

Eisenstein congruence at prime-square level

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Let $N, p \geq 5$ be primes, and let $E_{2,N} \in M_2(\Gamma_0(N))$ denote the unique normalized Eisenstein series of weight 2 and level $\Gamma_0(N)$. We are interested in mod- p congruences between $E_{2,N}$ and cusp forms of weight 2 and N -power level.

We begin by defining the spaces that parametrize such congruences. Write $H'(N^i)$ for the anemic Hecke algebra acting on $M_2(\Gamma_0(N^i))$, which is generated as a $\mathbb{Z}_p$ -algebra by T_ℓ for all primes $\ell \nmid N$. Let $\mathfrak{m}'$ denote the Eisenstein maximal ideal in $H'(N^i)$; that is, $\mathfrak{m}'$ is generated by p and $T_\ell - \ell - 1$ for all primes ℓ . Similarly, write $H(N^i)$ for the full Hecke algebra acting on $M_2(\Gamma_0(N^i))$, which is generated as a $\mathbb{Z}_p$ -algebra by T_ℓ for all primes ℓ . Let $\mathfrak{m}$ denote the maximal ideal of $H(N^i)$ generated by p, T_N , and $T_\ell - \ell - 1$. Set $\mathbb{T}(N) := H'(N)_{\mathfrak{m}'}$ and $\mathbb{T}(N^2) := H(N^2)_{\mathfrak{m}}$. Let $r := \mathrm{rk}_{\mathbb{Z}_p} \mathbb{T}(N)$, which is the number of normalized Hecke eigenforms in $M_2(\Gamma_0(N))$ that are congruent to $E_{2,N}$ modulo a prime above p . Mazur proved the following fundamental result about such congruences.

Theorem 1 (Mazur [4]). *We have $r \geq 2$ (equivalently, $S_2(\Gamma_0(N))_{\mathfrak{m}'} \neq 0$) if and only if $N \equiv 1 \pmod{p}$.*

The key fact used to prove Mazur's theorem is that we know that the constant term of $E_{2,N}$ equals $\frac{N-1}{24}$. Henceforth assume that $N \equiv 1 \pmod{p}$, and set $\Delta := (\mathbb{Z}/N\mathbb{Z})^\times, p\text{-part}$. Let $\mathbb{Z}_p[\Delta]$ be the group ring of Δ and $\mathbb{Z}_p[\Delta]^+$ the subring of $\mathbb{Z}_p[\Delta]$ that is fixed under the involution that inverts group-like elements. The goal of the talk is to explain two related theorems, the first of which is the following.

Theorem 2 (L.–Pollack–Wake [1]).

- (1) $r \geq 2$ (without using that $a_0(E_{2,N}) = \frac{N-1}{24}$).
- (2) $r \geq 3$ if and only if $S_2(\Gamma_0(N^2))_{\mathfrak{m}}^{\mathrm{new}} \neq 0$.
- (3) $\mathbb{T}(N^2)$ is a $\mathbb{Z}_p[\Delta]^+$ -algebra, and it is a free $\mathbb{Z}_p[\Delta]^+$ -module of rank r .

The last statement of Theorem 2 easily implies the first two by counting Eisenstein series.

To state the second theorem, let $B_2(x) = x^2 - x + 1/6$ be the second Bernoulli polynomial. We define

$$\zeta := \frac{-N}{2} \sum_{i=1}^{N-1} B_2(i/N)[i] \in \mathbb{Z}_p[(\mathbb{Z}/N\mathbb{Z})^\times].$$

This interpolates Dirichlet L -values in the sense that for any mod- N Dirichlet character χ we have

$$\chi(\zeta) = \begin{cases} L(-1, \chi) & \chi \neq 1 \\ (1-N)L(-1, 1) & \chi = 1. \end{cases}$$

Let $\bar{\zeta} \in \mathbb{F}_p[(\mathbb{Z}/N\mathbb{Z})^\times]$ denote the reduction of ζ , and let $\mathcal{I}$ be the augmentation ideal in $\mathbb{F}_p[(\mathbb{Z}/N\mathbb{Z})^\times]$. It is easy to see that $\bar{\zeta} \in \mathcal{I}$. Let $\text{ord}(\bar{\zeta})$ denote the largest integer m for which $\bar{\zeta} \in \mathcal{I}^m$. The second theorem is Theorem 3 below; the first statement was first proved by Merel [5]. It was later reproved and generalized by Lecouturier, who also proved the second statement using his theory of higher Eisenstein elements [3]. We give new, uniform, and transparent proofs of their theorems, in addition to proving the third point.

Theorem 3 (L.–Müller–Palvannan [2]).

- (1) (Merel [5]) $\text{ord}(\bar{\zeta}) = 1$ if and only if $r = 2$.
- (2) (Lecouturier [3]) $\text{ord}(\bar{\zeta}) = 2$ if and only if $r = 3$.
- (3) Suppose that exactly one of $\text{ord}(\bar{\zeta})$ and $r - 1$ equals 3. For simplicity, assume also that $p^2 \nmid N - 1$. Then $\text{rk}_{\mathbb{Z}_p} S_2(\Gamma_0(N^2))_{\mathfrak{m}}^{\text{new}} = \frac{(p-1)(r-2)}{2}$, and all eigenforms in this space are Galois conjugate.

We explain our proof of Theorem 2 and our proof of the first part of Theorem 3. The proof of Theorem 3 relies crucially on the freeness result of Theorem 2.

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