimes n}(X_n),$$

where we have replaced $\rho(\mathcal{N})$ with the Lebesgue measure in the bulk and have grouped the error estimate from Corollary 6.3 into

$$\varepsilon_e = C \begin{cases} |\bar{\mathrm{n}} - n| + C_\beta \epsilon L^{\mathsf{d}} & \text{in case } (1) \\ |n - \bar{\mathrm{n}}| + \left(\frac{h^2}{\tilde{\ell}} + \tilde{\ell} + h^{-\gamma}\right) L^{\mathsf{d}-1} + C_\beta \left(\frac{L^2}{\tilde{\ell}} + L\right) M L^{\mathsf{d}} h^{\mathsf{s}_+ - \mathsf{d} - 1} & \text{in case } (2) \end{cases}$$

with a as in (6.25). Notice that this computation is valid for outer screening as well with appropriately modified screening errors, since we only modify the configuration in the bulk.

We will now make use of Jensen's inequality coupled with a tilt as in the proof of Proposition 5.4. The argument is exactly as in [AS21, Proposition 4.2] and [Pei24, Proposition 3.5]. Integrating against $\tilde{\mu}$ instead of Lebesgue measure, we can rewrite the interior integral as

$$I = \int_{\mathcal{N}_\eta^{\bar{n}-n_{\mathcal{O}}}} \exp\Bigg(-\beta C \left(\sum_{(i,j)\in J} \mathsf{g}(x_i - z_j) + \mathsf{F}(Z_{\bar{n}-n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h,h]) \right) - \sum_{i=1}^{\bar{n}-n_{\mathcal{O}}} \log \tilde{\mu}(z_i) \Bigg)\, d\tilde{\mu}|_{\mathcal{N}_\eta}^{\otimes \bar{n}-n_{\mathcal{O}}}(Z_{\bar{n}-n_{\mathcal{O}}}).$$

Applying Jensen's inequality, we then have

$$I \geq \tilde{\mu}(\mathcal{N}_\eta)^{\bar{n}-n_{\mathcal{O}}} \exp(A + B + C)$$

with

$$A = \tilde{\mu}(\mathcal{N}_\eta)^{n_{\mathcal{O}}-\bar{n}} \int_{\mathcal{N}_\eta^{\bar{n}-n_{\mathcal{O}}}} -\beta C \sum_{(i,j)\in J} \mathsf{g}(x_i - z_j)\, d\tilde{\mu}|_{\mathcal{N}_\eta}^{\otimes \bar{n}-n_{\mathcal{O}}}(Z_{\bar{n}-n_{\mathcal{O}}})$$

$$B = \tilde{\mu}(\mathcal{N}_\eta)^{n_{\mathcal{O}}-\bar{n}} \int_{\mathcal{N}_\eta^{\bar{n}-n_{\mathcal{O}}}} -\beta C \mathsf{F}(Z_{\bar{n}-n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h,h])\, d\tilde{\mu}|_{\mathcal{N}_\eta}^{\otimes \bar{n}-n_{\mathcal{O}}}(Z_{\bar{n}-n_{\mathcal{O}}})$$

$$C = \tilde{\mu}(\mathcal{N}_\eta)^{n_{\mathcal{O}}-\bar{n}} \int_{\mathcal{N}_\eta^{\bar{n}-n_{\mathcal{O}}}} - \sum_{i=1}^{\bar{n}-n_{\mathcal{O}}} \log \tilde{\mu}(z_i)\, d\tilde{\mu}|_{\mathcal{N}_\eta}^{\otimes \bar{n}-n_{\mathcal{O}}}(Z_{\bar{n}-n_{\mathcal{O}}}).$$

Recall that $\tilde{\mu}(\mathcal{N}_\eta) = \bar{n} - n_{\mathcal{O}}$. B is dealt with immediately by Proposition 5.4, and A and C are dealt with as in [AS21, Proposition 4.2] and [Pei24, Proposition 3.5], namely

$$A \gtrsim -\beta \# I_\partial \gtrsim \frac{-\beta \mathcal{C}_\beta h L^{\mathsf{d}-1}}{\tilde{\ell}}, \qquad C \gtrsim \tilde{\ell}^{\mathsf{d}}$$

where we have used $S(X_n, w, h) \leq \mathcal{C}_\beta h L^{\mathsf{d}-1}$ (see (6.31)) to control $\# I_\partial \lesssim \frac{S(X_n,w,h)}{\tilde{\ell}}$.

Applying a mean value argument, we then find that for some X_n^0 we have

$$\bar{n}^{\bar{n}} \mathsf{K}_{\beta,L}(\Omega,\mu,\zeta) \geq \frac{\eta}{C\tilde{\ell}} \int_{\mathcal{G}_\Omega} \exp\left(-\beta \mathsf{G}_{a,h}^{\mathrm{inn}}(X_n, \Omega \times [-L,L]) - C\beta\varepsilon_e \right)\, d\rho^{\otimes n}(X_n) \times \frac{\binom{\bar{n}}{n_{\mathcal{O}}(X_n^0)} (\bar{n} - n_{\mathcal{O}})^{\bar{n}-n_{\mathcal{O}}}}{\binom{n}{n_{\mathcal{O}}(X_n^0)} |\mathcal{N}(X_n^0)|^{n-n_{\mathcal{O}}(X_n^0)}} \exp\left(-C\beta \left(\tilde{\ell}^{\mathsf{d}} + \frac{\mathcal{C}_\beta h L^{\mathsf{d}-1}}{\tilde{\ell}} \right) - C\tilde{\ell}^{\mathsf{d}} \right).$$

Rearranging and absorbing $\tilde{\ell}^{\mathsf{d}} + \frac{\mathcal{C}_\beta h L^{\mathsf{d}-1}}{\tilde{\ell}}$ into the definition of ε_e yields

$$\int_{\mathcal{G}_\Omega} \exp(-\beta \mathsf{G}_{a,h}^{\mathrm{inn}}(X_n, \Omega \times [-L,L]))\, d\rho^{\otimes n}(X_n) \leq \bar{n}^{\bar{n}} \mathsf{K}_{\beta,L}(\Omega,\mu,\zeta) \exp\left(\beta C \varepsilon_e + \varepsilon_v \right)$$

with

$$\varepsilon_e = \begin{cases} |\bar{n} - n| + C_\beta \epsilon L^{\mathsf{d}} + \tilde{\ell}^{\mathsf{d}} + \frac{\mathcal{C} L^{\mathsf{d}}}{\tilde{\ell}} & \text{in case (1)} \\ |\bar{n} - n| + \tilde{\ell}^{\mathsf{d}} + \frac{\mathcal{C} h L^{\mathsf{d}-1}}{\tilde{\ell}} + \left(\frac{h^2}{\tilde{\ell}} + \tilde{\ell} + h^{-\gamma} \right) L^{\mathsf{d}-1} + C_\beta \left(\frac{L^2}{\tilde{\ell}} + L \right) M L^{\mathsf{d}} h^{\mathsf{s}_+ - \mathsf{d} - 1} & \text{in case (2)} \end{cases}$$

where we have inserted the definition of a from (6.25) and

$$\varepsilon_v = C\tilde{\ell}^{\mathsf{d}} + \log \frac{C\tilde{\ell}}{\eta} + \log \left(\frac{n!(\bar{n} - n_{\mathcal{O}})! |\mathcal{N}|^{n-n_{\mathcal{O}}}}{\bar{n}!(n - n_{\mathcal{O}})!(\bar{n} - n_{\mathcal{O}})^{\bar{n}-n_{\mathcal{O}}}} \right).$$

We need now to make sure the volume error is not too big. Computing directly with Stirling's formula as in [AS21, Proposition 4.2] we find

$$\varepsilon_v = C\tilde{\ell}^{\mathsf{d}} + \log \frac{C\tilde{\ell}}{\eta} + n\log n - \bar{\mathrm{n}}\log\bar{\mathrm{n}} - (n-n_{\mathcal{O}})\log\frac{n-n_{\mathcal{O}}}{|\mathcal{N}|} + \frac{1}{2}\log\frac{n(\bar{\mathrm{n}}-n_{\mathcal{O}})}{\bar{\mathrm{n}}(n-n_{\mathcal{O}})}.$$

Pulling out the $n\log n - \bar{\mathrm{n}}\log\bar{\mathrm{n}}$ from ε_v, we now have

$$\int_{\mathcal{G}_\Omega} \exp(-\beta \mathsf{G}^{\mathrm{inn}}_{a,h}(X_n, \Omega\times[-2h,2h]))\, d\rho^{\otimes n}(X_n) \le n^n \mathsf{K}_{\beta,L}(\Omega,\mu,\zeta)\exp\left(\beta C\varepsilon_e + \varepsilon_v\right)$$

with ε_e unchanged and

$$\varepsilon_v = C\tilde{\ell}^{\mathsf{d}} + \log\frac{\mathcal{C}_\beta\tilde{\ell}}{\eta} - (n-n_{\mathcal{O}})\log\frac{n-n_{\mathcal{O}}}{|\mathcal{N}|} + \frac{1}{2}\log\frac{n(\bar{\mathrm{n}}-n_{\mathcal{O}})}{\bar{\mathrm{n}}(n-n_{\mathcal{O}})}$$

Let $\alpha = \tilde{\mu}(\mathcal{N})$ and $\alpha' = |\mathcal{N}|$. Then, observe that $n - n_{\mathcal{O}} = \alpha + n - \bar{\mathrm{n}}$ since $\alpha = \bar{\mathrm{n}} - n_{\mathcal{O}}$ and so we can write

$$\begin{aligned}\varepsilon_v &= C\tilde{\ell}^{\mathsf{d}} + \log\frac{\mathcal{C}_\beta\tilde{\ell}}{\eta} + (\bar{\mathrm{n}}-n-\alpha)\log\frac{\alpha+n-\bar{\mathrm{n}}}{\alpha'} + \frac{1}{2}\log\frac{n\alpha}{\bar{\mathrm{n}}(\alpha+n-\bar{\mathrm{n}})}\\ &= C\tilde{\ell}^{\mathsf{d}} + \log\frac{\mathcal{C}_\beta\tilde{\ell}}{\eta} + (\bar{\mathrm{n}}-n-\alpha)\log\frac{\alpha}{\alpha'} + \left(\bar{\mathrm{n}}-n-\alpha-\frac{1}{2}\right)\log\left(1+\frac{n-\bar{\mathrm{n}}}{\alpha}\right) + \frac{1}{2}\log\frac{n}{\bar{\mathrm{n}}},\end{aligned}$$

as desired.

$\square$

Remark 6.5. *The error contributions over Ω where n points fall and Ω^c where $N-n$ point fall add up to a well-bounded error. To be precise, if α, α' and γ, γ' are two pairs as in Proposition 6.4, we have*

$$\begin{aligned}&(\bar{\mathrm{n}}-n-\alpha)\log\frac{\alpha}{\alpha'} + \left(\bar{\mathrm{n}}-n-\alpha-\frac{1}{2}\right)\log\left(1+\frac{n-\bar{\mathrm{n}}}{\alpha}\right) + \frac{1}{2}\log\frac{n}{\bar{\mathrm{n}}}\\ &\qquad + (\bar{\mathrm{n}}-n-\gamma)\log\frac{\gamma}{\gamma'} + \left(\bar{\mathrm{n}}-n-\gamma-\frac{1}{2}\right)\log\left(1+\frac{n-\bar{\mathrm{n}}}{\gamma}\right) + \frac{1}{2}\log\frac{n}{\bar{\mathrm{n}}} \lesssim \tilde{\ell}^{\mathsf{d}}.\end{aligned}$$

Proof. The proof is exactly as in [Ser24, Remark 8.4] and [Pei24, Remark 3.8]. $\square$

6.3. **Conclusion.** We now have all of the tools to complete the proof of Proposition 6.1. We specialize to Case 1 of the previous results, focusing on the local laws bootstrap. Coupling Proposition 6.4 with Remark 6.5 and a subadditivity argument yields first the following control on the level of exponential moments.

Proposition 6.6. *Keep the same assumptions as in Corollary 6.3 and above results, and let $\mathcal{G}$ denote the good event from Corollary 6.3. Denote by $\mathcal{G}^n$ the configurations in $\mathcal{G}$ who have n points in Ω. Let u be as in (5.7). Then, if $\mathcal{C}_\beta$ is chosen large enough depending on $\mathsf{d}, \mathsf{s}, \|\mu\|_{L^\infty}, \varepsilon$ (and β for $\beta \le 1$),*

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_{\beta,\frac{L}{K}}(U,\mu,\zeta)}\left(\exp\left(\frac{\beta}{2}\left(\mathsf{F}^{\Omega\times[-L,L]}(X_N,\mu,\Lambda) + 2C_0\#I_\Omega\right)\right)\mathbf{1}_{\mathcal{G}^n}\right)\\ &\qquad\qquad \le C\frac{\mathsf{K}_{\frac{\beta}{2},\frac{L}{K}}(\Omega,\mu,\zeta)}{\mathsf{K}_{\beta,\frac{L}{K}}(\Omega,\mu,\zeta)}\exp\left(\beta\frac{\mathcal{C}_\beta}{8}L^{\mathsf{d}} + C\beta|\bar{\mathrm{n}}-n|\right)\end{aligned}$$

for some constant C dependent only on the upper and lower bounds of μ.

Proof. We recall that $\Omega = Q_L \cap \Lambda$ and we denote $\Omega_L = \Omega \times [-L, L]$ and $\mathring{\Omega}_L = Q_{L-2\tilde{\ell}} \cap \Lambda$ where $Q_{L-2\tilde{\ell}}$ is the hyperectangle with same center as Q_L.

The proof relies on the following superadditivity computation based on (5.15), using also (5.16) and (5.40) (here we drop the μ dependence in the notation for F).

$$\begin{aligned}\frac{\beta}{2}\mathsf{F}^{\mathring{\Omega}_L}(X_N,\Lambda) - \beta\mathsf{F}(X_N,\Lambda) &\le \frac{\beta}{2}\mathsf{F}^{\mathring{\Omega}_L}(X_N,\Lambda) - \beta\mathsf{F}^{\Omega_L}(X_N,\Lambda) - \beta\mathsf{F}^{\Lambda\setminus\Omega_L}(X_N,\Lambda)\\ &\le \frac{\beta}{2}\mathsf{F}^{\mathring{\Omega}_L}(X_N,\Lambda) - \frac{\beta}{2}\mathsf{F}^{\Omega_L}(X_N,\Lambda) - \frac{\beta}{2}\mathsf{F}^{\Omega_L}(X_N,\Lambda) - \beta\mathsf{F}^{\Lambda\setminus\Omega_L}(X_N,\Lambda)\\ &\le -\frac{\beta}{2}\mathsf{F}^{\Omega_L\setminus\mathring{\Omega}_L}(X_N,\Lambda) - \frac{\beta}{2}\mathsf{F}^{\Omega_L}(X_N,\Lambda) - \beta\mathsf{F}^{\Lambda\setminus\Omega_L}(X_N,\Lambda)\\ &\le -\frac{\beta}{2}\mathsf{F}^{\Omega_L}(X_N,\Lambda) - \beta\mathsf{F}^{\Lambda\setminus\Omega_L}(X_N,\Lambda) + \frac{\beta}{2}C_0 n\\ &\le -\frac{\beta}{2}\mathsf{G}^{\mathrm{inn}}_{a,\frac{L}{K}}(X_N|_\Omega,\Omega_L) - \beta\mathsf{G}^{\mathrm{ext}}_{a,\frac{L}{K}}(X_N|_{\Omega^c},\Lambda\setminus\Omega_L) + \frac{\beta}{2}C_0 n\end{aligned}$$

with $a = C_\beta\epsilon L^{\mathsf{d}-1}$ as in (6.25). We can then compute using Proposition 6.4 exactly as in [Pei24, Proposition 3.7]

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_{\beta,h}(U,\mu,\zeta)}\left(\exp\left(\frac{\beta}{2}\mathsf{F}^{\mathring{\Omega}_L}(\cdot,\Lambda)\right)\mathbf{1}_{\mathcal{G}_n}\right)\\ &\le \frac{1}{N^N\mathsf{K}_{\beta,\frac{L}{K}}(U,\mu,\zeta)}\binom{N}{n}\int_{\Omega^n\cap\mathcal{G}^\Omega}\exp\left(-\frac{\beta}{2}\mathsf{G}^{\mathrm{inn}}_{a,\frac{L}{K}}(X_n,\Omega_L) + \frac{\beta}{2}C_0 n\right)\,d\rho^{\otimes n}(X_n)\\ &\times\int_{(\Omega^c)^{N-n}\cap\mathcal{G}^{\Omega^c}}\exp\left(-\beta\mathsf{G}^{\mathrm{ext}}_{a,\frac{L}{K}}(X_{N-n},\Omega_L^c)\right)\,d\rho^{\otimes N-n}(X_{N-n})\\ &\le \frac{\binom{N}{n}}{N^N\mathsf{K}_{\beta,\frac{L}{K}}(U,\mu,\zeta)}n^n(N-n)^{N-n}\mathsf{K}_{\frac{\beta}{2},L/K}(\Omega,\mu,0)\,\mathsf{K}_{\beta,L/K}(\Omega^c,\mu,0)\exp\left(\beta\varepsilon_e + \varepsilon_v + \frac{\beta}{2}C_0 n\right)\end{aligned}$$

where ε_e and ε_v denote the energy and volume errors

$$\varepsilon_e = C(C_\beta\epsilon L^{\mathsf{d}} + \tilde{\ell}^{\mathsf{d}} + |\bar{\mathrm{n}} - n|) \qquad \varepsilon_v = C\tilde{\ell}^{\mathsf{d}} + \log\frac{\mathcal{C}_\beta\tilde{\ell}}{\eta},$$

where we have dropped the $\frac{L^{\mathsf{d}}}{\tilde{\ell}}$ error because it is controlled by the remaining terms. The superadditivity of the Neumann partition functions (5.21) then yields

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_{\beta,\frac{L}{K}}(U,\mu,\zeta)}\left(\exp\left(\frac{\beta}{2}\mathsf{F}^{\mathring{\Omega}_L}(\cdot,\Lambda)\right)\mathbf{1}_{\mathcal{G}_n}\right)\\ &\qquad\le \frac{\bar{\mathrm{n}}!(N-\bar{\mathrm{n}})!n^n(N-n)^{N-n}}{n!(N-n)!\bar{\mathrm{n}}^{\bar{\mathrm{n}}}(N-\bar{\mathrm{n}})^{N-\bar{\mathrm{n}}}}\frac{\mathsf{K}_{\frac{\beta}{2},\frac{L}{K}}(\Omega,\mu,\zeta)}{\mathsf{K}_{\beta,\frac{L}{K}}(\Omega,\mu,\zeta)}\exp\left(\beta\varepsilon_e + \varepsilon_v + \frac{\beta}{2}C_0 n\right).\end{aligned}$$

Stirling's formula yields

$$\mathbb{E}_{\mathbb{Q}_{\beta,h}(U,\mu,\zeta)}\left(\exp\left(\frac{\beta}{2}\mathsf{F}^{\mathring{\Omega}_L}(\cdot,\Lambda)\right)\mathbf{1}_{\mathcal{G}^n}\right) \lesssim \frac{\mathsf{K}_{\frac{\beta}{2},\frac{L}{K}}(\Omega,\mu,\zeta)}{\mathsf{K}_{\beta,\frac{L}{K}}(\Omega,\mu,\zeta)}\exp\left(\beta\varepsilon_e + \varepsilon_v + \frac{\beta}{2}C_0 n\right).$$

We then write $C_0 n \le C_0 \bar{n} + C_0|n-\bar{n}|$, and assume (without loss of generality) that $\mathcal{C}_\beta \ge 8\|\mu\|_{L^\infty} C_0$, which ensures that $C_0\bar{n} \le \frac{\mathcal{C}}{8}L^{\mathsf{d}}$. Choosing $\eta = \frac{\ell m}{6\|\mu\|_{L^\infty}}$, ϵ small enough in Proposition 6.2 and K_1 defining $\tilde{\ell} = \frac{L}{K_1}$ large enough in Corollary 6.3 makes ε_e small enough and we obtain the result. □

6.3.1. *Proof of Proposition 6.1.* We establish that

$$\mathbb{E}_{\mathbb{Q}_{\beta,h}(U,\mu,\zeta)}\left(\exp\left(\frac{\beta}{2}\left(\mathsf{F}^{\Omega_L}(\cdot,\Lambda)+2C_0\#I_\Omega\right)\right)\mathbf{1}_{\mathcal{G}}\right) \le \exp\left(\mathcal{C}_\beta\frac{\beta}{4}L^{\mathsf{d}}\right). \tag{6.34}$$

To prove this, we sum the control in Proposition 6.6 over all possible n. First, we can restrict the number of n needed to consider by using discrepancy estimates. Namely, using Proposition 3.5 coupled with Theorem 1 at the length scale $2L$ yields that either $|n-\bar{n}| \lesssim L^{\mathsf{d}-1}$ or

$$|\bar{n}-n| \le K\max\left(\mathcal{C}_\beta^{1/2}L^{\frac{\mathsf{d}+\mathsf{s}}{2}}, \mathcal{C}_\beta^{1/3}L^{\frac{2\mathsf{d}+\mathsf{s}}{3}}\right) \le K\mathcal{C}_\beta^{1/2}L^{\frac{2\mathsf{d}+\mathsf{s}}{3}},$$

for some $K>0$, since $\mathsf{s}<\mathsf{d}$. Thus,

$$\begin{aligned}&\sum_{|\bar{n}-n|\le K\mathcal{C}_\beta^{1/2}L^{\frac{2\mathsf{d}+\mathsf{s}}{3}}}\mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\frac{\beta}{2}\left(\mathsf{F}^{\Omega_L}(\cdot,\Lambda)+2C_0\#I_\Omega\right)\right)\mathbf{1}_{\mathcal{G}^n}\right)\\ &\quad\lesssim \sum_{|\bar{n}-n|\le K\mathcal{C}_\beta^{1/2}L^{\frac{2\mathsf{d}+\mathsf{s}}{3}}}\frac{\mathsf{K}_{\frac{\beta}{2},\frac{L}{K}}(\Omega,\mu,\zeta)}{\mathsf{K}_{\beta,\frac{L}{K}}(\Omega,\mu,\zeta)}\exp\left(\beta\left(\frac{\mathcal{C}_\beta}{8}L^{\mathsf{d}}+C_0\bar{n}+CC_0K\mathcal{C}_\beta^{1/2}L^{\frac{2\mathsf{d}+\mathsf{s}}{3}}\right)\right).\end{aligned}$$

Using Proposition 5.4, we can control the ratio of partition functions uniformly by $e^{C(1+\beta)L^{\mathsf{d}}}$. The remaining error terms are bounded at strictly smaller order. Making $\mathcal{C}_\beta$ larger (but still scale independent and β-independent for $\beta \ge 1$) if necessary, in particular $\mathcal{C}_\beta \ge 1$, we find (6.34). A Chernoff bound immediately implies

$$\begin{aligned}&\mathbb{P}_{N,\beta}\left(\left\{\mathsf{F}^{\Omega_L}(\cdot,\Lambda)+2C_0\#I_\Omega \ge \mathcal{C}_\beta L^{\mathsf{d}}\right\}\cap\mathcal{G}\right)\\ &\quad\le e^{-\frac{\beta}{2}\mathcal{C}_\beta L^{\mathsf{d}}}\mathbb{E}_{\mathbb{P}_{N,\beta}}\left(\exp\left(\frac{\beta}{2}\left(\mathsf{F}^{\Omega_L}(\cdot,\Lambda)+2C_0\#I_\Omega\right)\mathbf{1}_{\mathcal{G}}\right)\right) \le e^{-\frac{\beta}{4}\mathcal{C}_\beta L^{\mathsf{d}}} \le e^{-\frac{\beta}{4}L^{\mathsf{d}}},\end{aligned}\tag{6.35}$$

with $\mathcal{G}_L = \mathcal{G}$ as in (6.8). Coupling this with the bound in (6.8) and absorbing one constant into the other establishes Proposition 6.1, and as discussed after the statement of that proposition Theorem 1.

7. Expansion of the Next Order Partition Functions

The goal of this section is to leverage the local laws of Theorem 1 to establish a quantitative approximation of the local Neumann partition functions (5.17) in the form (4.24). This approximation is crucial in the approach in Section 4 to improve the error estimates in Theorem 2 and obtain the CLT, Theorem 3.

7.1. **Almost Additivity of the Next-Order Partition Functions.** We start with an almost additivity that generalizes the result for the Coulomb gas from [Ser24, Proposition 8.10] to nonCoulomb Riesz gases. The proof is similar in spirit, using the updated terminology and local laws for the Riesz gas alongside of the Riesz screening procedure Proposition 5.7. We recall the definitions of partition functions in (5.17) and (5.19).

Proposition 7.1. *Assume* $\mathsf{d} \geq 1$ *and* $\mathsf{s} \in (\mathsf{d}-2, \mathsf{d})$. *Let* $U \subset \mathbb{R}^{\mathsf{d}}$ *be a bounded, open set with piecewise* C^1 *boundary, and suppose as well that* μ *is bounded above and below in* U. *Assume that Theorem 1 holds for* μ *down to a minimal order one lengthscale* ρ_β.

Assume that $U \subset \hat{\Sigma}'$ *can be written as a disjoint union of* p *hyperrectangles* Q_i *with sidelengths in* $[L, 2L]$ *with* $L \geq \rho_\beta$ *such that* $\mu(Q_i) = N_i$ *integer. Let* $h_1, \dots, h_p \in [L, R]$. *Then, for any event* $\mathcal{G}$ *there is some constant* $C > 0$, *depending only on* d *and* μ *such that*

$$\left|\log \mathsf{K}^{\mathcal{G}}_{\beta,\infty}(\mathbb{R}^{\mathsf{d}}, \mu, \zeta) - \left(\log \mathsf{K}^{\mathrm{ext},\mathcal{G}_U}_{\beta,R}(U^c, \mu, \zeta) + \sum_{i=1}^{p} \log \mathsf{K}^{\mathcal{G}_{Q_i}}_{\beta,h_i}(Q_i, \mu, \zeta)\right)\right| \lesssim_\beta R^{\mathsf{d}} \mathscr{E}(L) \tag{7.1}$$

where $\mathcal{G}_\Omega$ *denotes the set of configurations in* Ω *that are restrictions of a configuration in* $\mathcal{G}$ *and the error rate* $\mathscr{E}(L) = o(1)$ *is given by*

$$\mathscr{E}(L) := \begin{cases} L^{-1+\frac{4}{4+\mathsf{d}-\mathsf{s}_+}} \log L & \text{if } \mathsf{d} \leq 5 \text{ or } \mathsf{s} < \mathsf{d}-1 \\ L^{-1+\frac{\mathsf{d}-1}{(\mathsf{d}-1)+(\mathsf{d}-\mathsf{s})}} \log L & \text{otherwise.} \end{cases} \tag{7.2}$$

Additionally, we have

$$\left|\log \mathsf{K}_{\beta,R}(U, \mu, \zeta) - \sum_{i=1}^{p} \log \mathsf{K}_{\beta,h_i}(Q_i, \mu, \zeta)\right| \lesssim_\beta R^{\mathsf{d}} \mathscr{E}(L). \tag{7.3}$$

As we will see below, a key ingredient in optimizing the error rate $\mathcal{R}$ is that we have local laws valid down to the microscale, which will allow us to make use of Remark 5.8 to screen at smaller heights than we were able to in the proof of Theorem 1.

Proof. We give the proof for $\mathcal{G} = \mathbb{R}^{\mathsf{d}N}$, it is a straightforward modification to extend to general $\mathcal{G}$. As in the proof of [Ser24, Proposition 8.10], it suffices to prove upper bounds since the corresponding lower bounds follow from the superadditivity result (5.21) and Stirling's formula; shrinking the heights from h to h_i is a consequence of the same argument as in the proof of (5.20). Namely, we can decrease h by appending a zero electric field for $|y| > h_i$ in the extended dimension to the electric field defining $\mathsf{F}(X_N|_{Q_i}, \mu, Q_i \times [-h_i, h_i])$ and applying Lemma 5.1.

The key input is that we now have local laws down to the minimal length scale in $\hat{\Sigma}'$.

Step 1: restricting to a good event. Let N denote the number of points in U, n_i be the number of points in Q_i, and $\bar{\mathrm{n}}_i = \mu(Q_i)$. We also set

$$\hat{Q}_i := \{x \in Q_i : \mathrm{dist}(x, \partial Q_i) \leq r\}$$

for r to be determined, and

$$\mathcal{G} = \Bigg\{X_N \in (\mathbb{R}^{\mathsf{d}})^N : |n_i - \bar{\mathrm{n}}_i| \leq \epsilon \ \forall i, \ \sup_x \int_{(\hat{Q}_1 \cap \square_r(x)) \times [-r,r]} |y|^\gamma |\nabla u_{\mathsf{r}}|^2 \leq \mathcal{C}_\beta r^{\mathsf{d}}, \\ e(X_n, u, r) \leq C_\beta M L^{\mathsf{d}} r^{(\mathsf{s})_+ - \mathsf{d} - 1}\Bigg\} \tag{7.4}$$

with $\epsilon, \tilde{\ell}, M, K$ and r to be determined and u the potential (5.7) for $\Lambda = \mathbb{R}^{\mathsf{d}+1}$.

The supremum condition is satisfied by $O\left(\frac{L^{\mathsf{d}-1}}{r^{\mathsf{d}-1}}\right)$ applications of the local law Theorem 1, each whose complement has probability bounded by $\exp\left(-C\beta r^{\mathsf{d}}\right)$. The control on $e(X_n, w, r)$

follows from Proposition 6.2 on a further restricted event for an appropriate C_β; applying Proposition 6.2, and using from Theorem 1 that $\mathbb{P}_{N,\beta}(\mathcal{G}_r^c) \le \left(\frac{L}{r}\right)^{\mathsf{d}} e^{-\beta r^{\mathsf{d}}}$, we find

$$\mathbb{P}_{N,\beta}(\mathcal{G}^c) \le \left(\frac{L}{r}\right)^{\mathsf{d}} e^{-Cr^{\mathsf{d}}} + M^{-\frac{\mathsf{d}}{2}} L^{\mathsf{d}} h^{\frac{\mathsf{d}(\mathsf{d}-\mathsf{s}_+-2)}{2}} e^{-CM} + \sum_{i=1}^{p} \mathbb{P}_{N,\beta}(|n_i - \bar{\mathrm{n}}_i| > \epsilon)$$

redefining C.

Let's next analyze the event with bad discrepancy. Applying the discrepancy estimate (3.29)–(3.30), covering $\Omega_\delta \setminus \Omega$, where $\Omega = Q_i$ by $\frac{L^{\mathsf{d}-1}\delta}{\mathfrak{R}^{\mathsf{d}}}$ balls of size $\mathfrak{R} > \rho_\beta$, and using Theorem 1 at scale $\mathfrak{R}$ one finds for each i that

$$|n_i - \bar{\mathrm{n}}_i| \lesssim L^{\mathsf{d}-1}\delta + \sqrt{L^{2\mathsf{d}-1+\gamma}\frac{\mathfrak{R}}{\delta}} = \epsilon \tag{7.5}$$

off of an event of size

$$\mathbb{P}_{N,\beta}(|n_i - \bar{\mathrm{n}}_i| > \epsilon) \lesssim \frac{L^{\mathsf{d}-1}\delta}{\mathfrak{R}^{\mathsf{d}}} e^{-C\beta\mathfrak{R}^{\mathsf{d}}}.$$

Hence we find

$$\mathbb{P}_{N,\beta}(\mathcal{G}^c) \le \left(\frac{L}{r}\right)^{\mathsf{d}} e^{-Cr^{\mathsf{d}}} + L^{\mathsf{d}} r^{\frac{\mathsf{d}(\mathsf{s}_+-(\mathsf{d}-2))}{2}} e^{-CM} + Cp\frac{L^{\mathsf{d}-1}\delta}{\mathfrak{R}^{\mathsf{d}}} e^{-C\mathfrak{R}^{\mathsf{d}}} \tag{7.6}$$

again redefining C. The third term in (7.6) will be smaller than $\frac{1}{4}$ as soon as we take $\mathfrak{R}^{\mathsf{d}} \gtrsim \log L$ for large enough constant; comparing the terms in the discrepancy bound (7.5) then and optimizing leads to taking $\delta = L^{\frac{1+\gamma}{3}}$, and

$$|n_i - \bar{\mathrm{n}}_i| \lesssim L^{\mathsf{d}-1+\frac{1+\gamma}{3}} \log^{\frac{1}{2\mathsf{d}}} L := \epsilon. \tag{7.7}$$

We will optimize the remaining parameters in the above in such a way that the probability in (7.6) is no more than $\frac{1}{2}$.

Step 2: superadditivity and screening. First, we write

$$N^{-N}\int_{\mathcal{G}^c} \exp\left(-\beta \mathsf{F}\left(X_N, \mu, \mathbb{R}^{\mathsf{d}+1}\right)\right)\, d\rho^{\otimes N} = \mathsf{K}_{\beta,\infty}\left(\mathbb{R}^{\mathsf{d}}, \mu, \zeta\right) \mathbb{P}_{N,\beta}(\mathcal{G}^c) \le \frac{1}{2}\mathsf{K}_{\beta,\infty}\left(\mathbb{R}^{\mathsf{d}}, \mu, \zeta\right)$$

where we are again using $\rho(x)\, dx$ as notation for the measure $e^{-\beta\zeta(x)}\, dx$, under the assumption that we have tuned the parameters in (7.6) to guarantee $\mathbb{P}_{N,\beta}(\mathcal{G}^c) \le \frac{1}{2}$.

Then, via the superadditivity (5.15) and (5.40),

$$\begin{aligned}
\frac{N^N}{2}\mathsf{K}_{\beta,\infty}\left(\mathbb{R}^{\mathsf{d}}, \mu, \zeta\right) &\le \int_{\mathcal{G}} \exp\left(-\beta \mathsf{F}\left(X_N, \mu, \mathbb{R}^{\mathsf{d}+1}\right)\right)\, d\rho^{\otimes N} \\
&\le \sum_{\substack{|n_i - \bar{\mathrm{n}}_i| \le \epsilon, \\ \sum n_i = N}} \binom{N}{n_1,\, n_2, \ldots, n_p} \prod_{i=1}^{p} \int_{\mathcal{G}} \exp\left(-\beta \mathsf{G}_{b,r}^{\mathrm{inn}}(\cdot, Q_i \times [-h_i, h_i])\right)\, d\rho^{\otimes n}(X_n) \\
&\qquad \times \int_{\mathcal{G}} \exp\left(-\beta \mathsf{G}_{b,r}^{\mathrm{ext}}\left(\cdot, (U \times [-R, R])^c\right)\right)\, d\rho^{\otimes N-n}(X_{N-n}).
\end{aligned}$$

where b denotes the expression $C_\beta M L^{\mathsf{d}} r^{\mathsf{s}_+-\mathsf{d}-1}$ controlling $e(X_n, u, r)$ and we have replaced $Q_i \times [-L, L]$ in the equation above with any $h_i \in [L, R]$ using Remark 5.8.

We next replace the energies G in the exponents with the corresponding Neumann energies (5.8) using the screening on the level of partition functions, Proposition 6.4. Applying Remark 6.5, we obtain

(7.8)
$$\mathsf{K}_{\beta,\infty}\left(\mathbb{R}^{\mathsf{d}},\mu,\zeta\right)\le 2\prod_{i=1}^{p}\mathsf{K}_{\beta,h_i}(Q_i,\mu,\zeta)\mathsf{K}^{\mathrm{ext}}_{\beta,R}(U^c,\mu,\zeta)\sum_{\substack{|n_i-\bar{\mathrm{n}}_i|\le\epsilon,\\ \sum n_i=N}}\binom{N}{\bar{\mathrm{n}}_1,\ \bar{\mathrm{n}}_2,\ldots,\bar{\mathrm{n}}_p}N^{-N}\prod_{i=1}^{p}\bar{\mathrm{n}}_i^{\bar{\mathrm{n}}_i}$$
$$\times\exp\left(Cp\left(\epsilon+\tilde{\ell}^{\mathsf{d}}+\left(\frac{r^2}{\tilde{\ell}}+\tilde{\ell}+r^{-\gamma}\right)L^{\mathsf{d}-1}+C_\beta\frac{L}{\min(r,\tilde{\ell})}\left(\frac{L^2}{\tilde{\ell}}+L\right)ML^{\mathsf{d}}r^{\mathsf{s}_+-\mathsf{d}-1}\right)\right)$$

using $r<L$, provided (6.28) holds and, if $\mathsf{s}\ge\mathsf{d}-1$, $r\gg L^{\frac{\mathsf{d}-1}{\mathsf{d}-\gamma}}$. An application of Stirling's formula yields
$$\binom{N}{\bar{\mathrm{n}}_1,\ \bar{\mathrm{n}}_2,\ldots,\bar{\mathrm{n}}_p}N^{-N}\prod_{i=1}^{p}\bar{\mathrm{n}}_i^{\bar{\mathrm{n}}_i}\le C\sqrt{\frac{N}{\prod_{i=1}^{p}\bar{\mathrm{n}}_i}}\le C$$
and so the summands are controlled by $C\epsilon p$.

Step 3: error optimization. We now need to optimize the choice of r, with competing terms that prefer r smaller or larger. Observe that we want to take $M\ge1$ as small as possible while keeping the estimate in (7.6) small.

Notice that (7.6) can be made smaller than $\frac12$ for any r at order L^ν and $M\gtrsim\log L$. So, we may as well optimize

$$L^{\mathsf{d}-1+\frac{1+\gamma}{3}}\log^{\frac{1}{2\mathsf{d}}}L+\tilde{\ell}^{\mathsf{d}}+\left(\frac{r^2}{\tilde{\ell}}+\tilde{\ell}+r^{-\gamma}\right)L^{\mathsf{d}-1}+C_\beta\frac{L}{\min(r,\tilde{\ell})}\left(\frac{L^2}{\tilde{\ell}}\right)ML^{\mathsf{d}}r^{\mathsf{s}_+-\mathsf{d}-1}\tag{7.9}$$

in (7.8) using (7.7) subject to

$$C_\beta\frac{L^2}{\tilde{\ell}^2}Mr^{\mathsf{s}+\mathsf{s}_+-2\mathsf{d}}\le\mathsf{c},\qquad r\gg L^{\frac{\mathsf{d}-1}{\mathsf{d}-\gamma}}\text{ if }\mathsf{s}\ge\mathsf{d}-1,\tag{7.10}$$

where we have dropped L in the parenthesis in (7.9) because $\tilde{\ell}<L$ implies the $\frac{L^{2\mathsf{d}}}{\tilde{\ell}^{2\mathsf{d}-1}}$ term is dominant. Comparing the optimization problems for $\tilde{\ell}<r$ and $r<\tilde{\ell}$, one sees in either case that the optimum choice is as the parameters approach r. So, we select $\ell=\tilde{\ell}=r$, and choose $M=C\log L$ for C large enough. Making the simplification $\tilde{\ell}=r$, we see that we need to optimize (absorbing some terms)

$$L^{\mathsf{d}-1+\frac{1+\gamma}{3}}\log^{\frac{1}{2\mathsf{d}}}L+rL^{\mathsf{d}-1}+L^{\mathsf{d}+3}r^{\mathsf{s}_+-\mathsf{d}-3}M\tag{7.11}$$

subject to

$$C_\beta ML^2r^{\mathsf{s}+\mathsf{s}_+-2\mathsf{d}-2}\le\mathsf{c},\qquad r\gg L^{\frac{\mathsf{d}-1}{\mathsf{d}-\gamma}}\text{ if }\mathsf{s}\ge\mathsf{d}-1,\tag{7.12}$$

where we have absorbed the error term $\ell^{1-\mathsf{d}}r^{-\gamma}L^{\mathsf{d}-1}=r^{-\mathsf{s}}L^{\mathsf{d}-1}$ into $rL^{\mathsf{d}-1}$ since $\mathsf{s}>-1$. Balancing the second and third terms in (7.11) requires

$$r^{\mathsf{d}+4-\mathsf{s}_+}=ML^4.\tag{7.13}$$

The first term in (7.11) is always smaller with this choice of r, since $\frac{1+\gamma}{3}=\frac{\mathsf{s}+2-\mathsf{d}}{3}$ and
$$\frac{4}{\mathsf{d}+4-\mathsf{s}_+}>\frac{\mathsf{s}+2-\mathsf{d}}{3}\iff 12>(\mathsf{s}+2-\mathsf{d})(\mathsf{d}-\mathsf{s}_++4)$$

which follows from $\mathsf{s} \in (\mathsf{d}-2, \mathsf{d})$. Notice also that this choice of r satisfies (7.12). Indeed, for the first condition we observe that, with $\gamma = \mathsf{s}+1-\mathsf{d}$ and this choice of r,

$$C_\beta M L^2 r^{\mathsf{s}+\mathsf{s}_+-2\mathsf{d}-2} = \frac{C_\beta M L^2}{r^{\mathsf{d}+4-\mathsf{s}_+} r^{\mathsf{d}-2-\mathsf{s}}} = \frac{C_\beta r^{\mathsf{s}-(\mathsf{d}-2)}}{L^2} \ll 1$$

since $r < L$ and $\mathsf{s} > \mathsf{d}-2$. For the second condition, since $M = C\log L$ it is sufficient to check that $\frac{4}{4+(\mathsf{d}-\mathsf{s})} \geq \frac{\mathsf{d}-1}{\mathsf{d}-\gamma}$ (since $\mathsf{s} \geq \mathsf{d}-1 \implies \mathsf{s} \geq 0 \implies \mathsf{s}_+ = \mathsf{s}$). Indeed, since $\gamma = \mathsf{s}+1-\mathsf{d}$ this is true if and only if

$$\frac{4}{4+(\mathsf{d}-\mathsf{s})} \geq \frac{\mathsf{d}-1}{(\mathsf{d}-1)+(\mathsf{d}-\mathsf{s})}.$$

which is true for $\mathsf{d} \leq 5$ because for $a > 0$, the quantity $\frac{x}{x+a}$ is increasing in x. For $\mathsf{d} > 5$, we instead need to take then $r = CL^{\frac{\mathsf{d}-1}{\mathsf{d}-\gamma}} \log L$.

We conclude that

$$\mathsf{K}_{\beta,\infty}\left(\mathbb{R}^{\mathsf{d}}, \mu, \zeta\right) \leq C\epsilon p \prod_{i=1}^{p} \mathsf{K}_{\beta,h_i}(Q_i,\mu,\zeta)\mathsf{K}^{\mathrm{ext}}_{\beta,R}(U^c,\mu,\zeta)\exp\left(C_\beta L^{\mathsf{d}}\mathscr{E}(L)\right) \tag{7.14}$$

where $\mathscr{E}(L)$ is as in (7.2). Taking logarithms yields the result.

The proof of (7.3) is the same using the local laws for the measure associated to $\mathsf{K}_{\beta,R}(U,\mu,\zeta)$ □

The goal is now to use this to derive a precise asymptotic expansion of local partition functions. First, we prove the analogue of [AS21, Proposition 6.2] at constant density. It is largely a corollary of the previous proposition.

Lemma 7.2. *Assume $\mathsf{d} \geq 1$ and $\mathsf{s} \in (\mathsf{d}-2, \mathsf{d})$. Then, there exists a function $f_{\mathsf{d},\mathsf{s}} : (0,\infty) \to \mathbb{R}$ and a constant $C > 0$ depending only on d such that if $L^{\mathsf{d}} \in \mathbb{Z}$ we have*

$$\left|\frac{\log \mathsf{K}_{\beta,L}(\square_L, 1, 0)}{\beta|\square_L|} + f_{\mathsf{d},\mathsf{s}}(\beta)\right| \lesssim_\beta \mathscr{E}(L), \tag{7.15}$$

with $\mathscr{E}$ as in (7.2).

Proof. The proof is exactly as in [AS21, Proposition 6.2] using Proposition 7.1, with our new error term. One first uses superadditivity (5.20) of the partition function to derive

$$\frac{1}{\beta}\log \mathsf{K}_{\beta,2L}\left(\square_{2L},1,0\right) \geq O\left(\frac{\log N}{\beta}\right) + \frac{2^{\mathsf{d}}}{\beta}\log \mathsf{K}_{\beta,L}\left(\square_L,1,0\right)$$

and so, setting $\phi(L) := \frac{\log \mathsf{K}_{\beta,L}(\square_L,1,0)}{\beta L^{\mathsf{d}}}$ one has

$$\phi(2L) \geq \phi(L) + O\left(\frac{\log L}{\beta L^{\mathsf{d}}}\right).$$

Iterating the previous estimate, we find

$$\phi(\infty) \geq \phi(L) + O\left(\sum_{k=1}^{\infty} \frac{\log L}{\beta\left(2^k L\right)^{\mathsf{d}}}\right)$$

where $\phi(\infty)$ is defined by $\limsup_{L\to\infty}\phi(L)$, which simplifying yields

$$\phi(L) \leq \phi(\infty) + O\left(\frac{\log L}{\beta L^{\mathsf{d}}}\right)$$

as in [AS21]. On the other hand, using Proposition 7.1 we can write

$$\phi(2L) \leq \phi(L) + C_\beta \mathscr{E}(L)$$

which we can sum to find $\phi(\infty) \leq \phi(L) + C_\beta \mathscr{E}(L)$. Defining $f_{\mathsf{d},\mathsf{s}}(\beta) := -\phi(\infty)$ yields the result. □

This Lemma allows us to compute the next-order partition function for other constant densities as well; indeed, by scaling and setting $Q_L' = m^{\frac{1}{\mathsf{d}}} Q_L$, we find

$$\begin{aligned}\frac{\log \mathsf{K}_{\beta,\infty}(Q_L, m, 0)}{\beta |Q_L|} &= m^{1+\frac{\mathsf{s}}{\mathsf{d}}} \frac{\log \mathsf{K}_{\beta m^{\frac{\mathsf{s}}{\mathsf{d}}},\infty}(Q_L', 1, 0)}{\beta m^{\frac{\mathsf{s}}{\mathsf{d}}} |Q_L'|} - \frac{m}{\beta} \log m + \frac{1}{2\mathsf{d}} (m \log m) \mathbf{1}_{\mathsf{s}=0} \\ &= -m^{1+\frac{\mathsf{s}}{\mathsf{d}}} f_{\mathsf{d},\mathsf{s}}\left(\beta m^{\frac{\mathsf{s}}{\mathsf{d}}}\right) - \frac{m}{\beta} \log m + \frac{m}{2\mathsf{d}} (\log m) \mathbf{1}_{\mathsf{s}=0} + O\left(\mathscr{E}(L)\right)\end{aligned} \tag{7.16}$$

In what follows, we will need the assumption (1.45).

7.2. **Comparison of Partition Functions by Transport.** We would now like to leverage the formula (7.16) to obtain expansions for inhomogeneous densities μ. In this section we return to normal coordinates, since the expansions we will derive will be leveraged to obtain results about fluctuations of linear statistics stated in usual coordinates. Just as in [Ser24, Chapter 9], (5.17) has the counterpart at the usual scale

$$\mathsf{K}_{N,\beta,\ell}(U, \mu, \zeta) := \int_{U^N} \exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}} \left(\mathsf{F}_N(X_N, \mu, U \times [-\ell, \ell]) + N \sum_{i=1}^N \zeta(x_i)\right)\right) \, dX_N \tag{7.17}$$

where $\mathsf{F}_N(X_N, \mu, U \times [-\ell, \ell])$ and $N\zeta$ are analogous to $\mathsf{F}\left(X_N', \mu', UN^{\frac{1}{\mathsf{d}}} \times \left[-\ell N^{\frac{1}{\mathsf{d}}}, \ell N^{\frac{1}{\mathsf{d}}}\right]\right)$ and ζ' as in Section 5.1. A superscript $\mathcal{G}$ in (7.17) will indicate a restriction of the integral to some good event $\mathcal{G} \subset U^N$. As in (3.2), when $U = \mathbb{R}^{\mathsf{d}}$ and $h = \infty$ we abbreviate (7.17) by $\mathsf{K}_{N,\beta}$. We also define a Gibbs measure associated to this next-order partition function by

$$d\mathbb{Q}_{N,\beta,\ell}(U, \mu, \zeta) := \frac{1}{\mathsf{K}_{N,\beta,\ell}(U, \mu, \zeta)} \exp\left(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}} \left(\mathsf{F}_N(X_N, \mu, U \times [-\ell, \ell]) + N \sum_{i=1}^N \zeta(x_i)\right)\right) \, dX_N. \tag{7.18}$$

If μ is sufficiently smooth then it cannot vary too much in a small cube of sidelengths $\ell \ll 1$. In particular then, we should be able to compare $\mathsf{K}_{N,\beta,\ell}(Q_\ell, \mu, \zeta_V)$ to that previously obtained for constant densities, more precisely $\mathsf{K}_{N,\beta,\ell}\left(Q_\ell, \fint_{Q_\ell} \mu, \zeta\right)$. We will make this comparison explicit by a transport as in [Ser24, Chapter 9]. First, let us recall some information about transporting Neumann energies in cubes from [Ser24]. Let $\psi_t : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}^{\mathsf{d}}$ be a Lipschitz vector field depending continuously on a parameter $t \in [0, 1]$. Let us define the flow $\Phi_t : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}^{\mathsf{d}}$ to be the solution to

$$\begin{cases} \dfrac{d\Phi_t}{dt}(x) = \psi_t(\Phi_t(x)) \\ \Phi_0(x) = x. \end{cases} \tag{7.19}$$

This flow is well-defined for $t \in [0, 1]$ by standard ODE theory. Moreover, it is standard to check that if μ is a probability density then the push-forward

$$\mu_t := \Phi_t \# \mu \tag{7.20}$$

solves

$$\partial_t \mu_t + \mathrm{div}\,(\psi_t \mu_t) = 0. \tag{7.21}$$

In [Ser24], the following lemma is shown.

Lemma 7.3. *Let μ_0 and μ_1 be two probability densities on $\mathbb{R}^{\mathsf{d}}$ and let $\mu_s = (1-s)\mu_0 + s\mu_1$ be their linear interpolant. Assume that for $s \in [0,1]$, ψ_s is a Lipschitz vector field on $\mathbb{R}^{\mathsf{d}}$, depending continuously on s, and satisfying*

$$-\mathrm{div}\,(\psi_s \mu_s) = \mu_1 - \mu_0. \tag{7.22}$$

Then defining Φ_s as in (7.19), we have that $\mu_s = \Phi_s \# \mu_0$.

Lemma 7.4. *Assume $\ell N^{\frac{1}{\mathsf{d}}} > \rho_\beta$, and let Q_ℓ be a hyperrectangle of sidelengths in $(\ell, 2\ell)$. Let μ_0, μ_1 be two Lipschitz densities bounded above and below by positive constants in Q_ℓ, a hyperrectangle of sidelengths in $[\ell, 2\ell]$ with $N\mu_0(Q_\ell) = N\mu_1(Q_\ell) = \bar{\mathrm{n}}$ an integer. Let $h > 0$ and let $\mathcal{G}$ denote an event on which the local laws hold for $\mathbb{Q}_{N,\beta,h}(Q_\ell, \mu_s, 0)$. Then*

$$\begin{aligned} &|\log \mathsf{K}^{\mathcal{G}}_{N,\beta,h}(Q_\ell, \mu_1, 0) - \log \mathsf{K}_{N,\beta,h}(Q_\ell, \mu_0, 0) + N(\mathrm{Ent}_{Q_\ell}(\mu_1) - \mathrm{Ent}_{Q_\ell}(\mu_0))| \\ &\qquad \lesssim_\beta (1+\beta) N \ell^{\mathsf{d}} \left(\ell^2 (|\mu_0|_{C^1} + |\mu_1 - \mu_0|_{C^1}) |\mu_1 - \mu_0|_{C^1} + \ell |\mu_1 - \mu_0|_{C^1} \right), \end{aligned} \tag{7.23}$$

where C depends only on d and a lower bound for μ_0 and μ_1, and where $\mathrm{Ent}_{Q_\ell}(\mu)$ is the entropy restricted to Q_ℓ, i.e. $\int_{Q_\ell} \mu \log \mu$.

Proof. Since we are working with the Neumann energy in a cube, we need to find a transport that preserves the cube and solves (7.22). For that we let φ solve

$$\begin{cases} -\Delta \varphi = \mu_1 - \mu_0 & \text{in } Q_\ell \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial Q_\ell. \end{cases} \tag{7.24}$$

By elliptic regularity and scaling we have

$$|\varphi|_{C^1} \le C\ell^2 |\mu_1 - \mu_0|_{C^1}, \quad |\varphi|_{C^2} \le C\ell |\mu_1 - \mu_0|_{C^1}.$$

Define, for $0 \le s \le 1$, the linear interpolant $\mu_s = (1-s)\mu_0 + s\mu_1$. Setting

$$\psi_s := \frac{\nabla \varphi}{\mu_s},$$

we thus have $-\mathrm{div}\,(\psi_s \mu_s) = \mu_1 - \mu_0$ i.e. (7.22) is satisfied, thus $\mu_1 = \Phi_1 \# \mu_0$. Moreover, by simple estimates

$$\begin{aligned} |\psi_s|_{C^1} &\le C \left(\left\| \frac{1}{\mu_s} \right\|^2_{L^\infty} |\mu_s|_{C^1} |\varphi|_{C^1} + \left\| \frac{1}{\mu_s} \right\|_{L^\infty} |\varphi|_{C^2} \right) \\ &\le C \left(\ell^2 \left(|\mu_0|_{C^1} + |\mu_1 - \mu_0|_{C^1} \right) |\mu_1 - \mu_0|_{C^1} + \ell |\mu_1 - \mu_0|_{C^1} \right), \end{aligned} \tag{7.25}$$

where C depends only on d and a lower bound for μ_0 and μ_1. Computing as in (3.32), we have

$$\log \mathsf{K}^{\mathcal{G}}_{N,\beta,h}(Q_\ell,\mu_1,0)-\log \mathsf{K}_{N,\beta,h}(Q_\ell,\mu_0,0)+N\left(\mathrm{Ent}_{Q_\ell}(\mu_1)-\mathrm{Ent}_{Q_\ell}(\mu_0)\right)$$
$$=\log \mathbb{E}_{\mathbb{Q}_{N,\beta,h}(Q_\ell,\mu_0,0)}\exp\Big(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\left(\mathsf{F}_N\left(\Phi_1(X_N),\mu_1,Q_\ell\times[-h,h]\right)-\mathsf{F}_N\left(X_N,\mu_0,Q_\ell\times[-h,h]\right)\right)$$
$$+\,\mathrm{Fluct}_{\mu_0}(\log\det D\phi_1)\Big)\mathbf{1}_{\mathcal{G}}.$$

The first term in the exponent is given by (cf. (3.36))

$$-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\int_0^1 \mathsf{A}_1(\Phi_s(X_N),\mu_s,\psi\circ\Phi_s^{-1})ds$$

which we can control using Proposition 3.8 and the local laws on $\mathcal{G}$ coupled with Proposition 3.11 by

$$\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\int_0^1 \left|\mathsf{A}_1(\Phi_s(X_N),\mu_s,\psi\circ\Phi_s^{-1})\right|\,ds\lesssim_\beta \beta N\ell^{\mathsf{d}}\|\psi\|_{C^1}$$

The second term is similarly well-controlled; (3.31) and the local laws yield

$$|\mathrm{Fluct}_\mu(\log\det D\phi_1)|\lesssim_\beta N\ell^{\mathsf{d}}\|\psi\|_{C^1}.$$

With (7.25) we deduce the result. $\square$

We now are able to write an expansion of the next-order partition functions as in (4.24).

7.3. **Relative Expansion of Next-Order Partition Functions.** The idea for obtaining (4.24) is based on tools from [Ser23], with the added difficulty that the transport (2.16) is nonlocal. We will use the almost additivity Proposition 7.1 to split the consideration between $U=Q_R$ for some $R>\ell$ to be determined and Q_R^c, applied to Q_i with sidelengths in $[r,2r]$. We will compare $\mathsf{K}_{N,\beta,r}(Q_i,\mu,\zeta)$ to that of $\mathsf{K}_{N,\beta,r}(Q_i,\fint\mu,\zeta)$ in Lemma 7.2 using Lemma 7.4, and optimize over the choice of r to improve our error bounds. Our analysis of $\mathsf{K}^{\mathrm{ext}}_{N,\beta,R}(Q_R,\mu,\zeta)$ will rely on the decay of (2.16) away from Q_ℓ.

Proposition 7.5 (Comparison of Partition Functions)**.** *Assume $\mathsf{s}\in(\mathsf{d}-2,\mathsf{d})$ and that $\ell N^{\frac{1}{\mathsf{d}}}>\rho_\beta$. Assume $\mathrm{supp}\,\varphi\subset\square_\ell\subset\square_{2\varepsilon}\subset\Sigma$. Let ψ be given by Proposition 2.2, and $\mu_t:=(\mathrm{Id}+t\psi)\#\mu_V$. Let $\mathcal{G}_\ell$ be the event of Theorem 2, and set*

$$\kappa:=\begin{cases}1-\frac{4}{4+\mathsf{d}-\mathsf{s}_+} & \text{if } \mathsf{d}\le 5 \text{ or } \mathsf{s}<\mathsf{d}-1\\ 1-\frac{\mathsf{d}-1}{(\mathsf{d}-1)+(\mathsf{d}-\mathsf{s})} & \text{otherwise.}\end{cases}\tag{7.26}$$

Then, for all $|t|\ell^{\mathsf{s}-\mathsf{d}}$ is small enough, we have

$$\log \mathsf{K}^{\mathcal{G}_\ell}_{N,\beta,\infty}\left(\mathbb{R}^{\mathsf{d}},\mu_t,\zeta_V\circ\Phi_t^{-1}\right)-\log\mathsf{K}_{N,\beta,\infty}\left(\mathbb{R}^{\mathsf{d}},\mu_0,\zeta_V\right)+N(\mathrm{Ent}(\mu_t)-\mathrm{Ent}(\mu_0))$$
$$=\mathcal{Z}(\beta,\mu_t)-\mathcal{Z}(\beta,\mu_0)+O_\beta((1+\beta)N\ell^{\mathsf{d}}\mathcal{R}_t)$$

with

$$\mathcal{Z}(\beta,\mu)=-N\beta\int\mu^{1+\frac{\mathsf{s}}{\mathsf{d}}}f_{\mathsf{d},\mathsf{s}}(\beta\mu^{\frac{\mathsf{s}}{\mathsf{d}}})+\frac{N\beta}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\int\mu\log\mu\tag{7.27}$$

and

$$\mathcal{R}_t = \max\left((|t|\ell^{\mathsf{s}-\mathsf{d}})^{\frac{\mathsf{d}}{2\mathsf{d}-\mathsf{s}}} \left(\mathscr{E}\left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}} \right) \right)^{\frac{\mathsf{d}-\mathsf{s}}{2\mathsf{d}-\mathsf{s}}}, |t|\varepsilon^{\mathsf{s}-\mathsf{d}} \right) \tag{7.28}$$

where $\mathscr{E}$ is as in (7.2) and ε is as in (1.27).

Together with (7.30) this will provide (4.24).

Proof. Before we start, we replace $\log \mathsf{K}_{N,\beta,\infty}\left(\mathbb{R}^{\mathsf{d}}, \mu_0, \zeta_V\right)$ with $\log \mathsf{K}^{\mathcal{G}_\ell}_{N,\beta,\infty}\left(\mathbb{R}^{\mathsf{d}}, \mu_0, \zeta_V\right)$; since the difference between the two is just $\log \mathbb{P}_{N,\beta}(\mathcal{G}_\ell) = \log(1 + O(e^{-C\beta N\ell^{\mathsf{d}}})) = O(e^{-C\beta N\ell^{\mathsf{d}}})$ so exponentially small, we will be able to absorb it into the error terms we generate below.

If ℓ is large enough, we just use simple additivity at some scale R. Otherwise, it is better to include Q_ℓ into some larger cube Q_R with R to be determined.

Step 1: Cutoff via Almost Additivity. Since $\operatorname{supp}\varphi \subset \hat{\Sigma}$, we may include $\operatorname{supp}\varphi$ in a hyperrectangle Q_R at distance $\geq \varepsilon > 0$ from $\partial\Sigma$, and such that $\mu_V(Q_R)$ is an integer, for some $R \leq \varepsilon$.

First, we apply almost additivity (Proposition 7.1) on $U_1 = Q_R$ and $U_2 = \Phi_t(Q_R)$. Note that if $t\ell^{\mathsf{s}-\mathsf{d}}$ is sufficiently small, $|\Phi_t - I| \leq t\|\psi\|_{L^\infty}$ is small as well, and thus both U_1 and U_2 are at distance $\geq \varepsilon$ from $\partial\Sigma$, hence sets where μ_V and μ_t are bounded below and ζ_V, $\zeta_V \circ \Phi_t$ vanish, and local laws hold.

We will partition U_1 and U_2 into sets Q_i with sidelengths in $[r, 2r]$ with r to be optimized; this yields

$$\begin{aligned}
&\log \mathsf{K}^{\mathcal{G}_\ell}_{N,\beta,\infty}\left(\mathbb{R}^{\mathsf{d}}, \mu_t, \zeta_V \circ \Phi_t^{-1}\right) - \log \mathsf{K}^{\mathcal{G}_\ell}_{N,\beta,\infty}\left(\mathbb{R}^{\mathsf{d}}, \mu_0, \zeta_V\right) = \\
&\quad \sum_{i=1}^{p} \left(\log \mathsf{K}^{\mathcal{G}_{\Phi_t(Q_i)}}_{N,\beta,r}\left(\Phi_t(Q_i), \mu_t, 0\right) - \log \mathsf{K}^{\mathcal{G}_{Q_i}}_{N,\beta,r}\left(Q_i, \mu_0, 0\right) \right) + NR^{\mathsf{d}} O_\beta\left(\mathscr{E}(rN^{\frac{1}{\mathsf{d}}}) \right) \\
&\qquad + \log \mathsf{K}^{\mathrm{ext}, \mathcal{G}_{\Phi_t(Q_R^c)}}_{N,\beta,R}\left(\Phi_t(Q_R^c), \mu_t, \zeta_V \circ \Phi_t^{-1}\right) - \log \mathsf{K}^{\mathrm{ext}, \mathcal{G}_{Q_R^c}}_{N,\beta,R}\left(Q_r^c, \mu_0, \zeta_V\right)
\end{aligned} \tag{7.29}$$

where $\mathscr{E}$ is as in (7.2) and $\mathcal{G}_\Omega$ denotes the set of configurations in Ω that are restrictions of a configuration in $\mathcal{G}_\ell$.

Since $\mu_t = (I + t\psi)\#\mu_0$, a computation based on (4.28) and (2.17) yields that if $t\ell^{\mathsf{s}-\mathsf{d}}$ is small enough,

$$\|\mu_t\|_{C^1} \lesssim |t|\|\psi\|_{C^2} \lesssim \ell^{\mathsf{d}-\mathsf{s}} \frac{1}{\ell^{\mathsf{d}-\mathsf{s}+1}} = \ell^{-1}. \tag{7.30}$$

Step 2: Comparison of Local Partition Functions. We next analyze

$$\sum_{i=1}^{p} \log \mathsf{K}^{\mathcal{G}_{Q_i}}_{N,\beta,r}\left(Q_i, \mu, 0\right)$$

with μ either μ_0 or μ_t above. The idea is reduce to Lemma 7.2 using Lemma 7.4. Using the scaling identity from (7.16) we find for each i that

$$\begin{aligned}
&\log \mathsf{K}^{\mathcal{G}_{Q_i}}_{N,\beta,r}\left(Q_i, \mu, 0\right) + N\mathrm{Ent}_{Q_i}(\mu) \\
&\quad = -\beta N|Q_i| \left(\overline{\mu}^{1+\frac{\mathsf{s}}{\mathsf{d}}} f_{\mathsf{d},\mathsf{s}}\left(\beta \overline{\mu_i}^{\frac{\mathsf{s}}{\mathsf{d}}}\right) - \frac{1}{2\mathsf{d}} \overline{\mu} \log \overline{\mu} \mathbf{1}_{\mathsf{s}=0} \right) + \left(\frac{\beta}{2\mathsf{d}} \bar{\mathrm{n}} \log N \right) \mathbf{1}_{\mathsf{s}=0} \\
&\qquad + (1+\beta)Nr^{\mathsf{d}} O_\beta\left(r\|\mu\|_{C^1} + r^2\|\mu\|^2_{C^1} \mathbf{1}_{\mathsf{s}=0} + \mathscr{E}\left(rN^{\frac{1}{\mathsf{d}}}\right) \right)
\end{aligned}$$

with $\bar{\mathrm{n}} = N \int_{Q_U} \mu$, and $\overline{\mu}_i = \fint_{Q_i} \mu$. Running the same computation as in the proof of [Ser23, Lemma 6.1] to replace μ with $\overline{\mu}$ using (1.45), and summing over i, we obtain

(7.31)
$$\sum_{i=1}^{p} \log \mathsf{K}_{N,\beta,r}^{\mathcal{G}_{Q_i}}(Q_i, \mu, 0) + N\mathrm{Ent}_{Q_i}(\mu) = -\beta N \int_U \mu^{1+\frac{\mathsf{s}}{\mathsf{d}}} f_{\mathsf{d},\mathsf{s}}\left(\beta \mu^{\frac{\mathsf{s}}{\mathsf{d}}}\right) - \frac{\beta}{2\mathsf{d}} N \mathbf{1}_{\mathsf{s}=0} \int_U \mu \log \mu + \left(\frac{\beta}{2\mathsf{d}} \bar{\mathrm{n}} \log N\right) \mathbf{1}_{\mathsf{s}=0} + (1+\beta) N R^{\mathsf{d}} O_\beta \left(r \|\mu\|_{C^1} + r^2 \|\mu\|_{C^1}^2 \mathbf{1}_{\mathsf{s}=0} + \mathscr{E}\left(r N^{\frac{1}{\mathsf{d}}}\right)\right)$$

with $\mathscr{E}$ as in (7.2). Requiring $r\|\mu\|_{C^1} \lesssim 1$, equivalently $r < \ell$ in view of (7.30), to absorb some errors, we need to optimize $\mathscr{E}(rN^{\frac{1}{\mathsf{d}}}) + r\|\mu\|_{C^1}$ over choices of $r < \ell$. Letting $-\kappa$ with $\kappa > 0$ denote the exponent on L in (7.2), we see that the right choice of r is

$$r = N^{-\frac{\kappa}{\mathsf{d}(\kappa+1)}} \|\mu\|_{C^1}^{-\frac{1}{\kappa+1}}.$$

With this choice, we have $r\|\mu\|_{C^1} \lesssim 1$ thanks to the fact that $\ell \geq \rho_\beta N^{-1/\mathsf{d}}$ and $\|\mu\|_{C^1} \lesssim \ell^{-1}$. Thus, the above optimization is the correct one, moreover $r \lesssim \ell$, and up to multiplying r by a constant, we have $r < \ell \leq R$, hence the partitioning of Q_R into size r hyperrectangles makes sense (at least provided R/ℓ is large enough). If R/r is large enough, we use [Ser24, Lemma 5.13] to partitioning Q_R into hyperrectangles of size comparable to r and with quantized μ-mass, i.e. $N\mu(Q_i)$ integer. If R/r is not, then we do not partition further Q_R. In all cases this yields

(7.32)
$$\sum_{i=1}^{p} \log \mathsf{K}_{N,\beta,r}^{\mathcal{G}_{Q_i}}(Q_i, \mu, 0) + N\mathrm{Ent}_{Q_i}(\mu) = -\beta N \int_U \mu^{1+\frac{\mathsf{s}}{\mathsf{d}}} f_{\mathsf{d},\mathsf{s}}\left(\beta \mu^{\frac{\mathsf{s}}{\mathsf{d}}}\right) - \frac{\beta}{2\mathsf{d}} N \mathbf{1}_{\mathsf{s}=0} \int_U \mu \log \mu + \left(\frac{\beta}{2\mathsf{d}} \bar{\mathrm{n}} \log N\right) \mathbf{1}_{\mathsf{s}=0} + O_\beta\left((1+\beta) N R^{\mathsf{d}} \mathscr{E}\left((N^{\frac{1}{\mathsf{d}}} \|\mu\|_{C^1}^{-1})^{\frac{1}{\kappa+1}}\right)\right).$$

Step 3: External estimate. We turn to estimating

$$\log \mathsf{K}_{N,\beta,R}^{\mathrm{ext},\mathcal{G}_{\Phi_t(Q_R^c)}}\left(\Phi_t(Q_R^c), \mu_t, \zeta_V \circ \Phi_t^{-1}\right) - \log \mathsf{K}_{N,\beta,R}^{\mathrm{ext},\mathcal{G}_{Q_R^c}}(Q_r^c, \mu_0, \zeta_V)$$

in (7.29). By the same computation as in (3.32) we obtain

$$\begin{aligned}
&\log \mathsf{K}_{N,\beta,R}^{\mathrm{ext},\mathcal{G}_{\Phi_t(Q_R^c)}}(\Phi_t(Q_R^c), \Phi_t \# \mu_V, \zeta_V \circ \Phi_t^{-1}) - \log \mathsf{K}_{N,\beta,R}^{\mathrm{ext},\mathcal{G}_{Q_R^c}}(Q_R^c, \mu_V, \zeta_V) \\
&\qquad + N\left(\mathrm{Ent}_{\Phi_t(Q_R^c)}(\Phi_t \# \mu_V) - \mathrm{Ent}_{Q_R^c}(\mu_V)\right) \\
&= \log \mathbb{E}_{\mathbb{Q}_{N,\beta,R}^{\mathrm{ext},\mathcal{G}_{Q_R}}(Q_R^c,\mu_V,\zeta_V)} \exp\Big(-\beta N^{-\frac{\mathsf{s}}{\mathsf{d}}}\Big(\mathsf{F}_N\left(\Phi_t(X_N), \Phi_t \#(\mu_{V|Q_R^c}), (\Phi_t(Q_R) \times [-R,R])^c\right) \\
&\qquad - \mathsf{F}_N\left(X_N, \mu_{V|Q_R^c}, (Q_R \times [-R,R])^c\right) + \mathrm{Fluct}_{\mu_V}(\log\det D\phi_t)\Big)
\end{aligned}$$

The first term in the exponent is equal by (3.36) to

$$\int_0^t \mathsf{A}_1(\Phi_s(X_N), \Phi_s \#(\mu_{|Q_R^c}), \psi \circ \Phi_s^{-1})\, ds$$

which we can control, using the local laws and modifying the proof of Corollary 3.12, by

$$\int_0^t \mathsf{A}_1(\Phi_s(X_N), \Phi_s\#(\mu_{|Q_R^c}), \psi\circ\Phi_s^{-1})\, ds \lesssim_\beta |t|N\|\varphi_0\|_{C^4}\ell^{\mathsf{s}} \sum_{k\geq \log_2 \frac{R}{\ell}} \frac{1}{(2^{\mathsf{d}-\mathsf{s}})^k}$$
$$\lesssim |t|N\|\varphi_0\|_{C^4}\ell^{\mathsf{s}}\left(\frac{\ell}{R}\right)^{\mathsf{d}-s} \lesssim_\beta |t|N\|\varphi_0\|_{C^4}\ell^{\mathsf{d}}R^{\mathsf{s}-\mathsf{d}}$$

The second term is similarly well-controlled; (3.31) and (2.17) yield also

$$|\mathrm{Fluct}_{\mu_V}(\log\det D\phi_t)| \lesssim_\beta |t|\|\varphi_0\|_{C^4}\sum_{k\geq\log_2\frac{R}{\ell}} \frac{\ell^{\mathsf{d}}N\left(2^k\ell\right)^{\mathsf{d}}}{\left(2^k\ell\right)^{2\mathsf{d}-\mathsf{s}}} \lesssim_\beta |t|N\|\varphi_0\|_{C^4}\ell^{\mathsf{d}}R^{\mathsf{s}-\mathsf{d}}.$$

We thus have the estimate

(7.33)
$$\log\mathsf{K}_{N,\beta,R}^{\mathrm{ext},\mathcal{G}_{\Phi_t(Q_R^c)}}\left(\Phi_t(Q_R^c),\mu_t,\zeta_V\circ\Phi_t^{-1}\right) - \log\mathsf{K}_{N,\beta,R}^{\mathrm{ext},\mathcal{G}_{Q_R^c}}\left(Q_r^c,\mu_0,\zeta_V\right) \lesssim_\beta |t|N\|\varphi_0\|_{C^4}\ell^{\mathsf{d}}R^{\mathsf{s}-\mathsf{d}}.$$

Step 4: Conclusion. We apply (7.32) to μ_t and to μ_V, use that $\|\mu_t\|_{C^1}\lesssim \ell^{-1}$, insert into (7.29) and add the bounds of the previous step to obtain

$$\log\mathsf{K}_{N,\beta,\infty}^{\mathcal{G}_\ell}\left(\mathbb{R}^{\mathsf{d}},\mu_t,\zeta_V\circ\Phi_t^{-1}\right) - \log\mathsf{K}_{N,\beta,\infty}\left(\mathbb{R}^{\mathsf{d}},\mu_0,\zeta_V\right) + N\left(\mathrm{Ent}(\mu_t)-\mathrm{Ent}(\mu_V)\right) =$$
$$-N\beta\int_{Q_R}\mu_t^{1+\frac{\mathsf{s}}{\mathsf{d}}}f_{\mathsf{d},\mathsf{s}}(\beta\mu_t^{\frac{\mathsf{s}}{\mathsf{d}}}) + \frac{N\beta}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\int_{Q_R}\mu_t\log\mu_t + N\beta\int_{Q_R}\mu_V^{1+\frac{\mathsf{s}}{\mathsf{d}}}f_{\mathsf{d},\mathsf{s}}(\beta\mu_V^{\frac{\mathsf{s}}{\mathsf{d}}}) - \frac{N\beta}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\int_{Q_R}\mu_V\log\mu_V$$
$$+O_\beta\left(|t|N\|\varphi_0\|_{C^4}\ell^{\mathsf{d}}R^{\mathsf{s}-\mathsf{d}}\right) + O_\beta\left((1+\beta)NR^{\mathsf{d}}\mathscr{E}\left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}}\right)\right).$$

We can insert the remaining integral over Q_R^c into the above as well as in the computation in (4.26). In particular, one has

$$\Bigg| -N\beta\int_{Q_R^c}\mu_t^{1+\frac{\mathsf{s}}{\mathsf{d}}}f_{\mathsf{d},\mathsf{s}}(\beta\mu_t^{\frac{\mathsf{s}}{\mathsf{d}}}) + \frac{N\beta}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\int_{Q_R^c}\mu_t\log\mu_t$$
$$+N\beta\int_{Q_R^c}\mu^{1+\frac{\mathsf{s}}{\mathsf{d}}}f_{\mathsf{d},\mathsf{s}}(\beta\mu^{\frac{\mathsf{s}}{\mathsf{d}}}) - \frac{N\beta}{2\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\int_{Q_R^c}\mu_V\log\mu_V\Bigg| \lesssim_\beta N\beta\int_{Q_R^c}|t||D\psi| \lesssim_\beta |t|N\beta\ell^{\mathsf{d}}R^{\mathsf{s}-\mathsf{d}}$$

using (2.17). Incorporating this into the error terms we find

$$\log\mathsf{K}_{N,\beta,\infty}^{\mathcal{G}_\ell}\left(\mathbb{R}^{\mathsf{d}},\mu_t,\zeta_V\circ\Phi_t^{-1}\right) - \log\mathsf{K}_{N,\beta,\infty}\left(\mathbb{R}^{\mathsf{d}},\mu_0,\zeta_V\right) + N\left(\mathrm{Ent}(\mu_t)-\mathrm{Ent}(\mu_V)\right)$$
$$=\mathcal{Z}(\beta,\mu_t)-\mathcal{Z}(\beta,\mu_0) + O_\beta\left(|t|N\|\varphi_0\|_{C^4}\ell^{\mathsf{d}}R^{\mathsf{s}-\mathsf{d}}\right) + O_\beta\left((1+\beta)NR^{\mathsf{d}}\mathscr{E}\left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}}\right)\right).$$

We now have to optimize the sum of the errors over $R\leq\varepsilon$. Equating the two terms leads to the choice

$$R=\min\left((|t|\ell^{\mathsf{d}})^{\frac{1}{2\mathsf{d}-\mathsf{s}}}\left(\mathscr{E}\left((N^{\frac{1}{\mathsf{d}}}\ell)^{\frac{1}{\kappa+1}}\right)\right)^{-\frac{1}{2\mathsf{d}-\mathsf{s}}},\varepsilon\right)$$

and an error rate $\max((|t|\ell^{\mathsf{s}-\mathsf{d}})^{\frac{\mathsf{d}}{2\mathsf{d}-\mathsf{s}}}\mathscr{E}^{\frac{\mathsf{d}-\mathsf{s}}{2\mathsf{d}-\mathsf{s}}}, |t|\varepsilon^{\mathsf{s}-\mathsf{d}})$.

$\square$

8. Fractional harmonic extension

In this section we state and prove existence and regularity of the fractional harmonic extension, i.e. the solution φ^Σ to

$$\begin{cases} \varphi^\Sigma = \varphi & \text{in } \Sigma \\ (-\Delta)^\alpha \varphi^\Sigma = 0 & \text{in } \Sigma^c. \end{cases} \tag{8.1}$$

for $\Sigma \subset \mathbb{R}^{\mathsf{d}}$, as well as provide estimates for it. We could not find a succinct statement of the existence result for solutions of fractional elliptic problems in unbounded domains in the literature, although it is certainly known. For completeness we offer its proof here. Interior and boundary regularity for solutions of equations like (8.1) are well-studied, and indeed the optimal regularity is understood to be C^α, see [ROS14], [ROW24, Theorem 1.1] and references therein. The regularity of $\frac{\varphi^\Sigma - \varphi}{\mathrm{dist}(x,\Sigma)^\alpha}$ and its scaling properties then becomes an interesting question, which we rely crucially on here.

Lemma 8.1. *Let U be a neighborhood of Σ. Let $0 < \alpha < 1$ and let $\varphi \in \dot{H}^\alpha(\mathbb{R}^{\mathsf{d}})$, where $\dot{H}^\alpha$ is the Sobolev space defined via (1.12). Suppose that $\partial\Sigma$ is a $C^{1,1}$ boundary. Then, there exists a unique function $\varphi^\Sigma \in \dot{H}^\alpha(\mathbb{R}^{\mathsf{d}})$ solving (1.36). If $(-\Delta)^\alpha \varphi \in L^\infty(\mathbb{R}^{\mathsf{d}})$ we also have the boundary estimate*

$$\left\| \frac{\varphi^\Sigma - \varphi}{\mathrm{dist}(x,\Sigma)^\alpha} \right\|_{C^\sigma(\overline{U\setminus\Sigma})} \lesssim \|(-\Delta)^\alpha \varphi\|_{L^\infty(U\setminus\Sigma)} \tag{8.2}$$

for all $\sigma \in (0,\alpha)$. Furthermore, if $\operatorname{supp}\varphi \subset \square_\ell$, for some cube $\square_\ell$ of size ℓ included in $\hat{\Sigma}$, then

$$\left\| \frac{\varphi^\Sigma - \varphi}{\mathrm{dist}(x,\Sigma)^\alpha} \right\|_{C^\sigma(\overline{U\setminus\Sigma})} \lesssim \|\varphi\|_{L^\infty} \ell^{\mathsf{d}} \tag{8.3}$$

for all $\sigma \in (0,\alpha)$.

If in addition we assume that $(-\Delta)^\alpha \varphi \in C^k(\mathbb{R}^{\mathsf{d}})$ and $\partial\Sigma$ is C^{k+1}, we also have

$$\left\| \frac{\varphi^\Sigma - \varphi}{\mathrm{dist}(x,\Sigma)^\alpha} \right\|_{C^\sigma(\overline{U\setminus\Sigma})} \lesssim \|(-\Delta)^\alpha \varphi\|_{C^{(\sigma-\alpha)}(U\setminus\Sigma)} \tag{8.4}$$

for all $\sigma \in (\alpha,k)$ and $\sigma, \sigma \pm \alpha \notin \mathbb{N}$. (8.3) holds as well in the mesoscopic interior case under the assumption that $\partial\Sigma$ is C^{k+1}; no additional regularity on the test function φ is needed.

The regularity assumptions on $(-\Delta)^\alpha\varphi$ and $\partial\Sigma$ are so that we can apply the local regularity results [ROS16, Theorem 1.2] and [ARO20a, Theorem 1.4].

Proof of Lemma 8.1. First, let us prove existence of φ^Σ in $\dot{H}^\alpha$. The proof requires us to solve a fractional Dirichlet problem on the unbounded domain Σ^c. Existence is standard, and follows for instance the variational technique used on bounded domains in [RO16] (and [FKV15] for more general equations). For completeness and to clarify that it applies to unbounded domains in our situation, we sketch a proof here.

Let

$$\Lambda(u) := \frac{1}{2}\int_{\mathbb{R}^{2\mathsf{d}}} (u(x) - u(z))^2 \frac{c_{\mathsf{d},\alpha}}{|z-x|^{\mathsf{d}+2\alpha}} \, dx dz, \tag{8.5}$$

where $c_{\mathsf{d},\alpha}$ is the constant associated to the fractional Laplace kernel in (1.10). Notice that $\Lambda(u)$ is an equivalent definition of $[u]^2_{\dot{H}^\alpha}$, where $[u]^2_{\dot{H}^\alpha}$ is as in (1.12) (see [DNPV12, Chapter 2] for a discussion of the kernel definition of fractional Sobolev spaces). Since $\mathsf{d} \geq 2 > 2\alpha$, $\dot{H}^\alpha$ continuously imbeds into L^q for all $q \in \left[p, \frac{2\mathsf{d}}{\mathsf{d}-2\alpha}\right]$ [DNPV12, Theorem 6.5]. $\Lambda(u)$ is in fact defined then for all $u \in \dot{H}^\alpha$ and is equivalent to $\|u\|^2_{\dot{H}^\alpha}$.

Let us minimize $\Lambda(u)$ over the set $\dot{H}^\alpha_\varphi = \{u \in \dot{H}^\alpha : u = \varphi \text{ on } \Sigma\}$; first, $\Lambda(u)$ is bounded below on $\dot{H}^\alpha_\varphi$, and there is at least one element of $\dot{H}^\alpha_\varphi$ for which $\Lambda(u)$ is finite (namely φ). Thus, $\inf_{\dot{H}^\alpha_\varphi} \Lambda(u)$ exists and is finite. Now, consider a minimizing sequence u_n in $\dot{H}^\alpha_\varphi$, with $\Lambda(u_n) = \|u_n\|^2_{\dot{H}^\alpha} \to \inf_{\dot{H}^\alpha_\varphi} \Lambda(u)$. $\dot{H}^\alpha$ is complete, so there is some $u \in \dot{H}^\alpha$ such that $u_n \to u$ and

$$\Lambda(u) = \|u\|^2_{\dot{H}^\alpha} = \lim_{n\to\infty} \|u_n\|^2_{\dot{H}^\alpha} = \lim_{n\to\infty} \Lambda(u_n) = \inf_{\dot{H}^\alpha_\varphi} \Lambda(u).$$

Furthermore, $u \in \dot{H}^\alpha_\varphi$: for any n and $\varphi \in C^\infty_c(\Sigma)$ we have

$$\left|\int (\varphi - u)\varphi\right| = \left|\int (u_n - u)\varphi\right| \leq \|u_n - u\|_{L^2}\|\varphi\|_{L^2} \leq \|u_n - u\|_{H^\alpha}\|\varphi\|_{L^2} \to 0$$

and so $u = \varphi$ a.e. on Σ. It remains to see that u solves (1.36). Let $\varphi \in C^\infty_c(\Sigma^c)$ be an arbitrary test function. Then, $u + \varepsilon\varphi \in \dot{H}^\alpha_\varphi$ and since u is a minimizer of Λ we have

$$\begin{aligned} 0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Lambda(u+\varepsilon\varphi) &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{1}{2}\int_{\mathbb{R}^{2\mathsf{d}}}(((u(x)-u(z)) + \varepsilon(\varphi(x)-\varphi(z)))^2 \frac{c_{\mathsf{d},\alpha}}{|z-x|^{\mathsf{d}+2\alpha}}\,dxdz \\ &= \int_{\mathbb{R}^{2\mathsf{d}}}(u(x)-u(z))(\varphi(x)-\varphi(z))\frac{c_{\mathsf{d},\alpha}}{|z-x|^{\mathsf{d}+2\alpha}}\,dxdz. \end{aligned}$$

So,

$$\int_{\mathbb{R}^{2\mathsf{d}}}(u(x)-u(z))\varphi(x)\frac{c_{\mathsf{d},\alpha}}{|z-x|^{\mathsf{d}+2\alpha}} = \int_{\mathbb{R}^{2\mathsf{d}}}(u(x)-u(z))\varphi(z)\frac{c_{\mathsf{d},\alpha}}{|z-x|^{\mathsf{d}+2\alpha}}$$

and thus by symmetry

$$\begin{aligned} \int_{\mathbb{R}^{2\mathsf{d}}}(u(x)-u(z))\varphi(z)\frac{c_{\mathsf{d},\alpha}}{|z-x|^{\mathsf{d}+2\alpha}} &= \int_{\mathbb{R}^{\mathsf{d}}}\varphi(z)\left(\int_{\mathbb{R}^{\mathsf{d}}}(u(x)-u(z))\frac{c_{\mathsf{d},\alpha}}{|z-x|^{\mathsf{d}+2\alpha}}\,dx\right)dz \\ &= \int \varphi(z)(-\Delta)^\alpha u = 0. \end{aligned}$$

Since $\varphi \in C^\infty_c(\Sigma^c)$ was arbitrary, we conclude that $(-\Delta)^\alpha u = 0$ a.e. in Σ^c as desired, and we set $\varphi^\Sigma = u$.

We now apply the regularity result of [ROS16, Theorem 1.2] to

$$\tilde{u} = \varphi^\Sigma - \varphi$$

solving

$$\begin{cases} \tilde{u} = 0 & \text{in } \Sigma \\ (-\Delta)^\alpha \tilde{u} = -(-\Delta)^\alpha\varphi & \text{in } \Sigma^c \end{cases}$$

since Σ has a $C^{1,1}$ boundary and $(-\Delta)^\alpha\varphi \in L^\infty$ to conclude the boundary estimate (8.2).

Now, let us examine the case where $\varphi = \varphi_0\left(\frac{\cdot - z}{\ell}\right)$. Let

$$\frac{\varphi^\Sigma - \varphi}{\mathrm{dist}(x,\partial\Sigma)^\alpha} := f, \tag{8.6}$$

which we have just seen is C^σ regular, with C^σ norm controlled by $\|(-\Delta)^\alpha \varphi\|_{L^\infty(U\setminus\Sigma)}$. The bound (8.3) then follows from a careful analysis of (8.2). The second term on the right hand side can be carefully analyzed from the definition of the fractional Laplacian;

$$(-\Delta)^\alpha \varphi(x) = \text{P.V.} \int (\varphi(x)-\varphi(y)) \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|x-y|^{\mathsf{d}+2\alpha}}\, dy = -\int \varphi(y) \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|x-y|^{\mathsf{d}+2\alpha}}\, dy \lesssim \|\varphi\|_{L^\infty} \ell^{\mathsf{d}}.$$

The second equation follows from $x \notin \operatorname{supp} \varphi$, and the bounds in the integral come from the fact that $|x-y|$ is bounded from below at order 1 and φ is only supported on a set of size ℓ. This is (8.3).

For higher regularity, we use [ARO20a, Theorem 1.4]. There, if $\sigma \in (\alpha, k)$ (the k is because $\partial\Sigma$ is C^{k+1} regular by assumption) is such that $\sigma, \sigma \pm \alpha \notin \mathbb{N}$ then, with f as in (8.6)

$$\|f\|_{C^\sigma(\overline{U\setminus\Sigma})} \lesssim \|(-\Delta)^\alpha \varphi\|_{C^{\sigma-\alpha}(\overline{U\setminus\Sigma})} + \|\varphi^\Sigma - \varphi\|_{L^\infty(\mathbb{R}^{\mathsf{d}})} \lesssim \|(-\Delta)^\alpha \varphi\|_{C^{\sigma-\alpha}(\overline{\Sigma^c})}$$

where we have used [ROS16, Theorem 1.2] to control the L^∞ term. We have $(-\Delta)^\alpha \varphi \in C^{\sigma-\alpha}(\Sigma^c)$ since we assume $\sigma < k$ and $\varphi \in C^{k+\alpha}(\mathbb{R}^{\mathsf{d}})$; this yields (8.4).

Specializing now to the mesoscopic case, it is slightly easier to differentiate

$$(-\Delta)^\alpha \varphi = \text{P.V.} \int \frac{\varphi(x)-\varphi(y)}{|x-y|^{\mathsf{d}+2\alpha}}\, dy = \int_\Sigma \frac{-\varphi(y)}{|x-y|^{\mathsf{d}+2\alpha}}\, dy$$

since $\varphi(x) = 0$ for $x \in \Sigma^c$. A computation shows that for $m \le k$ we have, for $x \in \Sigma^c$, using $\operatorname{dist}(\operatorname{supp}\varphi, \partial\Sigma) \ge \varepsilon > 0$,

$$|\nabla^{\otimes m} (-\Delta)^\alpha \varphi(x)| \lesssim \int_\Sigma \frac{|\varphi(y)|}{|x-y|^{\mathsf{d}+2\alpha+m}}\, dy \lesssim \|\varphi\|_{L^\infty} \ell^{\mathsf{d}}. \tag{8.7}$$

Hölder interpolation then yields the result (8.3). □

We will also need more precise information about the decay of φ^Σ at infinity; this is given by the following.

Lemma 8.2. *Let φ be as in the assumptions of Proposition 2.17. For any $m \le k$, if $x \in U^c$, we have*

$$\left|\nabla^{\otimes m} \varphi^\Sigma(x)\right| \lesssim \frac{\|(-\Delta)^\alpha \varphi\|_{L^\infty}}{|x-z|^{\mathsf{s}+m+2}}. \tag{8.8}$$

Additionally, if $\operatorname{supp}\varphi \subset \hat\Sigma$, *then we have for any $m \le k$ that, for $x \in U^c$,*

$$\left|\nabla^{\otimes m} \varphi^\Sigma(x)\right| \lesssim \frac{\ell^{\mathsf{d}+2} \|(-\Delta)^\alpha \varphi\|_{L^\infty} + \ell^{\mathsf{d}} \|\varphi\|_{L^\infty}}{|x|^{\mathsf{s}+m+2}}. \tag{8.9}$$

Proof. Without loss of generality, let us assume that $z = 0$ (thus $0 \in \Sigma$). Let us start with the case $\ell = 1$. Following the proof of Lemma A.8 with $v = \varphi^\Sigma - \varphi$, we may write

$$\operatorname{dist}(x, \partial\Sigma)^\alpha \left((-\Delta)^\alpha \varphi^\Sigma(x) - (-\Delta)^\alpha \varphi(x) \right) = \overline{c_\alpha} c_\circ + o\left(\operatorname{dist}(x,\partial\Sigma)^\alpha\right)$$

in Σ, where

$$|c_\circ| \le \left\| \frac{\varphi^\Sigma - \varphi}{\operatorname{dist}(x,\partial\Sigma)^\alpha} \right\|_{L^\infty(U\setminus\Sigma)} \le \|(-\Delta)^\alpha \varphi\|_{L^\infty(U\setminus\Sigma)}; \tag{8.10}$$

the second inequality follows from (8.2). As a result, we may expand

$$(-\Delta)^\alpha \varphi^\Sigma(x) = (-\Delta)^\alpha \varphi(x) + \frac{\overline{c_\alpha} c_\circ}{\operatorname{dist}(x,\partial\Sigma)^\alpha} + o(1) \tag{8.11}$$

in Σ. Now, the key is that, as discussed in (2.11), we can recover $\varphi^\Sigma = \mathsf{g} * (-\Delta)^\alpha \varphi^\Sigma$, i.e.

$$\varphi^\Sigma(x) = \int \frac{(-\Delta)^\alpha \varphi^\Sigma(y)}{|x-y|^{\mathsf{s}}}\, dy,$$

where $(-\Delta)^\alpha \varphi^\Sigma(y)$ is only supported in Σ. Taylor expanding the denominator to second order, we find for $x \in \Sigma^c$,

$$\varphi^\Sigma(x) = \int (-\Delta)^\alpha \varphi^\Sigma(y) \left(\frac{1}{|x|^{\mathsf{s}}} - \mathsf{s} \sum_i \frac{x_i y_i}{|x|^{\mathsf{s}+2}} + O\left(\frac{|y|^2}{|x|^{\mathsf{s}+2}} \right) \right)\, dy \tag{8.12}$$

$$= \frac{1}{|x|^{\mathsf{s}}} \int (-\Delta)^\alpha \varphi^\Sigma(y)\, dy - \mathsf{s} \sum_i \frac{x_i}{|x|^{\mathsf{s}+2}} \int y_i (-\Delta)^\alpha \varphi^\Sigma(y)\, dy + O\left(\frac{1}{|x|^{\mathsf{s}+2}} \int_\Sigma |(-\Delta)^\alpha \varphi^\Sigma(y)||y|^2\, dy \right).$$

Since the fractional Laplacian is a mean zero operator, the first term cancels. Furthermore, for all i we have

$$\int y_i (-\Delta)^\alpha \varphi^\Sigma(y)\, dy = \int (-\Delta)^\alpha y_i \varphi^\Sigma = 0$$

via fractional integration by parts, where we have used the odd symmetry of y_i to conclude that $(-\Delta)^\alpha y_i = 0$. Integrating (8.11) over Σ and using (8.10) to control $c_\circ$, it follows that

$$\left| \varphi^\Sigma(x) \right| \lesssim \frac{1}{|x|^{\mathsf{s}+2}} \int_\Sigma |y|^2 |(-\Delta)^\alpha \varphi^\Sigma(y)|\, dy \lesssim \frac{\|(-\Delta)^\alpha \varphi\|_{L^\infty}}{|x|^{\mathsf{s}+2}}.$$

The argument for the derivative is exactly analogous; passing the derivative through the integral, we find for any multiindex γ of length $|\gamma| = m$,

$$D^\gamma \varphi^\Sigma(x) = \int (-\Delta)^\alpha \varphi^\Sigma(y) D^\gamma \left(\frac{1}{|x-y|^{\mathsf{s}}} \right)\, dy.$$

Taylor expanding $D^\gamma \left(\frac{1}{|x-y|^{\mathsf{s}}} \right)$ about x and using that

$$\left| \partial_i \partial_j D^\gamma \left(\frac{1}{|x|^{\mathsf{s}}} \right) \right| \lesssim \frac{1}{|x|^{\mathsf{s}+m+2}}$$

yields the result via the same argument as above.

Next, consider the mesoscopic case $\ell < 1$. We may write the same expansion (8.11), although we now have from (8.3) that

$$|c_\circ| \le \left\| \frac{\varphi^\Sigma - \varphi}{\mathrm{dist}(x, \partial\Sigma)^\alpha} \right\|_{L^\infty(U \setminus \Sigma)} \lesssim \ell^{\mathsf{d}} \|\varphi\|_{L^\infty}. \tag{8.13}$$

We expand as in (8.12); the integral of the $\frac{\overline{c_\alpha} c_\circ}{\mathrm{dist}(x,\partial\Sigma)^\alpha}$ term in (8.11) now yields

$$O\left(\frac{1}{|x|^{\mathsf{s}+2}} \int_\Sigma \left| \frac{\overline{c_\alpha} c_\circ}{\mathrm{dist}(x, \partial\Sigma)^\alpha} \right| |y|^2\, dy \right) = O\left(\frac{\ell^{\mathsf{d}} \|\varphi\|_{L^\infty}}{|x|^{\mathsf{s}+2}} \right)$$

using (8.13). The contribution from $(-\Delta)^\alpha \varphi$ in (8.11) now yields

$$\frac{1}{|x|^{\mathsf{s}+2}} \int_\Sigma |y|^2 |(-\Delta)^\alpha \varphi(y)|\, dy$$

which we split into two contributions: those in $\square_{2\ell}$ and those away. In $\square_{2\ell}$, we find

$$\frac{1}{|x|^{\mathsf{s}+2}} \int_{\square_{2\ell}} |y|^2 |(-\Delta)^\alpha \varphi(y)|\, dy \lesssim \frac{\ell^{2+\mathsf{d}} \|(-\Delta)^\alpha \varphi\|_{L^\infty}}{|x|^{\mathsf{s}+2}}.$$

Away from $\square_{2\ell}$ we seek to estimate the decay of $(-\Delta)^\alpha \varphi$. Since we are at distance $\geq \ell$ from $\operatorname{supp}\varphi$, we can write

$$|(-\Delta)^\alpha \varphi(x)| = \left| \int \frac{-\varphi(y)}{|x-y|^{\mathsf{d}+2\alpha}}\, dy \right| \lesssim \frac{\ell^{\mathsf{d}} \|\varphi\|_{L^\infty}}{|x-z|^{\mathsf{d}+2\alpha}}$$

since $|x-y| \gtrsim |x-z|$ due to $\operatorname{dist}(x, \operatorname{supp}\varphi) \gtrsim \ell$. Thus,

$$\frac{1}{|x|^{\mathsf{s}+2}} \int_{\Sigma \setminus \square_{2\ell}} |y|^2 |(-\Delta)^\alpha \varphi(y)|\, dy \lesssim \frac{\ell^{\mathsf{d}} \|\varphi\|_{L^\infty}}{|x|^{\mathsf{s}+2}} \int_\ell^1 \frac{r^{\mathsf{d}-1}}{r^{\mathsf{d}+2\alpha-2}}\, dr \lesssim \frac{\ell^{\mathsf{s}+2}\|\varphi\|_{L^\infty}}{|x|^{\mathsf{s}+2}}.$$

Combining these estimates, we have

$$\frac{1}{|x|^{\mathsf{s}+2}} \int_\Sigma |y|^2 |(-\Delta)^\alpha \varphi(y)|\, dy \lesssim \frac{\ell^{\mathsf{d}+2}\|(-\Delta)^\alpha \varphi\|_{L^\infty}}{|x|^{\mathsf{s}+2}} + \frac{\ell^{\mathsf{s}+2}\|\varphi\|_{L^\infty}}{|x|^{\mathsf{s}+2}}.$$

Combining with the $c_\circ$ term contribution, we get the announced estimate for $m = 0$. A similar argument as in the $\ell = 1$ case recovers the analogous estimates for the behavior of $\nabla^{\otimes m}\varphi^\Sigma$ as $|x| \to +\infty$. Since $\mathsf{s} + 2 > \mathsf{d}$ and $\ell < 1$, we conclude with the result.

□

We are now able to turn to our goal of finding controls on $(-\Delta)^\alpha \varphi^\Sigma$ near $\partial\Sigma$, quantitative in φ. The L^∞ case is a direct consequence of Lemma A.8; for derivatives, we will need the following.

Lemma 8.3. *Let $\Omega \subset \mathbb{R}^{\mathsf{d}}$, and let $v \in L^\infty(\mathbb{R}^{\mathsf{d}})$ be a solution to*

$$\begin{cases} (-\Delta)^\alpha v = f & \text{in } \Omega \cap B_1 \\ v = 0 & \text{in } B_1 \setminus \Omega. \end{cases}$$

with $0 \in \partial\Omega$, and suppose that $\partial\Omega$ and $\frac{v}{\operatorname{dist}(x,\partial\Omega)^\alpha}$ are both C^k. Then,

$$\|\nabla^{\otimes k}(\operatorname{dist}(x,\partial\Omega)^\alpha (-\Delta)^\alpha v)\|_{L^\infty(B_{1/2}\setminus\Omega)} \leq C \left\| \frac{v}{\operatorname{dist}(x,\partial\Omega)^\alpha} \right\|_{C^k(B_1)},$$

where the constant depends on $\|f\|_{L^\infty}$.

Proof. The case $k = 0$ is Lemma A.8. Let us show the argument for $k = 1$; the result for $k > 1$ follows by differentiating in the same manner.

Choose coordinates such that $\vec{n}$, the outward normal at $0 \in \partial\Omega$, is the d-th coordinate vector e_{d}. Then, there is a function $g(x)$ such that $v(x) = g(x)(x_{\mathsf{d}})_+^\alpha$ in $\Omega \cap B_1$ by [ROS16, Theorem 1.2]; by our assumptions, $g(x)$ is C^k, and notice as well that it must decay like $|x|^{-\alpha}$ at infinity since $v \in L^\infty$.

We first differentiate $\operatorname{dist}(x,\partial\Omega)^\alpha(-\Delta)^\alpha v$ in the e_{d} direction. Let $x = -te_{\mathsf{d}}$ for some $t < 0$ (so that $x \in B_1 \setminus \Omega$). Then, $v(x) = 0$ and so by definition, see (1.10), we have

$$\operatorname{dist}(x,\partial\Omega)^\alpha(-\Delta)^\alpha v(x) = t^\alpha \int_{y_{\mathsf{d}}>0} \frac{-g(y)y_d^\alpha}{|x-y|^{\mathsf{d}+2\alpha}}\, dy = -t^\alpha \int_{y_{\mathsf{d}}>0} \frac{g(y)y_{\mathsf{d}}^\alpha}{\left(\sum_{i=1}^{\mathsf{d}-1} y_i^2 + (t+y_{\mathsf{d}})^2\right)^{\frac{\mathsf{d}}{2}+\alpha}}\, dy.$$

Factoring out t, we find with the substitution $y = tu$

$$\operatorname{dist}(x,\partial\Omega)^\alpha(-\Delta)^\alpha v(x) = \frac{-t^\alpha}{t^{\mathsf{d}+2\alpha}} \int_{y_{\mathsf{d}}>0} \frac{g(y)y_{\mathsf{d}}^\alpha}{\left(\sum_{i=1}^{d-1} \frac{y_i^2}{t^2} + (1+\frac{y_{\mathsf{d}}}{t})^2\right)^{\frac{\mathsf{d}}{2}+\alpha}}\, dy = -\int_{u_{\mathsf{d}}>0} \frac{g(tu)u_{\mathsf{d}}^\alpha}{|x'-u|^{\mathsf{d}+2\alpha}}\, du$$

with $x' = -e_{\mathsf{d}}$. Differentiating in the e_{d} direction amounts to differentiating in t, which yields

$$\left|\partial_{\mathsf{d}} \mathrm{dist}(x,\partial\Omega)^\alpha (-\Delta)^\alpha v(x) = \int_{u_{\mathsf{d}}>0} \frac{\sum \partial_i g(tu) u_i u_{\mathsf{d}}^\alpha}{|x'-u|^{\mathsf{d}+2\alpha}}\, du\right| \lesssim \|Dg\|_{L^\infty} \left|\int_{u_{\mathsf{d}}>0} \frac{|u| u_{\mathsf{d}}^\alpha}{|x'-u|^{\mathsf{d}+2\alpha}}\, du\right| \lesssim \|Dg\|_{L^\infty}$$

establishing the desired control. The same argument works for the normal derivative at other values of $x \in \Omega \cap B_{1/2}$ by choosing coordinates with origin at a different $z \in \partial\Omega$.

The tangential derivatives are easier since we have chosen coordinates for which $\mathrm{dist}(x,\partial\Omega)^\alpha$ only varies with e_{d}. Passing the derivative through the definition of the fractional Laplacian yields

$$\partial_i \left(\mathrm{dist}(x,\partial\Omega)^\alpha(-\Delta)^\alpha v(x)\right) = \mathrm{dist}(x,\partial\Omega)^\alpha(-\Delta)^\alpha \partial_i v(x) = \mathrm{dist}(x,\partial\Omega)^\alpha(-\Delta)^\alpha(\partial_i g(x)(x_{\mathsf{d}})_+^\alpha).$$

Using the L^∞ control given by the proof of Lemma A.8 we find

$$|\partial_i \left(\mathrm{dist}(x,\partial\Omega)^\alpha(-\Delta)^\alpha v(x)\right)| \lesssim \|Dg\|_{L^\infty},$$

establishing the desired bound. $\square$

We next state a result that controls the fractional Laplacian in terms of the derivatives of a function.

Lemma 8.4. *Assume φ is supported in some cube $\square_\ell$ of sidelength ℓ and (2.15) holds for $k \geq 2$. Then*

$$\|(-\Delta)^\alpha \varphi\|_{L^\infty} \lesssim \mathsf{M}\ell^{-2\alpha}. \tag{8.14}$$

Proof. Using the integral definition of the fractional Laplacian (1.10), we need to compute

$$\lim_{\eta\downarrow 0} \int_{B_\eta(0)^c} (\varphi(x) - \varphi(x+y)) \frac{c_{\mathsf{d},\alpha}}{|y|^{\mathsf{d}+2\alpha}}\, dy.$$

First consider $x \in \square_{2\ell}$. For $y \in \square_{2\ell}$, Taylor expanding, we find that the integrand is given by

$$\left|\int_{B_{4\ell}(0)\setminus B_\eta(0)} \left(-\nabla\varphi\cdot y + |y|^2 O\left(D^2\varphi\right)\right) \frac{c_{\mathsf{d},\alpha}}{|y|^{\mathsf{d}+2\alpha}}\, dy\right|$$
$$= \left|\int_{B_{4\ell}(0)\setminus B_\eta(0)} \left(|y|^2 O\left(D^2\varphi\right)\right) \frac{c_{\mathsf{d},\alpha}}{|y|^{\mathsf{d}+2\alpha}}\, dy\right| \lesssim \|\varphi\|_{C^2} \int_0^\ell r^{1-2\alpha}\, dr \lesssim \|\varphi\|_{C^2}\ell^{2-2\alpha}$$

by spherical symmetry of the order y term. For $|y| > 4\ell$ we estimate roughly,

$$\left|\int_{B_{4\ell}(0)^c} (\varphi(x) - \varphi(x+y)) \frac{c_{\mathsf{d},\alpha}}{|y|^{\mathsf{d}+2\alpha}}\, dy\right| \lesssim \|\varphi\|_{L^\infty} \int_{4\ell}^\infty r^{-1-2\alpha}\, dr \lesssim \|\varphi\|_{L^\infty}\ell^{-2\alpha}.$$

Assembling these results, we find that for $x \in \square_{2\ell}$,

$$|(-\Delta)^\alpha\varphi(x)| \lesssim \ell^{2-2\alpha}\|\varphi\|_{C^2} + \ell^{\mathsf{s}}\|\varphi\|_{L^\infty} \lesssim \mathsf{M}\ell^{-2\alpha}. \tag{8.15}$$

We next turn to the case $x \in \mathbb{R}^{\mathsf{d}}\setminus\square_{2\ell}$. Since φ vanishes outside $\square_\ell$, we don't need to use the principal value in the definition of the fractional Laplacian, and instead have

$$(-\Delta)^\alpha\varphi(x) = -\int_{x+y\in\square_\ell} \varphi(x+y)\frac{c_{\mathsf{d},\alpha}}{|y|^{\mathsf{d}+2\alpha}}\, dy.$$

We note that for $x \notin \square_{2\ell}$, $x+y \in \square_\ell$ implies that $|y| \ge \ell$. The integral is thus bounded by $\|\varphi\|_{L^\infty}\ell^{-2\alpha} \le \mathsf{M}\ell^{-2\alpha}$. Combining with (8.15), the conclusion follows.

$\square$

We can now state the scaling that we need for the behavior of the fractional Laplacian of the fractional harmonic extension in Σ.

Lemma 8.5. *Let φ be as in the assumptions of Lemma 8.2. Let $m \le k$. If* $\operatorname{supp}\varphi \subset \square_\ell \subset \hat{\Sigma}$ *with $\ell \le 1$, then*

(8.16)
$$\left|\nabla^{\otimes m}\left(\operatorname{dist}(x,\partial\Sigma)^\alpha(-\Delta)^\alpha\varphi^\Sigma(x)\right)\right| \lesssim \ell^{\mathsf{d}+2}\|(-\Delta)^\alpha\varphi\|_{L^\infty} + \begin{cases} \|\varphi\|_{C^{2+m}}\ell^{2+\mathsf{s}-\mathsf{d}} & \text{if } x \in \square_{2\ell} \\ \|\varphi\|_{L^\infty}\frac{\ell^{\mathsf{d}}}{|x-z|^{2\mathsf{d}-\mathsf{s}+m}} & \text{if } x \in \Sigma\setminus\square_{2\ell}\end{cases}$$

Proof. Let us first consider $m=0$. Note that by Lemma 8.1, we have

$$\varphi - \varphi^\Sigma = \operatorname{dist}(\cdot,\partial\Sigma)^\alpha f, \quad \|f\|_{C^\sigma(\overline{U\setminus\Sigma})} \lesssim \|\varphi\|_{L^\infty}\ell^{\mathsf{d}}. \tag{8.17}$$

Next, we argue similarly as in the proof of Lemma 8.4. First, take $x \in \square_{2\ell}$. Using the integral definition of the fractional Laplacian (1.10), we need to compute

$$\lim_{\eta\downarrow 0}\int_{B_\eta(0)^c}\left(\varphi^\Sigma(x)-\varphi^\Sigma(x+y)\right)\frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\,dy.$$

For $|y| \le 4\ell$, $\varphi^\Sigma = \varphi$ in the domain of integration. Taylor expanding, we find that the integrand is given by

$$\begin{aligned}&\left|\int_{B_{4\ell}(0)\setminus B_\eta(0)}\left(-\nabla\varphi\cdot y + |y|^2 O\left(D^2\varphi\right)\right)\frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\,dy\right| \\ &\qquad = \left|\int_{B_{4\ell}(0)\setminus B_\eta(0)}\left(|y|^2 O\left(D^2\varphi\right)\right)\frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\,dy\right| \lesssim \|\varphi\|_{C^2}\int_0^\ell r^{1-2\alpha}\,dr \lesssim \|\varphi\|_{C^2}\ell^{2-2\alpha}\end{aligned}$$

by spherical symmetry of the order y term. For $|y| > 4\ell$ we estimate roughly. First,

$$\left|\int_{B_{4\ell}(0)^c}\left(\varphi^\Sigma(x)-\varphi^\Sigma(x+y)\right)\frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\,dy\right| \le |\varphi(x)|\left|\int_{B_{4\ell}(0)^c}\frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\,dy\right| + \left|\int_{B_{4\ell}(0)^c}\frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}\varphi^\Sigma(x+y)}{|y|^{\mathsf{d}+2\alpha}}\,dy\right|$$

using $\varphi^\Sigma(x) = \varphi(x)$ for $x \in \square_{2\ell}$. The first term directly yields

$$|\varphi(x)|\left|\int_{B_{4\ell}(0)^c}\frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\,dy\right| \lesssim \|\varphi\|_{L^\infty}\int_\ell^\infty r^{-1-2\alpha}\,dr \lesssim \|\varphi\|_{L^\infty}\ell^{-2\alpha}.$$

On the other hand, using (8.17), for $x+y \in U$, we have

$$\left|\varphi^\Sigma(x+y)\right| \lesssim \|\varphi\|_{L^\infty}\ell^{\mathsf{d}}$$

since for $y \in B_{4\ell}(0)^c$, $\varphi(x+y) = 0$. Using (8.9) for $x+y \notin U$, we obtain

$$\begin{aligned}\left|\int_{B_{4\ell}(0)^c}\varphi^\Sigma(x+y)\frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\,dy\right| &\lesssim \|\varphi\|_{L^\infty}\ell^{\mathsf{d}}\int_\ell^\infty r^{-1-2\alpha}\,dr \\ &\quad + \left(\ell^{\mathsf{d}}\|\varphi\|_{L^\infty} + \ell^{\mathsf{d}+2}\|(-\Delta)^\alpha\varphi\|_{L^\infty}\right)\int_1^\infty r^{-\mathsf{s}-3-2\alpha}dr \\ &\lesssim \left(\ell^{\mathsf{s}}\|\varphi\|_{L^\infty} + \ell^{\mathsf{d}+2}\|(-\Delta)^\alpha\varphi\|_{L^\infty}\right)\end{aligned}$$

where we used that $2\alpha = \mathsf{d} - \mathsf{s}$. Assembling these results, we find that for $x \in \square_{2\ell}$,

$$\left|(-\Delta)^\alpha \varphi^\Sigma(x)\right| \lesssim \ell^{2+\mathsf{s}-\mathsf{d}} \|\varphi\|_{C^2} + \ell^{\mathsf{s}-\mathsf{d}} \|\varphi\|_{L^\infty} + \ell^{\mathsf{d}+2} \|(-\Delta)^\alpha \varphi\|_{L^\infty}. \tag{8.18}$$

We next turn to the case $x \in \Sigma \backslash \square_{2\ell}$. Since $\varphi^\Sigma(x) = 0$, we don't need to use the principal value in the definition of the fractional Laplacian, and instead have

$$(-\Delta)^\alpha \varphi^\Sigma(x) = -\int \varphi^\Sigma(x+y) \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\, dy.$$

We split this integral into three parts: $x+y \in \Sigma$, $x+y \in \Sigma^c \cap U$ and $x+y \in \mathbb{R}^{\mathsf{d}} \backslash U$. For $x+y \in \Sigma$, since $\varphi^\Sigma(x+y) = \varphi(x+y)$ vanishes unless $x+y \in Q_\ell$, we have

$$-\int_{x+y\in\Sigma} \varphi^\Sigma(x+y) \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\, dy \lesssim \|\varphi\|_{L^\infty} \frac{\ell^{\mathsf{d}}}{|x-z|^{\mathsf{d}+2\alpha}}.$$

For $x+y \in \Sigma^c \cap U$, notice that we have $|y| \geq \mathrm{dist}(x, \partial\Sigma)$ and $|y| \geq \mathrm{dist}(x+y, \partial\Sigma)$. Since $\varphi = 0$ in Σ^c, using (8.17), we have

$$\begin{aligned}
-\int_{x+y\in\Sigma^c\cap U} \varphi^\Sigma(x+y) \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\, dy &= -\int_{x+y\in\Sigma^c\cap U} f(x+y)\mathrm{dist}(x+y, \partial\Sigma)^\alpha \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\, dy \\
\lesssim \ell^{\mathsf{d}} \|\varphi\|_{L^\infty} \int_{x+y\in\Sigma^c\cap U} \frac{|y|^\alpha}{|y|^{\mathsf{d}+2\alpha}}\, dy &\lesssim \ell^{\mathsf{d}} \|\varphi\|_{L^\infty} \int_{\mathrm{dist}(x,\partial\Sigma)}^\infty r^{-1-\alpha} \lesssim \ell^{\mathsf{d}} \|\varphi\|_{L^\infty} \mathrm{dist}(x, \partial\Sigma)^{-\alpha}.
\end{aligned}$$

Finally, if $x+y \in U^c$, this implies that $|y|$ is bounded below by some positive $\varepsilon > 0$. Using (8.9), we then find

$$\begin{aligned}
-\int_{x+y\in U^c} \varphi^\Sigma(x+y) \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha}}\, dy &\lesssim (\ell^{\mathsf{d}+2} \|(-\Delta)^\alpha \varphi\|_{L^\infty} + \ell^{\mathsf{d}} \|\varphi\|_{L^\infty}) \int_{x+y\in U^c} \frac{1}{|y|^{2\mathsf{d}-\mathsf{s}}} \frac{1}{|x+y|^{\mathsf{s}+2}} dy \\
&\lesssim (\ell^{\mathsf{d}+2} \|(-\Delta)^\alpha \varphi\|_{L^\infty} + \ell^{\mathsf{d}} \|\varphi\|_{L^\infty}) \int_\varepsilon^\infty r^{-\mathsf{d}-3} dr \lesssim (\ell^{\mathsf{d}+2} \|(-\Delta)^\alpha \varphi\|_{L^\infty} + \ell^{\mathsf{d}} \|\varphi\|_{L^\infty}).
\end{aligned}$$

This yields

$$\left|(-\Delta)^\alpha \varphi^\Sigma(x)\right| \lesssim \|\varphi\|_{L^\infty} \left(\frac{\ell^{\mathsf{d}}}{|x-z|^{2\mathsf{d}-\mathsf{s}}} + \ell^{\mathsf{d}} \mathrm{dist}(x, \partial\Sigma)^{-\alpha} \right) + (\ell^{\mathsf{d}+2} \|(-\Delta)^\alpha \varphi\|_{L^\infty} + \ell^{\mathsf{d}} \|\varphi\|_{L^\infty})$$

in this case.

In view of (8.18), this concludes the proof of (8.16) for $m = 0$.

Now, let us turn to derivatives, i.e. $m \geq 1$. The same argument as above works for the estimate inside of $\square_{2\ell}$. For the decay, we start with $x \in \hat{\Sigma}$; there, $\mathrm{dist}(x, \partial\Sigma)^\alpha$ and its derivatives are bounded so we need only consider $(-\Delta)^\alpha \varphi^\Sigma$. It is again easier to use the definition

$$(-\Delta)^\alpha \varphi^\Sigma = \mathrm{P.V.} \int \frac{\varphi^\Sigma(x) - \varphi^\Sigma(y)}{|x-y|^{\mathsf{d}+2\alpha}}\, dy = \int \frac{-\varphi^\Sigma(y)}{|x-y|^{\mathsf{d}+2\alpha}}\, dy$$

since $\varphi^\Sigma(x) = 0$ for $x \in \hat{\Sigma} \setminus \square_{2\ell}$. A computation shows that for any mutiindex γ with $|\gamma| = m \leq k$ we have

$$\left|D^\gamma (-\Delta)^\alpha \varphi^\Sigma\right| \lesssim \int \frac{|\varphi^\Sigma(y)|}{|x-y|^{\mathsf{d}+2\alpha+m}}\, dy$$

We again split this integral into three parts: $y \in \Sigma$, $y \in U \setminus \Sigma$, and $y \in U^c$. For $y \in \Sigma$, since $\varphi^\Sigma = \varphi$ there we have

$$\int_{y\in\Sigma} \left|\varphi^\Sigma(y)\right| \frac{1}{|x-y|^{\mathsf{d}+2\alpha+m}}\, dy \lesssim \|\varphi\|_{L^\infty} \int_{O(\ell)} \frac{O(1)}{|x-z|^{\mathsf{d}+2\alpha+m}}\, dy \lesssim \frac{\ell^{\mathsf{d}}\|\varphi\|_{L^\infty}}{|x-z|^{\mathsf{d}+2\alpha+m}}$$

since the integral is only defined for $y \in \operatorname{supp}(\varphi)$, which forces $|x-y| \gtrsim |x-z|$. For $y \in U\setminus\Sigma$, we have using (8.17) that

$$\int_{y\in U\setminus\Sigma} \left|\varphi^\Sigma(y)\right| \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha+m}}\, dy = \int_{y\in U\setminus\Sigma} |f(y)|\, \mathrm{dist}(y,\partial\Sigma)^\alpha \frac{1}{|y|^{\mathsf{d}+2\alpha+m}}\, dy$$

$$\lesssim \ell^{\mathsf{d}}\|\varphi\|_{L^\infty} \int_{y\in U\setminus\Sigma} \frac{|y|^\alpha}{|y|^{\mathsf{d}+2\alpha+m}}\, dy \lesssim \ell^{\mathsf{d}}\|\varphi\|_{L^\infty} \int_{\mathrm{dist}(x,\partial\Sigma)}^\infty r^{-1-\alpha-m} \lesssim \ell^{\mathsf{d}}\|\varphi\|_{L^\infty}\mathrm{dist}(x,\partial\Sigma)^{-\alpha-m}$$

since y needs to be order $\mathrm{dist}(x,\partial\Sigma)^\alpha$ for $x+y \in \Sigma^c$. Since we are inside of the bulk, $\mathrm{dist}(x,\partial\Sigma) \sim 1$ and thus the integral contribution $y \in \Sigma$ dominates. Finally, if $y \in U^c$, this implies that $|y|$ is bounded below by some positive $\varepsilon > 0$. Using (8.9), we then find

$$-\int_{y\in U^c} |\varphi^\Sigma(y)| \frac{\mathsf{c}_{\mathsf{d},\mathsf{s}}}{|y|^{\mathsf{d}+2\alpha+m}}\, dy \lesssim (\ell^{\mathsf{d}+2}\|(-\Delta)^\alpha\varphi\|_{L^\infty} + \ell^{\mathsf{d}}\|\varphi\|_{L^\infty}) \int_{x+y\in U^c} \frac{1}{|y|^{2\mathsf{d}-\mathsf{s}+m}} \frac{1}{|y|^{\mathsf{s}+2}} dy$$
$$\lesssim (\ell^{\mathsf{d}+2}\|(-\Delta)^\alpha\varphi\|_{L^\infty} + \ell^{\mathsf{d}}\|\varphi\|_{L^\infty}) \int_\varepsilon^\infty r^{-\mathsf{d}-3-m} dr \lesssim \ell^{\mathsf{d}+2}\|(-\Delta)^\alpha\varphi\|_{L^\infty} + \ell^{\mathsf{d}}\|\varphi\|_{L^\infty},$$

and again the contribution from $y \in \Sigma$ dominates, yielding the required decay bound.

Finally, let us establish the desired control in $\Sigma \setminus \hat{\Sigma}$. We apply Lemma 8.3 to $v = \varphi^\Sigma$ with $f = 0$ to obtain

$$\left|\nabla^{\otimes m}\left(\mathrm{dist}(x,\partial\Sigma)^\alpha(-\Delta)^\alpha\varphi^\Sigma(x)\right)\right| \lesssim \left\|\frac{\varphi^\Sigma}{\mathrm{dist}(x,\partial\Sigma)^\alpha}\right\|_{C^m(U)},$$

which, by (8.3) is controlled by $\ell^{\mathsf{d}}\|\varphi\|_{L^\infty}$. This establishes the result. □

Finally, we establish the formula for the limit variance in the mesoscopic case.

Lemma 8.6. *Assume that* $\operatorname{supp}\varphi \subset \hat{\Sigma}$ *and* $\varphi = \varphi_0(\frac{\cdot - z}{\ell})$ *for some fixed function* φ_0. *As* $\ell \to 0$ *we have*

$$\ell^{-\mathsf{s}}\|\varphi^\Sigma\|^2_{\dot{H}^{\frac{\mathsf{d}-\mathsf{s}}{2}}} \to \|\varphi_0\|^2_{\dot{H}^{\frac{\mathsf{d}-\mathsf{s}}{2}}}. \tag{8.19}$$

and

$$\ell^{-\mathsf{s}} \int_\Sigma (-\Delta)^\alpha\varphi^\Sigma(\log\mu_V) \to 0. \tag{8.20}$$

Proof. Let us start with (8.19); we will use the equivalent characterization of the homogeneous Sobolev norm from the proof of Lemma 8.1, namely

$$\|u\|^2_{\dot{H}^{\frac{\mathsf{d}-\mathsf{s}}{2}}} = \frac{1}{2}\int_{\mathbb{R}^{2\mathsf{d}}} (u(x)-u(y))^2 \frac{c_{\mathsf{d},\alpha}}{|y-x|^{2\mathsf{d}-\mathsf{s}}}\, dxdz \tag{8.21}$$

using $2\alpha = \mathsf{d}-\mathsf{s}$. Notice that for $\varphi = \varphi_0\left(\frac{\cdot - z}{\ell}\right)$,

$$(-\Delta)^\alpha\varphi = \ell^{-2\alpha}\left((-\Delta)^\alpha\varphi_0\right)\left(\frac{\cdot - z}{\ell}\right)$$

and hence $\varphi^\Sigma(\cdot) = \varphi_0^{\frac{\Sigma - z}{\ell}}\left(\frac{\cdot - z}{\ell}\right)$. We will separate

$$\ell^{-\mathsf{s}}\|\varphi^\Sigma\|_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} = I_1 + I_2$$

where

$$I_1 := \frac{c_{\mathsf{d},\alpha}\ell^{-\mathsf{s}}}{2}\int_{\Sigma\times\Sigma} \frac{\left(\varphi_0^{\frac{\Sigma-z}{\ell}}\left(\frac{x-z}{\ell}\right) - \varphi_0^{\frac{\Sigma-z}{\ell}}\left(\frac{y-z}{\ell}\right)\right)^2}{|x-y|^{2\mathsf{d}-\mathsf{s}}}\,dxdy,$$

$$I_2 := \frac{c_{\mathsf{d},\alpha}\ell^{-\mathsf{s}}}{2}\int_{(\Sigma\times\Sigma)^c} \frac{\left(\varphi^\Sigma(x) - \varphi^\Sigma(y)\right)^2}{|x-y|^{2\mathsf{d}-\mathsf{s}}}\,dxdy.$$

Notice that via a change of variables we may write

$$I_1 = \frac{c_{\mathsf{d},\alpha}\ell^{-\mathsf{s}}}{2}\int_{\Sigma\times\Sigma} \frac{\left(\varphi_0^{\frac{\Sigma-z}{\ell}}\left(\frac{x-z}{\ell}\right) - \varphi_0^{\frac{\Sigma-z}{\ell}}\left(\frac{y-z}{\ell}\right)\right)^2}{|x-y|^{2\mathsf{d}-\mathsf{s}}}\,dxdy$$

$$= \frac{c_{\mathsf{d},\alpha}}{2}\int_{\frac{\Sigma-z}{\ell}\times\frac{\Sigma-z}{\ell}} \frac{\left(\varphi_0^{\frac{\Sigma-z}{\ell}}(u) - \varphi_0^{\frac{\Sigma-z}{\ell}}(w)\right)^2}{|u-w|^{2\mathsf{d}-\mathsf{s}}}\,dudw = \frac{c_{\mathsf{d},\alpha}}{2}\int_{\frac{\Sigma-z}{\ell}\times\frac{\Sigma-z}{\ell}} \frac{\left(\varphi_0(u) - \varphi_0(w)\right)^2}{|u-w|^{2\mathsf{d}-\mathsf{s}}}\,dudw$$

and so the difference $\ell^{-\mathsf{s}}\|\varphi^\Sigma\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} - \|\varphi_0\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}}$ is merely

$$\ell^{-\mathsf{s}}\|\varphi^\Sigma\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} - \|\varphi_0\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} = I_2 - \frac{c_{\mathsf{d},\alpha}}{2}\int_{\left(\frac{\Sigma-z}{\ell}\times\frac{\Sigma-z}{\ell}\right)^c} \frac{\left(\varphi_0(u) - \varphi_0(w)\right)^2}{|u-w|^{2\mathsf{d}-\mathsf{s}}}\,dudw. \tag{8.22}$$

Notice that the same change of variables implies

$$\ell^{-\mathsf{s}}\|\varphi^\Sigma\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} = \left\|\varphi_0^{\frac{\Sigma-z}{\ell}}\right\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}}.$$

Since $\varphi_0^{\frac{\Sigma-z}{\ell}}$ is defined as the function coinciding with φ_0 on $\frac{\Sigma-z}{\ell}$ and minimizing $\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}$ norm, it follows that

$$\ell^{-\mathsf{s}}\|\varphi^\Sigma\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} - \|\varphi_0\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} = \left\|\varphi_0^{\frac{\Sigma-z}{\ell}}\right\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} - \|\varphi_0\|^2_{\dot H^{\frac{\mathsf{d}-\mathsf{s}}{2}}} \le 0.$$

It is thus sufficient in (8.22) to show that

$$\left|\frac{c_{\mathsf{d},\alpha}}{2}\int_{\left(\frac{\Sigma-z}{\ell}\times\frac{\Sigma-z}{\ell}\right)^c} \frac{\left(\varphi_0(u) - \varphi_0(w)\right)^2}{|u-w|^{2\mathsf{d}-\mathsf{s}}}\,dudw\right| \to 0$$

since $I_2 \ge 0$. Notice that since $2\operatorname{supp}\varphi_0 \subset \frac{\Sigma-z}{\ell}$, we may rewrite it via symmetry as

$$\left|-c_{\mathsf{d},\alpha}\int_{(\operatorname{supp}\varphi_0)\times\left(\frac{\Sigma-z}{\ell}\right)^c} \frac{\varphi_0(u)^2}{|u-w|^{2\mathsf{d}-\mathsf{s}}}\,dudw\right| \lesssim \|\varphi_0\|^2_{L^\infty}\int_{\operatorname{supp}\varphi_0}\int_{\operatorname{dist}\left(\operatorname{supp}\varphi_0,\partial\left(\frac{\Sigma-z}{\ell}\right)\right)}^\infty \frac{r^{\mathsf{d}-1}}{r^{2\mathsf{d}-\mathsf{s}}}\,drdu$$

$$\lesssim \|\varphi_0\|^2_{L^\infty}\operatorname{dist}\left(\operatorname{supp}\varphi_0, \partial\left(\frac{\Sigma-z}{\ell}\right)\right)^{\mathsf{s}-\mathsf{d}} \lesssim \|\varphi_0\|^2_{L^\infty}\ell^{\mathsf{d}-\mathsf{s}} \to 0$$

and we conclude the result.

Now for (8.20) it is easier to use the equivalent characterization of the mean from the proof of (4.2), namely

$$\ell^{-\mathsf{s}}\int_\Sigma (-\Delta)^\alpha \varphi^\Sigma(\log\mu_V) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\ell^{-\mathsf{s}}\int_{\mathbb{R}^\mathsf{d}}(\operatorname{div}\psi)\mu_V = \mathsf{c}_{\mathsf{d},\mathsf{s}}\ell^{-\mathsf{s}}\int_\Sigma(\operatorname{div}\psi)\mu_V$$

where ψ is the transport map defined by (2.16). Integrating by parts and using that $\psi\mu_V\equiv 0$ on $\partial\Sigma$,

$$\mathsf{c}_{\mathsf{d},\mathsf{s}}\ell^{-\mathsf{s}}\int_\Sigma(\operatorname{div}\psi)\mu_V = -\mathsf{c}_{\mathsf{d},\mathsf{s}}\ell^{-\mathsf{s}}\int_\Sigma \psi\cdot\nabla\mu_V.$$

In the bulk, we use that $\nabla\mu_V$ is bounded below and (2.17) to bound

$$\begin{aligned}\left|-\mathsf{c}_{\mathsf{d},\mathsf{s}}\ell^{-\mathsf{s}}\int_{\hat\Sigma}\psi\cdot\nabla\mu_V\right| &\lesssim \ell^{-\mathsf{s}}\int_{\square_{2\ell}}\frac{\|\varphi_0\|_{C^3}}{\ell^{\mathsf{d}-\mathsf{s}-1}} + \ell^{-\mathsf{s}}\int_{\hat\Sigma\setminus\square_{2\ell}}\frac{\ell^\mathsf{d}\|\varphi_0\|_{C^3}}{|x-z|^{2\mathsf{d}-\mathsf{s}-1}}\\ &\lesssim \ell\|\varphi_0\|_{C^3} + \ell^{\mathsf{d}-\mathsf{s}}\|\varphi_0\|_{C^3}\int_\ell^1 \frac{r^{\mathsf{d}-1}}{r^{2\mathsf{d}-\mathsf{s}-1}}\,dr \lesssim \ell\|\varphi_0\|_{C^3}\to 0.\end{aligned}$$

Outside of $\hat\Sigma$, we use that $\nabla\mu_V$ decays like $\operatorname{dist}(x,\partial\Sigma)^{-\alpha}$ coupled with (2.17) for $|x-z|$ at order one to see that

$$\left|-\mathsf{c}_{\mathsf{d},\mathsf{s}}\ell^{-\mathsf{s}}\int_{\Sigma\setminus\hat\Sigma}\psi\cdot\nabla\mu_V\right| \lesssim \ell^{-\mathsf{s}}\int_{\Sigma\setminus\hat\Sigma}\|\varphi_0\|_{C^3}\ell^\mathsf{d}\operatorname{dist}(x,\partial\Sigma)^{-\alpha}\lesssim \|\varphi_0\|_{C^3}\ell^{\mathsf{d}-\mathsf{s}}\to 0.$$

since $\operatorname{dist}(x,\partial\Sigma)^{-\alpha}$ is an integrable singularity. Coupling the estimates in $\hat\Sigma$ and $\Sigma\setminus\hat\Sigma$ yields (8.20).

□

9. Proof of the Screening Result - Proposition 5.7

We focus on the proof of outer screening; we will discuss the main idea of inner screening and what computational changes are necessary at the end of this section. From this point on, we denote $E_\mathsf{r} = \nabla w_\mathsf{r}$ to emphasize that we are considering the electric field.

9.1. **The Setup.** The first part of the proof consists of using the energy bounds to find a good boundary outside of which to construct the screened configuration. We have two separate cases, depending on which screenability condition is satisfied.

If the first condition is satisfied in the minimum in (5.36), then using a mean value argument as in [PS17, Section 6.2, Step 1] and [AS21, Section C.1], we can find a $T\in[R-2\tilde\ell+2, R-\tilde\ell-2]$ such that

$$\int_{(Q_{T+2}\setminus Q_{T-2})\times[-h,h]}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2 \le \frac{S(X_n,w,h)}{\tilde\ell} := M \tag{9.1}$$

$$\int_{\partial Q_T\times[-h,h]}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2\lesssim M \tag{9.2}$$

where $Q_T\subset Q_R$ and $Q_T\in\mathcal{Q}_T$ (i.e. in particular $\mu(Q_T)$ is integer). We then take $\Gamma := \partial Q_T$, which in one dimension is simply two points on the axis.

Otherwise, the second condition is satisfied in (5.36). Then, via a mean-value argument analogous to [Ser24, Appendix A], we can find some $t\in[R-2\tilde\ell, R-\tilde\ell-\ell]$ such that

$$\int_{(Q_{t+\ell}\setminus Q_t)\times[-h,h]}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2\le C\frac{S(X_n,w,h)\ell}{\tilde\ell}.$$

We then apply a mean-value argument in the strip $Q_{t+\ell} \setminus Q_t$ and find a piecewise affine boundary Γ in $Q_{t+\ell} \setminus Q_t$ with faces parallel to those of Q_R and sidelengths of order ℓ such that

$$\int_{\Gamma\times[-h,h]} |y|^\gamma |E_{\hat{\mathsf{r}}}|^2 \leq C\frac{S(X_n,w,h)}{\tilde{\ell}}, \qquad \sup_x \int_{(\Gamma\cap\square_\ell(x))\times[-h,h]} |y|^\gamma |E_{\hat{\mathsf{r}}}|^2 \leq CS'(X_n,w,h) \tag{9.3}$$

and

$$\int_{\Gamma_1\times[-h,h]} |y|^\gamma |E_{\hat{\mathsf{r}}}|^2 \leq C\frac{S(X_n,w,h)}{\tilde{\ell}}$$

where Γ_1 denotes the 1-neighborhood of Γ. In both cases we let $M = \frac{S(X_n,w,h)}{\tilde{\ell}}$ and in the latter case $M_\ell = CS'(X_n,w,h)$.

Γ encloses a set $\mathcal{O}$, in which we will keep X_n and the associated electric field E unchanged. The modifications to the electric field will take place in the set $\mathcal{N} := Q_L \setminus \mathcal{O}$. We let

$$\mathcal{N}_\eta = \{x \in \mathcal{N} : \operatorname{dist}(x,\Gamma) \geq \eta\} \tag{9.4}$$

and keep $\mathcal{N}\backslash\mathcal{N}_\eta$ as a buffer region where we will refrain from placing points. We also let M_0^+ and M_0^- be averaged electric fluxes on the top and bottom of our hyperrectangular region:

$$M_0^+ := \frac{h^{-\gamma}}{|\mathcal{N}|}\int_{\mathcal{O}\times\{h\}} |y|^\gamma E_{\mathsf{r}} \cdot \vec{n}, \tag{9.5}$$

$$M_0^- := \frac{h^{-\gamma}}{|\mathcal{N}|}\int_{\mathcal{O}\times\{-h\}} |y|^\gamma E_{\mathsf{r}} \cdot \vec{n}, \tag{9.6}$$

and set

$$M_0 = h^\gamma M_0^+ + h^\gamma M_0^-. \tag{9.7}$$

With this region in tow, we partition our space (as in [PS17, Section 6.2, Step 2]) into the following subregions, and solve elliptic problems in each:

(1) $D_0 := \mathcal{O} \times [-h,h]$
(2) $D_\partial := \mathcal{N} \times [-h,h]$
(3) $D_1 := (Q_R \times [-\max(R,H), \max(R,H)]) \setminus (D_0 \cup D_\partial)$.

where H is the height of $\Lambda \subset \mathbb{R}^{\mathsf{d}+1}$ in the case where we are proving local laws in Λ. We partition $Q_R \setminus \mathcal{O}$ into regions H_k with piecewise affine boundary and sidelengths at scale ℓ, in $\left[\frac{\ell}{C}, \ell C\right]$. Let $\tilde{H}_k$ denote the cells $H_k \times [-h,h]$, and set

$$H_k^\eta := \{x \in H_k : \operatorname{dist}(x,\Gamma) \geq \eta\} \tag{9.8}$$

with $\ell > 2\eta$.

Observe that the delineation of our points into old and new sets might intersect some of the “smeared" points; these smeared regions will have to be modified appropriately. We let I_∂ denote the set of charges that are smeared by the boundary Γ, i.e.

$$I_\partial := \{i : B(x_i, \mathsf{r}_i) \cap \Gamma \neq \emptyset\}.$$

Set

$$n_k := \mathsf{c}_{\mathsf{d},\mathsf{s}} \int_{\tilde{H}_k} \sum_{i\in I_\partial} \delta_{x_i}^{(\mathsf{r}_i)} \tag{9.9}$$

to be the amount of smear in a region $\tilde{H}_k$. We let $n_{\mathcal{O}}$ denote the number of smeared charges and the number of charges we want wholly unchanged in $\mathcal{O}$, i.e.

$$n_{\mathcal{O}} := \#I_\partial + \#(\{i : x_i \in \mathcal{O}\} \setminus I_\partial).$$

The goal will be to place $\bar{\mathrm{n}} - n_{\mathcal{O}}$ sampled points in $\mathcal{N} := Q_R \setminus \mathcal{O}$, where $\bar{\mathrm{n}} = \mu(Q_R)$, while leaving a point-free zone of η-thickness. For each k, choose constants m_k such that

$$\mathsf{c}_{\mathsf{d},\mathsf{s}} m_k |H_k^\eta| = \int_{\partial D_0 \cap \partial \tilde{H}_k} |y|^\gamma E_{\mathsf{r}} \cdot \vec{n} + M_0 |H_k| - n_k + \mathsf{c}_{\mathsf{d},\mathsf{s}} \int_{H_k \setminus H_k^\eta} d\mu \tag{9.10}$$

If m_k is small enough, namely $|m_k| \leq \frac{1}{2} m$ (where m is a lower bound for μ), then we can guarantee $\int_{H_k} \mu + m_k |H_K^\eta| \in \mathbb{N}$.

Define

$$\tilde{\mu} := \mu \mathbf{1}_{\mathrm{dist}(x,\Gamma)\geq \eta} + \sum_k m_k \mathbf{1}_{H_k^\eta}.$$

On the other hand, we have immediately from the divergence theorem and (5.29)

$$\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\partial D_0} |y|^\gamma E_{\mathsf{r}} \cdot \vec{n} = \int_{\mathcal{O}} d\mu - n_{\mathcal{O}} + \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \sum_k n_k.$$

Hence, by definition (9.10),

$$\begin{aligned}
\tilde{\mu}(\mathcal{N}) &= \tilde{\mu}(\mathcal{N}_\eta) = \mu(\mathcal{N}_\eta) + \sum_k m_k |H_k^\eta| \\
&= \bar{\mathrm{n}} - \mu(\mathcal{O}) - \mu(\mathcal{N} \setminus \mathcal{N}_\eta) + \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\partial D_0 \setminus (\mathcal{O} \times \{-h,h\})} |y|^\gamma E_{\mathsf{r}} \cdot \vec{n} \\
&+ \frac{M_0}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \sum_k |H_k| - \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \sum_k n_k + \mu(\mathcal{N} \setminus \mathcal{N}_\eta) \\
&= \bar{\mathrm{n}} - \mu(\mathcal{O}) + \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\partial D_0} |y|^\gamma E_{\mathsf{r}} \cdot \vec{n} - \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \sum_k n_k \\
&= \bar{\mathrm{n}} - n_{\mathcal{O}}.
\end{aligned}$$

With all of these quantities defined, we are in a position to construct a new screened field outside of D_0.

9.2. **Defining the Electric Field.** We define the screened electric field in each of the different subregions.

First we have E_1, which completes the smeared charges, defined by

$$E_1 := \sum_k \mathbf{1}_{\tilde{H}_k} \nabla h_{1,k},$$

where $h_{1,k}$ solves

$$\begin{cases}
-\mathrm{div}\,(|y|^\gamma \nabla h_{1,k}) = \mathsf{c}_{\mathsf{d},\mathsf{s}} \sum_{i \in I_\partial} \delta_{x_i}^{(\mathsf{r}_i)} & \text{in } \tilde{H}_k \\
\frac{\partial h_{1,k}}{\partial n} = 0 & \text{on } \partial \tilde{H}_k \setminus \partial D_0 \\
\frac{\partial h_{1,k}}{\partial n} = -\frac{n_k}{\int_{F_k} |y|^\gamma} & \text{on } F_k,
\end{cases}$$

where F_k is the face of $\partial \tilde{H}_k$ touching ∂D_0, if it exists. This is solvable by (9.9).

E_2 balances the top region, D_1 and is defined by

$$E_2 := \sum_k \mathbf{1}_{\tilde{H}_k} \nabla h_{2,k},$$

where $h_{2,k}$ solves

$$\begin{cases} -\mathrm{div}\left(|y|^\gamma h_{2,k}\right) = \mathsf{c}_{\mathsf{d},\mathsf{s}}(m_k \mathbf{1}_{H_k^\eta} - \mu \mathbf{1}_{H_k \setminus H_k^\eta})\delta_{\mathbb{R}^\mathsf{d}} & \text{in } \tilde{H}_k \\ \frac{\partial h_{2,k}}{\partial n} = -M_0^+ & \text{on } H_k \times \{h\} \\ \frac{\partial h_{2,k}}{\partial n} = -M_0^- & \text{on } H_k \times \{-h\} \\ \frac{\partial h_{2,k}}{\partial n} = g_k & \text{on the rest of } \partial \tilde{H}_k, \end{cases}$$

where $g_k \equiv 0$ if H_k doesn't touch Γ, and $g_k = -E_\mathsf{r} \cdot \vec{n} + \frac{n_k}{\int_{F_k} |y|^\gamma}$ otherwise, with $\vec{n}$ throughout the outward normal from D_0. This is solvable by (9.10).

E_3 gives us the sampled configuration $Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}}$ in $\mathcal{N}_\eta$ and is defined by

$$E_3 = \nabla h_3 \mathbf{1}_{D_\partial},$$

where h_3 solves the Neumann problem

$$\begin{cases} -\mathrm{div}\left(|y|^\gamma \nabla h_3\right) = \mathsf{c}_{\mathsf{d},\mathsf{s}} \left(\sum_{j=1}^{\bar{\mathrm{n}}-n_{\mathcal{O}}} \delta_{z_j} - \tilde{\mu}\delta_{\mathbb{R}^\mathsf{d}}\right) & \text{in } \mathcal{N}_\eta \times [-h,h] \\ \frac{\partial h_3}{\partial n} = 0 & \text{on } \partial\left(\mathcal{N}_\eta \times [-h,h]\right). \end{cases}$$

Finally, E_4 gives us the screened electric field in D_1 and is defined by

$$E_4 := \nabla h_4,$$

where h_4 solves

$$\begin{cases} -\mathrm{div}\left(|y|^\gamma \nabla h_4\right) = 0 & \text{in } D_1 \\ \frac{\partial h_4}{\partial n} = -\phi & \text{on } \partial D_1, \end{cases}$$

where

$$\phi := \mathbf{1}_{\partial D_1 \cap \partial D_0} E \cdot \vec{n} - \mathbf{1}_{\partial D_1 \cap \partial D_\partial \cap \{y>0\}} M_0^+ - \mathbf{1}_{\partial D_1 \cap \partial D_\partial \cap \{y<0\}} M_0^-.$$

Now, set $E_{\hat{\mathsf{r}}}^{\mathrm{scr}} := (E_1 + E_2 + E_3)\mathbf{1}_{D_\partial} + E_4 \mathbf{1}_{D_1} + E_{\hat{\mathsf{r}}} \mathbf{1}_{D_0}$ and add back in the truncations, by setting

$$E^{\mathrm{scr}} := E_{\hat{\mathsf{r}}}^{\mathrm{scr}} + \sum_{i=1}^{\bar{\mathrm{n}}} \nabla \mathsf{f}_{\bar{\mathsf{r}}_i}(x - y_i),$$

where y_i correspond to the points (in $\mathbb{R}^\mathsf{d}$) of the new configuration $Y_{\bar{\mathrm{n}}} = (\{X_n\} \cap \mathcal{O}) \cup Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}}$, and $\bar{\mathsf{r}}$ are the (possibly changed) minimal distances, blown up versions of (5.6) for the new configuration $Y_{\bar{\mathrm{n}}}$. Due to the Neumann condition, no divergence is created across boundaries when we set E^{scr} to vanish outside of our region. By definition, we have

$$\begin{cases} -\mathrm{div}(|y|^\gamma E^{\mathrm{scr}}) = \mathsf{c}_{\mathsf{d},\mathsf{s}} \Big(\sum_{i \in Y_{\bar{\mathrm{n}}}} \delta_{y_i} - \mu \delta_{\mathbb{R}^\mathsf{d}}\Big) & \text{in } Q_R \times [-R,R] \\ E^{\mathrm{scr}} \cdot \vec{n} = 0 & \text{on } \partial(Q_R \times [-R,R]). \end{cases}$$

9.3. **Estimating Constants.** Instead of estimating M_0^+ and M_0^- using Cauchy-Schwarz immediately as in [PS17], we instead carry these constants through our calculations. This will allow us to be as precise as possible in our estimates of M_0. As we discussed above, the screening process requires that $|m_k| \le \frac{m}{2}$. It will be convenient in the proof of the local laws to have a bound $\left\|\frac{\mu-\tilde{\mu}}{\tilde{\mu}}\right\|_{L^\infty(\mathcal{N}_\eta)} \le C < 1$; in order to obtain this, we will actually need a bit more than $|m_k| \le \frac{m}{2}$. So, we seek $|m_k| \le \frac{m}{3}$.

First observe, using the bound (9.1) and the discrepancy estimates (3.28) applied on balls B_α of radius 1 (at blown-up scale) near Γ, we have

$$n_{k,\alpha} \lesssim \|\mu\|_{L^\infty} + \left(\int_{H_k\cap B_\alpha} |y|^\gamma |E_{\hat{\mathsf{r}}}|^2\right)^{\frac{1}{2}} \tag{9.11}$$

and summing over α (choosing the $O(\ell^{\mathsf{d}-1})$ balls to form a finite covering of the desired region), we find

$$n_k \lesssim \|\mu\|_{L^\infty}\ell^{\mathsf{d}-1} + \ell^{\frac{\mathsf{d}-1}{2}}(\min(M, M_\ell))^{\frac{1}{2}}.$$

We also have

$$n_k^2 = \left(\sum_\alpha n_{k,\alpha}\right)^2 \lesssim \ell^{\mathsf{d}-1}\sum_\alpha n_{k,\alpha}^2 \lesssim \|\mu\|_{L^\infty}\ell^{2\mathsf{d}-2} + \ell^{\mathsf{d}-1}\int_{H_k\cap[Q_{T+2}\setminus Q_{T-2}]\times[-h,h]} |y|^\gamma|E_{\hat{\mathsf{r}}}|^2$$

using (9.11), which yields

$$\sum n_k^2 \lesssim R^{\mathsf{d}-1}\ell^{\mathsf{d}-1}\|\mu\|_{L^\infty}^2 + M\ell^{\mathsf{d}-1} \tag{9.12}$$

since the number of H_k intersecting Γ is bounded by $O\left(\frac{R^{\mathsf{d}-1}}{\ell^{\mathsf{d}-1}}\right)$. We will make use of this estimate below. In the same way $\#I_\partial \lesssim M + R^{\mathsf{d}-1}$. Hence, by definition (9.10), using the above bounds, Cauchy-Schwarz and (9.3), we have

$$\begin{aligned}
|m_k| &\le C\left(\frac{1}{|H_k|}\int_{\partial D_0\cap\partial\tilde{H}_k} |y|^\gamma|E_{\hat{\mathsf{r}}}\cdot\vec{n}| + M_0 + \frac{n_k}{|H_k|}\right) + \frac{\eta}{\ell}\|\mu\|_{L^\infty} \\
&\le C\left(M_0 + \ell^{-\mathsf{d}}\int_{\partial D_0\cap\partial\tilde{H}_k}|y|^\gamma|E_{\hat{\mathsf{r}}}| + \ell^{-\mathsf{d}}(\min(M,M_\ell))^{\frac{1}{2}}\ell^{\frac{\mathsf{d}-1}{2}} + \|\mu\|_{L^\infty}\ell^{\mathsf{d}-1}\right) + \frac{\eta}{\ell}\|\mu\|_{L^\infty} \\
&\le C\left(M_0 + \ell^{-\mathsf{d}}\sqrt{M}\sqrt{|\partial D_0\cap\partial\tilde{H}_k|h^\gamma} + (\min(M,M_\ell))^{\frac{1}{2}}\ell^{-\frac{\mathsf{d}+1}{2}} + \|\mu\|_{L^\infty}\ell^{-1}\right) + \frac{\eta}{\ell}\|\mu\|_{L^\infty} \\
&\le C\left(M_0 + \ell^{-\mathsf{d}}\ell^{\frac{\mathsf{d}-1}{2}}h^{\frac{1+\gamma}{2}}(\min(M,M_\ell))^{\frac{1}{2}}\right) + \frac{C}{\ell}\|\mu\|_{L^\infty},
\end{aligned}$$

after absorbing some terms, using $\eta < 1$ and $h > 1$. To obtain that this is less than $\frac{m}{3}$, squaring, rearranging and using the bounds on M we reduce to the sufficient conditions that

$$\frac{C}{\ell}\|\mu\|_{L^\infty} < \frac{m}{6}, \qquad M_0^2 + \frac{h^{1+\gamma}\min(M,M_\ell)}{\ell^{\mathsf{d}+1}} \le \mathsf{c} \tag{9.13}$$

for some fixed constant c (depending on m), as long as ℓ is large enough, $\eta < 1$ and $h > 1$. The first condition is satisfied trivially for large enough $\ell > 1$. Inserting the definition of M and M_ℓ, this corresponds to the second part of the screenability condition (5.36). Furthermore,

using Cauchy-Schwarz and the definition (5.32), we have
(9.14)
$$(M_0^+)^2 = \left(\frac{h^{-\gamma}}{|\mathcal{N}|}\int_{\mathcal{O}\times\{h\}} |y|^\gamma E\cdot\vec{n}\right)^2 \le \frac{|\mathcal{O}|}{|\mathcal{N}|^2} h^{-\gamma} e(X_n,w,h) \lesssim \frac{R^{\mathsf{d}}}{R^{2(\mathsf{d}-1)}\tilde{\ell}^2} h^{-\gamma} e(X_n,w,h)$$

and the same for M_0^-, hence

$$M_0^2 \le \frac{1}{R^{\mathsf{d}-2}\tilde{\ell}^2} h^\gamma e(X_n,w,h).$$

Substituting this into the above yields the screenability condition (5.36).

Notice that this condition yields a nice L^∞ bound

$$\left\|\frac{\mu-\tilde{\mu}}{\tilde{\mu}}\right\|_{L^\infty(H_k^\eta)} = \left\|\frac{m_k}{\mu+m_k}\right\|_{L^\infty(H_k^\eta)} \le \frac{\frac{m}{3}}{m-\frac{m}{3}} = \frac{1}{2}.$$

Since $\tilde{\mu}$ is defined separately on the H_k, we then also have

$$\left\|\frac{\mu-\tilde{\mu}}{\tilde{\mu}}\right\|_{L^\infty(\mathcal{N}_\eta)} \le \frac{1}{2}.$$

9.4. **Estimating the screened field.** We first estimate E_1; the idea is to use an analog of [Ser24, Lemma A.2]. An examination of the proof shows that the $(\#I)^2 a^{2-\mathsf{d}}$ term there is obtained from applying the Coulomb version of [PS17, Lemma 6.4] to

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma \nabla v) = c\frac{|F_k|}{\tilde{H}_k}\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\tilde{H}_k \setminus \partial D_0 \\ \frac{\partial v}{\partial n} = c & \text{on } F_k, \end{cases}$$

with $c = -\frac{n_k}{\int_{F_k}|y|^\gamma}$. We instead just apply [PS17, Lemma 6.4] to control this term; however, since $\tilde{H}_k$ has a height of length h, an examination of the proof induces an aspect ratio $\frac{h}{\ell}$. Hence,

$$\int_{\tilde{H}_k} |y|^\gamma |\nabla h_{1,k,\hat{\mathsf{r}}}|^2 \lesssim \frac{h n_k^2}{\int_{F_k}|y|^\gamma} + \sum_{i\in H_k} \mathsf{g}(\hat{\mathsf{r}}_i). \tag{9.15}$$

Using (3.22) we can bound the sum over all $\mathsf{g}(\mathsf{r}_i)$ in D_∂ by $M + \#I_\partial \lesssim M + R^{\mathsf{d}-1}$ as seen above, and so

$$\begin{aligned}\int_{D_\partial} |y|^\gamma |E_{1,\hat{\mathsf{r}}}|^2 \lesssim h\sum_k \frac{n_k^2}{\int_{F_k}|y|^\gamma} + M + R^{\mathsf{d}-1} &\lesssim \frac{1}{\ell^{\mathsf{d}-1}h^\gamma}\left(\ell^{\mathsf{d}-1}R^{\mathsf{d}-1} + \ell^{\mathsf{d}-1}M\right) + M + R^{\mathsf{d}-1} \\ &\lesssim (1+h^{-\gamma})\left(M + R^{\mathsf{d}-1}\right)\end{aligned}$$

using (9.12) and

$$\int_{F_k} |y|^\gamma \sim |\partial H_k| \int_{-h}^h |y|^\gamma \sim \ell^{\mathsf{d}-1} h^{1+\gamma}.$$

We next turn to E_2. which bounding similarly with [PS17, Lemma 6.4] yields

$$\begin{aligned}\int_{D_\partial}|y|^\gamma|E_{2,\hat{\mathsf{r}}}|^2 &\lesssim \frac{h}{\ell}\ell\sum_k\left(\int_{F_k}|y|^\gamma|g_k|^2+(M_0^+)^2h^\gamma\ell^{\mathsf{d}}+(M_0^-)^2h^\gamma\ell^{\mathsf{d}}\right)\\ &\lesssim h\int_{\Gamma\times[-h,h]}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2+h\sum_k\frac{n_k^2}{\int_{F_k}|y|^\gamma}+h\left((M_0^+)^2h^\gamma\ell^{\mathsf{d}}+(M_0^-)^2h^\gamma\ell^{\mathsf{d}}\right)\frac{R^{\mathsf{d}-1}\tilde{\ell}}{\ell^{\mathsf{d}}}\\ &\lesssim hM+Mh^{-\gamma}+R^{\mathsf{d}-1}h^{-\gamma}+\tilde{\ell}R^{\mathsf{d}-1}h^{1-\gamma}M_0^2\end{aligned}$$

where we have borrowed from above the estimate on $h\sum_k\frac{n_k^2}{\int_{F_k}|y|^\gamma}$. Since $\gamma<1$ and $h>1$, we may rewrite this as

$$\int_{D_\partial}|y|^\gamma|E_{2,\hat{\mathsf{r}}}|^2\lesssim hM+R^{\mathsf{d}-1}h^{-\gamma}+\tilde{\ell}R^{\mathsf{d}-1}h^{1-\gamma}M_0^2.$$

For E_3, we obtain directly from the definition of $\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}},\tilde{\mu},\mathcal{N}_\eta\times[-h,h])$ (see 5.8) and the fact that f_η is uniformly bounded in L^1 for small η by (3.17) that

$$\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\mathcal{N}_\eta\times[-h,h]}|y|^\gamma|\nabla h_{3,\hat{\mathsf{r}}}|^2\le\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}},\tilde{\mu},\mathcal{N}_\eta\times[-h,h])+\mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{j=1}^{\bar{\mathrm{n}}-n_{\mathcal{O}}}\mathsf{g}(\hat{\mathsf{r}}_j)+C(\bar{\mathrm{n}}-n_{\mathcal{O}}).$$

Finally, to bound the top field E_4, we use [PS17, Lemma 6.4] at scale R. This yields

$$\begin{aligned}\int_{D_1}|E_4|^2\lesssim R\int_{\partial D_1}|y|^\gamma|\phi|^2 &\le R|\mathcal{N}|h^{-\gamma}M_0^2+R\int_{\square_{T+1}\times\{-h,h\}}|y|^\gamma|\nabla w|^2\\ &\le R\tilde{\ell}R^{\mathsf{d}-1}h^{-\gamma}M_0^2+Re(X_n,w,h)\\ &\lesssim R^{\mathsf{d}}\tilde{\ell}h^{-\gamma}M_0^2+Re(X_n,w,h)\end{aligned}$$

using the definition of ϕ; the multiplication by $|\mathcal{N}|$ comes from integrating the constants M_0^+ and M_0^- over where they are supported in the definition of ϕ, namely $\mathcal{N}\times\{h\}$ and $\mathcal{N}\times\{-h\}$, respectively. It remains to put it all together and obtain the requisite screening estimate. We kept the original electric field fixed in D_0, so combining the above estimates allows us to write

$$\begin{aligned}\int_{Q_R\times[-R,R]}|E_{\hat{\mathsf{r}}}^{\mathrm{scr}}|^2 \le \int_{D_0}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2+ChM+CR^{\mathsf{d}-1}+R^{\mathsf{d}-1}h^{-\gamma}+CR^{\mathsf{d}}\tilde{\ell}h^{-\gamma}M_0^2+CRe(X_n,w,h)\\ +C(\bar{\mathrm{n}}-n_{\mathcal{O}})+C\left(2\mathsf{c}_{\mathsf{d},\mathsf{s}}\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}},\tilde{\mu},\mathcal{N}_\eta\times[-h,h])+\mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{i=1}^{\bar{\mathrm{n}}-n_{\mathcal{O}}}\mathsf{g}(\hat{\mathsf{r}}_j)\right).\end{aligned}$$

where we have used $h\le R$ to absorb $\tilde{\ell}R^{\mathsf{d}-1}h^{1-\gamma}M_0^2$ from E_2 into the $\tilde{\ell}R^{\mathsf{d}}h^{-\gamma}M_0^2$ error above. Using Lemma 5.1 we can replace the screened electric field with the gradient defining

$\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, Q_R \times [-R,R])$, we find with a uniform bound on f_η in L^1 for small η that

$$\begin{aligned}
&\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, Q_R \times [-R,R]) - \left(\int_{Q_L \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 - \mathsf{c_d} \sum_{i=1}^n \mathsf{g}(\mathsf{r}_i)\right) \\
&\qquad \leq -\frac{1}{2\mathsf{c_{d,s}}} \int_{\mathcal{N} \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 + \frac{1}{2} \sum_{\{i \in \{1,\dots,n\}: x_i \notin \mathcal{O}\}} \mathsf{g}(\hat{\mathsf{r}}_i) \\
&\quad + C \sum_{j=1}^{\bar{\mathrm{n}} - n_{\mathcal{O}}} \mathsf{g}(\hat{\mathsf{r}}_j) + ChM + CR^{\mathsf{d}-1} + R^{\mathsf{d}-1} h^{-\gamma} + C\tilde{\ell} R^{\mathsf{d}} h^{-\gamma} M_0^2 + CRe(X_n, w, h) + \\
&\qquad C\mathsf{F}(Z_{\bar{\mathrm{n}} - n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h,h]) + C \sum_{i,j} \mathsf{g}(x_i - z_j) + C|n - \bar{\mathrm{n}}| + C(\bar{\mathrm{n}} - n_{\mathcal{O}}).
\end{aligned}$$

Using (9.14) we can rewrite the above bound as

$$\begin{aligned}
&\mathsf{F}(Y_{\bar{\mathrm{n}}}, \tilde{\mu}, Q_R \times [-R,R]) - \left(\int_{Q_R \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 - \mathsf{c_d} \sum_{i=1}^n \mathsf{g}(\hat{\mathsf{r}}_i)\right) \\
&\qquad \leq -\frac{1}{2\mathsf{c_{d,s}}} \int_{\mathcal{N} \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 + \frac{1}{2} \sum_{\{i \in \{1,\dots,n\}: x_i \notin \mathcal{O}\}} \mathsf{g}(\hat{\mathsf{r}}_i) + C \sum_{j=1}^{\bar{\mathrm{n}} - n_{\mathcal{O}}} \mathsf{g}(\hat{\mathsf{r}}_j) \\
&\quad + ChM + CR^{\mathsf{d}-1} + R^{\mathsf{d}-1} h^{-\gamma} + C\left(\frac{R^2}{\tilde{\ell}} + R\right) e(X_n, w, h) + C\mathsf{F}(Z_{\bar{\mathrm{n}} - n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta \times [-h,h]) \\
&\qquad + C \sum_{i,j} \mathsf{g}(x_i - z_j) + C|n - \bar{\mathrm{n}}| + C(\bar{\mathrm{n}} - n_{\mathcal{O}}).
\end{aligned}$$

Next, we would like to control $\frac{1}{2} \sum_{\{i \in \{1,\dots,n\}: x_i \notin \mathcal{O}\}} \mathsf{g}(\hat{\mathsf{r}}_i) - \frac{1}{2\mathsf{c_{d,s}}} \int_{\mathcal{N} \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2$ by the number of points not in $\mathcal{O}$, but the possible blowup of $\mathsf{g}(\hat{\mathsf{r}}_i)$ presents an issue. We adjust the truncation parameter and apply [Ser24, Lemma 4.13], but need to shrink $\mathcal{O}$ a tad in order to guarantee that it does not intersect $B(x_i, \frac{1}{4})$ for all $x_i \notin \mathcal{O}$. To do this, we simply observe that $\square_{T-4} \subset \mathcal{O}$ and write

$$\begin{aligned}
&\frac{1}{2} \sum_{\{i \in \{1,\dots,n\}: x_i \notin \mathcal{O}\}} \mathsf{g}(\hat{\mathsf{r}}_i) - \frac{1}{2\mathsf{c_{d,s}}} \int_{\mathcal{N} \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 = \frac{1}{2} \sum_{\{i \in \{1,\dots,n\}: x_i \notin \mathcal{O}\}} \mathsf{g}(\hat{\mathsf{r}}_i) \\
&- \frac{1}{2\mathsf{c_{d,s}}} \int_{(\square_R \setminus \square_{T-4}) \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 + \frac{1}{2\mathsf{c_{d,s}}} \int_{(\mathcal{O} \setminus \square_{T-4}) \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 \\
&\leq \frac{1}{2} \sum_{\{i \in \{1,\dots,n\}: x_i \notin \square_{T-4}\}} \mathsf{g}(\tilde{\mathsf{r}}_i) - \frac{1}{2\mathsf{c_{d,s}}} \int_{(\square_R \setminus \square_{T-4}) \times [-R,R]} |y|^\gamma |\nabla w_{\tilde{\mathsf{r}}}|^2 - \sum_{\{i: x_i \in \mathcal{O} \setminus \square_{T-4}\}} \mathsf{g}(\hat{\mathsf{r}}_i) \\
&+ \frac{1}{2\mathsf{c_{d,s}}} \int_{(\mathcal{O} \setminus \square_{T-4}) \times [-R,R]} |y|^\gamma |\nabla w_{\hat{\mathsf{r}}}|^2 + \sum_{\{i: x_i \notin \square_{T-4}\}} \int_{\square_R \setminus \square_{T-4}} (\mathsf{f}_{\tilde{\mathsf{r}}_i} - \mathsf{f}_{\hat{\mathsf{r}}_i})(x - x_i)\, d\mu \\
&\leq C(n - n_{\mathcal{O}}) + CM,
\end{aligned}$$

where $\tilde{\mathsf{r}}_i$ is defined to be $\frac{1}{4}$ for $x_i \notin \mathcal{O}$ and is kept fixed otherwise. This allows us to cancel all contributions of f and g for $x_i \in \mathcal{O} \setminus \square_{T-4}$, and bound the remaining contributions of f and g by $C(n - n_{\mathcal{O}})$ since $\tilde{\mathsf{r}}_i$ is bounded below for such i and f_η is again controlled uniformly in L^1 for small η. We have bounded negative contributions of L^2 energy by zero, and $\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{(\mathcal{O}\setminus\square_{T-4})\times[-R,R]}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \le CM$. Finally, using Proposition 3.4 we can bound

$$\begin{aligned}\sum_{j=1}^{\bar{\mathrm{n}}-n_{\mathcal{O}}}\mathsf{g}(\hat{\mathsf{r}}_j) &\le C\left(\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}},\tilde{\mu},\mathcal{N}_\eta\times[-h,h])+\bar{\mathrm{n}}-n_{\mathcal{O}}\right)\\ &\le C\left(\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}},\tilde{\mu},\mathcal{N}_\eta\times[-h,h])+\tilde{\ell}R^{\mathsf{d}-1}\right),\end{aligned}$$

controlling $\bar{\mathrm{n}}-n_{\mathcal{O}}=\tilde{\mu}(\mathcal{N})\lesssim\tilde{\ell}R^{\mathsf{d}-1}$ by our L^∞ control on $\tilde{\mu}$, completing the argument.

9.5. **Outer Screening.** Let us first comment on the changes to the setup that are required for screening in $(Q_R\times[-R,R])^c$. First, we choose our good boundary Γ exterior to Q_R. If the first case of 5.36 is satisfied, we find $T\in[R+\tilde{\ell}+2,R+2\tilde{\ell}-2]$ such that

$$\begin{aligned}M:=\int_{(Q_{T+4}\setminus Q_{T-4})\times[-h,h]^k}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2 &\le \frac{S(X_n,w,h)}{\tilde{\ell}}\\ \int_{\partial Q_T\times[-h,h]^k}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2 &\lesssim M\end{aligned}\tag{9.16}$$

and again take $\Gamma=\partial Q_T$.

Otherwise, the second condition is satisfied in (5.36). Then, via a mean-value argument analogous to [Ser24, Appendix A], we can find some $t\in[R+\tilde{\ell}+\ell,R+2\tilde{\ell}]$ such that

$$\int_{(Q_t\setminus Q_{t-\ell})\times[-h,h]}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2\le C\frac{S(X_n,w,h)\ell}{\tilde{\ell}}.$$

We then apply a mean-value argument in the strip $Q_t\setminus Q_{t-\ell}$ and find a piecewise affine boundary Γ in $Q_t\setminus Q_{t-\ell}$ with faces parallel to those of Q_L and sidelengths of order ℓ such that

$$\int_{\Gamma\times[-h,h]}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2\le C\frac{S(X_n,w,h)}{\tilde{\ell}},\qquad \sup_x\int_{(\Gamma\cap\square_\ell(x))\times[-h,h]}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2\le CS'(X_n,w,h)$$

and

$$\int_{\Gamma_1\times[-h,h]}|y|^\gamma|E_{\hat{\mathsf{r}}}|^2\le C\frac{S(X_n,w,h)}{\tilde{\ell}}$$

where Γ_1 denotes the 1-neighborhood of Γ. In both cases we let $M=C\frac{S(X_n,w,h)}{\tilde{\ell}}$ and in the latter case $M_\ell=CS'(X_n,w,h)$.

We will leave the configuration unchanged in $\mathcal{O}=Q_T^c$, and only place new points in $\mathcal{N}=Q_R^c\setminus Q_T^c$. We now partition space so that we only change the field near $\partial(Q_R\times[-R,R])$. Namely, we define

(1) $D_0:=(Q_T\times[-(R+h),(R+h)])^c$
(2) $D_\partial:=\mathcal{N}\times[-h,h]$
(3) $D_1:=(\square_T\times[-(R+h),(R+h)])\setminus(\Omega\times[-R,R]\cup D_\partial)$.

We partition $\mathcal{N}$ into cells H_k with sidelengths at scale ℓ, in $\left[\frac{\ell}{C}, \ell C\right]$, and let $\tilde{H}_k$ denote the rectangles $H_k \times [-h,h]$. Then, we set

$$M_0^+ := \frac{h^{-\gamma}}{|\mathcal{N}|}\int_{(\partial D_1 \cap \{y>0\})\setminus \partial(D_\partial \cup (Q_R\times[-R,R]))} E\cdot \vec{n}$$
$$M_0^- := \frac{h^{-\gamma}}{|\mathcal{N}|}\int_{(\partial D_1 \cap \{y<0\})\setminus \partial(D_\partial \cup (Q_R\times[-R,R))} E\cdot \vec{n}$$

and denote by M_0 the sum $h^\gamma M_0^+ + h^\gamma M_0^-$. The sets and quantities $\mathcal{N}_\eta$, H_k^η, n_k, I_∂, $n_{\mathcal{O}}$, m_k and $\tilde{\mu}$ are then all defined analogously to the outer screening, as is the screenability condition. E_1, E_2, E_3 and E_4 are defined in exactly the same manner as in the outer screening. Setting $E_{\hat{\mathsf{r}}}^{\mathrm{scr}} := (E_1+E_2+E_3)\mathbf{1}_{D_\partial} + E_4\mathbf{1}_{D_1} + E_{\hat{\mathsf{r}}}\mathbf{1}_{D_0}$ and adding back the truncations we have

$$E^{\mathrm{scr}} := E_{\hat{\mathsf{r}}}^{\mathrm{scr}} + \sum_{i=1}^{\bar{\mathrm{n}}} \nabla \mathsf{f}_{\bar{\mathsf{r}}_i}(x-y_i),$$

where $Y_{\bar{\mathrm{n}}} = (\{X_n\}\cap\mathcal{O})\cup Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}}$, and $\bar{\mathsf{r}}$ are the (possibly changed) minimal distances for the new configuration $Y_{\bar{\mathrm{n}}}$. Due to the Neumann condition, no divergence is created across boundaries when we set E^{scr} to vanish outside of our region. By definition then, we have

$$\begin{cases} -\mathrm{div}(|y|^\gamma E^{\mathrm{scr}}) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i\in Y_{\bar{\mathrm{n}}}}\delta_{y_i} - \mu\right) & \text{in } (Q_R\times[-R,R])^c \\ E^{\mathrm{scr}}\cdot\vec{n} = 0 & \text{on } \partial(Q_R\times[-R,R]). \end{cases}$$

Since the geometry of D_∂ is unchanged and the equations are the same as with outer screening, all of the estimates on E_1, E_2, E_3 are the same. The sidelengths of D_1 are not necessarily of the same order, so the estimate on E_4 needs to be multiplied by an aspect ratio of $\frac{R}{\min(h,\tilde{\ell})}$.

Thus,

$$\int_{(Q_R\times[-R,R])^c} |y|^\gamma |E_{\hat{\mathsf{r}}}^{\mathrm{scr}}|^2 \le \int_{D_0}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 + ChM + C\frac{R}{\min(h,\tilde{\ell})}\left(R^{\mathsf{d}}\tilde{\ell}h^{-\gamma}M_0^2 + Re(X_n,w,h)\right)$$
$$+ CR^{\mathsf{d}-1} + R^{\mathsf{d}-1}h^{-\gamma} + C(\bar{\mathrm{n}}-n_{\mathcal{O}}) + C\left(2\mathsf{c}_{\mathsf{d},\mathsf{s}}\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}},\tilde{\mu},\mathcal{N}_\eta\times[-h,h]) + \mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{i=1}^{\bar{\mathrm{n}}-n_{\mathcal{O}}}\mathsf{g}(\hat{\mathsf{r}}_i)\right).$$

We find exactly as before then that

$$\mathsf{F}(Y_{\bar{\mathrm{n}}},\tilde{\mu},(Q_R\times[-R,R])^c) - \left(\int_{(Q_R\times[-R,R])^c}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 - \mathsf{c}_{\mathsf{d}}\sum_{i=1}^n \mathsf{g}(\hat{\mathsf{r}}_i)\right)$$
$$\le -\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{D_\partial\cup D_1}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 + \frac{1}{2}\sum_{\{i\in\{1,\dots,n\}:x_i\notin\mathcal{O}\}}\mathsf{g}(\mathsf{r}_i) + C\sum_{j=1}^{\bar{\mathrm{n}}-n_{\mathcal{O}}}\mathsf{g}(\hat{\mathsf{r}}_j) + CR^{\mathsf{d}-1} + R^{\mathsf{d}-1}h^{-\gamma}$$
$$+ ChM + C\frac{R}{\min(h,\tilde{\ell})}\left(\frac{R^2}{\tilde{\ell}} + R\right)e(X_n,w,h) + C\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}},\tilde{\mu},\mathcal{N}_\eta\times[-h,h]) + C\sum_{i,j}\mathsf{g}(x_i-z_j)$$
$$+ C|n-\bar{\mathrm{n}}| + C(\bar{\mathrm{n}}-n_{\mathcal{O}}).$$

Next, we would like to control $\frac{1}{2}\sum_{\{i\in\{1,\dots,n\}:x_i\notin\mathcal{O}\}}\mathsf{g}(\hat{\mathsf{r}}_i) - \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{D_\partial\cup D_1}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2$ by the number of points not in $\mathcal{O}$, but the possible blowup of $\mathsf{g}(\mathsf{r}_i)$ again presents an issue. We adjust the truncation parameter and apply [Ser24, Lemma 4.13], but again need to shrink $\mathcal{O}$ a tad in order to guarantee that it does not intersect $B(x_i, \frac{1}{4})$ for all $x_i \notin \mathcal{O}$. To do this, we simply observe that $\square_{T+4}^c \subset \mathcal{O}$ and write

$$\begin{aligned}
&\frac{1}{2}\sum_{\{i\in\{1,\ldots,n\}:x_i\notin\mathcal{O}\}} \mathsf{g}(\hat{\mathsf{r}}_i) - \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{D_\partial\cup D_1}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \le \frac{1}{2}\sum_{\{i\in\{1,\ldots,n\}:x_i\notin\mathcal{O}\}} \mathsf{g}(\hat{\mathsf{r}}_i) - \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{D_\partial}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \\
&= \frac{1}{2}\sum_{\{i\in\{1,\ldots,n\}:x_i\notin\mathcal{O}\}} \mathsf{g}(\hat{\mathsf{r}}_i) - \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{(\square_R^c\setminus\square_{T+4}^c)\times[-h,h]}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \\
&\qquad\qquad + \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{(\square_{T+4})\setminus\mathcal{N}\times[-h,h]}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 \\
&\le \frac{1}{2}\sum_{\{i\in\{1,\ldots,n\}:x_i\notin\square_{T+4}^c\}} \mathsf{g}(\tilde{\mathsf{r}}_i) - \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{(\square_R^c\setminus\square_{T+4}^c)\times[-h,h]}|y|^\gamma|\nabla w_{\tilde{\mathsf{r}}}|^2 - \sum_{\{i:x_i\in\mathcal{O}\setminus\square_{T+4}^c\}} \mathsf{g}(\hat{\mathsf{r}}_i) \\
&+ \frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{(\square_{T+4}\setminus\mathcal{N})\times[-h,h]}|y|^\gamma|\nabla w_{\hat{\mathsf{r}}}|^2 + \sum_{\{i:x_i\notin\square_{T+4}^c\}}\int_{\square_R^c\setminus\square_{T+4}^c}(\mathsf{f}_{\tilde{\mathsf{r}}_i} - \mathsf{f}_{\hat{\mathsf{r}}_i})(x-x_i)\,d\mu \\
&\le C(n-n_{\mathcal{O}}) + CM,
\end{aligned}$$

where $\tilde{\mathsf{r}}_i$ is defined to be $\frac{1}{4}$ for $x_i \notin \mathcal{O}$ and is kept fixed otherwise. This allows us to cancel all contributions of f and g for $x_i \in \mathcal{O} \setminus \square_{T+4}^c$, and bound the remaining contributions of f and g by $C(n-n_{\mathcal{O}})$ since $\tilde{\mathsf{r}}_i$ is bounded below for such i and f_η is again controlled uniformly in L^1 for small η. We have bounded negative contributions of L^2 energy by zero, and $\frac{1}{2\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{(\square_{T+4}\setminus\mathcal{N})\times[-h,h]}|\nabla w_{\mathsf{r}}|^2 \le CM$. Finally, using Proposition 3.4 we can bound

$$\begin{aligned}
\sum_{j=1}^{\bar{\mathrm{n}}-n_{\mathcal{O}}} \mathsf{g}(\hat{\mathsf{r}}_j) &\le C\left(\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta\times[-h,h]) + \bar{\mathrm{n}} - n_{\mathcal{O}}\right) \\
&\le C\left(\mathsf{F}(Z_{\bar{\mathrm{n}}-n_{\mathcal{O}}}, \tilde{\mu}, \mathcal{N}_\eta\times[-h,h]) + \tilde{\ell}R^{d-1}\right),
\end{aligned}$$

controlling $\bar{\mathrm{n}} - n_{\mathcal{O}} = \tilde{\mu}(\mathcal{N}) \lesssim \tilde{\ell}R^{d-1}$ by our L^∞ control on $\tilde{\mu}$, completing the argument.

Appendix A. Some remarks about the fractional obstacle problem - by Xavier Ros-Oton

Let $s \in (0,1)$, and consider the operator

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} (u(x) - u(x+y)) \frac{dy}{|y|^{n+2s}}, \tag{A.1}$$

We want to study the **obstacle problem**

$$\min\{(-\Delta)^s u,\, u - \psi\} = 0 \quad \text{in} \quad B_1 \subset \mathbb{R}^n. \tag{A.2}$$

The function u is assumed to be bounded in $\mathbb{R}^n$. In some cases, we want to consider the *global* problem, which for $n > 2s$ is

$$\begin{aligned} \min\{(-\Delta)^s u,\, u - \psi\} &= 0 \quad \text{in} \quad \mathbb{R}^n \\ u &\longrightarrow 0 \quad \text{at} \quad \infty. \end{aligned} \tag{A.3}$$

When $n = 2s = 1$ the global problem becomes

$$\begin{aligned} \min\{\sqrt{-\Delta} u,\, u - \psi\} &= 0 \quad \text{in} \quad \mathbb{R} \\ \frac{u(x)}{-\log|x|} &\longrightarrow \kappa \quad \text{at} \quad \infty, \end{aligned} \tag{A.4}$$

while when $n = 1 < 2s$ we have

$$\begin{aligned} \min\{(-\Delta)^s u,\, u - \psi\} &= 0 \quad \text{in} \quad \mathbb{R} \\ \frac{u(x)}{-|x|^{1-2s}} &\longrightarrow \kappa \quad \text{at} \quad \infty, \end{aligned} \tag{A.5}$$

for some constant $\kappa > 0$. Notice that in that case we need $\psi \ll -\log|x|$ or $\psi \ll -|x|^{1-2s}$ at ∞.

Moreover, the constant κ is the total mass of $(-\Delta)^s u$ in $\mathbb{R}$.

A.1. **Known results.** The optimal regularity of solutions was first established in [CSS08b] for $\psi \in C^{2,1}$, and these arguments were refined in [CDSS17b] in order to establish[4] the same result under minimal assumptions on the obstacle ψ.

Theorem 4 ([CDSS17b]). *Let $\psi \in C^{1+s+\delta}(B_1)$ for some $\delta > 0$, and u be any viscosity solution of (A.2). Then, u is C^{1+s} in $B_{1/2}$, with*

$$\|u\|_{C^{1+s}(B_{1/2})} \le C\left(\|\psi\|_{C^{1+s+\delta}(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right).$$

The constant C depends only on n, s, and δ.

The regularity of free boundaries was also established for the first time in [CSS08b] for $\psi \in C^{2,1}$, and under weaker assumptionson the obstacle in [CDSS17b,ROTLW25]. Combining the results in [CSS08b, CDSS17b] with those in [ROS17b] (see also [CROS17]) and in [FROS23], we get the following. Here, the function d denotes the distance to the contact set $\{u = 0\}$.

Theorem 5 ([CSS08b,CDSS17b,ROS17b,ROTLW25]). *Let ψ be such that $\psi \in C^{1+2s+\delta}(B_1)$ for some $\delta > 0$, and u be any solution of (A.2). Then, for every free boundary point $x_\circ \in \{u > \psi\} \cap B_{1/2}$ we have:*

[4]There is actually an important detail that was omitted in [CDSS17b], and is the fact that one needs to prove that blow-ups are *convex*. This was not done in [CDSS17b], but it follows from the results in [FRJ21, ROTLW25, CDV22].

(i) either
$$(u-\psi)(x) = c_{x_\circ} d^{1+s}(x) + O(|x-x_\circ|^{1+s+\alpha}),$$
with $c_{x_\circ} > 0$.

(ii) or
$$(u-\psi)(x) = O(|x-x_\circ|^{1+s+\alpha}).$$

Moreover, in case (i) the free boundary is $C^{1,\alpha}$ *in a neighborhood of* $x_\circ$.

Furthermore, we have
$$\|(u-\psi)/d^{1+s}\|_{C^\alpha(\overline{\{u>\psi\}}\cap B_{1/2})} \le C\left(\|(-\Delta)^s\nabla\psi\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right).$$
The constants C *and* $\alpha > 0$ *depend only on* n, s, *and* δ.

Notice that this gives a dichotomy between **regular points** (i), and **degenerate/singular points** (ii). We have no information a priori on how many regular or singular points there could be.

Notice also that at any regular point $x_\circ$ we have
$$c_{x_\circ} = \lim_{\Omega\ni x\to x_\circ} \frac{u-\psi}{d^{1+s}},$$
while at degenerate/singular points this limit is zero.

Concerning the *generic* regularity of free boundaries, the best known results are those in [FRRO21, FRTL23, CC24] and [CF25, KM00].

Theorem 6 ([CC24]). *Let* $s\in(0,1)$ *with* $n>2s$ *and* $\psi\in C_c^5(B_1)$. *Let* u_t *be the family of solutions of* (A.3) *with* $\psi_t := \psi + t$, *for* $t\in(-1,1)$. *Assume in addition* $n\le 3$.

Then, for almost every t, *all points of the free boundary* $\partial\{u_t>0\}$ *are regular, in the sense of (i) above.*

On the other hand, we also have a nondegeneracy condition:

Proposition A.1 ([BFRO18]). *Let* $s\in(0,1)$, $n>2s$, *and* $\psi\in C_c^{3,\gamma}(\mathbb{R}^n)$ *satisfying*
$$\Delta\psi\le 0 \quad \text{in} \quad \{\psi>0\}.$$
Let u *be the global solution of* (A.3). *Then, for any free boundary point* $x_\circ$ *we have*
$$\|u-\psi\|_{B_r(x_\circ)} \ge c_1 r^2 > 0$$
for all $r\in(0,1)$. *The constant* c_1 *might depend on* u, *but not on the point* $x_\circ$.

Finally, the following result shows that without the sign assumption on $\Delta\psi$, such nondegeneracy may fail at *every* free boundary point.

Proposition A.2 ([FRRO21]). *Let* $s\in(0,1)$ *and* $m\in\{1,2,3,\ldots\}$. *Given any bounded* C^∞ *domain* $\Omega\subset\mathbb{R}^n$, *there exists an obstacle* $\psi\in C^\infty(\mathbb{R}^n)$ *with* $\psi\to 0$ *at* ∞, *and a global solution to the obstacle problem* (A.3), *such that the contact set is exactly* $\{u=\psi\}=\Omega$, *and moreover at every free boundary point* $x_\circ$ *we have*
$$(u-\psi)(x) = c_\circ d^{m+s}(x) + O(|x-x_\circ|^{m+1+s}),$$
where d *is the distance to* Ω.

In particular, when $m\ge 2$, *all free boundary points are of the type (ii) above.*

Notice that, thanks to the results in [FRRO21], such type of degenerate solutions ($m\ge 2$) are very rare, in the sense that generically we expect $m=1$.

A.2. **Some new results: nondegeneracy.** Our first goal is to prove some quantitative nondegeneracy results for the global obstacle problem (A.3). More precisely, we want to get uniform lower bounds for the constants $c_{x_\circ}$ in Theorem 5(i).

Proposition A.3. *Let $s \in (0,1)$, $n > 2s$, and $\psi \geq 0$ satisfying*

$$\|\psi\|_{L^1} = 1, \qquad \operatorname{supp}\psi \subset B_M, \qquad \|\psi\|_{C^{2+\gamma}(B_\rho(x_\circ))} \leq M, \qquad \Delta\psi \leq 0 \quad \textit{in} \quad \{\psi > 0\},$$

with $\gamma > s$. Let u be global solution to (A.3), and assume that $x_\circ \in \partial\{u > \psi\}$ and that the free boundary can be written as a Lipschitz graph in $B_\rho(x_\circ)$, with Lipschitz constant bounded by M.

Then, $x_\circ$ is a regular point, and the constant in Theorem 5(i) satisfies

$$c_{x_\circ} \geq \delta > 0,$$

where δ depends only on n, s, M, and ρ.

Remark A.4. Notice that the main two assumptions in this result are:

- The global assumption $\Delta\psi \leq 0$
- The local assumption on the geometry of the free boundary

Both assumptions are somewhat necessary, in the following sense. Without the first one we may have solutions with smooth free boundaries, for which all points are degenerate (recall Proposition A.2). Without the second one, we may still have solutions with degenerate points (of order 2).

In case $n \leq 2s$ we have the following analogue result.
Here, we denote $\Gamma(x) = -\log|x|$ if $s = \frac{1}{2}$, and $\Gamma(x) = -|x|^{1-2s}$ if $s > \frac{1}{2}$.

Proposition A.5. *Let $n = 1$, $s \in [\frac{1}{2}, 1)$, $\gamma > s$, and u be global solution to (A.4) or (A.5) with ψ satisfying*

$$\|\psi\|_{C^{2+\gamma}([x_\circ-\rho,x_\circ+\rho]))} \leq M, \quad \psi'' \leq 0 \;\; \textit{for} \;\; |x| < M, \quad \frac{u-\psi}{\kappa\Gamma} \geq \rho \quad \textit{for} \quad |x| > M.$$

Assume that $x_\circ \in \partial\{u > \psi\}$ with $u > \psi$ in $(x_\circ, x_\circ + \rho)$ and $u = \psi$ in $(x_\circ - \rho, x_\circ]$.

Then, $x_\circ$ is a regular point, and the constant in Theorem 5(i) satisfies

$$c_{x_\circ} \geq \delta > 0,$$

where δ depends only on s, M, and ρ.

To prove Propositions A.3 and A.5, we will first need the following.

Lemma A.6. *Let $s \in (0,1)$, $n \geq 1$, and ψ as in Proposition A.3 or A.5. Let u be global solution to (A.3) or (A.4) or (A.5). Then, for any free boundary point $x_\circ \in B_{\rho/2}(z)$ we have*

$$\|u - \psi\|_{L^\infty(B_r(x_\circ))} \geq c_M r^2 > 0 \tag{A.6}$$

for all $r \in (0,1)$, where c_M depends only on n, s, κ, ρ, δ, and M.

Proof. Case 1. Assume first $n > 2s$.

As in [BFRO18], we consider $w := (-\Delta)^s u \geq 0$ in $\mathbb{R}^n$. Since $\operatorname{supp} w \subset \{u = \psi\}$, then for any $x_1 \in \{u > \psi\} \cap B_{M+1}$ we have

$$-(-\Delta)^{1-s} w(x_1) = c_{n,s} \int_{\mathbb{R}^n} \frac{w(z)dz}{|x_1 - z|^{n+2(1-s)}} \geq c \int_{B_M} w = c \int_{\mathbb{R}^n} |w|.$$

Now, by classical estimates for Riesz potentials (see [Ste70]), it follows that

$$\int_{\mathbb{R}^n} |(-\Delta)^s u| \geq c\|u\|_{L^{\frac{n}{n-2s}}_{\mathrm{weak}}(\mathbb{R}^n)} \geq c\|\psi\|_{L^{\frac{n}{n-2s}}_{\mathrm{weak}}(\mathbb{R}^n)} \geq c_M > 0,$$

where we used that $u \geq \psi \geq 0$ and that $\|\psi\|_{L^1} = 1$.

Since u is a global solution, we deduce

$$\Delta u = -(-\Delta)^{1-s} w \geq c_M > 0 \quad \text{in} \quad \{u > \psi\} \cap B_{M+1}.$$

We now observe that at every free boundary point $x_\circ \in B_{\rho/2}(z)$ and for any $r \in (0, \rho/2)$ we have $u(x_\circ) = \psi(x_\circ)$ and

$$0 = (-\Delta)^s u(x_\circ) \leq (-\Delta)^s \psi(x_\circ) \leq Cr^{2-2s}\|\psi\|_{C^{1,1}(B_r(x_\circ))} + C\int_{B_r^c(x_\circ)} \frac{\psi(x_\circ) - \psi(x_\circ + y)}{|y|^{n+2s}}.$$

Rearranging terms and using the assumptions on ψ, we get

$$c\|\psi\|_{L^1} - Cr^{2-2s} \leq Cr^{-2s}\psi(x_\circ).$$

Choosing a small $r > 0$ (depending only on n, s, ρ, and M), we deduce that

$$\psi(x_\circ) \geq c_M > 0$$

for some constant c_M. Since $x_\circ$ was an arbitrary free boundary point, and by regularity of ψ, it follows that there exists $r_M > 0$ such that

$$\operatorname{dist}\big(\{\psi = 0\} \cap B_{\rho/2}(z),\, \{u = \psi\} \cap B_{\rho/2}(z)\big) \geq r_M > 0.$$

Then, the proof of (A.6) finishes exactly as in [BFRO18, Proof of Lemma 3.1].

CASE 2. Assume now $n = 1 \leq 2s$. Then, $w := (-\Delta)^s u \geq 0$ satisfies $\operatorname{supp} w \subset \{u = \psi\}$, and then for any $x_1 \in \{u > \psi\}$ with $|x_1| < M + 1$

$$-(-\Delta)^{1-s} w(x_1) = c_{1,s}\int_{\mathbb{R}^n} \frac{w(z)dz}{|x_1 - z|^{1+2(1-s)}} \geq c\int_{B_M} w = c\kappa > 0.$$

Hence,

$$u'' = -(-\Delta)^{1-s} w \geq c\kappa > 0 \quad \text{in} \quad \{u > \psi\} \cap \{|x| < M + 1\}.$$

Thus, around such free boundary point $x_\circ \in \{|x| < M - \rho\}$ we have $u'' \geq c\kappa > 0$ and $\psi'' \leq 0$ in $(x_\circ - \rho, x_\circ + \rho)$, and in particular $(u - \psi)(x_\circ \pm r) \geq c\kappa > 0$. □

We now prove the following.

Proof of Propositions A.3 and A.5. By Lemma A.6 and Theorem 5 above, it follows that $x_\circ$ is a regular point, and the free boundary is a $C^{1,\alpha}$ graph in $B_{\rho/2}(x_\circ)$, for some $\alpha > 0$. Moreover, thanks to [ARO20b, Theorem 1.2] this implies that the free boundary is a $C^{2+\gamma-s}$ graph in $B_{\rho/4}(x_\circ)$, and a quick inspection of the proof shows that its $C^{2+\gamma-s}$ norm is bounded by a constant depending only on n, s, γ, M, and ρ. Then, by [ARO20b, Theorem 1.4] (applied to the derivatives of $u - \psi$) we deduce that

$$\left\| \frac{\partial_i(u - \psi)}{d^s} \right\|_{C^{1+\gamma-s}(B_{\rho/8}(x_\circ))} \leq C$$

and therefore

$$|u - \psi - c_{x_\circ} d^{1+s}| \leq C|x - x_\circ|^{2+\gamma},$$

with C depending only on n, s, M, and ρ.

Combining this with the nondegeneracy condition (A.6), we get

$$c_M r^2 - c_{x_\circ} r^{1+s} \le \left\| (u-\psi) - c_{x_\circ} d^{1+s} \right\|_{L^\infty(B_r(x_\circ))} \le C r^{2+\gamma}.$$

Hence, we deduce

$$c_M r^{1-s} - C r^{\gamma+1-s} \le c_{x_\circ}.$$

Choosing $r > 0$ such that $Cr^\gamma = \frac{1}{2} c_M$, the result follows. □

A.3. **Some new results: behavior of $(-\Delta)^s u$ near the free boundary.** We next want to prove a new result concerning the local behavior of $(-\Delta)^s u$ near the free boundary.

Proposition A.7. *Let ψ be such that $\psi \in C^{1+2s+\delta}(B_1)$ for some $\delta > 0$, and u be any solution of (A.2). Define*

$$d(x) = \operatorname{dist}(x, \{u = \psi\}) \qquad \textit{and} \qquad d_-(x) := \operatorname{dist}(x, \{u > \psi\})$$

Assume that $x_\circ \in \{u > \psi\} \cap B_{1/2}$ is a regular free boundary point, i.e., we have

$$(u-\psi)(x) = c_{x_\circ} d^{1+s}(x) + O(|x - x_\circ|^{1+s+\alpha}),$$

with $c_{x_\circ} > 0$, and the free boundary is $C^{1,\alpha}$ in a neighborhood of $x_\circ$.

Then, we have

$$\left| (-\Delta)^s u(x) - \bar{c}_s c_\circ d_-^{1-s}(x) \right| \le C|x - x_\circ|^{1-s+\alpha} \quad \textit{in} \quad \{u = \psi\} \cap B_{1/2},$$

where $\kappa_s := \bar{c}_s/(1-s)$ and $\bar{c}_s$ is given by (A.7). The constant C depends only on n, s, δ, α, $\|(-\Delta)^s \nabla\psi\|_{L^\infty}$, and the $C^{1,\alpha}$ norm of the free boundary in $B_{1/2}$.

We will first need the following result, similar to [FRRO24, Lemma 2.6].

Lemma A.8. *Let $s \in (0,1)$, and $\Omega \subset \mathbb{R}^n$ be any $C^{1,\alpha}$ domain, with $0 \in \partial\Omega$. Let $d(x) = \operatorname{dist}(x, \partial\Omega)$. Assume that $f \in L^\infty(\Omega \cap B_1)$ and v solves*

$$\begin{cases} (-\Delta)^s v &= f & \text{in } B_1 \cap \Omega \\ v &= 0 & \text{in } B_1 \setminus \Omega, \end{cases}$$

and let us define $c_\circ$ as the unique constant such that

$$v(x) = c_\circ d^s(x) + O(|x|^{s+\alpha}) \quad \textit{in} \quad \Omega.$$

Then,

$$\left| (-\Delta)^s v(x) - \kappa_s c_\circ d^{-s}(x) \right| \le C|x|^\alpha \, d^{-s}(x) \quad \textit{in} \quad B_{1/2} \setminus \Omega,$$

where $\bar{c}_s$ is given by (A.7) below, and C depends only on n, s, Ω, $\|f\|_{L^\infty}$.

Proof. First, it follows from [FRRO24, Lemma 2.6] that for any $e \in \mathbb{S}^{n-1}$

$$(-\Delta)^s (x_n)_+^s = \bar{c}_s (x_n)_-^{-s} \quad \text{with} \quad \bar{c}_s = -\frac{\Gamma(1+s)}{\Gamma(1-s)}. \tag{A.7}$$

Then, by [ROS17b], we know that $v/d^s \in C^\alpha(\overline{\Omega} \cap B_{1/2})$. Thus, if we define $c_\circ := (v/d^s)(0)$ we then have

$$\left| (v/d^s)(x) - c_\circ \right| \le C|x|^\alpha,$$

and multiplying this by d^s we get

$$\left| v(x) - c_\circ d^s(x) \right| \le C|x|^{s+\alpha} \quad \text{in} \quad \Omega.$$

Notice that such expansion also implies that

$$\left| v(x) - c_\circ (x_n)_+^s \right| \le C|x|^{s+\alpha},$$

where we assume that $\nu = e_n$ is the normal vector to $\partial\Omega$ at the origin.

Now, thanks to the previous expansion, and since $v \equiv 0$ in $\Omega^c \cap B_1$, we find that for $x = -te_n \in \Omega^c$, with $t > 0$,

$$(-\Delta)^s v(x) = c_\circ(-\Delta)^s (x_n)_+^s + O(t^{-s+\alpha}) = \bar{c}_s c_\circ t^{-s} + O(t^{-s+\alpha}),$$

where we used (A.7). Since this can be done not only at the origin but at every boundary point $z \in \partial\Omega \cap B_{1/2}$, we deduce that for every $x = z - t\nu_z \in \Omega^c \cap B_{1/2}$

$$(-\Delta)^s u(x) = \bar{c}_s c_z t^{-s} + O(t^{-s+\alpha}).$$

For each $x \in \Omega^c$ we can choose $z \in \partial\Omega$ such that $|x - z| = d(x)$, and then we deduce

$$(-\Delta)^s u(x) = \bar{c}_s c_z d^{-s}(x) + O(d^{-s+\alpha}(x)).$$

Since $c_z = c_\circ + O(|z|^\alpha) = c_\circ + O(|x|^\alpha)$, we finally get

$$(-\Delta)^s u(x) = \bar{c}_s c_\circ d^{-s}(x) + O(|x|^\alpha)\, d^{-s}(x),$$

as wanted. □

We can now give the:

Proof of Proposition A.7. We want to apply Lemma A.8 to the functions $\partial_i(u - \psi)$. More precisely, let $x \in \{u = \psi\} \cap B_{1/2}$, and let z be its closest free boundary point, and denote $x = z - t_\circ\nu$ with $\nu \in \mathbb{S}^{n-1}$ and $t_\circ > 0$. Up to a rotation, we may assume $\nu = e_n$. Then, it follows from Lemma A.8 (applied to $v = \partial_n(u - \psi)$) that

$$\left|(-\Delta)^s v(x) - \bar{c}_s c_\circ t_\circ^{-s}\right| \le C t_\circ^{\alpha - s}.$$

Moreover, the same holds for any point y in the segment joining x and z: if $y = z - te_n$, with $t \in [0, t_\circ]$, then

$$\left|\partial_n(-\Delta)^s(u - \psi)(z - te_n) - \bar{c}_s c_\circ t^{-s}\right| \le C t^{\alpha - s}.$$

where we used the definition of v. Integrating in t, and using that —thanks to (A.2) and the fact that $(-\Delta)^s(u - \psi)$ is continuous—

$$(-\Delta)^s(u - \psi)(z) = 0,$$

we deduce

$$\left|(-\Delta)^s(u - \psi)(z - te_n) - \frac{\bar{c}_s}{1 - s} c_\circ t^{1-s}\right| \le C t^{1 - s\alpha}.$$

The result follows by recalling that $t = d_-(x)$ and the fact that $x \in \{u = \psi\} \cap B_{1/2}$ was arbitrary. □

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