University of California
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# Images of Galois representations associated to $p$-adic families of modular forms

A dissertation submitted in partial satisfaction
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by

**Jaclyn Ann Lang**

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Abstract of the Dissertation

# Images of Galois representations associated to $p$-adic families of modular forms

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Doctor of Philosophy in Mathematics

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Fix a prime $p > 2$. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{I})$ be the Galois representation coming from a non-CM irreducible component $\mathbb{I}$ of Hida's $p$-ordinary Hecke algebra. Assume the residual representation $\bar{\rho}$ is absolutely irreducible. Under a minor technical condition, we identify a subring $\mathbb{I}_0$ of $\mathbb{I}$ containing $\mathbb{Z}_p[[T]]$ such that the image of $\rho$ is large with respect to $\mathbb{I}_0$. That is, $\operatorname{Im}\rho$ contains $\ker(\mathrm{SL}_2(\mathbb{I}_0) \to \mathrm{SL}_2(\mathbb{I}_0/\mathfrak{a}))$ for some non-zero $\mathbb{I}_0$-ideal $\mathfrak{a}$. This work builds on recent work of Hida, who showed that the image of such a Galois representation is large with respect to $\mathbb{Z}_p[[T]]$. Our result is an $\mathbb{I}$-adic analogue of the description of the image of the Galois representation attached to a non-CM classical modular form obtained by Ribet and Momose in the 1980s. In addition, we discuss further questions related to determining the largest $\mathbb{I}_0$-ideal $\mathfrak{c}_0$ for which $\operatorname{Im}\rho$ contains $\ker(\mathrm{SL}_2(\mathbb{I}_0) \to \mathrm{SL}_2(\mathbb{I}_0/\mathfrak{c}_0))$.

The dissertation of Jaclyn Ann Lang is approved.

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*To the strong, independent, Lang women*

*(and men, Slausons, and Mawrters)*

*who raised me*

*to follow my dreams*

# Table of Contents

Acknowledgments

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| | |
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# Publications

M. Daub, J. Lang, M. Merling, N. Pitiwan, A. Pacell, and M. Rosen, *Function fields with class number indivisible by a prime $\ell$*, Acta Arith., Vol. 150, No. 4 (2011), 339–359.

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# CHAPTER 1

# Introduction

## 1.1 The question and heuristic

The main question to be studied in this thesis is a special case of the following broad question, which has been of interest to number theorists for many decades:

**Question.** *Given a Galois representation (usually arising from an arithmetic object such as an elliptic curve, modular form, or motive), what is its image?*

The following heuristic is a rough guide for how we expect the images of such Galois representations to behave.

**Heuristic.** *The image of a Galois representation that arises from a geometric object (such as an elliptic curve, modular form, or motive) should be as large as possible, subject to the symmetries of the geometric object from which it arises.*

The idea behind the above heuristic is that (most of) the Galois action on the geometric object will have to commute with the geometric symmetries. Thus, the more symmetries (or endomorphisms) the geometric object has, the more matrices the image of the associated Galois representation will have to commute with. Since large subgroups of $\mathrm{GL}_n$ are highly non-commutative, the image of the associated Galois representation must be small.

Of course, this is quite vague as terms like "symmetries", "large", and "small" have not been rigorously defined. We shall make theses notions more precise in Chapter 3. For now we give an example that illustrates this philosophy.

*Example* 1. Let $E$ be the elliptic curve with Weierstrass equation $y^2 = x^3 + x$. The curve $E$ has CM by $\mathbb{Q}(i)$; that is, its endomorphism ring is an order in $\mathbb{Q}(i)$, which is larger than the usual endomorphism ring $\mathbb{Z}$. Indeed, $i$ acts on $E(\overline{\mathbb{Q}})$ by sending $x$ to $-x$ and $y$ to $-iy$. One can check that this endomorphism commutes with the group law on $E$ and hence $i$ acts on the $p$-adic Tate module $E[p^\infty]$ of $E$ for any prime $p$.

Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $(x, y) \in E(\overline{\mathbb{Q}})$. Then

$$i \circ \sigma(x, y) = i(x^\sigma, y^\sigma) = (-x^\sigma, -iy^\sigma)$$

while

$$\sigma \circ i(x, y) = \sigma(-x, -iy) = (-x^\sigma, -i^\sigma y^\sigma).$$

Thus, the actions of $i$ and $\sigma$ commute if and only if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(i))$. The fact that $i$ has order four and does not commute with complex conjugation shows that the action of $i$ on $E[p^\infty]$ has two distinct eigenvalues. Thus, there is some basis of $E[p^\infty]$ for which the matrix $A$ representing the endomorphism $i$ is diagonal and non-scalar. Writing $\rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_p)$ for the $p$-adic Galois representation of $E$, we see that, in the chosen basis, $\mathrm{Im}\, \rho_{E,p}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(i))}$ is contained in the diagonal matrices. Finally, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(i))$ is a subgroup of index two in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so the set of diagonal matrices in $\mathrm{Im}\, \rho_{E,p}$ form an open subset of $\mathrm{Im}\, \rho_{E,p}$. From this it follows that $\mathrm{Im}\, \rho_{E,p}$ cannot be "large". For example, $\mathrm{Im}\, \rho_{E,p}$ cannot contain

$$\{\mathbf{x} \in \mathrm{SL}_2(\mathbb{Z}_p) : \mathbf{x} \equiv 1 \bmod p^r\}$$

for any $r > 0$.

## 1.2 History of the problem

Serre initiated the study of images of Galois representations in 1968. He showed that if an elliptic curve defined over $\mathbb{Q}$ does not have CM, then the image of the associated $p$-adic Galois representation is open for any prime $p$ [40]. Furthermore, he showed that the $p$-adic Galois representation associated to such an elliptic curve is surjective (onto $\mathrm{GL}_2(\mathbb{Z}_p)$) for almost all primes $p$ [40].

The study of the image of the Galois representation attached to a modular form, and showing that it is large in the absence of CM, was first carried out by Serre [41] and Swinnerton-Dyer [43] in the early 1970s. They studied the Galois representation attached to a modular form of level one with integral coefficients. In the 1980s, Ribet [38, 39] and Momose [34] generalized the work of Serre and Swinnerton-Dyer to cover all Galois representations coming from classical modular forms. Ribet's work dealt with the weight two case, and Momose proved the general case. A key innovation of Ribet in generalizing the work of Serre and Swinnerton-Dyer was to introduce a new type of symmetry of modular forms that we shall call *conjugate self-twists*. These new symmetries, which can be viewed as a weak version of the CM-type symmetry, allowed Ribet and Momose to define what "as large as possible" should mean in the above heuristic. The main theorem in this thesis is an analogue of their results in the $\mathbb{I}$-adic setting.

In the 1980s, Hida developed his theory of $p$-adic families of ordinary modular forms [13] and the Galois representations attached to them [12]. This theory will be reviewed in more detail in Chapter 2. For now, it suffices to say that the Galois representation $\rho_F$ arising from a Hida family $F$ take values in $\mathrm{GL}_2(\mathbb{I})$, where $\mathbb{I}$ is an integral domain that is finite flat over $\Lambda = \mathbb{Z}_p[[T]]$. Shortly after Hida constructed these Galois representations, Mazur and Wiles [30] showed that if $\mathbb{I} = \Lambda$ and the image of the residual representation $\bar{\rho}_F$ contains $\mathrm{SL}_2(\mathbb{F}_p)$ then $\mathrm{Im}\,\rho_F$ contains $\mathrm{SL}_2(\Lambda)$. Under the assumptions that $\mathbb{I}$ is a power series ring in one variable and the image of the residual representation $\bar{\rho}_F$ contains $\mathrm{SL}_2(\mathbb{F}_p)$, our main result was proved by Fischman [8]. Fischman's work is the only previous work that considers the effect of conjugate self-twists on $\mathrm{Im}\,\rho_F$.

Hida has shown [19] under some technical hypotheses that, if $F$ does not have CM then $\mathrm{Im}\,\rho_F$ is "large" with respect to the ring $\Lambda$, even when $\mathbb{I} \supsetneq \Lambda$. That is, there is a non-zero $\Lambda$-ideal $\mathfrak{a}$ such that $\mathrm{Im}\,\rho_F$ contains $\ker(\mathrm{SL}_2(\Lambda) \to \mathrm{SL}_2(\Lambda/\mathfrak{a}))$ (but it is possible that $\mathrm{Im}\,\rho_F \not\supseteq \mathrm{SL}_2(\Lambda)$). The methods he developed play an important role in this thesis.

The local behavior of $\rho_F$ at $p$ was studied by Ghate, Vatsal [9] and later by Hida [18].

Let $D_p$ denote the decomposition group at $p$ in $G_{\mathbb{Q}}$. They showed, under some assumptions that were later removed by Zhao [49], that $\rho_F|_{D_p}$ is indecomposable, a result that we make use of in this thesis.

The main result of this thesis is to identify a $\Lambda$-subalgebra $\mathbb{I}_0$ of $\mathbb{I}$ such that $\operatorname{Im}\rho_F$ is large with respect to $\mathbb{I}_0$ in a sense analogous to that of [19]. In some sense, we are $p$-adically interpolating the results of Ribet [38, 39] and Momose [34] about images of Galois representations associated to (ordinary) classical modular forms. As such, their results are a key input into my proof.

Finally, there has been some work in other settings. Hida and Tilouine have shown that certain $\mathrm{GSp}_4$-representations associated to $p$-adic families of Siegel modular forms have large image [21]. Conti, Iovita, and Tilouine have investigated the non-ordinary version of Hida's [19] and obtained results analogous to those presented in this thesis [3].

Detailed information about images of Galois representations is important for other methods in number theory. For example, the Euler systems recently constructed by Lei, Loeffler, and Zerbes require big image results [27, 28]. Furthermore, precise descriptions of images of Galois representations often lead to new solutions to the inverse Galois problem [48, 50].

## 1.3 Structure of the dissertation

In Chapter 2 we will review Hida's theory of ordinary $p$-adic families of modular forms and the Galois representations associated to them. In particular, we will define Hida's big Hecke algebra, use the Hecke algebra to define Hida families, and discuss a duality between the two. We will also show how Wiles' definition of a Hida family [47] is equivalent to that of Hida. Finally, we give a brief exposition of the theory of newforms for Hida families.

Chapter 3 is the main original mathematical content of the dissertation. We prove that the image of a Galois representation associated to a non-CM Hida family is large, under some assumptions. Chapter 3 is essentially my paper [25]. See Section 3.1 for a more detailed summary of the internal structure of Chapter 3.

Chapter 4 contains a purely automorphic proof of some special cases of Theorem 3.2.1. The statement of Theorem 3.2.1 is purely automorphic, but the proof we give in Section 3.2 uses deformation theory. Chapter 4 is a proof-of-concept that suggests one may be able to avoid the use of deformation theory in the proof.

In Chapter 5 we prove some small results that may be of independent interest and have some relationship with the problems and conjectures discussed in Chapter 6.

Finally, Chapter 6 serves as my rough research plan going forward. I discuss problems and conjectures related to the main theorem and techniques in Chapter 3 that I hope to make some progress on in the next several years.

# CHAPTER 2

# Background

The purpose of this chapter is to give a self-contained summary of the parts of Hida theory that will be used in Chapter 3. We begin in Section 2.1 by introducing Hida's notion of $p$-adic families of modular forms and his associated Hecke algebras, following [13]. We quote some basic results from that paper, such as the duality between modular forms and Hecke algebras, without proof. While one can develop the theory in great generality, we focus on a particular congruence subgroup and only describe the theory for cusp forms as that is all that will be needed in the rest of the thesis. In Section 2.2 we discuss two properties of the Hecke algebra that are used in Chapter 3. First we sketch the equivalence of Wiles' definition of a Hida family with that introduced in Section 2.1. We also note that the Hecke algebra is étale over $\Lambda$ at arithmetic points, a key geometric fact that is critical to the argument in Section 3.2. Finally, in Section 2.3, we give a brief account of the theory of newforms for Hida families, without proofs.

## 2.1 Defining big Hecke algebras and Hida families

Fix a prime $p > 2$ and an integer $N = N_0 p^r$, with $(N_0, p) = 1$ and $r \geq 1$. Fix a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}.$$

### 2.1.1 Classical modular forms

For an integer $k \geq 2$, let $S_k(N,\chi)$ denote the space of classical modular forms of weight $k$, level $\Gamma_0(N)$, and nebentypus $\chi$. That is, $f \in S_k(N,\chi)$ if $f : \{z \in \mathbb{C} : \Im z > 0\} \to \mathbb{C}$ is homomorphic and satisfies the following two properties:

1. for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$ for all $z \in \mathbb{C}$ such that $\Im z > 0$
2. for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have $\lim_{y\to\infty, y\in\mathbb{R}} f\left(\frac{aiy+b}{ciy+d}\right) = 0$ (so $f$ vanishes at all cusps).

It is well known that these two properties imply that any such $f$ has a Fourier expansion of the form

$$f = \sum_{n=1}^{\infty} a(n,f)q^n,$$

where $q = e^{2\pi i z}$.

There are Hecke operators acting on $S_k(N,\chi)$ given by the following formulae. For a prime $\ell$,

$$a(m, f|T(\ell)) = \begin{cases} a(m\ell, f) + \chi(\ell)q^{k-1}a(m/\ell, f) & \ell \nmid N \\ a(m\ell, f) & \ell | N, \end{cases} \tag{2.1}$$

where $a(m/\ell, f) = 0$ if $m/\ell$ is not an integer. We sometimes write $U(\ell)$ for $T(\ell)$ when $\ell|N$. Let $h_k(N,\chi)$ denote the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(N,\chi))$ generated by $\{T(\ell)\}_{\ell \text{ prime}}$.

It turns out that $S_k(N,\chi)$ has a basis of eigenforms. If $f$ is an eigenform normalized such that $a(1,f) = 1$, then $f|T(n) = a(n,f)$ for all $n$. From this, together with the fact that $S_k(N,\chi)$ is a finite dimensional $\mathbb{C}$-vector space, one can conclude that $\mathbb{Q}(a(n,f) : n \in \mathbb{Z}^+)$ is a finite extension of $\mathbb{Q}$. Let $\mathbb{Z}[\chi]$ denote the ring generated by the values of $\chi$. For any subring $A$ of $\mathbb{C}$, let $S_k(N,\chi;A) = \{f \in S_k(N,\chi) : a(n,f) \in A, \forall n \in \mathbb{Z}^+\}$, which is a finite type $A$-module. It is a deep fact that $S_k(N,\chi;\mathbb{Z}[\chi,\frac{1}{6N}]) \otimes_{\mathbb{Z}[\chi,\frac{1}{6N}]} \mathbb{C} \cong S_k(N,\chi)$.

### 2.1.2 The space of $p$-adic modular forms

Fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ as well as embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Fix a finite extension $K$ of $\mathbb{Q}_p$ (sufficiently large) with ring of integers $\mathcal{O}$. Let $K_0 \subset K$ be a finite extension of $\mathbb{Q}$ that is dense in $K$ under the $p$-adic topology. Define

$$\mathcal{S}_k(N,\chi;K) = S_k(N,\chi;K_0) \otimes_{K_0} K.$$

Let $\Omega$ be the $p$-adic completion of the fixed algebraic closure of $\overline{\mathbb{Q}}_p$. Consider the formal power series ring $\Omega[[q]]$, which comes with the following $p$-adic norm:

$$|\sum_{n=0}^{\infty} a_n q^n|_p = \sup_n |a_n|_p.$$

The norm is finite on $S_k(N,\chi;K_0)$, and $\mathcal{S}_k(N,\chi;K)$ may be identified with the completion of $S_k(N,\chi;K_0)$ in $\Omega[[q]]$ under the $p$-adic norm. Define

$$\mathcal{S}_k(N,\chi;\mathcal{O}) = \{f \in \mathcal{S}_k(N,\chi;K) : |f|_p \leq 1\} = \mathcal{S}(N,\chi;K) \cap \mathcal{O}[[q]].$$

The next step is to put all the $p$-adic modular forms of different weights together into a big space. Let $A$ be either $K$ or $\mathcal{O}$. For $j > 0$, define

$$\mathcal{S}^j(N,\chi;A) = \oplus_{k=1}^{j} \mathcal{S}_k(N,\chi;A).$$

Define

$$\mathcal{S}(N,\chi;A) = \bigcup_{j>0} \mathcal{S}^j(N,\chi;A).$$

One can check that one can embed $\mathcal{S}(N,\chi;A) \hookrightarrow A[[q]]$ via $q$-expansions. The *space of cuspidal $p$-adic modular forms of level $\Gamma_0(N)$ and nebentypus $\chi$ with coefficients in $A$* is then defined to be the $p$-adic completion of $\mathcal{S}(N,\chi;A)$ in $A[[q]]$. The space will be denoted $\bar{\mathcal{S}}(N,\chi;A)$.

### 2.1.3 The $p$-adic Hecke algebra

Define Hecke operators on $\bar{\mathcal{S}}(N,\chi;A)$ by the formulae given in (2.1). One can check that for all $j > 0$, the space $\mathcal{S}^j(N,\chi;A)$ is stable under all of the above Hecke operators. Let

$h^j(N,\chi;A)$ denote the $A$-subalgebra of $\mathrm{End}_{A-\mathrm{Mod}}(\mathcal{S}^j(N,\chi;A))$ generated by $\{T(\ell)\}_{\ell \text{ prime}}$. Note that for $j' > j$ the restriction map gives a natural morphism $h^{j'}(N,\chi;A) \to h^j(N,\chi;A)$. Define

$$h(N,\chi;\mathcal{O}) = \varprojlim_j h^j(N,\chi;\mathcal{O}).$$

By definition, this acts on $\mathcal{S}(N,\chi;\mathcal{O}) = \varinjlim_j \mathcal{S}^j(N,\chi;\mathcal{O})$. One can easily check that the action of $h(N,\chi;\mathcal{O})$ on $\mathcal{S}(N,\chi;\mathcal{O})$ is uniformly continuous and therefore $h(N,\chi;\mathcal{O})$ acts naturally on $\bar{\mathcal{S}}(N,\chi;\mathcal{O})$.

Unfortunately, $h(N,\chi;\mathcal{O})$ is too big to study all at once, so Hida introduced the *ordinary* part of the space. We begin by defining the ordinary projector on $h^j(N,\chi;\mathcal{O})$ by

$$e_j = \lim_{n\to\infty} T(p)^{n!}.$$

This limit converges [14, Lemma 7.2.1] to an idempotent in $h^j(N,\chi;\mathcal{O})$. These idempotents are compatible with the projection maps $h^j(N,\chi;\mathcal{O}) \to h^i(N,\chi;\mathcal{O})$ for $j \geq i$, and so we define the ordinary projector [13, (1.17b)] by

$$e = \varprojlim_j e_j \in h(N,\chi;\mathcal{O}).$$

Again, $e$ is an idempotent. For any $h(N,\chi;\mathcal{O})$-module $M$, we write $M^{\mathrm{ord}}$ for $eM$, which is called the *ordinary part* of $M$.

**Definition 2.1.1.** A *Hida family* is an irreducible component of $\operatorname{Spec} h^{\mathrm{ord}}(N,\chi;\mathcal{O})$, which we sometimes identify with the corresponding ring.

## 2.2 Important properties of the big Hecke algebra

In this section we discuss some properties of Hecke algebras and Hida families that will be used in Chapter 3.

### 2.2.1 Duality between Hecke algebras and modular forms

As above, let $A$ denote either $K$ or $\mathcal{O}$. We begin by defining a bilinear form

$$h^j(N,\chi;A) \times \mathcal{S}^j(N,\chi;A) \to A \tag{2.2}$$

$$(h,f)^j_A \mapsto a(1,f|h), \tag{2.3}$$

where $a(1,f|h)$ denotes coefficient of $q$ in the $q$-expansion of $f|h$.

Furthermore, set $\mathcal{S}^j(N,\chi;K/\mathcal{O}) = \mathcal{S}^j(N,\chi;K)/\mathcal{S}^j(N,\chi;\mathcal{O})$. Then we can define

$$\mathcal{S}(N,\chi;K/\mathcal{O}) = \mathcal{S}(N,\chi;K)/\mathcal{S}(N,\chi;\mathcal{O}) = \varinjlim_j \mathcal{S}^j(N,\chi;K/\mathcal{O})$$

which we equip with the discrete topology. Define a pairing

$$h^j(N,\chi;\mathcal{O}) \times \mathcal{S}(N,\chi;K/\mathcal{O}) \to K/\mathcal{O}$$

$$(h,\bar{f})^j_{K/\mathcal{O}} \mapsto (h,f)^j \bmod \mathcal{O},$$

where $\bar{f}$ denotes the class in $\mathcal{S}(N,\chi;K/\mathcal{O})$ containing $f$. These pairings are compatible with the projective and injective limits in the definitions of $h(N,\chi;\mathcal{O})$ and $\mathcal{S}(N,\chi;K/\mathcal{O})$, respectively, and so we get a pairing

$$(\ ,\ )_{K/\mathcal{O}} : h(N,\chi;\mathcal{O}) \times \mathcal{S}(N,\chi;K/\mathcal{O}) \to K/\mathcal{O}. \tag{2.4}$$

**Theorem 2.2.1.** *[13]*

1. *The pairing $(\ ,\ )^j_A$ induces a natural isomorphism*

$$h^j(N,\chi;A) \cong \mathrm{Hom}_A(\mathcal{S}^j(N,\chi;A),A).$$

2. *The pairing $(\ ,\ )_{K/\mathcal{O}}$ induces a Pontrjagin duality between the compact module $h(N,\chi;\mathcal{O})$ and the discrete module $\mathcal{S}(N,\chi;K/\mathcal{O})$.*

3. *Furthermore, $(\ ,\ )_{K/\mathcal{O}}$ respects ordinarity. That is, $\mathcal{S}^{\mathrm{ord}}(N,\chi;K/\mathcal{O})$ is dual to $h^{\mathrm{ord}}(N,\chi;\mathcal{O})$ with respect to this pairing.*

### 2.2.2 The $\Lambda$-algebra structure of $h(N,\chi;\mathcal{O})$

Let $\Gamma = 1 + p\mathbb{Z}_p$ and for each positive integer $n$ set $\Gamma_n = 1 + p^n\mathbb{Z}_p$. Fix $\gamma \in \Gamma$ to be a topological generator; for example, one could choose $\gamma = 1 + p$. Define

$$\Lambda_K = \varprojlim_n \mathcal{O}[\Gamma/\Gamma_n] \cong \mathcal{O}[[T]].$$

The last (non-canonical isomorphism) is given by identifying $\gamma$ with $1 + T$. When $K = \mathbb{Q}_p$ we simply write $\Lambda$ for $\Lambda_K$.

We begin by defining an action of $\Gamma$ on $\mathcal{S}(N,\chi;K)$. For $f \in \mathcal{S}(N,\chi;K)$, we can write $f$ as a finite sum $f = \sum_{k=1}^{j} f_k$, with each $f_k \in \mathcal{S}_k(N,\chi;K)$. For $z \in \Gamma$, the action of $z$ on $f$ is defined by

$$f|z := \sum_{k=1}^{j} z^k \chi(z) f_k,$$

where we view $z \in (\mathbb{Z}/N\mathbb{Z})^\times$ as

$$(z \bmod p^r, 1) \in (\mathbb{Z}/p^r\mathbb{Z})^\times \times (\mathbb{Z}/N_0\mathbb{Z})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times.$$

It follows directly from the definition that $\mathcal{S}(N,\chi;\mathcal{O})$ is preserved under the $\Gamma$-action and that $\mathcal{S}(N,\chi;K/\mathcal{O})$ inherits a continuous $\Lambda_K$-action [13, §3].

By duality (Theorem 2.2.1), we see that $h(N,\chi;\mathcal{O})$ inherits a $\Lambda_K$-action, which can be shown to be continuous with respect to the projective limit topologies on $\Lambda_K$ and $h(N,\chi;\mathcal{O})$.

We now introduce some special prime ideals in the ring $\Lambda_K$. An *arithmetic prime* of $\Lambda_K$ is a prime ideal of the form

$$P_{k,\varepsilon} := (1 + T - \varepsilon(1+p)(1+p)^k)$$

for an integer $k \geq 2$ and character $\varepsilon : 1 + p\mathbb{Z}_p \to \mathcal{O}^\times$ of $p$-power order. We shall write $r(\varepsilon)$ for the non-negative integer such that $p^{r(\varepsilon)}$ is the order of $\varepsilon$. If $R$ is a finite extension of $\Lambda_K$, then we say a prime of $R$ is *arithmetic* if it lies over an arithmetic prime of $\Lambda_K$.

Furthermore, let $\omega$ denote the $p$-adic Teichmüller character. Let $\chi_1$ be the product of $\chi|_{(\mathbb{Z}/N_0\mathbb{Z})^\times}$ with the tame $p$-part of $\chi$. The following theorem summarizes the essence of

Hida's theory of ordinary families of $p$-adic modular forms and their Hecke algebras. The content of this theorem is used both explicitly and implicitly throughout the dissertation.

**Theorem 2.2.2.** *1. [13, Theorem 3.1] The ordinary Hecke algebra $h^{\text{ord}}(N, \chi; \mathcal{O})$ is free of finite rank over $\Lambda_K$.*

*2. [19]The ordinary Hecke algebra $h^{\text{ord}}(N, \chi; \mathcal{O})$ satisfies the following specialization property: for every arithmetic prime $P_{k,\varepsilon}$ of $\Lambda_K$, there is an isomorphism*

$$h^{\text{ord}}(N, \chi; \mathcal{O})/P_{k,\varepsilon}h^{\text{ord}}(N, \chi; \mathcal{O}) \cong h_k^{\text{ord}}(Np^{r(\varepsilon)+1}, \chi\varepsilon\omega^{-k})$$

*sending $T(\ell)$ to $T(\ell)$ for all primes $\ell$.*

*3. [16, Proposition 3.78] The ordinary Hecke algebra $h^{\text{ord}}(N, \chi; \mathcal{O})$ is étale over all arithmetic points of $\Lambda_K$.*

### 2.2.3 Wiles' definition of Hida families

In [47], Wiles gave a different definition of Hida families that will be used in Chapter 3. In this section we give Wiles' definition and explain how to translate between the two definitions.

Fix a finite flat integral domain $\mathbb{I}$ over $\Lambda_{\mathbb{Q}_p(\chi)}$. A formal power series $G = \sum_{n=1}^{\infty} a(n, G)q^n$ is an $\mathbb{I}$-*adic cusp form* of level $N$ and character $\chi$ if for almost all arithmetic primes $\mathfrak{P}$ of $\mathbb{I}$, the specialization of $G$ at $\mathfrak{P}$ gives the $q$-expansion of an element $g_{\mathfrak{P}}$ of $S_k(Np^{r(\varepsilon)}, \varepsilon\chi\omega^{-k})$, where $\mathfrak{P}$ lies over $P_{k,\varepsilon}$. Such a form is *ordinary* if $A_p \in \mathbb{I}^{\times}$. Let $\mathbb{S}(N, \chi; \mathbb{I})$ be the $\mathbb{I}$-submodule of $\mathbb{I}[[q]]$ spanned by all $\mathbb{I}$-adic cusp forms of level $N$ and character $\chi$, and let $\mathbb{S}^{\text{ord}}(N, \chi; \mathbb{I})$ denote the $\mathbb{I}$-submodule of $\mathbb{S}(N, \chi; \mathbb{I})$ spanned by all ordinary $\mathbb{I}$-adic cusp forms. Equivalently, we could define $\Lambda_{\mathbb{Q}_p(\chi)}$-adic forms as above, and then for any finite flat integral domain $\mathbb{I}$ over $\Lambda_{\mathbb{Q}_p(\chi)}$ we would have $\mathbb{S}(N, \chi; \mathbb{I}) = \mathbb{S}(N, \chi; \Lambda_{\mathbb{Q}_p(\chi)}) \otimes_{\Lambda_{\mathbb{Q}_p(\chi)}} \mathbb{I}$ and similarly for the ordinary part. Wiles proved that $\mathbb{S}^{\text{ord}}(N, \chi; \Lambda_{\mathbb{Q}_p(\chi)})$ is free of finite rank over $\Lambda_{\mathbb{Q}_p(\chi)}$ [14, Theorem 7.3.1]. Furthermore, $\mathbb{S}^{\text{ord}}(N, \chi; \Lambda_{\mathbb{Q}_p(\chi)})$ satisfies the same interpolation property as $h^{\text{ord}}(N, \chi; \mathcal{O})$ in the previous section. That is, for any arithmetic prime $P_{k,\varepsilon}$, we have [14, Theorem 7.3.3]

$$\mathbb{S}^{\text{ord}}(N, \chi; \Lambda_{\mathbb{Q}_p(\chi)})/P_{k,\varepsilon}\mathbb{S}^{\text{ord}}(N, \chi; \Lambda_{\mathbb{Q}_p(\chi)}) \cong \mathcal{S}_k^{\text{ord}}(Np^{r(\varepsilon)}, \varepsilon\chi\omega^{-k}). \tag{2.5}$$

We now wish to define Hecke operators on $\mathbb{S}(N,\chi;\mathbb{I})$. Recall that for $p>2$ we have a canonical decomposition $\mathbb{Z}_p^\times \cong \mu_{p-1}\times\Gamma$. For $s\in\mathbb{Z}_p$, let $\langle s\rangle$ denote its projection to $\Gamma$. Define the character $\kappa:\Gamma\to\Lambda^\times$ by, for any $s\in\mathbb{Z}_p$,

$$\kappa(\gamma^s)=(1+T)^s=\sum_{n=0}^{\infty}\binom{s}{n}T^n. \tag{2.6}$$

For $F=\sum_{n=1}^{\infty}a(n,F)q^n\in\mathbb{S}(N,\chi;\mathbb{I})$ and a prime $\ell$, we define

$$a(m,F|T(\ell))=\begin{cases}a(m\ell,F)+\kappa(\langle\ell\rangle)\chi(\ell)\ell^{-1}a(m/\ell,F) & \ell\nmid N\\ a(m\ell,F) & \ell|N,\end{cases} \tag{2.7}$$

where, as in the classical case, $a(m/\ell,F)$ is defined to be 0 whenever $m/\ell$ is not an integer. It is straightforward to check (see [14, §7.3]) that, modulo arithmetic primes, these formulae induce the classical formulae for Hecke operators given in (2.1). Therefore the Hecke operators preserve $\mathbb{S}(N,\chi;\mathbb{I})$ as well as the ordinary subspace $\mathbb{S}^{\mathrm{ord}}(N,\chi;\mathbb{I})$. Let $\mathbf{h}^{\mathrm{ord}}(N,\chi;\mathbb{I})$ denote the $\mathbb{I}$-submodule of $\mathrm{End}_{\mathbb{I}}(\mathbb{S}^{\mathrm{ord}}(N,\chi;\mathbb{I}))$ generated by $\{T(\ell)\}_{\ell\text{ prime}}$. As usual, for each integer $e\geq 1$ we inductively define

$$T(\ell^{e+1})=\begin{cases}T(\ell)T(\ell^e)-\chi(\ell)\kappa(\langle\ell\rangle)\ell^{-1}T(\ell^{e-1}) & \ell\nmid N\\ T(\ell)^{e+1} & \ell|N\end{cases}$$

For any integer $m=\prod_\ell \ell^{e_\ell}$, we define $T(m)=\prod_\ell T(\ell^{e_\ell})$.

As in (2.2), we can define a pairing

$$\mathbf{h}^{\mathrm{ord}}(N,\chi;\mathbb{I})\times\mathbb{S}^{\mathrm{ord}}(N,\chi;\mathbb{I})\to\mathbb{I} \tag{2.8}$$

$$(h,F)_{\mathbb{I}}\mapsto a(1,F|h). \tag{2.9}$$

Just as in the previous section, there is a duality between the space of ordinary $\mathbb{I}$-adic cusp forms and the Hecke algebra.

**Theorem 2.2.3.** *The pairing* (2.8) *induces isomorphisms* $\mathrm{Hom}_{\mathbb{I}}(\boldsymbol{h}^{\mathrm{ord}}(N,\chi;\mathbb{I}),\mathbb{I})\cong\mathbb{S}^{\mathrm{ord}}(N,\chi;\mathbb{I})$ *and* $\mathrm{Hom}_{\mathbb{I}}(\mathbb{S}^{\mathrm{ord}}(N,\chi;\mathbb{I}),\mathbb{I})\cong\boldsymbol{h}^{\mathrm{ord}}(N,\chi;\mathbb{I})$.

We now explain how to translate between $\mathbb{I}$-adic cusp forms and Hida families.

**Proposition 2.2.4.** *Every Hida family gives rise to an $\mathbb{I}$-adic cusp form.*

*Proof.* Start with a Hida family, that is, an irreducible component $\operatorname{Spec}\mathbb{I}$ of $\operatorname{Spec} h^{\mathrm{ord}}(N,\chi;\mathcal{O})$. Then there is a surjective ring morphism $\pi_{\mathbb{I}} : h^{\mathrm{ord}}(N,\chi;\mathcal{O}) \to \mathbb{I}$ induced by the natural inclusion of spectra. Define

$$F = \sum_{n=1}^{\infty} a(n,F)q^n \in \mathbb{I}[[q]],$$

where $a(n,F) = \pi_{\mathbb{I}}(T(n))$. To see that $F$ is an $\mathbb{I}$-adic cusp form, let $\mathfrak{P}$ be an arithmetic prime of $\mathbb{I}$ lying over $P_{k,\varepsilon}$. By Theorem 2.2.2 we get an algebra homomorphism

$$h_k^{\mathrm{ord}}(Np^{r(\varepsilon)}, \chi\varepsilon\omega^{-k}) \cong h^{\mathrm{ord}}(N,\chi;\mathcal{O})/P_{k,\varepsilon}h^{\mathrm{ord}}(N,\chi;\mathcal{O}) \twoheadrightarrow \mathbb{I}/P_{k,\varepsilon}\mathbb{I} \to \mathbb{I}/\mathfrak{P}.$$

By duality (Theorem 2.2.1), the above homomorphism corresponds to en eigenform $f_{\mathfrak{P}} \in \mathcal{S}_k^{\mathrm{ord}}(Np^{r(\varepsilon)}, \varepsilon\chi\omega^{-k};\mathcal{O})$. Since $\mathcal{S}_k^{\mathrm{ord}}(Np^{r(\varepsilon)}, \varepsilon\chi\omega^{-k},\mathcal{O}) = S_k^{\mathrm{ord}}(Np^{r(\varepsilon)}, \varepsilon\omega^{-k};\mathcal{O}_{K_0}) \otimes_{\mathcal{O}_{K_0}} \mathcal{O}$, and $S_k^{\mathrm{ord}}(Np^{r(\varepsilon)}, \varepsilon\chi\omega^{-k};\mathcal{O}_{K_0})$ has a basis of eigenforms, we have $f_{\mathfrak{P}} \in S_k^{\mathrm{ord}}(Np^{r(\varepsilon)}, \varepsilon\chi\omega^{-k};\mathcal{O}_{K_0})$. Hence, $F$ is an $\mathbb{I}$-adic cusp form in the sense of Wiles. The fact that $a(p,F) \in \mathbb{I}^{\times}$ follows from the fact that, by definition of ordinarity, the $U(p)$-operator is a unit in $h^{\mathrm{ord}}(N,\chi;\mathcal{O})$. Hence $F \in \mathbb{S}^{\mathrm{ord}}(N,\chi;\mathbb{I})$, as desired. □

**Proposition 2.2.5.** *Let $F$ be an $\mathbb{I}$-adic cusp form of level $N$ and character $\chi$ that is also a Hecke eigenform. Without loss of generality, assume that $\mathbb{I} = \Lambda[a(n,F) : n \in \mathbb{Z}^+]$. Then $\mathbb{I}$ is a Hida family; that is,* $\operatorname{Spec}\mathbb{I}$ *is an irreducible component of $h^{\mathrm{ord}}(N,\chi;\mathcal{O})$.*

*Proof.* The key point in the proof is that both $\mathbf{h}^{\mathrm{ord}}(N,\chi;\mathbb{I})$ and $h^{\mathrm{ord}}(N,\chi;\mathcal{O})$ $p$-adically interpolate the weight $k$ $p$-adic Hecke algebras, and there is essentially only one way to do that since arithmetic primes are dense in $\Lambda$.

Since $\mathbb{I}$ is an integral domain, $\operatorname{Spec}\mathbb{I}$ is irreducible. Therefore, it suffices to construct a ring homomorphism $h^{\mathrm{ord}}(N,\chi;\mathcal{O}) \to \mathbb{I}$. By Theorem 2.2.3 we know that there is an $\mathbb{I}$-algebra homomorphism $\lambda_F : \mathbf{h}^{\mathrm{ord}}(N,\chi;\mathbb{I}) \to \mathbb{I}$ such that $\lambda_F(T(n)) = a(n,F)$ for all integers $n$. Thus, it suffices to show that $\mathbf{h}^{\mathrm{ord}}(N,\chi;\Lambda_{\mathcal{O}}) \cong h^{\mathrm{ord}}(N,\chi;\mathcal{O})$.

Define a $\Lambda_{\mathcal{O}}$-algebra map

$$\varphi : \mathbf{h}^{\mathrm{ord}}(N,\chi;\Lambda_{\mathcal{O}}) \to h^{\mathrm{ord}}(N,\chi;\mathcal{O})$$

by sending $T(n)$ to $T(n)$ for all positive integers $n$. This is well-defined since if $r$ were any $\Lambda_{\mathcal{O}}$-formal linear combination of $T(n)$'s such that $r = 0 \in \mathbf{h}^{\text{ord}}(N, \chi; \Lambda_{\mathcal{O}})$, it follows that for any arithmetic prime $P_{k,\varepsilon}$, the formal $\Lambda_{\mathcal{O}}/P_{k,\varepsilon}$-linear combination $r$ would be equal to zero in $h_k^{\text{ord}}(Np^{r(\varepsilon)}, \varepsilon\chi\omega^{-k}; \mathcal{O}[\varepsilon])$ by the interpolation property of $\mathbf{h}^{\text{ord}}(N, \chi; \Lambda_{\mathcal{O}})$ ((2.5) and Theorem 2.2.3). Since arithmetic primes are dense in $\Lambda_{\mathcal{O}}$, it follows that $\varphi(r) = 0$, and so $\varphi$ is well-defined.

The surjectivity of $\varphi$ follows from the fact that, by definition, $h^{\text{ord}}(N, \chi; \mathcal{O})$ is generated by the $T(n)$'s. To see that $\varphi$ is injective, let $P_{k,\varepsilon}$ be any arithmetic prime of $\Lambda_{\mathcal{O}}$. Then by the interpolation property of $\mathbf{h}^{\text{ord}}(N, \chi; \Lambda_{\mathcal{O}})$ ((2.5) and Theorem 2.2.3), it follows that

$$\ker \varphi = P_{k,\varepsilon} \ker \varphi \subseteq P_{k,\varepsilon} \mathbf{h}^{\text{ord}}(N, \chi; \Lambda_{\mathcal{O}}).$$

The above holds for all arithmetic primes $P_{k,\varepsilon}$, and such primes are dense in $\Lambda$. Since $\mathbf{h}^{\text{ord}}(N, \chi; \Lambda_{\mathcal{O}})$ is finite over $\Lambda_{\mathcal{O}}$, it follows that $\ker \varphi = 0$, as desired. □

## 2.3 Theory of newforms

We briefly review the theory of newforms for both classical modular forms and Hida families.

### 2.3.1 Classical newforms

The content in this section is taken from [33, Section 4.6]. Let $\mathcal{H}$ denote the upper half complex plane. For $z \in \mathcal{H}$, we write $z = x + iy$ with $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. Recall that for $f, g \in S_k(N, \chi)$ we have the Hermitian Petersson inner product given by the following formula:

$$\langle f, g \rangle = \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)}\Im(z)^{k-2} dx dy,$$

where the bar over $g(z)$ denotes complex conjugation.

The space of old forms in $S_k(N, \chi)$ is generated by forms coming from lower level. More precisely, let $c(\chi)$ denote the conductor of $\chi$. Let $S_k^{\text{old}}(N, \chi)$ be the $\mathbb{C}$-subspace of $S_k(N, \chi)$

generated by

$$\bigcup_M \bigcup_d \{f(dz) : f(z) \in S_k(M, \chi)\},$$

where $M$ runs over all positive integers such that $c(\chi)|M|N$ and $M \neq N$, and $d$ runs over all positive integral divisors of $N/M$, including 1 and $N/M$. The space of *newforms*, denoted $S_k^{\text{new}}(N, \chi)$, is defined to be the orthogonal complement of $S_k^{\text{old}}(N, \chi)$ in $S_k(N, \chi)$ with respect to the Petersson inner product [33, p. 162]. The spaces of new and old forms are stable under the Hecke operators $T(\ell)$ for all primes $\ell \nmid N$. We say $f \in S_k^{\text{new}}(N, \chi)$ is a *primitive form of conductor* $N$ if $f$ is an eigenfunction for $T(\ell)$ for all primes $\ell$ such that $\ell \nmid N$ and the first term in the $q$-expansion of $f$ is equal to 1. Every Hecke eigenform has a unique associated primitive form. Furthermore, primitive forms are the object to which one can attach Galois representations. The following theorem makes these two statements more precise.

**Theorem 2.3.1.** *1. [33, Corollary 4.6.14] If $f \in S_k(N, \chi)$ is an eigenform for $T(\ell)$ with eigenvalue $a(\ell, f)$ for every prime $\ell$ with $\ell \nmid N$, then there is an integer $M$ such that $c(\chi)|M|N$ and a unique primitive form $f_0 \in S_k^{\text{new}}(M, \chi)$ such that $f_0|T(\ell) = a(\ell, f)f_0$ for all primes $\ell$ such that $\ell \nmid N$.*

*2. [4] Let $f = \sum_{n=1}^{\infty} a(n, f)q^n$ be a primitive form, and let $K$ be the field extension of $\mathbb{Q}$ generated by $\{a(n, f) : n \in \mathbb{Z}^+\}$, which is well known to be a finite extension of $\mathbb{Q}$. Let $K_{\mathfrak{p}}$ be the completion of $K$ with respect to a prime lying over $p$. There is a continuous representation $\rho_{f,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{\mathfrak{p}})$ that is unramified outside $Np$ and for all primes $\ell \nmid Np$,*

$$\operatorname{tr} \rho_{f,p}(\mathrm{Frob}_\ell) = a(\ell, f),$$

*where $\mathrm{Frob}_\ell$ denotes a conjugacy class of a Frobenius element at $\ell$ in $G_{\mathbb{Q}}$.*

These are the Galois representations studied by Ribet and Momose.

### 2.3.2 Newforms in $p$-adic families

Let $F \in \mathbb{I}[[q]]$ be a Hida family in the sense of Wiles. Assume that $F$ is an eigenform for Hida's Hecke algebra. We say that $F$ is *primitive* if, for every arithmetic prime $\mathfrak{P}$ of $\mathbb{I}$, the specialization $f_{\mathfrak{P}}$ of $F$ at $\mathfrak{P}$ satisfies one of the following two properties:

1. $f_{\mathfrak{P}}$ is a primitive form in the sense of the previous section

2. the level of $f_{\mathfrak{P}}$ and the level of the associated primitive form to $f_{\mathfrak{P}}$ from Theorem 2.3.1 differ only by a power of $p$.

If $F$ has level $N$ and $N_0$ is the prime-to-$p$ part of $N$, we shall refer to the above two conditions simultaneously by saying that $f_{\mathfrak{P}}$ is $N_0$-*new*. It turns out that $F$ is primitive if and only if there *exists* an arithmetic prime $\mathfrak{P}$ of $\mathbb{I}$ for which $f_{\mathfrak{P}}$ is $N_0$-new.

Let $f \in S_k(\Gamma_0(N), \chi)$ be a classical primitive form of weight $k \geq 2$ whose level is prime to $p$. Assume that $f$ is ordinary at $p$ (so $a(p, f)$ is a $p$-adic unit). Then the polynomial $x^2 - a(p, f)x + \chi(p)p^{k-1} = (x - \alpha)(x - \beta)$ has exactly one root that is a $p$-adic unit, say $\alpha$. The *$p$-stabilization* of $f$ is defined to be

$$f_\alpha(z) := f(z) - \beta f(pz).$$

It is a $p$-ordinary normalized eigenform of level $Np$, and $f_\alpha | U(p) = \alpha f_\alpha$.

A key fact in the theory of newforms in $p$-adic families is the following theorem of Hida.

**Theorem 2.3.2.** *[13, Corollary 3.7] Let $f$ be a primitive classical form of weight at least* 2 *and level $N$. Let $f_0$ be $f$ if $p|N$ and $f_0$ be the $p$-stabilization of $f$ if $p \nmid N$. Then there is a unique primitive Hida family $F \in \mathbb{I}[[q]]$ of level $N$ such that $f_0 = f_{\mathfrak{P}}$ for some arithmetic prime $\mathfrak{P}$ of* $\mathbb{I}$.

The idea of the proof of the above theorem is as follows. One counts the number of classical forms of a fixed weight that are arithmetic specializations of a given primitive Hida family of level $N_0$. One then counts the number of classical forms $f$ of weight $k$ whose associated $f_0$ as in the above theorem has conductor divisible by $N_0$. The first number is bounded above by the second number, and one uses Hida's control theorem [13, Theorem 3.1] to conclude that the two numbers must be equal. Thus, each $f_0$ necessarily belongs to a unique primitive Hida family.

Using Theorem 2.3.2, together with the rest of the theory of newforms developed in [13], one can prove the first part of the following analogue of Theorem 2.3.1.

**Theorem 2.3.3.** *1. If $F \in \mathbb{S}^{\mathrm{ord}}(N,\chi)$ is an eigenform for $T(\ell)$ with eigenvalue $a(\ell,F)$ for every prime $\ell$ with $\ell \nmid N$, then there is an integer $M$ such that $c(\chi)|M|N$ and a unique primitive form $F_0 \in \mathbb{S}^{\mathrm{ord}}(M,\chi)$ such that $F_0|T(\ell) = a(\ell,F)F_0$ for all primes $\ell$ such that $\ell \nmid N$.*

*2. [12] Let $F = \sum_{n=1}^{\infty} a(n,F)q^n$ be a primitive Hida family, and let $\mathbb{K}$ be the field extension of $Q(\Lambda)$ generated by $\{a(n,F) : n \in \mathbb{Z}^+\}$, which is well known to be a finite extension of $Q(\Lambda)$. There is a continuous representation $\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{K})$ that is unramified outside $Np$ and for all primes $\ell \nmid Np$,*

$$\operatorname{tr} \rho_F(\mathrm{Frob}_\ell) = a(\ell,F),$$

*where* Frob$_\ell$ *denotes a conjugacy class of a Frobenius element at $\ell$ in $G_{\mathbb{Q}}$.*

It is these representations that we are interested in studying in this thesis.

# CHAPTER 3

# The Main Result

## 3.1 Main theorems and structure of paper

We begin by fixing notation that will be in place throughout this chapter. Let $p > 2$ be prime. Fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ as well as an embedding $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. Let $\mathbb{Z}^+$ denote the set of positive integers. Fix $N_0 \in \mathbb{Z}^+$ prime to $p$; it will serve as our tame level. Let $N = N_0 p^r$ for some fixed $r \in \mathbb{Z}^+$. Fix a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ which will serve as our nebentypus. Let $\chi_1$ be the product of $\chi|_{(\mathbb{Z}/N_0\mathbb{Z})^\times}$ with the tame $p$-part of $\chi$, and write $c(\chi)$ for the conductor of $\chi$. During the proof of the main theorem we will assume that the order of $\chi$ is a power of 2 and that $2c(\chi)|N$. The fact that we can assume these restrictions on $\chi$ for the purpose of demonstrating $\mathbb{I}_0$-fullness is shown in Proposition 3.2.9.

For a valuation ring $W$ over $\mathbb{Z}_p$, let $\Lambda_W = W[[T]]$. Let $\mathbb{Z}_p[\chi]$ be the extension of $\mathbb{Z}_p$ generated by the values of $\chi$. When $W = \mathbb{Z}_p[\chi]$ we write $\Lambda_\chi$ for $\Lambda_W$. When $W = \mathbb{Z}_p$ then we let $\Lambda = \Lambda_{\mathbb{Z}_p}$. For a commutative ring $R$, we use $Q(R)$ to denote the total ring of fractions of $R$. Let $\operatorname{Spec}\mathbb{I}$ be an irreducible component of $\operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(N_0, \chi; \Lambda_\chi)$. Assume further that $\mathbb{I}$ is primitive in the sense of section 2.3.2. Let $\lambda_F : \mathbf{h}^{\mathrm{ord}}(N_0, \chi; \Lambda_\chi) \to \mathbb{I}$ be the natural $\Lambda_\chi$-algebra homomorphism coming from the inclusion of spectra. Let

$$\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(Q(\mathbb{I}))$$

be the associated Galois representation from Theorem 2.3.3. So $\rho_F$ is unramified outside $N$ and satisfies

$$\operatorname{tr}\rho_F(\mathrm{Frob}_\ell) = \lambda_F(T(\ell))$$

for all primes $\ell$ not dividing $N$.

Henceforth for any $n \in \mathbb{Z}^+$ we shall let $a(n, F)$ denote $\lambda_F(T(n))$. Let $F$ be the formal power series in $q$ given by

$$F = \sum_{n=1}^{\infty} a(n, F) q^n.$$

Let $\mathbb{I}' = \Lambda_\chi[\{a(\ell, F) : \ell \nmid N\}]$ which is an order in $Q(\mathbb{I})$ since $F$ is primitive. We shall consider the Hida family $F$ and the associated ring $\mathbb{I}'$ to be fixed throughout the chapter. For a local ring $R$ we will use $\mathfrak{m}_R$ to denote the unique maximal ideal of $R$. Let $\mathbb{F} := \mathbb{I}'/\mathfrak{m}_{\mathbb{I}'}$ the residue field of $\mathbb{I}'$. We exclusively use the letter $\mathfrak{P}$ to denote a prime of $\mathbb{I}$, and $\mathfrak{P}'$ shall always denote $\mathfrak{P} \cap \mathbb{I}'$. Conversely, we exclusively use $\mathfrak{P}'$ to denote a prime of $\mathbb{I}'$ in which case we are implicitly fixing a prime $\mathfrak{P}$ of $\mathbb{I}$ lying over $\mathfrak{P}'$.

Recall that Hida has shown [12] that there is a well defined residual representation $\bar{\rho}_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}/\mathfrak{m}_{\mathbb{I}})$ of $\rho_F$. Throughout this thesis we impose the following assumption.

$$\text{Assume that } \bar{\rho}_F \text{ is absolutely irreducible.} \tag{abs}$$

By the Chebotarev density theorem, we see that $\mathrm{tr}\, \bar{\rho}_F$ is valued in $\mathbb{F}$. Under (abs) we may use pseudo representations to find a $\mathrm{GL}_2(\mathbb{I}')$-valued representation that is isomorphic to $\rho_F$ over $Q(\mathbb{I})$. Thus we may (and do) assume that $\rho_F$ takes values in $\mathrm{GL}_2(\mathbb{I}')$.

**Definition 3.1.1.** Let $g = \sum_{n=1}^{\infty} a(n, g) q^n$ be either a classical Hecke eigenform or a Hida family of such forms. Let $K$ be the field generated by $\{a(n, g) : n \in \mathbb{Z}^+\}$ over either $\mathbb{Q}$ in the classical case or $Q(\Lambda_\chi)$ in the $\Lambda_\chi$-adic case. We say a pair $(\sigma, \eta_\sigma)$ is a *conjugate self-twist* of $g$ if $\eta_\sigma$ is a Dirichlet character, $\sigma$ is an automorphism of $K$, and

$$\sigma(a(\ell, g)) = \eta_\sigma(\ell) a(\ell, g)$$

for all but finitely many primes $\ell$. If there is a non-trivial character $\eta$ such that $(1, \eta)$ is a conjugate self-twist of $g$, then we say that $g$ has *complex multiplication* or *CM*. Otherwise, $g$ does not have CM.

If a modular form does not have CM then a conjugate self-twist is uniquely determined by the automorphism.

We shall always assume that our fixed Hida family $F$ does not have CM. Let

$$\Gamma = \{\sigma \in \mathrm{Aut}(Q(\mathbb{I})) : \sigma \text{ is a conjugate self-twist of } F\}.$$

Under the assumption (abs) it follows from a lemma of Carayol and Serre [15, Proposition 2.13] that if $\sigma \in \Gamma$ then $\rho_F^\sigma \cong \rho_F \otimes \eta_\sigma$ over $\mathbb{I}'$. As $\rho_F$ is unramified outside $N$ we see that in fact $\sigma(a(\ell, F)) = \eta_\sigma(\ell)a(\ell, F)$ for all primes $\ell$ not dividing $N$. Therefore $\sigma$ restricts to an automorphism of $\mathbb{I}'$. Let $\mathbb{I}_0 = (\mathbb{I}')^\Gamma$. Define

$$H_0 := \bigcap_{\sigma \in \Gamma} \ker \eta_\sigma$$

and

$$H := H_0 \cap \ker(\det(\overline{\rho}_F)).$$

These open normal subgroups of $G_{\mathbb{Q}}$ play an important role in our proof.

For a commutative ring $B$ and ideal $\mathfrak{b}$ of $B$, write

$$\Gamma_B(\mathfrak{b}) := \ker(\mathrm{SL}_2(B) \to \mathrm{SL}_2(B/\mathfrak{b})).$$

We call $\Gamma_B(\mathfrak{b})$ a *congruence subgroup* of $\mathrm{GL}_2(B)$ if $\mathfrak{b} \neq 0$. We can now define what we mean when we say a representation is "large" with respect to a ring.

**Definition 3.1.2.** Let $G$ be a group, $A$ a commutative ring, and $r : G \to \mathrm{GL}_2(A)$ a representation. For a subring $B$ of $A$, we say that $r$ is *$B$-full* if there is some $\gamma \in \mathrm{GL}_2(A)$ such that $\gamma(\mathrm{Im}\, r)\gamma^{-1}$ contains a congruence subgroup of $\mathrm{GL}_2(B)$.

Let $D_p$ be the decomposition group at $p$ in $G_{\mathbb{Q}}$. That is, $D_p$ is the image of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ under the embedding $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$ induced by $\iota_p$. Recall that over $Q(\mathbb{I})$ the local representation $\rho_F|_{D_p}$ is isomorphic to $\left(\begin{smallmatrix} \varepsilon & u \\ 0 & \delta \end{smallmatrix}\right)$ [17, Theorem 4.3.2]. Let $\bar{\varepsilon}$ and $\bar{\delta}$ denote the residual characters of $\varepsilon$ and $\delta$, respectively.

**Definition 3.1.3.** For any open subgroup $G_0 \leq G_{\mathbb{Q}}$ we say that $\overline{\rho}_F$ is *$G_0$-regular* if $\bar{\varepsilon}|_{D_p \cap G_0} \neq \bar{\delta}|_{D_p \cap G_0}$.

The main result of this thesis is the following.

**Theorem 3.1.4.** *Assume $p > 2$ and let $F$ be a primitive non-CM $p$-adic Hida family. Assume $|\mathbb{F}| \neq 3$ and that the residual representation $\overline{\rho}_F$ is absolutely irreducible and $H_0$-regular. Then $\rho_F$ is $\mathbb{I}_0$-full.*

The strategy of the proof is to exploit the results of Ribet [38, 39] and Momose [34]. Since an arithmetic specialization of a non-CM Hida family cannot be CM, their work implies that if $\mathfrak{P}'$ is an arithmetic prime of $\mathbb{I}'$ then there is a certain subring $\mathcal{O} \subseteq \mathbb{I}'/\mathfrak{P}'$ for which $\rho_F \bmod \mathfrak{P}'$ is $\mathcal{O}$-full. To connect their ring $\mathcal{O}$ with $\mathbb{I}_0$, in section 3.5 we show that $Q(\mathcal{O}) = Q(\mathbb{I}_0/\mathcal{Q})$, where $\mathcal{Q} = \mathbb{I}_0 \cap \mathfrak{P}'$. The proof that $Q(\mathcal{O}) = Q(\mathbb{I}_0/\mathcal{Q})$ relies on establishing a relationship between conjugate self-twists of $F$ and conjugate self-twists of the arithmetic specializations of $F$. As this may be of independent interest we state the result here.

**Theorem 3.1.5.** *Let $\mathfrak{P}$ be an arithmetic prime of $\mathbb{I}$ and $\sigma$ be a conjugate self-twist of $f_{\mathfrak{P}}$ that is also an automorphism of the local field $\mathbb{Q}_p(\{a(n, f_{\mathfrak{P}}) : n \in \mathbb{Z}^+\})$. Then $\sigma$ can be lifted to $\tilde{\sigma} \in \Gamma$ such that $\tilde{\sigma}(\mathfrak{P}') = \mathfrak{P}'$, where $\mathfrak{P}' = \mathfrak{P} \cap \mathbb{I}'$.*

The proof, in section 3.2, uses a combination of abstract deformation theory and automorphic techniques. Deformation theory is used to lift $\sigma$ to an automorphism of the universal deformation ring of $\bar{\rho}_F$. Then we use automorphic methods to show that this lift preserves the irreducible component $\operatorname{Spec} \mathbb{I}$. The key technical input is that $\mathbf{h}^{\mathrm{ord}}(N, \chi; \Lambda_\chi)$ is étale over arithmetic points of $\Lambda$.

The remainder of the chapter consists of a series of reduction steps that allow us to deduce our theorem from the aforementioned results of Ribet and Momose. Our methods make it convenient to modify $\rho_F$ to a related representation $\rho : H \to \mathrm{SL}_2(\mathbb{I}_0)$ and show that $\rho$ is $\mathbb{I}_0$-full. We axiomatize the properties of $\rho$ at the beginning of section 3.3 and use $\rho$ in the next three sections to prove Theorem 3.1.4. Then in section 3.6 we explain how to show the existence of $\rho$ with the desired properties.

The task of showing that $\rho$ is $\mathbb{I}_0$-full is done in three steps. In section 3.3 we consider the projection of $\operatorname{Im} \rho$ to $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$, where $P$ is an arithmetic prime of $\Lambda$ and $\mathcal{Q}$ runs over all primes of $\mathbb{I}_0$ lying over $P$. We show that if the image of $\operatorname{Im}\rho$ in $\prod_{\mathcal{Q}|P}\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open then $\rho$ is $\mathbb{I}_0$-full. This uses Pink's theory of Lie algebras for $p$-profinite subgroups of $\mathrm{SL}_2$ over $p$-profinite semilocal rings [35] and the related techniques developed by Hida [19].

In section 3.4 we show that if the image of $\operatorname{Im}\rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is $\mathbb{I}_0/\mathcal{Q}$-full for all primes $\mathcal{Q}$ of $\mathbb{I}_0$ lying over $P$, then the image of $\operatorname{Im}\rho$ is indeed open in $\prod_{\mathcal{Q}|P}\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$. The argument is by contradiction and uses Goursat's Lemma. It was inspired by an argument of Ribet [36]. This is the only section where we make use of the assumption that $|\mathbb{F}| \neq 3$.

The final step showing that the image of $\operatorname{Im}\rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is $\mathbb{I}_0/\mathcal{Q}$-full for every $\mathcal{Q}$ lying over $P$ is done in section 3.5. The key input is Theorem 3.1.5 from section 3.2 together with the work of Ribet and Momose on the image of the Galois representation associated to a non-CM classical modular form. We give a brief exposition of their work and a precise statement of their result at the beginning of section 3.5. We reiterate the structure of the proof of Theorem 3.1.4 at the end of section 3.5.

## 3.2 Lifting twists

Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be (not necessarily distinct) arithmetic primes of $\mathbb{I}$, and let $\mathfrak{P}_i' = \mathfrak{P}_i \cap \mathbb{I}'$. We shall often view $\mathfrak{P}_i$ as a geometric point in $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$. Suppose there is an isomorphism $\sigma : \mathbb{I}/\mathfrak{P}_1 \cong \mathbb{I}/\mathfrak{P}_2$ and a Dirichlet character $\eta : G_{\mathbb{Q}} \to Q(\mathbb{I}/\mathfrak{P}_2)^\times$ such that

$$\sigma(a(\ell, f_{\mathfrak{P}_1})) = \eta(\ell)a(\ell, f_{\mathfrak{P}_2})$$

for all primes $\ell$ not dividing $N$. We may (and do) assume without loss of generality that $\eta$ is primitive since the above relation holds even when $\eta$ is replaced by its primitive character. In this section we show that $\sigma$ can be lifted to a conjugate self-twist of $F$.

**Theorem 3.2.1.** *Assume that $\eta$ takes values in $\mathbb{Z}_p[\chi]$ and that the order of $\chi$ is a power of 2. If $\eta$ is ramified at 2, assume further that $2c(\chi)|N$. Then there is an automorphism $\tilde{\sigma} : \mathbb{I}' \to \mathbb{I}'$ such that*

$$\tilde{\sigma}(a(\ell, F)) = \eta(\ell)a(\ell, F)$$

*for all but finitely many primes $\ell$ and $\sigma \circ \mathfrak{P}'_1 = \mathfrak{P}'_2 \circ \tilde{\sigma}$. In particular, $\mathfrak{P}'_1$ and $\mathfrak{P}'_2$ necessarily lie over the same prime of $\mathbb{I}_0$.*

*Remark* 1. The condition that the order of $\chi$ be a power of 2 looks restrictive. However in Proposition 3.2.9 we show that for the purpose of proving $\mathbb{I}_0$-fullness we may replace $F$ with a family whose nebentypus has order a power of 2. The same proposition shows that the condition that $2c(\chi)|N$ is not restrictive when proving $\mathbb{I}_0$-fullness.

There are two steps in the proof of Theorem 3.2.1. First we use abstract deformation theory to construct a lift $\Sigma$ of $\sigma$ to the universal deformation ring of $\bar{\rho}_F$ (or some base change of that ring). This allows us to show that $\eta$ is necessarily quadratic. Then we show that the induced map on spectra $\Sigma^*$ sends the irreducible component $\operatorname{Spec} \mathbb{I}'$ to another *modular* component of the universal deformation ring. Since $\sigma$ is an isomorphism between $\mathbb{I}/\mathfrak{P}_1$ and $\mathbb{I}/\mathfrak{P}_2$ it follows that the arithmetic point $\mathfrak{P}'_1$ lies on both $\operatorname{Spec} \mathbb{I}'$ and $\Sigma^*(\operatorname{Spec} \mathbb{I}')$. Since the Hecke algebra is étale over arithmetic points of $\Lambda$, it follows that $\Sigma^*(\operatorname{Spec} \mathbb{I}') = \operatorname{Spec} \mathbb{I}'$ and hence $\Sigma$ descends to the desired automorphism of $\mathbb{I}'$.

### 3.2.1 Lifting $\sigma$ to the universal deformation ring

Let $W$ be the ring of Witt vectors of $\mathbb{F}$. Let $\mathbb{Q}^N$ be the maximal subfield of $\overline{\mathbb{Q}}$ unramified outside $N$ and infinity, and let $G_{\mathbb{Q}}^N := \operatorname{Gal}(\mathbb{Q}^N/\mathbb{Q})$. Note that $\rho_F$ factors through $G_{\mathbb{Q}}^N$. For the remainder of this section we shall consider $G_{\mathbb{Q}}^N$ to be the domain of $\rho_F$ and $\bar{\rho}_F$.

We set up the notation for deformation theory. For our purposes universal deformation rings of pseudo representations are sufficient. However, since we are assuming that $\bar{\rho}_F$ is absolutely irreducible, we use universal deformation rings of representations to avoid introducing extra notation for pseudo representations.

Let $\mathcal{C}$ denote the category of complete local $p$-profinite $W$-algebras with residue field $\mathbb{F}$. Let $\bar{\pi} : G_{\mathbb{Q}}^N \to \mathrm{GL}_n(\mathbb{F})$ be an absolutely irreducible representation. We say an object $R_{\bar{\pi}} \in \mathcal{C}$ and representation $\bar{\pi}^{\mathrm{univ}} : G_{\mathbb{Q}}^N \to \mathrm{GL}_n(R_{\bar{\pi}})$ is a *universal couple* for $\bar{\pi}$ if: $\bar{\pi}^{\mathrm{univ}} \bmod \mathfrak{m}_{R_{\bar{\pi}}} \cong \bar{\pi}$ and for every $A \in \mathcal{C}$ and representation $r : G_{\mathbb{Q}}^N \to \mathrm{GL}_n(A)$ such that $r \bmod \mathfrak{m}_A \cong \bar{\pi}$, there exists a unique $W$-algebra homomorphism $\alpha(r) : R_{\bar{\pi}} \to A$ such that $r \cong \alpha(r) \circ \bar{\pi}^{\text{univ}}$. Mazur proved that a universal couple always exists (and is unique) when $\bar{\pi}$ is absolutely irreducible [31].

Since $\eta$ takes values in $\mathbb{Z}_p[\chi]$ which may not be contained in $W$, we need to extend scalars. Let $\mathcal{O} = W[\eta]$. We recommend the reader assume $\mathcal{O} = W$ on the first read. In fact, in Proposition 3.2.4 we will use deformation theory to conclude that $\eta$ is quadratic, but we cannot assume that from the start. For a commutative $W$-algebra $A$, let ${}^{\mathcal{O}}A := \mathcal{O} \otimes_W A$. It will be important that we are tensoring on the left by $\mathcal{O}$ as we will sometimes want to view ${}^{\mathcal{O}}A$ as a right $W$-algebra.

Let $\bar{\sigma}$ denote the automorphism of $\mathbb{F}$ induced by $\sigma$ and $\bar{\eta}$ the projection of $\eta$ to $\mathbb{F}$. The automorphism $\bar{\sigma}$ of $\mathbb{F}$ induces an automorphism $W(\bar{\sigma})$ on $W$. For any $W$-algebra $A$, let $A^{\bar{\sigma}} := A \otimes_{W(\bar{\sigma})} W$, where $W$ is considered as a $W$-algebra via $W(\bar{\sigma})$. Note that $A^{\bar{\sigma}}$ is a $W$-bimodule with different left and right actions. Namely there is the left action given by $w(a \otimes w') = aw \otimes w'$, which may be different from the right action given by $(a \otimes w')w = a \otimes ww'$. In particular, ${}^{\mathcal{O}}A^{\bar{\sigma}} = \mathcal{O} \otimes_W A \otimes_{W(\bar{\sigma})} W$. Let $\iota(\bar{\sigma}, A) : A \to A^{\bar{\sigma}}$ be the usual map given by $\iota(\bar{\sigma}, A)(a) = a \otimes 1$. It is an isomorphism of rings with inverse given by $\iota(\bar{\sigma}^{-1}, A)$. Furthermore, $\iota(\bar{\sigma}, A)$ is a *left* $W$-algebra homomorphism.

Consider the universal couples $(R_{\bar{\rho}_F}, \bar{\rho}_F^{\text{univ}})$, $(R_{\bar{\rho}_F^{\bar{\sigma}}}, (\bar{\rho}_F^{\bar{\sigma}})^{\text{univ}})$, and $(R_{\bar{\eta} \otimes \bar{\rho}_F}, (\bar{\eta} \otimes \bar{\rho}_F)^{\text{univ}})$. The next lemma describes the relationship between these deformation rings.

**Lemma 3.2.2.** *1. If $\bar{\rho}_F^{\bar{\sigma}} \cong \bar{\eta} \otimes \bar{\rho}_F$ then the universal couples $(R_{\bar{\eta} \otimes \bar{\rho}_F}, (\bar{\eta} \otimes \bar{\rho}_F)^{\text{univ}})$ and $(R_{\bar{\rho}_F^{\bar{\sigma}}}, (\bar{\rho}_F^{\bar{\sigma}})^{\text{univ}})$ are canonically isomorphic.*

*2. There is a canonical isomorphism $\varphi : R_{\bar{\rho}_F}^{\bar{\sigma}} \to R_{\bar{\rho}_F^{\bar{\sigma}}}$ of right $W$-algebras such that*

$$(\bar{\rho}_F^{\bar{\sigma}})^{\text{univ}} \cong \varphi \circ \iota(\bar{\sigma}, R_{\bar{\rho}_F}) \circ \bar{\rho}_F^{\text{univ}}.$$

*3. Viewing $(\bar{\eta} \otimes \bar{\rho}_F)^{\text{univ}}$ as a representation valued in $\mathrm{GL}_2({}^{\mathcal{O}}R_{\bar{\eta} \otimes \bar{\rho}_F})$ via the natural map $R_{\bar{\eta} \otimes \bar{\rho}_F} \to {}^{\mathcal{O}}R_{\bar{\eta} \otimes \bar{\rho}_F}$, there is a natural $W$-algebra homomorphism $\psi : R_{\bar{\eta} \otimes \bar{\rho}_F} \to {}^{\mathcal{O}}R_{\bar{\rho}_F}$ such that*

$$\eta \otimes \bar{\rho}_F^{\text{univ}} \cong (1 \otimes \psi) \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\text{univ}}.$$

*Proof.* The first statement follows directly from the definition of universal couples.

For the second point, we show that the right $W$-algebra $R_{\bar{\rho}_F}^{\bar{\sigma}}$ satisfies the universal property for $\bar{\rho}_F^{\bar{\sigma}}$. Let $A \in \mathcal{C}$ and $r : G_{\mathbb{Q}}^N \to \mathrm{GL}_2(A)$ be a deformation of $\bar{\rho}_F^{\bar{\sigma}}$. Then $\iota(\bar{\sigma}^{-1}, A) \circ r$ is a deformation of $\bar{\rho}_F$, viewing $A^{\bar{\sigma}^{-1}}$ as a right $W$-algebra. By universality there is a unique *right* $W$-algebra homomorphism $\alpha(\iota(\bar{\sigma}^{-1}, A) \circ r) : R_{\bar{\rho}_F} \to A^{\bar{\sigma}^{-1}}$ such that $\iota(\bar{\sigma}^{-1}, A) \circ r \cong \alpha(\iota(\bar{\sigma}^{-1}, A) \circ r) \circ \bar{\rho}_F^{\mathrm{univ}}$. Tensoring $\alpha(\iota(\bar{\sigma}^{-1}, A) \circ r)$ with $W$ over $W(\bar{\sigma})$ gives a homomorphism of right $W$-algebras $\alpha(\iota(\bar{\sigma}^{-1}, A) \circ r) \otimes_{W(\bar{\sigma})} 1 : R_{\bar{\rho}_F}^{\bar{\sigma}} \to A$ such that $r \cong (\alpha(\iota(\bar{\sigma}^{-1}, A) \circ r) \otimes_{W(\bar{\sigma})} 1) \circ \iota(\bar{\sigma}, R_{\bar{\rho}_F}) \circ \bar{\rho}_F^{\mathrm{univ}}$. This shows that the right $W$-algebra $R_{\bar{\rho}_F}^{\bar{\sigma}}$ satisfies the universal property for $\bar{\rho}_F^{\bar{\sigma}}$. With notation as above, when $r = (\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}}$ we set $\varphi = \alpha(\iota(\bar{\sigma}^{-1}, R_{\bar{\rho}_F^{\bar{\sigma}}}) \circ (\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}}) \otimes_{W(\bar{\sigma})} 1$, so

$$(\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}} \cong \varphi \circ \iota(\bar{\sigma}, R_{\bar{\rho}_F}) \circ \bar{\rho}_F^{\mathrm{univ}}. \tag{3.1}$$

In particular, $\varphi$ is a right $W$-algebra homomorphism.

Finally, let $i : R_{\bar{\eta}\otimes\bar{\rho}_F} \to {}^{\mathcal{O}}R_{\bar{\eta}\otimes\bar{\rho}_F}$ be the map given by $x \mapsto 1 \otimes x$. If $A$ is a $W$-algebra and $r : G_{\mathbb{Q}}^N \to \mathrm{GL}_2(A)$ is a deformation of $\bar{\rho}_F$ then $\eta \otimes r : G_{\mathbb{Q}}^N \to \mathrm{GL}_2({}^{\mathcal{O}}A)$ is a deformation of $\bar{\eta} \otimes \bar{\rho}_F$. Hence there is a unique $W$-algebra homomorphism $\alpha(\eta \otimes r) : R_{\bar{\eta}\otimes\bar{\rho}_F} \to {}^{\mathcal{O}}A$ such that $\eta \otimes r \cong \alpha(\eta \otimes r) \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}}$. We can extend $\alpha(\eta \otimes r)$ to an $\mathcal{O}$-algebra homomorphism $1 \otimes \alpha(\eta \otimes r) : {}^{\mathcal{O}}R_{\bar{\eta}\otimes\bar{\rho}_F} \to {}^{\mathcal{O}}A$ by sending $x \otimes y$ to $(x \otimes 1)\alpha(\eta \otimes r)(y)$. In particular, $\eta \otimes r \cong (1 \otimes \alpha(\eta \otimes r)) \circ i \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}}$. When $r = \bar{\rho}_F^{\mathrm{univ}}$, let $\psi$ denote $\alpha(\eta \otimes \bar{\rho}_F^{\mathrm{univ}})$, so

$$\eta \otimes \bar{\rho}_F^{\mathrm{univ}} \cong (1 \otimes \psi) \circ i \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}}.$$

□

Let $A$ be a $W$-algebra. We would like to define a ring homomorphism $m(\bar{\sigma}, A) : A^{\bar{\sigma}} \to A$ such that $m(\bar{\sigma}, A) \circ \iota(\bar{\sigma}, A)$ is a lift of $\bar{\sigma}$. When $A = \mathbb{F}$ we can do this by defining $m(\bar{\sigma}, \mathbb{F})(x \otimes y) = \bar{\sigma}(x)y$. Similarly, when $A = W$ we can define $m(\bar{\sigma}, W)(x \otimes y) = W(\bar{\sigma})(x)y$. If $A = W[T]$ or $W[[T]]$ then $A^{\bar{\sigma}} = W^{\bar{\sigma}}[T]$ or $W^{\bar{\sigma}}[[T]]$, and we can define $m(\bar{\sigma}, A)$ by simply applying $m(\bar{\sigma}, W)$ to the coefficients of the polynomials or power series. However, for a general $W$-algebra $A$ it is not necessarily possible to define $m(\bar{\sigma}, A)$ or to lift $\bar{\sigma}$. (If $A$

happens to be smooth over $W$ then it is always possible to lift $\bar{\sigma}$ to $A$.) Note that by Nakayama's Lemma, if $m(\bar{\sigma}, A)$ exists then $m(\bar{\sigma}, A) \circ \iota(\bar{\sigma}, A)$ is a ring automorphism of $A$.

Fortunately, we do not need $m(\bar{\sigma}, A)$ to exist for all $W$-algebras; just for $\mathbb{I}'$. Our strategy is to prove that if $\bar{\rho}_F^{\bar{\sigma}} \cong \bar{\eta} \otimes \bar{\rho}_F$, then the ring homomorphism $m(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F})$ exists.

**Lemma 3.2.3.** *If $\bar{\rho}_F$ is absolutely irreducible and $\bar{\rho}_F^{\bar{\sigma}} \cong \bar{\eta} \otimes \bar{\rho}_F$ then there is a ring homomorphism $m(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F}) : {}^{\mathcal{O}}R_{\bar{\rho}_F}^{\bar{\sigma}} \to {}^{\mathcal{O}}R_{\bar{\rho}_F}$ that is a lift of $m(\bar{\sigma}, \mathbb{F})$. In particular, $m(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F}) \circ \iota(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F})$ is a lift of $\bar{\sigma}$.*

*Proof.* With notation as in Lemma 3.2.2 define $m(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F}) = (1 \otimes \psi) \circ (1 \otimes \varphi)$. We will show that $1 \otimes \varphi$ induces $m(\bar{\sigma}, \mathbb{F})$ and $1 \otimes \psi$ induces the identity on $\mathbb{F}$. Note that $\mathbb{F}$ is the residue field of $\mathcal{O}$ since $\bar{\chi}$, and hence $\bar{\eta}$, takes values in $\mathbb{F}$. Therefore all of the tensor products with $\mathcal{O}$ residually disappear. Hence it suffices to show that $\varphi$ induces $m(\bar{\sigma}, \mathbb{F})$ and $\psi$ acts trivially on $\mathbb{F}$.

By definition $\mathbb{F}$ is generated by $\{\overline{a(\ell, F)} : \ell \nmid N\}$. Therefore it suffices to check that $\psi$ acts trivially on $\overline{a(\ell, F)}$ for any prime $\ell$ not dividing $N$. But $\psi \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\text{univ}} \cong \eta \otimes \bar{\rho}_F^{\text{univ}}$. Evaluating at $\text{Frob}_\ell$, taking traces, and reducing to the residue field shows that $\psi$ induces the identity on $\mathbb{F}$.

Let $\bar{\varphi} : \mathbb{F} \otimes_{\bar{\sigma}} \mathbb{F} \to \mathbb{F}$ be the residual map induced by $\varphi$. By reducing (3.1) to the residue field we find that $\bar{\sigma} \circ \bar{\rho}_F \cong \bar{\varphi} \circ \iota(\bar{\sigma}, \mathbb{F}) \circ \bar{\rho}_F$. By universality we conclude that $\bar{\sigma} = \bar{\varphi} \circ \iota(\bar{\sigma}, \mathbb{F})$. But $\bar{\sigma} = m(\bar{\sigma}, \mathbb{F}) \circ \iota(\bar{\sigma}, \mathbb{F})$ and hence $\bar{\varphi} = m(\bar{\sigma}, \mathbb{F})$, as desired. □

Define $\Sigma = (1 \otimes \psi) \circ (1 \otimes \varphi) \circ (1 \otimes \iota(\bar{\sigma}, R_{\bar{\rho}_F}))$. By the proof of Lemma 3.2.3 we see that $\Sigma$ is a lift of $\bar{\sigma}$ to ${}^{\mathcal{O}}R_{\bar{\rho}_F}$. In subsection 3.2.2 we use automorphic techniques to descend $\Sigma$ to $\mathbb{I}'$. In order to do so we need the following properties of $\Sigma$.

**Proposition 3.2.4.** 

1. *For all $w \in W$ we have $\Sigma(1 \otimes w) = 1 \otimes W(\bar{\sigma})(w)$.*
2. *For all $x \in \mathcal{O}$ we have $\Sigma(x \otimes 1) = x \otimes 1$.*
3. *The automorphism $\bar{\sigma}$ of $\mathbb{F}$ is necessarily trivial and hence, under the assumption that the order of $\chi$ is a power of 2 and $p \neq 2$, it follows that $\eta$ is a quadratic character.*

*4. The automorphism $\Sigma$ of $R_{\bar{\rho}_F}$ is a lift of $\sigma$.*

*Proof.* The first point is the most subtle. The key point is that $\varphi$ is a *right* $W$-algebra homomorphism. Let $w \in W$. Then

$$(1 \otimes \iota(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F}))(1 \otimes w) = 1 \otimes w \otimes 1 = 1 \otimes 1 \otimes W(\bar{\sigma})(w).$$

Since $\varphi$ is a right $W$-algebra homomorphism and $\psi$ is a $W$-algebra homomorphism we see that $\Sigma(1 \otimes w) = 1 \otimes W(\bar{\sigma})(w)$, as claimed.

The fact that $\Sigma(x \otimes 1) = x \otimes 1$ for all $x \in \mathcal{O}$ follows directly from the definition of $\Sigma$.

The first two facts imply that $W(\bar{\sigma})$ is trivial. Indeed, for any $w \in W$ we have $w \otimes 1 = 1 \otimes w \in {}^{\mathcal{O}}R_{\bar{\rho}_F}$. Therefore by the first two facts, in ${}^{\mathcal{O}}R_{\bar{\rho}_F}$ we have

$$w \otimes 1 = \Sigma(w \otimes 1) = \Sigma(1 \otimes w) = 1 \otimes W(\bar{\sigma})(w) = W(\bar{\sigma})(w) \otimes 1.$$

The ring homomorphism $\mathcal{O} \to {}^{\mathcal{O}}R_{\rho_F}$ is injective since $R_{\bar{\rho}_F}$ covers $\mathbb{I}'$ and $\mathbb{I}' \supset \mathcal{O}$. Therefore $W(\bar{\sigma})$ and hence $\bar{\sigma}$ must be trivial.

Therefore $\bar{\rho}_F \cong \bar{\eta} \otimes \bar{\rho}_F$. Taking determinants we find that $\det \bar{\rho}_F = \bar{\eta}^2 \det \bar{\rho}_F$ and hence $\bar{\eta}$ is quadratic. Therefore the values of $\eta$ are of the form $\pm\zeta$, where $\zeta$ is a $p$-power root of unity. But by assumption $\eta$ takes values in $\mathbb{Z}_p[\chi]$ and $\chi$ has 2-power order. Since $p \neq 2$ it follows that $\eta$ must be quadratic.

In view of the previous parts of the current proposition we see that $\mathcal{O} = W$ and hence ${}^{\mathcal{O}}R_{\bar{\rho}_F} = R_{\bar{\rho}_F}$. Furthermore, the first two maps in the definition of $\Sigma$ become trivial and hence $\Sigma = \psi$. By definition of $\psi$ we have $\psi \circ \bar{\rho}_F^{\text{univ}} \cong \eta \otimes \bar{\rho}_F^{\text{univ}}$. Let $\alpha = \alpha(\rho_F) : R_{\bar{\rho}_F} \to \mathbb{I}'$ and regard $\mathfrak{P}_i' : \mathbb{I}' \to \overline{\mathbb{Q}}_p$ as an algebra homomorphism. Since $\rho_1^{\sigma} \cong \eta \otimes \rho_2$ it follows from the definitions of all maps involved that

$$\sigma \circ \mathfrak{P}_1' \circ \alpha \circ \bar{\rho}_F^{\text{univ}} \cong \mathfrak{P}_2' \circ \alpha \circ \Sigma \circ \bar{\rho}_F^{\text{univ}}.$$

By universality $\sigma \circ \mathfrak{P}_1' \circ \alpha = \mathfrak{P}_2' \circ \alpha \circ \Sigma$ and thus $\Sigma$ is a lift of $\sigma$. $\square$

### 3.2.2 Descending $\Sigma$ to $\mathbb{I}'$ via automorphic methods

To prove Theorem 3.2.1 it now remains to show that $\Sigma$ descends to an automorphism of $\mathbb{I}'$. Let us describe the strategy of proof before proceeding. We begin by showing that the character $\eta$ is unramified at $p$. Once we know this, it is fairly straightforward to check that the irreducible component $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ is modular in the sense that it is an irreducible component of an ordinary Hecke algebra of *some* tame level and nebentypus. We then verify that the tame level and nebentypus of $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ match those for $\operatorname{Spec}\mathbb{I}'$, so we have two irreducible components of the same Hecke algebra. Finally, $\mathfrak{P}_1'$ is an arithmetic point on both $\operatorname{Spec}\mathbb{I}'$ and $\Sigma^*(\operatorname{Spec}\mathbb{I}')$. As the ordinary Hecke algebra is étale over $\Lambda$ at arithmetic points [16, Proposition 3.78], the two irreducible components $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ and $\operatorname{Spec}\mathbb{I}'$ must coincide. In other words, $\Sigma$ descends to $\mathbb{I}'$ as desired. There is a technical point that $\operatorname{Spec}\mathbb{I}'$ and $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ are only irreducible components of the algebra generated by Hecke operators away from $N$, so in order to use étaleness we must associate to $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ a primitive irreducible component $\operatorname{Spec}\mathbb{J}$ of the full Hecke algebra. See the discussion after Corollary 3.2.7.

**Lemma 3.2.5.** *Let $\rho_1, \rho_2 : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be ordinary representations such that the inertia group acts by an infinite order character on the kernel of the unique $p$-unramified quotient of each $\rho_i$. Assume there is an automorphism $\sigma \in G_{\mathbb{Q}_p}$ and a finite order character $\eta$ such that $\rho_1^\sigma \cong \eta \otimes \rho_2$. Then $\eta$ is unramified at $p$.*

*Proof.* Since $\rho_i$ is $p$-ordinary, by choosing bases appropriately we may assume $\rho_i = \left(\begin{smallmatrix} \varepsilon_i & * \\ 0 & \delta_i \end{smallmatrix}\right)$ with $\delta_i$ unramified. By assumption $\varepsilon_i|_{I_p}$ has infinite order. As $\rho_1^\sigma \cong \eta \otimes \rho_2$, it follows that for some $\mathbf{x} \in \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ we have $\rho_1^\sigma = \mathbf{x}(\eta \otimes \rho_2)\mathbf{x}^{-1}$. Write $\mathbf{x} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\eta \otimes \rho_2 = \left(\begin{smallmatrix} \eta\varepsilon_2 & u \\ 0 & \eta\delta_2 \end{smallmatrix}\right)$. A straightforward matrix computation shows that on $I_p$ we have

$$\begin{pmatrix} \varepsilon_1^\sigma & * \\ 0 & 1 \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} (ad\varepsilon_2 - bc)\eta - acu & * \\ c(d\eta(\varepsilon_2 - 1) - cu) & (ad - bc\varepsilon_2)\eta + acu \end{pmatrix}.$$

Hence either $c = 0$ or $cu = d\eta(\varepsilon_2 - 1)$.

If $c = 0$ then on $I_p$ we have

$$\begin{pmatrix} \varepsilon_1^\sigma & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \eta\varepsilon_2 & * \\ 0 & \eta \end{pmatrix},$$

and so $\eta|_{I_p} = 1$, as desired. If $cu = d\eta(\varepsilon_2 - 1)$ then on $I_p$ we have

$$\begin{pmatrix} \varepsilon_1^\sigma & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \eta & * \\ 0 & \eta\varepsilon_2 \end{pmatrix}.$$

Therefore we have $\varepsilon_1^\sigma|_{I_p} = \eta|_{I_p} = \varepsilon_2^{-1}|_{I_p}$. But this is impossible since $\varepsilon_i|_{I_p}$ has infinite order by assumption while $\eta$ has finite order. Therefore $\eta$ must be unramified. □

In what follows we use Wiles's interpretation of Hida families [47]. For each Dirichlet character $\psi$, we shall write $c(\psi) \in \mathbb{Z}^+$ for the conductor of $\psi$. Let $\psi : (\mathbb{Z}/L\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a Dirichlet character. Let $\eta$ be a primitive Dirichlet character with values in $\mathbb{Z}[\chi]$. (Every twist character of $F$ has this property by Lemma 3.2.11.) Denote by $M(\psi, \eta)$ the least common multiple of $L, c(\eta)^2$, and $c(\psi)c(\eta)$. By [42, Proposition 3.64], there is a linear map

$$\begin{aligned} R_{\psi,\eta} : S_k(\Gamma_0(M(\psi,\eta)), \psi) &\to S_k(\Gamma_0(M(\psi,\eta)), \eta^2\psi) \\ f = \sum_{n=1}^{\infty} a(n,f)q^n &\mapsto \eta f = \sum_{n=1}^{\infty} \eta(n)a(n,f)q^n. \end{aligned}$$

We now show that there is an analogous map in the $\mathbb{J}$-adic setting.

**Lemma 3.2.6.** *There is a well defined $\mathbb{J}$-linear map*

$$\begin{aligned} \mathbb{R}_{\chi,\eta} : \mathbb{S}(M(\chi,\eta), \chi; \mathbb{J}) &\to \mathbb{S}(M(\chi,\eta), \eta^2\chi; \mathbb{J}) \\ G = \sum_{n=1}^{\infty} a(n,G)q^n &\mapsto \eta G = \sum_{n=1}^{\infty} \eta(n)a(n,G)q^n. \end{aligned}$$

*If $p \nmid c(\eta)$ then $\mathbb{R}_{\chi,\eta}$ sends $\mathbb{S}^{\mathrm{ord}}(M(\chi,\eta),\chi;\mathbb{J})$ to $\mathbb{S}^{\mathrm{ord}}(M(\chi,\eta),\eta^2\chi;\mathbb{J})$.*

*Proof.* Let $\mathfrak{P}$ be an arithmetic prime of $\mathbb{J}$, and let $P_{k,\varepsilon}$ be the arithmetic prime of $\Lambda$ lying under $\mathfrak{P}$. If $G \in \mathbb{S}^{\mathrm{ord}}(M(\chi,\eta),\chi;\mathbb{J})$ then

$$g_{\mathfrak{P}} \in S_k(\Gamma_0(M(\chi,\eta)p^{r(\varepsilon)}), \varepsilon\chi\omega^{-k}).$$

Let $\psi = \varepsilon\chi\omega^{-k}$. One checks easily from the definitions that $M(\psi,\eta) = M(\chi,\eta)p^{r(\varepsilon)}$. Therefore

$$\eta g_{\mathfrak{P}} = R_{\psi,\eta}(g_{\mathfrak{P}}) \in S_k(\Gamma_0(M(\psi,\eta)), \eta^2\psi) = S_k(\Gamma_0(M(\chi,\eta)p^{r(\varepsilon)}), \eta^2\varepsilon\chi\omega^{-k}),$$

so $\eta G \in \mathbb{S}(M(\chi,\eta), \eta^2\chi; \mathbb{J})$.

For the statement about ordinarity, we may assume $G$ is a normalized eigenform, so $a(p,G)$ is the eigenvalue of $G$ under the $U(p)$ operator. If $G$ is ordinary then $a(p,G) \in \mathbb{J}^\times$. Hence $\eta(p)a(p,G) = a(p,\eta G) \in \mathbb{J}^\times$ if and only if $\eta(p) \neq 0$. □

**Corollary 3.2.7.** *The representation associated to* $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ *is modular of level* $M(\chi,\eta)$ *and nebentypus* $\chi$.

*Proof.* The representation associated to $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ is isomorphic to $\eta \otimes \rho_F$. Consider the formal $q$-expansion $\eta F := \sum_{n=1}^{\infty} \eta(n)a(n,F)q^n \in \mathbb{I}[[q]]$. By Lemma 3.2.6 and Lemma 3.2.5 we see that $\eta F$ is a Hida family of level $\Gamma_0(M(\chi,\eta))$ and nebentypus $\eta^2\chi$. Clearly the Galois representation of $\eta F$ is isomorphic to $\eta \otimes \rho_F$ since their traces on Frobenius elements agree on all but finitely many primes. Since $\eta \otimes \rho_F \cong \alpha \circ \Sigma \circ \bar{\rho}_F^{\mathrm{univ}}$, it follows that $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ is modular of level $M(\chi,\eta)$ and nebentypus $\eta^2\chi$. By Proposition 3.2.4 we know that $\eta$ is quadratic and hence $\eta^2\chi = \chi$. □

For any integer multiple $M$ of $N$, let $\mathbf{h}^{\mathrm{ord}}(M,\chi;\Lambda_\chi)'$ be the $\Lambda_\chi$-subalgebra of $\mathbf{h}^{\mathrm{ord}}(M,\chi;\Lambda_\chi)$ generated by $\{T(n) : (n,N) = 1\}$. Corollary 3.2.7 shows that $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ is an irreducible component of $\operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(M(\chi,\eta),\chi;\Lambda_\chi)'$. There is a natural map $\beta : \operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(M,\chi;\Lambda_\chi) \to \operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(M,\chi;\Lambda_\chi)'$ coming from the natural inclusion of algebras. An irreducible component $\operatorname{Spec}\mathbb{J}'$ of $\mathbf{h}^{\mathrm{ord}}(M,\chi;\Lambda_\chi)'$ essentially corresponds to the data of the fourier coefficients away from $N$. The preimage $\beta^{-1}(\operatorname{Spec}\mathbb{J}')$ is a union of irreducible components whose fourier coefficients agree with those of $\mathbb{J}'$ away from $N$. By the theory of newforms we know that there is a unique primitive irreducible component $\operatorname{Spec}\mathbb{J}$ of $\mathbf{h}^{\mathrm{ord}}(M,\chi;\Lambda_\chi)$ that projects to $\operatorname{Spec}\mathbb{J}'$ under $\beta$. Let $\operatorname{Spec}\mathbb{J}$ be the primitive component of $\mathbf{h}^{\mathrm{ord}}(M(\chi,\eta),\chi;\Lambda_\chi)$ that projects to $\Sigma^*(\operatorname{Spec}\mathbb{I}')$ under $\beta$. By the proof of Corollary 3.2.7, $\mathbb{J}$ is the primitive form associated to $\eta F$ and so $\rho_{\mathbb{J}} \cong \eta \otimes \rho_F$.

Since $N|M(\chi,\eta)$ there is a natural inclusion

$$\operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(N,\chi;\Lambda_\chi)\hookrightarrow\operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(M(\chi,\eta),\chi;\Lambda_\chi).$$

We wish to show that $\operatorname{Spec}\mathbb{J}$ is an irreducible component of $\operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(N,\chi;\Lambda_\chi)$. We do this locally by computing the level of $\operatorname{Spec}\mathbb{J}$ at each prime $\ell$. Let $v_\ell$ denote the usual $\ell$-adic valuation on the integers, normalized such that $v_\ell(\ell)=1$.

**Proposition 3.2.8.** *The primitive component* $\operatorname{Spec}\mathbb{J}$ *is an irreducible component of the Hecke algebra* $\operatorname{Spec}\boldsymbol{h}^{\mathrm{ord}}(N,\chi;\Lambda_\chi)$.

*Proof.* First note that if $\ell\nmid c(\eta)$ then $v_\ell(M(\chi,\eta))=v_\ell(N)$ since $c(\chi)|N$. In particular, by Lemma 3.2.5 we have $v_p(M(\chi,\eta))=v_p(N)$.

Fix a prime $\ell\neq p$ at which $\eta$ is ramified. For a pro-$p$ ring $A$ and representation $\pi:G_{\mathbb{Q}_\ell}\to\mathrm{GL}_2(A)$, let $C_\ell(\pi)$ denote the $\ell$-conductor of $\pi$. See [19, p. 659] for the precise definition. When $\pi$ is the representation associated to a classical form $f$, the $\ell$-conductor of $\pi$ is related to the level of $f$ by the proof of the local Langlands conjecture for $\mathrm{GL}_2$. Indeed, when $f$ is a classical newform of level $N$ we have $C_\ell(\rho_f)=\ell^{v_\ell(N)}$. If $f$ is new away from $p$ and $\ell\neq p$ then we still have $C_\ell(\rho_f)=\ell^{v_\ell(N)}$.

First suppose that $\rho_F|_{I_\ell}$ is not reducible indecomposable. Then $(\eta\otimes\rho_F)|_{I_\ell}$ is not reducible indecomposable either. Therefore $C_\ell(\rho_F)=C_\ell(\rho_{f_{\mathfrak{P}_1}})$ and $C_\ell(\eta\otimes\rho_F)=C_\ell(\eta\otimes\rho_{f_{\mathfrak{P}_2}})$ [19, Lemma 10.2(2)]. Since Galois action does not change conductors we have

$$C_\ell(\rho_F)=C_\ell(\rho_{f_{\mathfrak{P}_1}})=C_\ell(\rho^\sigma_{f_{\mathfrak{P}_1}})=C_\ell(\eta\otimes\rho_{f_{\mathfrak{P}_2}})=C_\ell(\eta\otimes\rho_F).$$

Since $F$ is a primitive form we have that $f_{\mathfrak{P}_1}$ is new away from $p$ and hence $C_\ell(\rho_{f_{\mathfrak{P}_1}})=\ell^{v_\ell(N)}$. On the other hand since $\mathbb{J}$ is primitive we have $C_\ell(\rho_{\mathbb{J}})=C_\ell(\eta\otimes\rho_F)$ is equal to the $\ell$-part of the level of $\mathbb{J}$, which gives the desired result at $\ell$.

Now assume that $\rho_F|_{I_\ell}$ is reducible indecomposable. By [19, Lemma 10.1(4)] we have a character $\psi:G_{\mathbb{Q}_\ell}\to\mathbb{I}^\times$ such that $\rho_F|_{G_{\mathbb{Q}_\ell}}\cong\left(\begin{smallmatrix}\mathcal{N}\psi & *\\ 0 & \psi\end{smallmatrix}\right)$, where $\mathcal{N}$ is the unramified cyclotomic character acting on $p$-power roots of unity and $\psi|_{I_\ell}$ has finite order. Note that since $\eta$ is a quadratic character, $c(\eta)$ is squarefree away from 2. Similarly, since $\chi$ has 2-power order it follows that $c(\chi)$ is a power of 2 times a product of distinct odd primes. Therefore, for odd primes $\ell$ it is enough to show that $c_\ell(\eta)^2 | N$. We use the description of the conductor of a locally reducible indecomposable representation given on [19, p. 660]. Let $\psi_1 = \psi \mod \mathfrak{P}_1$. Then $\rho_{f_{\mathfrak{P}_2}}|_{I_\ell} \cong \left(\begin{smallmatrix} \eta^{-1}\psi_1^\sigma & * \\ 0 & \eta^{-1}\psi_1^\sigma \end{smallmatrix}\right)$. If $\psi_1$ is unramified then $\eta^{-1}\psi_1^\sigma$ is ramified and hence

$$c_\ell(\eta)^2 = c_\ell(\eta^{-1})^2 = c_\ell(\eta^{-1}\psi_1^\sigma)^2 = C_\ell(\rho_{f_{\mathfrak{P}_2}}).$$

Since $\rho_{f_{\mathfrak{P}_2}}$ is a specialization of $F$ we have $C_\ell(\rho_{f_{\mathfrak{P}_2}}) | N$ giving the desired result. Now suppose that $\psi_1$ is ramified. Then $c_\ell(\eta) = \ell | c_\ell(\psi_1)$ and $C_\ell(\rho_{f_{\mathfrak{P}_1}}) = c_\ell(\psi_1)^2$. Again, since $\rho_{f_{\mathfrak{P}_1}}$ is a specialization of $\rho_F$ we see that $c_\ell(\eta)^2 | N$.

Finally the case $\ell = 2$ follows from the assumption that $2c(\chi) | N$. We are able to make this hypothesis by Proposition 3.2.9.

Therefore $\operatorname{Spec} \mathbb{J}$ is an irreducible component of $\mathbf{h}^{\text{ord}}(N, \chi; \Lambda_\chi)$, as desired. $\square$

We now summarize how the results in this section fit together to prove Theorem 3.2.1.

*Proof of Theorem 3.2.1.* We first lift $\sigma$ to an automorphism $\Sigma$ of ${}^{\mathcal{O}}R_{\bar{\rho}_F}$ by Lemma 3.2.3. We are able to use the definition of $\Sigma$ to show that ${}^{\mathcal{O}}R_{\bar{\rho}_F} = R_{\bar{\rho}_F}$ and that $\eta$ is quadratic in Proposition 3.2.4. By Proposition 3.2.8 we see that $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ is a component of $\mathbf{h}^{\text{ord}}(N, \chi; \Lambda_\chi)'$. Since $\rho_{f_{\mathfrak{P}_1}}^\sigma \cong \eta \otimes \rho_{f_{\mathfrak{P}_2}}$ it follows that the arithmetic point $\mathfrak{P}_1'$ is a point on both $\operatorname{Spec} \mathbb{I}'$ and $\Sigma^*(\operatorname{Spec} \mathbb{I}')$. We claim that in fact $\mathfrak{P}_1 \in \operatorname{Spec} \mathbb{I} \cap \operatorname{Spec} \mathbb{J}$.

Note that $\mathbb{J}$ is the primitive family passing through $f_{\mathfrak{P}_1}^\sigma$. (We know $f_{\mathfrak{P}_1}^\sigma$ is primitive since $f_{\mathfrak{P}_1}$ is an arithmetic specialization of the primitive family $F$, and Galois conjugation does not change the level.) Indeed, $\mathbb{J}$ is the primitive form of $\eta F$. Let $\mathfrak{P} \in \operatorname{Spec} \mathbb{J}$ such that $\mathbb{J} \mod \mathfrak{P} = f_{\mathfrak{P}_1}^\sigma$. On the other hand $f_{\mathfrak{P}_1}^\sigma = \sigma(F \mod \mathfrak{P}_1)$ and so the kernel of the specialization map giving rise to $f_{\mathfrak{P}_1}^\sigma$ is $\mathfrak{P}_1$. Therefore $\mathfrak{P} = \mathfrak{P}_1 \in \operatorname{Spec} \mathbb{I} \cap \operatorname{Spec} \mathbb{J}$.

Since $\mathbf{h}^{\text{ord}}(N, \chi; \Lambda_\chi)$ is étale over arithmetic points of $\Lambda$ [16, Proposition 3.78], it follows that the irreducible components $\operatorname{Spec} \mathbb{I}$ and $\operatorname{Spec} \mathbb{J}$ must coincide and hence $\Sigma^*(\operatorname{Spec} \mathbb{I}') = \operatorname{Spec} \mathbb{I}'$. Therefore $\Sigma$ descends to the desired automorphism $\tilde{\sigma}$ of $\mathbb{I}'$. The fact that $\tilde{\sigma}(a(\ell, F)) =$

$\eta(\ell)a(\ell, F)$ for almost all primes $\ell$ follows from specializing $\Sigma \circ \bar{\rho}_F^{\text{univ}} \cong \eta \otimes \bar{\rho}_F^{\text{univ}}$ to $\mathbb{I}'$ and taking traces. Finally, $\sigma \circ \mathfrak{P}_1' = \mathfrak{P}_2' \circ \tilde{\sigma}$ since $\Sigma$ is a lift of $\sigma$ by Proposition 3.2.4. $\square$

### 3.2.3 Nebentypus and twist characters

We end this section with some information about twist characters. In particular Proposition 3.2.9 shows that we may assume from the beginning that $\chi$ has 2-power order with $2c(\chi)|N$.

Note that the ring $\mathbb{I}_0$ depends on $F$. However, if $\psi$ is a character then $\psi F$ has the same group of conjugate self-twists as that of $F$, and thus the same fixed ring $\mathbb{I}_0$. Indeed, if $\sigma$ is a conjugate self-twist of $F$ with character $\eta$, then a straightforward calculation shows that $\psi^\sigma \eta \psi^{-1}$ is the twist character of $\sigma$ on $\psi F$.

**Proposition 3.2.9.** *There is a Dirichlet character $\psi$ such that the nebentypus $\psi^2\chi$ of $\psi F$ has order a power of 2 and $2c(\psi^2\chi)|M(\chi, \psi)$. Furthermore, $\rho_F$ is $\mathbb{I}_0$-full if and only if $\rho_{\psi F}$ is $\mathbb{I}_0$-full.*

*Proof.* It is well known that the nebentypus of $\psi F$ is $\psi^2\chi$ [42, Proposition 3.64]. Write $\chi = \chi_2\xi$, where $\chi_2$ is a character whose order is a power of 2 and $\xi$ is an odd order character. Let $2n-1$ denote the order of $\xi$. Then $\xi^{2n} = \xi$, so taking $\psi_{\text{odd}} = \xi^{-n}$ we see that $\psi_{\text{odd}}^2\chi = \chi_2\xi^{-2n}\xi = \chi_2$ is a character whose order is a power of 2.

Let $2^{t-1}$ be the order of $\psi_{\text{odd}}^2\chi$, and let $\psi_2 : (\mathbb{Z}/2^t\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be the associated primitive character. Let $\psi = \psi_2\psi_{\text{odd}}$. Then $2^{2t}|M(\chi, \psi)$ whereas $c_2(\psi^2\chi)|2^{t-1}$. Since $t \geq 1$ we see that

$$2c_2(\psi^2\chi)|2^t|M(\chi, \psi),$$

as desired.

Suppose that $\rho_{\psi F}$ is $\mathbb{I}_0$-full. Since $\psi$ is a finite order character, $\ker\psi$ is an open subgroup of $G_\mathbb{Q}$. Thus $\rho_{\psi F}|_{\ker\psi}$ is also $\mathbb{I}_0$-full. Note that $\rho_{\psi F}|_{\ker\psi} = \rho_F|_{\ker\psi}$. Thus $\rho_F$ is $\mathbb{I}_0$-full. $\square$

We finish this section by recalling a lemma of Momose that shows that twist characters are valued in $\mathbb{Z}_p[\chi]$. Thus Theorem 3.2.1 says that whenever a conjugate self-twist of a classical specialization $f_{\mathfrak{P}}$ of $F$ induces an automorphism of $\mathbb{Q}_p(f_{\mathfrak{P}})$, that conjugate self-twist can be lifted to a conjugate self-twist of the whole family $F$.

**Lemma 3.2.10** (Lemma 1.5, [34]). *If $\sigma$ is a conjugate self-twist of $f \in S_k(\Gamma_0(N), \chi)$, then $\eta_\sigma$ is the product of a quadratic character with some power of $\chi$. In particular, $\eta_\sigma$ takes values in $\mathbb{Z}[\chi]$.*

The proof of Lemma 3.2.10 is not difficult and goes through without change in the $\mathbb{I}$-adic setting. For completeness, we give the proof in that setting.

**Lemma 3.2.11.** *If $\sigma$ is a conjugate self-twist of $F$ then $\eta_\sigma$ is the product of a quadratic character with some power of $\chi$. In particular, $\eta_\sigma$ has values in $\mathbb{Z}[\chi]$.*

*Proof.* As $\bar{\rho}_F$ is absolutely irreducible, $\rho_F^\sigma \cong \eta_\sigma \otimes \rho_F$. Thus $\sigma(\det \rho_F) = \eta_\sigma^2 \det \rho_F$. Recall that for all primes $\ell$ not dividing $N$ we have

$$\det \rho_F(\mathrm{Frob}_\ell) = \chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1},$$

where $\kappa$ is the map defined in (2.6). Substituting this expression for $\det \rho_F$ into $\sigma(\det \rho_F) = \eta_\sigma^2 \det \rho_F$ yields $\eta_\sigma^2 = \chi^\sigma \chi^{-1}$.

Recall that $\chi^\sigma = \chi^\alpha$ for some integer $\alpha > 0$. To prove the result it suffices to show that there is some $i \in \mathbb{Z}$ such that $\eta_\sigma^2 = \chi^{2i}$. If $\chi$ has odd order then there is a positive integer $j$ for which $\chi = \chi^{2j}$. Thus $\eta_\sigma^2 = \chi^{\sigma-1} = \chi^{2j(\alpha-1)}$. If $\chi$ has even order then $\chi^\sigma$ also has even order since $\sigma$ is an automorphism. Thus $\alpha$ must be odd. Then $\alpha - 1$ is even and $\eta_\sigma^2 = \chi^\sigma \chi^{-1} = \chi^{\alpha-1}$, as desired. $\square$

## 3.3 Sufficiency of open image in product

Recall that $H_0 = \cap_{\sigma\in\Gamma} \ker(\eta_\sigma)$ and $H = H_0 \cap \ker(\det \bar{\rho}_F)$. For a variety of reasons, our methods work best for representations valued in $\mathrm{SL}_2(\mathbb{I}_0)$ rather than $\mathrm{GL}_2(\mathbb{I}')$. Therefore, for the next three sections we assume the following theorem, the proof of which is given in section 3.6.

**Theorem 3.3.1.** *Assume that $\bar{\rho}_F$ is absolutely irreducible and $H_0$-regular. If $V = \mathbb{I}'^2$ is the module on which $G_{\mathbb{Q}}$ acts via $\rho_F$, then there is a basis for $V$ such that all of the following happen simultaneously:*

1. *$\rho_F$ is valued in $\mathrm{GL}_2(\mathbb{I}')$;*
2. *$\rho_F|_{D_p}$ is upper triangular;*
3. *$\rho_F|_{H_0}$ is valued in $\mathrm{GL}_2(\mathbb{I}_0)$;*
4. *There is a matrix $\boldsymbol{j} = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$, where $\zeta$ and $\zeta'$ are roots of unity, such that $\boldsymbol{j}$ normalizes the image of $\rho_F$ and $\zeta \not\equiv \zeta' \bmod p$.*

Let $H' = \ker(\det \bar{\rho}_F)$. For any $h \in H'$ we have $\det \rho_F(h) \in 1 + \mathfrak{m}_{\mathbb{I}'}$. Since $p \neq 2$ and $\mathbb{I}'$ is $p$-adically complete, we have

$$\sqrt{\det \rho_F(h)} = \sum_{n=0}^{\infty} \binom{1/2}{n} (\det \rho_F(h) - 1)^n \in \mathbb{I}'^{\times}.$$

Since $\rho_F$ is a 2-dimensional representation $\rho_F|_{H'} \otimes \sqrt{\det \rho_F|_{H'}}^{-1}$ takes values in $\mathrm{SL}_2(\mathbb{I}')$. Restricting further it follows from Theorem 3.3.1 that

$$\rho := \rho_F|_H \otimes \sqrt{\det \rho_F|_H}^{-1}$$

takes values in $\mathrm{SL}_2(\mathbb{I}_0)$. Note that the image of $\rho$ is still normalized by the matrix $\mathbf{j}$ of Theorem 3.3.1 since we only modified $\rho_F$ by scalars, which commute with $\mathbf{j}$. In Proposition 3.3.10 we show that $\rho_F$ is $\mathbb{I}_0$-full if and only if $\rho$ is $\mathbb{I}_0$-full. The proof of Proposition 3.3.10 is postponed until the end of the current section since it uses the theory of Pink-Lie algebras developed below. In the next three sections we prove that $\rho$ is $\mathbb{I}_0$-full.

The purpose of the current section is to make the following reduction step in the proof of Theorem 3.1.4.

**Proposition 3.3.2.** *Assume there is an arithmetic prime $P$ of $\Lambda$ such that the image of $\mathrm{Im}\,\rho$ in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open in the product topology. Then $\rho$ (and hence $\rho_F$) is $\mathbb{I}_0$-full.*

In the proof we use a result of Pink [35] that classifies $p$-profinite subgroups of $\mathrm{SL}_2(A)$ for a complete semilocal $p$-profinite ring $A$. (Our assumption that $p > 2$ is necessary for Pink's theory.) We give a brief exposition of the relevant parts of his work for the sake of establishing notation. Define

$$\begin{aligned}\Theta : \mathrm{SL}_2(A) &\to \mathfrak{sl}_2(A)\\ \mathbf{x} &\mapsto \mathbf{x} - \frac{1}{2}\operatorname{tr}(\mathbf{x}),\end{aligned}$$

where we consider $\frac{1}{2}\operatorname{tr}(\mathbf{x})$ as a scalar matrix. Let $\mathcal{G}$ be a $p$-profinite subgroup of $\mathrm{SL}_2(A)$. Define $L_1(\mathcal{G})$ to be the closed subgroup of $\mathfrak{sl}_2(A)$ that is topologically generated by $\Theta(\mathcal{G})$. Let $L_1 \cdot L_1$ be the closed (additive) subgroup of $M_2(A)$ topologically generated by $\{\mathbf{xy} : \mathbf{x}, \mathbf{y} \in \mathcal{G}\}$. Let $C$ denote $\operatorname{tr}(L_1 \cdot L_1)$. Sometimes we will view $C \subset M_2(A)$ as a set of scalar matrices. For $n \geq 2$ define $L_n(\mathcal{G})$ to be the closed (additive) subgroup of $\mathfrak{sl}_2(A)$ generated by

$$[L_1(\mathcal{G}), L_{n-1}(\mathcal{G})] := \{\mathbf{xy} - \mathbf{yx} : \mathbf{x} \in L_1(\mathcal{G}), \mathbf{y} \in L_{n-1}(\mathcal{G})\}.$$

**Definition 3.3.3.** The *Pink-Lie algebra* of a $p$-profinite group $\mathcal{G}$ is $L_2(\mathcal{G})$. Whenever we write $L(\mathcal{G})$ without a subscript we shall always mean $L_2(\mathcal{G})$.

As an example one can compute that for an ideal $\mathfrak{a}$ of $A$, the $p$-profinite subgroup $\mathcal{G} = \Gamma_A(\mathfrak{a})$ has Pink-Lie algebra $L_2(\mathcal{G}) = \mathfrak{a}^2\mathfrak{sl}_2(A)$. This example plays an important role in what follows.

For $n \geq 1$, define

$$\begin{aligned}\mathcal{M}_n(\mathcal{G}) &= C \oplus L_n(\mathcal{G}) \subset M_2(A)\\ \mathcal{H}_n(\mathcal{G}) &= \{\mathbf{x} \in \mathrm{SL}_2(A) : \Theta(\mathbf{x}) \in L_n(\mathcal{G}) \text{ and } \operatorname{tr}(\mathbf{x}) - 2 \in C\}.\end{aligned}$$

Pink proves that $\mathcal{M}_n(\mathcal{G})$ is a closed $\mathbb{Z}_p$-Lie subalgebra of $M_2(A)$ and $\mathcal{H}_n = \mathrm{SL}_2(A) \cap (1 + \mathcal{M}_n)$ for all $n \geq 1$. Furthermore, write

$$\mathcal{G}_1 = \mathcal{G}, \mathcal{G}_{n+1} = (\mathcal{G}, \mathcal{G}_n),$$

where $(\mathcal{G}, \mathcal{G}_n)$ is the closed subgroup of $\mathcal{G}$ topologically generated by the commutators $\{g g_n g^{-1} g_n^{-1} : g \in \mathcal{G}, g_n \in \mathcal{G}_n\}$. Pink proves the following theorem.

**Theorem 3.3.4** (Pink [35]). *With notation as above, $\mathcal{G}$ is a closed normal subgroup of $\mathcal{H}_1(\mathcal{G})$. Furthermore, $\mathcal{H}_n(\mathcal{G}) = (\mathcal{G}, \mathcal{G}_n)$ for $n \geq 2$.*

There are two important functoriality properties of the correspondence $\mathcal{G} \mapsto L(\mathcal{G})$ that we will use. First, since $\Theta$ is constant on conjugacy classes of $\mathcal{G}$ it follows that $L_n(\mathcal{G})$ is stable under the adjoint action of the normalizer $N_{\mathrm{SL}_2(A)}(\mathcal{G})$ of $\mathcal{G}$ in $\mathrm{SL}_2(A)$. That is, for $\mathbf{g} \in N_{\mathrm{SL}_2(A)}(\mathcal{G}), \mathbf{x} \in L_n(\mathcal{G})$ we have $\mathbf{g}\mathbf{x}\mathbf{g}^{-1} \in L_n(\mathcal{G})$. If $\mathfrak{a}$ is an ideal of $A$ such that $A/\mathfrak{a}$ is $p$-profinite, then we write $\overline{\mathcal{G}}_{\mathfrak{a}}$ for the $p$-profinite group $\mathcal{G} \cdot \Gamma_A(\mathfrak{a})/\Gamma_A(\mathfrak{a}) \subseteq \mathrm{SL}_2(A/\mathfrak{a})$. The second functoriality property is that the canonical linear map $L(\mathcal{G}) \to L(\overline{\mathcal{G}}_{\mathfrak{a}})$ induced by $\mathbf{x} \mapsto \mathbf{x} \bmod \mathfrak{a}$ is surjective.

Let $\mathfrak{m}_0$ be the maximal ideal of $\mathbb{I}_0$, and let $\mathbb{G}$ denote the $p$-profinite group $\mathrm{Im}\,\rho \cap \Gamma_{\mathbb{I}_0}(\mathfrak{m}_0)$. The proof of Proposition 3.3.2 consists of showing that if $\overline{\mathbb{G}}_{P\mathbb{I}_0}$ is open in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ then $\mathbb{G}$ contains $\Gamma_{\mathbb{I}_0}(\mathfrak{a}_0)$ for some nonzero $\mathbb{I}_0$-ideal $\mathfrak{a}_0$. Let $L = L(\mathbb{G})$ be the Pink-Lie algebra of $\mathbb{G}$. Since $\overline{\mathbb{G}}_{P\mathbb{I}_0}$ is open, for every prime $\mathcal{Q}$ of $\mathbb{I}_0$ lying over $P$ there is a nonzero $\mathbb{I}_0/\mathcal{Q}$-ideal $\overline{\mathfrak{a}}_{\mathcal{Q}}$ such that

$$\overline{\mathbb{G}}_{P\mathbb{I}_0} \supseteq \prod_{\mathcal{Q}|P} \Gamma_{\mathbb{I}_0/\mathcal{Q}}(\overline{\mathfrak{a}}_{\mathcal{Q}}).$$

Thus $L(\overline{\mathbb{G}}_{P\mathbb{I}_0}) \supseteq \oplus_{\mathcal{Q}|P} \overline{\mathfrak{a}}_{\mathcal{Q}}^2 \mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q})$.

Recall from Theorem 3.3.1 that we have roots of unity $\zeta$ and $\zeta'$ such that $\zeta \not\equiv \zeta' \bmod p$ and the matrix $\mathbf{j} := \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$ normalizes $\mathbb{G}$. Let $\alpha = \zeta\zeta'^{-1}$. A straightforward calculation shows that the eigenvalues of $\mathrm{Ad}(\mathbf{j})$ acting on $\mathfrak{sl}_2(\mathbb{I}_0)$ are $\alpha, 1, \alpha^{-1}$. Note that since $\zeta \neq \zeta'$ either all of $\alpha, 1, \alpha^{-1}$ are distinct or else $\alpha = -1$. For $\lambda \in \{\alpha, 1, \alpha^{-1}\}$ let $L[\lambda]$ be the $\lambda$-eigenspace of $\mathrm{Ad}(\mathbf{j})$ acting on $L$. One computes that $L[1]$ is the set of diagonal matrices in $L$. If $\alpha = -1$ then $L[-1]$ is the set of antidiagonal matrices in $L$. If $\alpha \neq -1$ then $L[\alpha]$ is the set of upper nilpotent matrices in $L$, and $L[\alpha^{-1}]$ is the set of lower nilpotent matrices in $L$. Regardless of the value of $\alpha$, let $\mathfrak{u}$ denote the set of upper nilpotent matrices in $L$ and $\mathfrak{u}^t$ denote the set of lower nilpotent matrices in $L$. Let $\mathcal{L}$ be the $\mathbb{Z}_p$-Lie algebra generated by $\mathfrak{u}$ and $\mathfrak{u}^t$ in $\mathfrak{sl}_2(\mathbb{I}_0)$.

**Lemma 3.3.5.** *The matrix $\boldsymbol{J} := \left(\begin{smallmatrix} 1+T & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes $\mathrm{Im}\,\rho$, and $\mathcal{L}$ is a $\Lambda$-submodule of $\mathfrak{sl}_2(\mathbb{I}_0)$.*

*Proof.* First we show that $\mathcal{L}$ is a $\Lambda$-module assuming that $\mathbf{J}$ normalizes $\operatorname{Im}\rho$. Since $\mathcal{L}$ is a $\mathbb{Z}_p$-Lie algebra and $\Lambda = \mathbb{Z}_p[[T]]$, it suffices to show that $\mathbf{x} \in \mathcal{L}$ implies $T\mathbf{x} \in \mathcal{L}$. If $\mathbf{x} \in \mathfrak{u}$ then a simple computation shows that $\mathbf{J}\mathbf{x}\mathbf{J}^{-1} = (1+T)\mathbf{x}$. As $L$ is an abelian group it follows that $T\mathbf{x} = (1+T)\mathbf{x} - \mathbf{x} \in \mathfrak{u}$. Similarly, for $\mathbf{y} \in \mathfrak{u}^t$ we have $T\mathbf{y} \in \mathfrak{u}^t$. It follows that $T[\mathbf{x}, \mathbf{y}] = [T\mathbf{x}, \mathbf{y}] \in \mathcal{L}$. Any element in $\mathcal{L}$ can be written as a sum of elements in $\mathfrak{u}, \mathfrak{u}^t$, and $[\mathfrak{u}, \mathfrak{u}^t]$. Therefore $\mathcal{L}$ is a $\Lambda$-submodule of $\mathfrak{sl}_2(\mathbb{I}_0)$.

Now we show that $\mathbf{J}$ normalizes $\operatorname{Im}\rho$. The proof is nearly identical to the proof of [19, Lemma 1.4] except we do not require $\zeta, \zeta' \in \mathbb{Z}_p$. By [17, Theorem 4.3.2], there is an element $\boldsymbol{\tau} = \left(\begin{smallmatrix} 1+T & u \\ 0 & 1 \end{smallmatrix}\right) \in \operatorname{Im}\rho_F$. A straightforward matrix calculation shows that $\boldsymbol{\tau} \in \operatorname{Im}\rho_F|_H$. Writing $t = (1+T)^{1/2}$ and $u' = t^{-1}u$ we see that $\boldsymbol{\tau'} = \left(\begin{smallmatrix} t & u' \\ 0 & t^{-1} \end{smallmatrix}\right) \in \operatorname{Im}\rho$. Since $\rho_F|_H$ and $\rho$ differ only by a character, their images have the same normalizer. In particular, the matrix $\mathbf{j}$ from Theorem 3.3.1 normalizes $\operatorname{Im}\rho$. Hence the commutator $(\boldsymbol{\tau'}, \mathbf{j}) \in \operatorname{Im}\rho$ and we can compute

$$(\boldsymbol{\tau'}, \mathbf{j}) = \begin{pmatrix} 1 & u't(1-\alpha) \\ 0 & 1 \end{pmatrix}.$$

Let $\mathfrak{v} = \left\{x \in \mathbb{I}_0 : \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{u}\right\}$. Then $\mathfrak{v}$ is a $\mathbb{Z}_p[\alpha]$-module. Indeed, it is a $\mathbb{Z}_p$-module since we can raise unipotent matrices to $\mathbb{Z}_p$-powers, so it suffices to show that $\mathfrak{v}$ is closed under multiplication by $\alpha$. This follows by conjugating unipotent elements by $\mathbf{j}$. Since $\alpha \not\equiv 1 \bmod p$ we have that $1-\alpha$ is a unit in $\mathbb{Z}_p[\alpha]$. Therefore $u't \in \mathfrak{v}$. Let $\boldsymbol{\beta} = \boldsymbol{\tau'}^{-1}\left(\begin{smallmatrix} 1 & u't \\ 0 & 1 \end{smallmatrix}\right)\boldsymbol{\tau'} \in \operatorname{Im}\rho$. Then $t^{-1}\mathbf{J} = \boldsymbol{\tau'}\boldsymbol{\beta}^{-1}$ (and hence $\mathbf{J}$) normalizes $\operatorname{Im}\rho$. $\square$

The proof of Proposition 3.3.2 depends on whether or not $\alpha = -1$; it is easier when $\alpha \neq -1$.

*Proof of Proposition 3.3.2 when $\alpha \neq -1$.* We will show that the finitely generated $\Lambda$-module

$$X := \mathfrak{sl}_2(\mathbb{I}_0)/\mathcal{L}$$

is a torsion $\Lambda$-module. From this it follows that there is a nonzero $\Lambda$-ideal $\mathfrak{a}$ such that $\mathfrak{a}\mathfrak{sl}_2(\mathbb{I}_0) \subseteq \mathcal{L}$. Thus

$$(\mathfrak{a}\mathbb{I}_0)^2\mathfrak{sl}_2(\mathbb{I}_0) \subseteq \mathcal{L} \subseteq L$$

since $\mathbb{I}_0\mathfrak{sl}_2(\mathbb{I}_0) = \mathfrak{sl}_2(\mathbb{I}_0)$. But $(\mathfrak{a}\mathbb{I}_0)^2\mathfrak{sl}_2(\mathbb{I}_0)$ is the Pink-Lie algebra of $\Gamma_{\mathbb{I}_0}(\mathfrak{a}\mathbb{I}_0)$ and so $\Gamma_{\mathbb{I}_0}(\mathfrak{a}\mathbb{I}_0) \subseteq \mathbb{G}_2 \subseteq \mathbb{G}$, as desired.

To show that $X$ is a finitely generated $\Lambda$-module, recall that the arithmetic prime $P$ in the statement of Proposition 3.3.2 is a height one prime of $\Lambda$. By Nakayama's Lemma it suffices to show that $X/PX$ is $\Lambda/P$-torsion. The natural epimorphism $\mathfrak{sl}_2(\mathbb{I}_0)/P\mathfrak{sl}_2(\mathbb{I}_0) \twoheadrightarrow X/PX$ has kernel $\mathcal{L} \cdot P\mathfrak{sl}_2(\mathbb{I}_0)/P\mathfrak{sl}_2(\mathbb{I}_0)$, so

$$X/PX \cong \mathfrak{sl}_2(\mathbb{I}_0/P\mathbb{I}_0)/(\mathcal{L} \cdot P\mathfrak{sl}_2(\mathbb{I}_0)/P\mathfrak{sl}_2(\mathbb{I}_0)).$$

We use the following notation:

$$\begin{aligned} \overline{L} = L(\overline{\mathbb{G}}_{P\mathbb{I}_0}) &: \text{the Pink-Lie algebra of } \overline{\mathbb{G}}_{P\mathbb{I}_0} \\ \overline{L}[\lambda] &: \text{the } \lambda\text{-eigenspace of } \operatorname{Ad}(\mathbf{j}) \text{ on } \overline{L}, \text{ for } \lambda \in \{\alpha, 1, \alpha^{-1}\} \\ \overline{\mathcal{L}} &: \text{the } \mathbb{Z}_p\text{-algebra generated by } \overline{L}[\alpha] \text{ and } \overline{L}[\alpha^{-1}] \end{aligned}$$

The functoriality of Pink's construction implies that the canonical surjection $\mathbb{I}_0 \twoheadrightarrow \mathbb{I}_0/P\mathbb{I}_0$ induces surjections

$$L[\lambda] \twoheadrightarrow \overline{L}[\lambda]$$

for all $\lambda \in \{\alpha, 1, \alpha^{-1}\}$. Therefore the canonical linear map $\mathcal{L} \to \overline{\mathcal{L}}$ is also a surjection. That is, $\mathcal{L} \cdot P\mathfrak{sl}_2(\mathbb{I}_0)/P\mathfrak{sl}_2(\mathbb{I}_0) = \overline{\mathcal{L}}$ and so $X/PX \cong \mathfrak{sl}_2(\mathbb{I}_0/P\mathbb{I}_0)/\overline{\mathcal{L}}$. Since $\overline{\mathbb{G}}_{P\mathbb{I}_0} \supseteq \prod_{\mathcal{Q}|P} \Gamma_{\mathbb{I}_0/\mathcal{Q}}(\overline{\mathfrak{a}}_{\mathcal{Q}})$, it follows that

$$\begin{aligned} \overline{L}[\alpha] &\supseteq \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} | x \in \oplus_{\mathcal{Q}|P}\overline{\mathfrak{a}}_{\mathcal{Q}}^2 \right\} \\ \overline{L}[\alpha^{-1}] &\supseteq \left\{ \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} | x \in \oplus_{\mathcal{Q}|P}\overline{\mathfrak{a}}_{\mathcal{Q}}^2 \right\}. \end{aligned}$$

Since $\alpha \neq -1$ we have $\mathfrak{u} = \overline{L}[\alpha]$ and $\mathfrak{u}^t = \overline{L}[\alpha^{-1}]$. Therefore

$$\overline{\mathcal{L}} \supseteq \oplus_{\mathcal{Q}|P}\overline{\mathfrak{a}}_{\mathcal{Q}}^4\mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q}).$$

Since each $\overline{\mathfrak{a}}_{\mathcal{Q}}$ is a nonzero $\mathbb{I}_0/\mathcal{Q}$-ideal, it follows that $\oplus_{\mathcal{Q}|P}\mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q})/\overline{\mathfrak{a}}_{\mathcal{Q}}^4\mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q})$ is $\Lambda/P$-torsion. Finally, the inclusions

$$\oplus_{\mathcal{Q}|P}\overline{\mathfrak{a}}_{\mathcal{Q}}^4\mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q}) \subseteq \overline{\mathcal{L}} \subseteq \mathfrak{sl}_2(\mathbb{I}_0/P\mathbb{I}_0) \subseteq \oplus_{\mathcal{Q}|P}\mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q}\mathbb{I})$$

show that $\mathfrak{sl}_2(\mathbb{I}_0/P\mathbb{I}_0)/\overline{\mathcal{L}} \cong X/PX$ is $\Lambda/P$-torsion. □

Let

$$\mathfrak{v} = \left\{ v \in \mathbb{I}_0 : \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{u} \right\} \text{ and } \mathfrak{v}^t = \left\{ v \in \mathbb{I}_0 : \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in \mathfrak{u}^t \right\}.$$

**Definition 3.3.6.** A $\Lambda$-*lattice* in $Q(\mathbb{I}_0)$ is a finitely generated $\Lambda$-submodule $M$ of $Q(\mathbb{I}_0)$ such that the $Q(\Lambda)$-span of $M$ is equal to $Q(\mathbb{I}_0)$. If in addition $M$ is a subring of $\mathbb{I}_0$ then we say $M$ is a $\Lambda$-*order*.

*Proof of Proposition 3.3.2 when $\alpha = -1$.* We show in Lemmas 3.3.7 and 3.3.8 that $\mathfrak{v}$ and $\mathfrak{v}^t$ are $\Lambda$-lattices in $Q(\mathbb{I}_0)$. To do this we use the fact that the local Galois representation $\rho_F|_{D_p}$ is indecomposable [9, 49].

We then show in Proposition 3.3.9 that any $\Lambda$-lattice in $Q(\mathbb{I}_0)$ contains a nonzero $\mathbb{I}_0$-ideal. Let $\mathfrak{b}$ and $\mathfrak{b}^t$ be nonzero $\mathbb{I}_0$-ideals such that $\mathfrak{b} \subseteq \mathfrak{v}$ and $\mathfrak{b}^t \subseteq \mathfrak{v}^t$. Let $\mathfrak{a}_0 = \mathfrak{b}\mathfrak{b}^t$. Then from the definitions of $\mathfrak{v}, \mathfrak{v}^t$, and $\mathcal{L}$, we find that

$$\mathcal{L} \supseteq \mathfrak{a}_0^2 \mathfrak{sl}_2(\mathbb{I}_0).$$

By Pink's theory it follows that $\mathbb{G} \supseteq \Gamma_{\mathbb{I}_0}(\mathfrak{a}_0)$. □

Finally, we prove the three key facts used in the proof of Proposition 3.3.2 when $\alpha = -1$.

**Lemma 3.3.7.** *With notation as above, $\mathfrak{v}$ is a $\Lambda$-lattice in $Q(\mathbb{I}_0)$.*

*Proof.* Let $\overline{L} = L(\overline{\mathbb{G}}_{P\mathbb{I}_0})$. Recall that $L[1]$ surjects onto $\overline{L}[1]$. Now $\overline{L}[1]$ contains

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \oplus_{\mathcal{Q}|P} \overline{\mathfrak{a}}_{\mathcal{Q}}^2 \right\},$$

and $\oplus_{\mathcal{Q}|P} \overline{\mathfrak{a}}_{\mathcal{Q}}^2$ is a $\Lambda/P$-lattice in $Q(\mathbb{I}_0/P\mathbb{I}_0)$. It follows from Nakayama's Lemma that the set of entries in the matrices of $L[1]$ contains a $\Lambda$-lattice $\mathfrak{a}$ for $Q(\mathbb{I}_0)$.

By a theorem of Ghate-Vatsal [9] (later generalized by Hida [18] and Zhao [49]) we know that $\rho_F|_{D_p}$ is indecomposable. Hence there is a matrix in the image of $\rho$ whose upper right entry is nonzero. This produces a nonzero nilpotent matrix in $L_1$. Taking the Lie bracket of this matrix with a nonzero element of $L[1]$ produces a nonzero nilpotent matrix in $L$ which we will call $\left(\begin{smallmatrix} 0 & v \\ 0 & 0 \end{smallmatrix}\right)$. Note that for any $a \in \mathfrak{a}$ we have

$$\begin{pmatrix} 0 & 2av \\ 0 & 0 \end{pmatrix} = \left[ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \right] \in L.$$

Thus the lattice $\mathfrak{a}v$ is contained in $\mathfrak{v}$, so $Q(\Lambda)\mathfrak{v} = Q(\mathbb{I}_0)$. The fact that $\mathfrak{v}$ is finitely generated follows from the fact that $\Lambda$ is noetherian and $\mathfrak{v}$ is contained in the finitely generated $\Lambda$-module $\mathbb{I}_0$. □

**Lemma 3.3.8.** *With notation as above, $\mathfrak{v}^t$ is a $\Lambda$-lattice in $Q(\mathbb{I}_0)$.*

*Proof.* Let $\bar{c} \in \oplus_{\mathcal{Q}|P} \overline{\mathfrak{a}}_{\mathcal{Q}}^2$. Since $L[-1]$ surjects to $\overline{L}[-1]$ there is some $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in L$ such that $b \in P\mathbb{I}_0$ and $c \bmod P\mathbb{I}_0 = \bar{c}$. Since $\mathfrak{v}$ is a $\Lambda$-lattice in $Q(\mathbb{I}_0)$ by Lemma 3.3.7, it follows that there is some nonzero $\alpha \in \Lambda$ such that $\alpha b \in \mathfrak{v}$.

We claim that there is some nonzero $\beta \in \Lambda$ for which $\left(\begin{smallmatrix} 0 & \alpha b \\ \beta c & 0 \end{smallmatrix}\right) \in L$. Assuming the existence of $\beta$, since $\alpha b \in \mathfrak{v}$ it follows that $\beta c \in \mathfrak{v}^t$. That is, $c \in Q(\Lambda)\mathfrak{v}^t$. Since $\bar{c}$ runs over $\oplus_{\mathcal{Q}|P} \overline{\mathfrak{a}}_{\mathcal{Q}}^2$, it follows from Nakayama's Lemma that $\mathfrak{v}^t$ is a $\Lambda$-lattice in $Q(\mathbb{I}_0)$.

To see that $\beta$ exists, recall that $L$ is normalized by the matrix $\mathbf{J} = \left(\begin{smallmatrix} 1+T & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by Lemma 3.3.5. Thus

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & Tb \\ ((1+T)^{-1} - 1)c & 0 \end{pmatrix} = \begin{pmatrix} 1+T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \begin{pmatrix} (1+T)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in L.$$

Write $\alpha = f(T)$ as a power series in $T$. Since $(1+T)^{-1} - 1$ is divisible by $T$, we can evaluate $f$ at $(1+T)^{-1} - 1$ to get another element of $\mathbb{Z}_p[[T]]$. Taking $\beta = f((1+T)^{-1} - 1)$, the above calculation shows that

$$\begin{pmatrix} 0 & \alpha b \\ \beta c & 0 \end{pmatrix} \in L,$$

as desired. □

**Proposition 3.3.9.** *Every $\Lambda$-lattice in $Q(\mathbb{I}_0)$ contains a nonzero $\mathbb{I}_0$-ideal.*

*Proof.* Let $M$ be a $\Lambda$-lattice in $Q(\mathbb{I}_0)$. Define

$$R = \{x \in \mathbb{I}_0 : xM \subseteq M\}.$$

Then $R$ is a subring of $\mathbb{I}_0$ that is also a $\Lambda$-lattice for $Q(\mathbb{I}_0)$. Thus $R$ is a $\Lambda$-order in $\mathbb{I}_0$, and $M$ is an $R$-module. Therefore

$$\mathfrak{c} := \{x \in \mathbb{I}_0 : x\mathbb{I}_0 \subseteq R\}$$

is a nonzero $\mathbb{I}_0$-ideal. Note that $Q(R) = Q(\mathbb{I}_0) = Q(\Lambda)M$. Since $M$ is a finitely generated $\Lambda$-module there is some nonzero $r \in \mathbb{I}_0$ such that $rM \subseteq R$. As $rM$ is still a $\Lambda$-lattice for $Q(\mathbb{I}_0)$, by replacing $M$ with $rM$ we may assume that $M$ is an $R$-ideal.

Now consider $\mathfrak{a} = \mathfrak{c} \cdot (M\mathbb{I}_0)$, where $M\mathbb{I}_0$ is the ideal generated by $M$ in $\mathbb{I}_0$. Note that $\mathfrak{a}$ is a nonzero $\mathbb{I}_0$-ideal since both $\mathfrak{c}$ and $M\mathbb{I}_0$ are nonzero $\mathbb{I}_0$-ideals. To see that $\mathfrak{a} \subseteq M$, let $x \in \mathbb{I}_0$ and $c \in \mathfrak{c}$. Then $xc \in R$ by definition of $\mathfrak{c}$. If $a \in M$ then $xca \in M$ since $M$ is an $R$-ideal. Thus $xca \in M$, so $\mathfrak{a} \subseteq M$. □

*Remark* 2. Note that the only property of $\mathbb{I}_0$ that is used in the proof of Proposition 3.3.9 is that $\mathbb{I}_0$ is a $\Lambda$-order in $Q(\mathbb{I}_0)$. Thus, once we have shown that $\rho$ (or $\rho_F$) is $\mathbb{I}_0$-full, it follows that the representation is $R$-full for *any* $\Lambda$-order $R$ in $Q(\mathbb{I}_0)$. In particular, if $\tilde{\mathbb{I}}_0$ is the maximal $\Lambda$-order in $Q(\mathbb{I}_0)$ then $\rho_F$ is $\tilde{\mathbb{I}}_0$-full.

Finally, we show that for the purposes of proving $\mathbb{I}_0$-fullness it suffices to work with $\rho$ instead of $\rho_F$.

**Proposition 3.3.10.** *The representation $\rho_F$ is $\mathbb{I}_0$-full if and only if $\rho$ is $\mathbb{I}_0$-full.*

*Proof.* Note that $\operatorname{Im} \rho_F|_{H_0} \cap \mathrm{SL}_2(\mathbb{I}_0) \subseteq \operatorname{Im} \rho$ by definition. Thus if $\rho_F$ is $\mathbb{I}_0$-full then so is $\rho$.

Now assume that $\rho$ is $\mathbb{I}_0$-full. As in the proof of [19, Theorem 8.2], let $\Gamma = \{(1+T)^s : s \in \mathbb{Z}_p\}$ and

$$\mathbb{K} = \{x \in \rho_F(H_0) : \det x \in \Gamma\}.$$

Note that $\mathbb{K}$ is a finite index subgroup of $\operatorname{Im}\rho_F$. Since $F$ is ordinary and non-CM we can find an element of the form $\boldsymbol{\tau} = \left(\begin{smallmatrix} 1+T & u \\ 0 & 1 \end{smallmatrix}\right) \in \operatorname{Im}\rho_F$ [17, Theorem 4.3.2]. Let $n = [G_{\mathbb{Q}} : H_0]$. By replacing $\Gamma$ with $\{(1+T)^{ns} : s \in \mathbb{Z}_p\}$ and $\boldsymbol{\tau}$ with $\boldsymbol{\tau}^n$, we may assume that $\boldsymbol{\tau} \in \mathbb{K}$.

Let $\mathbb{S} = \mathbb{K} \cap \mathrm{SL}_2(\mathbb{I}_0)$ and $\mathcal{T} = \{\boldsymbol{\tau}^s : s \in \mathbb{Z}_p\}$. Then we can write $\mathbb{K}$ as a semidirect product

$$\mathbb{K} = \mathcal{T} \ltimes \mathbb{S}.$$

Indeed, given $\mathbf{x} \in \mathbb{K}$ there is a unique $s \in \mathbb{Z}_p$ such that $\det \mathbf{x} = (1+T)^s$. Thus we identify $\mathbf{x}$ with $(\boldsymbol{\tau}^s, \boldsymbol{\tau}^{-s}\mathbf{x}) \in \mathcal{T} \ltimes \mathbb{S}$.

Let $\mathbb{K}'$ be the image of $\mathbb{K}$ under the natural map

$$\begin{aligned} \Phi : \mathbb{K} &\to \operatorname{Im}\rho \\ \mathbf{x} &\mapsto \mathbf{x}(\det \mathbf{x})^{-1/2}. \end{aligned}$$

Then $\mathbb{K}'$ is a finite index subgroup of $\operatorname{Im}\rho$ and therefore contains $\Gamma_{\mathbb{I}_0}(\mathfrak{a})$ for some nonzero $\mathbb{I}_0$-ideal $\mathfrak{a}$ since $\rho$ is $\mathbb{I}_0$-full. Note that $\ker\Phi$ is precisely the set of scalar matrices in $\mathbb{K}$. Therefore, for some $0 \leq r \leq \infty$,

$$\ker\Phi \cong \{(1+T)^s : s \in p^r\mathbb{Z}_p\},$$

where $r = \infty$ means $\ker\Phi = \{1\}$. If $r \neq \infty$ then by passing to finite index subgroups of $\mathbb{K}, \mathbb{K}'$, and $\Gamma$ we may assume that $\ker\Phi = \Gamma$. Thus, given any $\mathbf{y} \in \Gamma_{\mathbb{I}_0}(\mathfrak{a})$ we can find $\mathbf{x} \in \mathbb{K}$ such that $\Phi(\mathbf{x}) = \mathbf{y}$. Let $s \in \mathbb{Z}_p$ such that $\det \mathbf{x} = (1+T)^{s/2}$. Then the scalar matrix $(1+T)^{-s/2}$ is in $\Gamma$ hence in $\mathbb{K}$. Hence $\mathbf{x}(1+T)^{-s/2} \in \mathbb{S}$ and $\Phi(\mathbf{x}(1+T)^{s/2}) = \mathbf{y}$. But $\Phi$ is the identity on $\mathbb{S}$, so $\mathbf{y} = \mathbf{x}(1+T)^{-s/2} \in \mathbb{S}$. Therefore $\Gamma_{\mathbb{I}_0}(\mathfrak{a}) \subseteq \mathbb{S}$ and $\rho_F$ is $\mathbb{I}_0$-full.

It remains to deal with the case when $\ker\Phi = \{1\}$. In this case $\Phi$ is an isomorphism onto $\mathbb{K}'$ and we can use $\Phi^{-1}$ to get a continuous group homomorphism from $\mathbb{K}'$ onto $\mathbb{Z}_p$:

$$s : \mathbb{K}' \cong \mathbb{K} \cong \mathcal{T} \ltimes \mathbb{S} \twoheadrightarrow \mathcal{T} \cong \mathbb{Z}_p.$$

Note that $\ker s = \mathbb{S}$, so we want to show that $\ker s$ is $\mathbb{I}_0$-full. By assumption there is a non-zero $\mathbb{I}_0$-ideal $\mathfrak{a}_0$ such that $\Gamma_{\mathbb{I}_0}(\mathfrak{a}_0) \subseteq \mathbb{K}'$. Let $\mathfrak{v} = \{b \in \mathfrak{a}_0 : \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \ker s\}$ and

$\mathfrak{v}^t = \{c \in \mathfrak{a}_0 : \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \in \ker s\}$. Both $\mathfrak{v}$ and $\mathfrak{v}^t$ are $\Lambda$-lattices in $Q(\mathbb{I}_0)$. We shall prove this for $\mathfrak{v}$; the proof for $\mathfrak{v}^t$ is similar. Note that $\mathfrak{v}$ is a $\mathbb{Z}_p$-module: if $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{S}$ then $\left(\begin{smallmatrix} 1 & sb \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)^s \in \mathbb{S}$ since $\mathbb{S}$ is closed (as it is the determinant one image of a Galois representation). To see that $\mathfrak{v}$ is a $\Lambda$-module, recall that $\mathbb{S}$ is normalized by $\boldsymbol{J} = \left(\begin{smallmatrix} 1+T & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by the proof of Lemma 3.3.5. Therefore conjugation by $\boldsymbol{J}$ gives an action of $T$ on $\mathfrak{v}$ as in the proof of Lemma 3.3.5. Now we consider the $\Lambda$-module $\mathfrak{a}_0/\mathfrak{v}$ which, as a group, is isomorphic to a closed subgroup of $\mathbb{Z}_p$. Therefore $\mathfrak{a}_0/\mathfrak{v}$ is a torsion $\Lambda$-module. Since $\mathfrak{a}_0$ is a $\Lambda$-lattice in $Q(\mathbb{I}_0)$ it follows that $\mathfrak{v}$ must also be a $\Lambda$-lattice in $Q(\mathbb{I}_0)$, as claimed.

We have shown that there are non-zero $\mathbb{I}_0$-ideals $\mathfrak{b} \subseteq \mathfrak{v}$ and $\mathfrak{b}^t \subseteq \mathfrak{v}^t$ such that the Pink-Lie algebra $L(\mathbb{S})$ contains

$$\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathfrak{b}\right\} \cup \left\{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in \mathfrak{b}^t\right\}.$$

By letting $\mathfrak{c} = \mathfrak{b}\mathfrak{b}^t$ and taking Lie brackets of the upper and lower nilpotent matrices above, we find that $L(\mathbb{S}) \supseteq \mathfrak{c}^2\mathfrak{sl}_2(\mathbb{I}_0)$. Therefore $\mathbb{S}$ is $\mathbb{I}_0$-full, as desired.

□

*Remark* 3. Here is another proof of Proposition 3.3.10. Assume $\rho$ is $\mathbb{I}_0$-full, so by Corollary 1 in [44] we see that $\operatorname{Im}\rho$ is a nontrivial subnormal subgroup of $\mathrm{SL}_2(\mathbb{I}_0)$. Let $G = \operatorname{Im}\rho_F|_H \cap \mathrm{SL}_2(\mathbb{I}_0)$. To see that $\rho_F$ is $\mathbb{I}_0$-full it suffices to show that $G$ is a nontrivial subnormal subgroup of $\mathrm{SL}_2(\mathbb{I}_0)$. Since $\operatorname{Im}\rho$ is nontrivial and subnormal and $G \subseteq \operatorname{Im}\rho$, it suffices to show that $G$ is normal in $\operatorname{Im}\rho$ and $G \neq 1$. The fact that $G$ is normal in $\operatorname{Im}\rho$ follows easily from the definition of $\rho$. If $G = 1$, then the determinant map on $\operatorname{Im}\rho_F|_H$ is an isomorphism onto its image. In particular, $\operatorname{Im}\rho_F|_H$ is abelian. Since $F$ is non-CM, this contradicts Ribet's results on the images of classical non-CM Galois representations [39].

## 3.4 Open image in product

The purpose of this section is to prove the following reduction step in the proof of Theorem 3.1.4.

**Proposition 3.4.1.** *Assume that $|\mathbb{F}| \neq 3$. Fix an arithmetic prime $P$ of $\Lambda$. Assume that for every prime $\mathcal{Q}$ of $\mathbb{I}_0$ lying over $P$, the image of $\operatorname{Im}\rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open. Then the image of $\operatorname{Im}\rho$ in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open in the product topology.*

Thus if we can show that there is some arithmetic prime $P$ of $\Lambda$ satisfying the hypothesis of Proposition 3.4.1, then combining the above result with Proposition 3.3.2 yields Theorem 3.1.4.

Fix an arithmetic prime $P$ of $\Lambda$ satisfying the hypothesis of Proposition 3.4.1. Note that $\mathbb{Z}_p$ does not contain any $p$-power roots of unity since $p > 2$. Therefore $P = P_{k,1}$ for some $k \geq 2$. Recall that $\mathbb{G} = \operatorname{Im}\rho \cap \Gamma_{\mathbb{I}_0}(\mathfrak{m}_0)$, and write $\overline{\mathbb{G}}$ for the image of $\mathbb{G}$ in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$. We begin our proof of Proposition 3.4.1 with the following lemma of Ribet which allows us to reduce to considering products of only two copies of $\mathrm{SL}_2$.

**Lemma 3.4.2** (Lemma 3.4, [36])**.** *Let $S_1, \ldots, S_t (t > 1)$ be profinite groups. Assume for each $i$ that the following condition is satisfied: for each open subgroup $U$ of $S_i$, the closure of the commutator subgroup of $U$ is open in $S_i$. Let $\mathcal{G}$ be a closed subgroup of $S = S_1 \times \cdots \times S_t$ that maps to an open subgroup of each group $S_i \times S_j (i \neq j)$. Then $\mathcal{G}$ is open in $S$.*

Apply this lemma to our situation with $\{S_1, \ldots, S_t\} = \{\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}) : \mathcal{Q}|P\}$ and $\mathcal{G} = \overline{\mathbb{G}}$. The lemma implies that it is enough to prove that for all primes $\mathcal{Q}_1 \neq \mathcal{Q}_2$ of $\mathbb{I}_0$ lying over $P$, the image $G$ of $\overline{\mathbb{G}}$ under the projection to $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ is open. We shall now consider what happens when this is not the case. Indeed, the reader should be warned that the rest of this section is a proof by contradiction.

**Proposition 3.4.3.** *Let $P$ be an arithmetic prime of $\Lambda$ satisfying the hypotheses of Proposition 3.4.1, and assume $|\mathbb{F}| \neq 3$. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be distinct primes of $\mathbb{I}_0$ lying over $P$. Let $\mathfrak{P}_i$ be a prime of $\mathbb{I}$ lying over $\mathcal{Q}_i$. If $G$ is not open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ then there is an isomorphism $\sigma : \mathbb{I}_0/\mathcal{Q}_1 \cong \mathbb{I}_0/\mathcal{Q}_2$ and a character $\varphi : H \to Q(\mathbb{I}_0/\mathcal{Q}_2)^\times$ such that*

$$\sigma(a(\ell, f_{\mathfrak{P}_1})) = \varphi(\ell) a(\ell, f_{\mathfrak{P}_2})$$

*for all primes $\ell$ for which $\mathrm{Frob}_\ell \in H$.*

*Proof.* Our strategy is to mimic the proof of [36, Theorem 3.5]. Let $G_i$ be the projection of $G$ to $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_i)$, so $G \subseteq G_1 \times G_2$. By hypothesis $G_i$ is open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_i)$. Let $\pi_i : G \to G_i$ be the projection maps and set $N_1 = \ker \pi_2$ and $N_2 = \ker \pi_1$. Though a slight abuse of notation, we view $N_i$ as a subset of $G_i$. Goursat's Lemma implies that the image of $G$ in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism

$$\alpha : G_1/N_1 \cong G_2/N_2.$$

Since $G$ is not open in $G_1 \times G_2$ by hypothesis, either $N_1$ is not open in $G_1$ or $N_2$ is not open in $G_2$. (Otherwise $N_1 \times N_2$ is open and hence $G$ is open.) Without loss of generality we may assume that $N_1$ is not open in $G_1$. From the classification of subnormal subgroups of $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1)$ in [44] it follows that $N_1 \subseteq \{\pm 1\}$ since $N_1$ is not open. If $N_2$ is open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ then $\alpha$ gives an isomorphism from either $G_1$ or $\mathrm{PSL}_2(\mathbb{I}_0/\mathcal{Q}_1)$ to the finite group $G_2/N_2$. Clearly this is impossible, so $N_2$ is not open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$. Again by [44] we have $N_2 \subseteq \{\pm 1\}$. Recall that $G_i$ comes from $\mathbb{G} = \mathrm{Im}\,\rho \cap \Gamma_{\mathbb{I}_0}(\mathfrak{m}_0)$ by reduction. In particular, $-1 \notin G_i$ since all elements of $\mathbb{G}$ reduce to the identity in $\mathrm{SL}_2(\mathbb{F})$. Thus we must have $N_i = \{1\}$. Hence $\alpha$ gives an isomorphism $G_1 \cong G_2$. We note that the Theorem in [44] requires $|\mathbb{F}| \neq 3$. This invocation of [44] is the only reason we assume $|\mathbb{F}| \neq 3$.

The isomorphism theory of open subgroups of $\mathrm{SL}_2$ over a local ring was studied by Merzljakov in [32]. (There is a unique theorem in his paper, and that is the result to which we refer. His theorem applies to more general groups and rings, but it is relevant in particular to our situation.) Although his result is stated only for automorphisms of open subgroups, his proof goes through without change for isomorphisms. His result implies that $\alpha$ must be of the form

$$\alpha(\mathbf{x}) = \eta(\mathbf{x})\mathbf{y}^{-1}\sigma(\mathbf{x})\mathbf{y}, \tag{3.2}$$

where $\eta \in \mathrm{Hom}(G_1, Q(\mathbb{I}_0/\mathcal{Q}_2)^\times)$, $\mathbf{y} \in \mathrm{GL}_2(Q(\mathbb{I}_0/\mathcal{Q}_2))$ and $\sigma : \mathbb{I}_0/\mathcal{Q}_1 \cong \mathbb{I}_0/\mathcal{Q}_2$ is a ring isomorphism. By $\sigma(\mathbf{x})$ we mean that we apply $\sigma$ to each entry of the matrix $\mathbf{x}$.

For any $\mathbf{g} \in G$ we can write $\mathbf{g} = (\mathbf{x}, \mathbf{y})$ with $\mathbf{x} \in G_1, \mathbf{y} \in G_2$. Since $G$ is the graph of $\alpha$ we have $\alpha(\mathbf{x}) = \mathbf{y}$. By definition of $G$ there is some $h \in H$ such that $\mathbf{x} = \mathfrak{P}_1(\rho(h))$ and

$\mathbf{y} = \mathfrak{P}_2(\rho(h))$. Recall that for almost all primes $\ell$ for which $\mathrm{Frob}_\ell \in H$ we have $\mathrm{tr}(\rho(\mathrm{Frob}_\ell)) = \sqrt{\det \rho_F(\mathrm{Frob}_\ell)}^{-1} a(\ell, F)$. Furthermore $\det \rho_F(\mathrm{Frob}_\ell) \bmod P = \chi(\ell)\ell^{k-1}$ since $P = P_{k,1}$. Using these facts together with equation (3.2) we see that for almost any $\mathrm{Frob}_\ell \in H$ we have

$$\sigma(a(\ell, f_{\mathfrak{P}_1})) = \varphi(\ell) a(\ell, f_{\mathfrak{P}_2}),$$

where

$$\varphi(\ell) := \eta^{-1}(\mathfrak{P}_1(\rho(\mathrm{Frob}_\ell))) \frac{\sigma(\sqrt{\chi(\ell)\ell^{k-1}})}{\sqrt{\chi(\ell)\ell^{k-1}}},$$

as claimed. □

To finish the proof of Proposition 3.4.1 we need to remove the condition that $\mathrm{Frob}_\ell \in H$ from the conclusion of Proposition 3.4.3. That is, we would like to show that there is an isomorphism $\tilde{\sigma} : \mathbb{I}'/\mathfrak{P}'_1 \cong \mathbb{I}'/\mathfrak{P}'_2$ extending $\sigma$ and a character $\tilde{\varphi} : G_{\mathbb{Q}} \to Q(\mathbb{I}'/\mathfrak{P}'_2)^\times$ extending $\varphi$ such that

$$\tilde{\sigma}(a(\ell, f_{\mathfrak{P}_1})) = \tilde{\varphi}(\ell) a(\ell, f_{\mathfrak{P}_2})$$

for almost all primes $\ell$. If we can do this, then applying Theorem 3.2.1 allows us to lift $\tilde{\sigma}$ to an element of $\Gamma$ that sends $\mathfrak{P}'_1$ to $\mathfrak{P}'_2$. (We also need to verify that $\tilde{\varphi}$ takes values in $\mathbb{Z}_p[\chi]$ in order to apply Theorem 3.2.1.) But this is a contradiction since $\mathfrak{P}'_1$ and $\mathfrak{P}'_2$ lie over different primes of $\mathbb{I}_0$. Hence it follows from Proposition 3.4.3 that $G$ must be open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ and Lemma 3.4.2 implies Proposition 3.4.1.

We show the existence of $\tilde{\sigma}$ and $\tilde{\varphi}$ using obstruction theory as developed in [15, §4.3.5]. For the sake of notation, we briefly recall the theory here. For the proofs we refer the reader to [15]. Let $K$ be a finite extension of $\mathbb{Q}_p$, $n \in \mathbb{Z}^+$, and $r : H \to \mathrm{GL}_n(K)$ be an absolutely irreducible representation. For all $g \in G_{\mathbb{Q}}$ define a twisted representation on $H$ by $r^g(h) := r(ghg^{-1})$. Assume the following condition:

$$r \cong r^g \text{ over } K \text{ for all } g \in G_{\mathbb{Q}}. \tag{3.3}$$

Under hypothesis (3.3) it can be shown that there is a function $c : G_{\mathbb{Q}} \to \mathrm{GL}_n(K)$ with the following properties:

1. $r = c(g)^{-1} r^g c(g)$ for all $g \in G_{\mathbb{Q}}$;

2. $c(hg) = r(h)c(g)$ for all $h \in H, g \in G_{\mathbb{Q}}$;

3. $c(1) = 1$.

As $r$ is absolutely irreducible, it follows that $b(g, g') := c(g)c(g')c(gg')^{-1}$ is a 2-cocycle with values in $K^{\times}$. In fact $b$ factors through $\Delta := G_{\mathbb{Q}}/H$ and hence represents a class in $H^2(\Delta, K^{\times})$. We call this class $\mathrm{Ob}(r)$. It is independent of the function $c$ satisfying the above three properties. The class $\mathrm{Ob}(r)$ measures the obstruction to lifting $r$ to a representation of $G_{\mathbb{Q}}$. We say a continuous representation $\tilde{r} : G_{\mathbb{Q}} \to \mathrm{GL}_n(K)$ is an *extension* of $r$ to $G_{\mathbb{Q}}$ if $\tilde{r}|_H = r$.

**Proposition 3.4.4.** *1. There is an extension $\tilde{r}$ of $r$ to $G_{\mathbb{Q}}$ if and only if $\mathrm{Ob}(r) = 0 \in H^2(\Delta, K^{\times})$.*

*2. If $\mathrm{Ob}(r) = 0$ and $\tilde{r}$ is an extension of $r$ to $G_{\mathbb{Q}}$, then all other extensions of $r$ to $G_{\mathbb{Q}}$ are of the form $\tilde{r} \otimes \psi$ for some character $\psi : \Delta \to K^{\times}$.*

For ease of notation we shall write $K_i = Q(\mathbb{I}/\mathfrak{P}_i)$ and $E_i = Q(\mathbb{I}_0/\mathcal{Q}_i)$. Write $\rho_i : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_i)$ for $\rho_{f_{\mathfrak{P}_i}}$. By Theorem 3.3.1 we see that $\rho_i|_H$ takes values in $\mathrm{GL}_2(E_i)$. Proposition 3.4.3 gives an isomorphism $\sigma : E_1 \cong E_2$ and a character $\varphi : H \to E_2^{\times}$ such that

$$\mathrm{tr}(\rho_1|_H^{\sigma}) = \mathrm{tr}(\rho_2|_H \otimes \varphi).$$

In order to use obstruction theory to show the existence of $\tilde{\sigma}$ and $\tilde{\varphi}$ we must show that all of the representations in question satisfy hypothesis (3.3).

**Lemma 3.4.5.** *Let $L_i$ be a finite extension of $K_i$. View $\rho_1$ as a representation over $L_1$ and $\rho_2|_H, \rho_1|_H^{\sigma}, \rho_2|_H \otimes \varphi$, and $\varphi$ as representations over $L_2$. Then $\rho_i|_H, \rho_1|_H^{\sigma}, \rho_2|_H \otimes \varphi$, and $\varphi$ all satisfy hypothesis* (3.3)*. Furthermore we have $\mathrm{Ob}(\rho_i|_H) = 0, \mathrm{Ob}(\rho_1|_H^{\sigma}) = \mathrm{Ob}(\rho_2|_H \otimes \varphi)$, and*

$$\mathrm{Ob}(\rho_2|_H \otimes \varphi) = \mathrm{Ob}(\rho_2|_H)\,\mathrm{Ob}(\varphi) \in H^2(\Delta, (L_2)^{\times}).$$

*Proof.* Recall that a continuous representation of a compact group over a field of characteristic 0 is determined up to isomorphism by its trace. Therefore to verify (3.3) it suffices to show that if $r$ is any of the representations listed in the statement of the lemma, then

$$\operatorname{tr} r = \operatorname{tr} r^g$$

for all $g \in G_{\mathbb{Q}}$. This is obvious when $r$ is $\rho_1|_H$ or $\rho_2|_H$ since both extend to representations of $G_{\mathbb{Q}}$ and hence

$$\operatorname{tr} \rho_i^g(h) = \operatorname{tr} \rho_i(g)\rho_i(h)\rho_i(g)^{-1} = \operatorname{tr} \rho_i(h).$$

Since $\rho_i$ is an extension of $\rho_i|_H$ and $L_i \supseteq K_i$ we have $\operatorname{Ob}(\rho_i|_H) = 0$.

When $r = \rho_1|_H^\sigma$, let $\tau : K_1 \hookrightarrow \overline{\mathbb{Q}}_p$ be an extension of $\sigma$. Then $\rho_1^\tau$ is an extension of $\rho_1|_H^\sigma$ and hence we can use the same argument as above to conclude that $\operatorname{tr} \rho_1|_H^\sigma = \operatorname{tr}(\rho_1|_H^\sigma)^g$. (Note that for this particular purpose, we do not care about the field in which $\tau$ takes values.)

When $r = \rho_2|_H \otimes \varphi$, recall that $\operatorname{tr} \rho_1|_H^\sigma = \varphi \operatorname{tr} \rho_2|_H$. Since both $\rho_1|_H^\sigma$ and $\rho_2|_H$ satisfy hypothesis (3.3) so does $\rho_2|_H \otimes \varphi$. Furthermore, $\operatorname{tr} \rho_1|_H^\sigma = \operatorname{tr}(\rho_2|_H \otimes \varphi)$ implies that $\rho_1|_H^\sigma \cong \rho_2|_H \otimes \varphi$ and hence $\operatorname{Ob}(\rho_1|_H^\sigma) = \operatorname{Ob}(\rho_2|_H \otimes \varphi)$.

Since $(\rho_1|_H^\sigma)^g \cong \rho_2|_H^g \otimes \varphi^g$ for any $g \in G_{\mathbb{Q}}$ and since both $\rho_i|_H$ satisfy (3.3) we see that

$$\varphi^g \operatorname{tr} \rho_2|_H = \varphi \operatorname{tr} \rho_2|_H. \tag{3.4}$$

Thus if we know $\operatorname{tr} \rho_2|_H$ is nonzero sufficiently often then we can deduce that $\varphi$ satisfies (3.3) . More precisely, let $m \in \mathbb{Z}^+$ be the conductor for $\varphi$, so $\varphi : (\mathbb{Z}/m\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$. Then we have a surjection $H \twoheadrightarrow \operatorname{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ with kernel $\kappa$. Choose a set $S$ of coset representatives of $\kappa$ in $H$, so $H = \sqcup_{s\in S} s\kappa$. If we can show that $\operatorname{tr} \rho_2(s\kappa) \neq \{0\}$ for all $s \in S$, then it follows from equation (3.4) that $\varphi^g = \varphi$ for all $g \in G_{\mathbb{Q}}$. Recall that $\rho_2$ is a Galois representation attached to a classical modular form, and so by Ribet [38, 39] and Momose's [34] result we know that its image is open. (See Theorem 3.5.1 for a precise statement of their result.) Then the restriction of $\rho_2$ to any open subset of $G_{\mathbb{Q}}$ also has open image and hence $\operatorname{tr} \rho_2$ is not identically zero. Each $s\kappa$ is open in $G_{\mathbb{Q}}$, so $\varphi^g = \varphi$.

Finally, note that if $c : G_{\mathbb{Q}} \to \mathrm{GL}_2(L_2)$ is a function satisfying conditions 1-3 above for $r = \rho_2|_H$ and $\eta : G_{\mathbb{Q}} \to L_2^\times$ is a function satisfying conditions 1-3 above for $\varphi$, then $\eta c$ is a function satisfying conditions 1-3 for $\rho_2|_H \otimes \varphi$. From this it follows that $\mathrm{Ob}(\rho_2|_H \otimes \varphi) = \mathrm{Ob}(\rho_2|_H)\,\mathrm{Ob}(\varphi)$. □

With $L_i$ as in the previous lemma, suppose there is an extension $\tilde{\sigma} : L_1 \cong L_2$ of $\sigma$ and an extension $\tilde{\varphi} : G_{\mathbb{Q}} \to L_2^\times$ of $\varphi$. We now show that this gives us the desired relation among traces.

**Lemma 3.4.6.** *If there exists extensions $\tilde{\sigma}$ of $\sigma$ and $\tilde{\varphi}$ of $\varphi$, then there exists a character $\eta : G_{\mathbb{Q}} \to L_2^\times$ that is also a lift of $\varphi$ such that $\rho_1^{\tilde{\sigma}} \cong \rho_2 \otimes \eta$.*

*Proof.* Note that since $F$ does not have CM, $\rho_1|_H$ and $\rho_2|_H$ are absolutely irreducible by results of Ribet [37]. For any absolutely irreducible representation $\pi : G_{\mathbb{Q}} \to \mathrm{GL}_2(L_2)$ Frobenius reciprocity gives

$$\langle \pi, \mathrm{Ind}_H^{G_{\mathbb{Q}}}(\rho_1|_H^\sigma)\rangle_{G_{\mathbb{Q}}} = \langle \pi|_H, \rho_1|_H^\sigma\rangle_H = \langle \pi|_H, \rho_2|_H \otimes \varphi\rangle_H. \tag{3.5}$$

Thus if $\pi$ is a 2-dimensional irreducible constituent of $\mathrm{Ind}(\rho_1|_H^\sigma)$ then $\rho_1|_H^\sigma$ is a constituent of $\pi|_H$. As both are 2-dimensional, it follows that $\rho_1|_H^\sigma \cong \pi|_H$ and thus $\pi$ is an extension of $\rho_1|_H^\sigma$. Since $\tilde{\sigma}$ exists by hypothesis, we know that $\rho_1^{\tilde{\sigma}}$ is also an extension of $\rho_1|_H^\sigma$.

Since $\tilde{\varphi}$ exists by hypothesis, we can take $\pi = \rho_2 \otimes \tilde{\varphi}$. Then (3.5) implies that $\pi$ is an irreducible constituent of $\mathrm{Ind}_H^G(\rho_1|_H^\sigma)$. By Proposition 3.4.4 there is a character $\psi : \Delta \to L_2^\times$ such that $\rho_2 \otimes \tilde{\varphi} \cong \rho_1^{\tilde{\sigma}} \otimes \psi$. That is,

$$\rho_1^{\tilde{\sigma}} \cong \rho_2 \otimes (\tilde{\varphi}\psi^{-1}).$$

Setting $\eta = \tilde{\varphi}\psi^{-1}$ gives the desired conclusion. □

Finally, we turn to showing the existence of $\tilde{\sigma}$ and $\tilde{\varphi}$. With notation as in Lemma 3.4.5, suppose there exists $\tilde{\sigma}^{-1} : L_2 \cong L_1$ that lifts $\sigma^{-1}$. Then $\tilde{\sigma}^{-1}$ induces an isomorphism $H^2(\Delta, L_2^\times) \cong H^2(\Delta, L_1^\times)$ that sends $\mathrm{Ob}(\rho_1|_H^\sigma)$ to $\mathrm{Ob}(\rho_1|_H)$. It follows from Lemma 3.4.5 that

$\mathrm{Ob}(\rho_1|_H^\sigma) = 1$ and hence $\mathrm{Ob}(\rho_2|_H \otimes \varphi) = 1$. But $1 = \mathrm{Ob}(\rho_2|_H \otimes \varphi) = \mathrm{Ob}(\rho_2|_H)\,\mathrm{Ob}(\varphi) = \mathrm{Ob}(\varphi)$, and thus we can extend $\varphi$ to $\tilde{\varphi} : G_{\mathbb{Q}} \to L_2^\times$.

The above argument requires that we find $L_i \supseteq K_i$ such that $L_1$ is isomorphic to $L_2$ via a lift of $\sigma$. We can achieve this as follows. Let $\tau : K_1 \hookrightarrow \overline{\mathbb{Q}}_p$ be an extension of $\sigma$. Let $L_2 = K_2\tau(K_1)$. Let $\tilde{\sigma}^{-1} : L_2 \hookrightarrow \overline{\mathbb{Q}}_p$ be an extension of $\tau^{-1}$ and set $L_1 = \tilde{\sigma}^{-1}(L_2)$. This construction satisfies the desired properties. Applying Lemma 3.4.6 we see that there is a character $\eta : G_{\mathbb{Q}} \to L_2^\times$ such that

$$\mathrm{tr}\,\rho_1^{\tilde{\sigma}} = \mathrm{tr}\,\rho_2 \otimes \eta. \tag{3.6}$$

This is almost what we want. Note that by (3.6) it follows that $\tilde{\sigma}$ restricts to an isomorphism from $(\mathbb{I}'/\mathfrak{P}_1')[\eta]$ to $(\mathbb{I}'/\mathfrak{P}_2')[\eta]$. The only problem is that $\tilde{\sigma}$ may not send $\mathbb{I}'/\mathfrak{P}_1'$ to $\mathbb{I}'/\mathfrak{P}_2'$ and $\eta$ may have values in $L_2$ that are not in $(\mathbb{I}'/\mathfrak{P}_2')^\times$. We shall show that this cannot be the case.

Recall that $\chi$ is the nebentypus of $F$ and $\mathfrak{P}_1$ and $\mathfrak{P}_2$ lie over the arithmetic prime $P_{k,1}$ of $\Lambda$. Thus for almost all primes $\ell$ we have $\det \rho_i(\mathrm{Frob}_\ell) = \chi(\ell)\ell^{k-1}$. Applying this to equation (3.6) we find that

$$\chi^{\tilde{\sigma}}(\ell)\ell^{k-1} = \eta^2(\ell)\chi(\ell)\ell^{k-1}.$$

Recall that $\chi(\ell)$ is a root of unity and hence $\chi^{\tilde{\sigma}}(\ell)$ is just a power of $\chi(\ell)$. Thus $\eta^2(\ell) \in \mathbb{Z}_p[\chi] \subseteq \mathbb{I}'/\mathfrak{P}_i'$ and hence $[(\mathbb{I}'/\mathfrak{P}_i')[\eta] : \mathbb{I}'/\mathfrak{P}_i'] \leq 2$. Thus we may assume that $L_2 = K_2[\eta]$, which is at most a quadratic extension of $K_2$.

Note that since $\eta^2$ takes values in $\mathbb{I}'/\mathfrak{P}_i'$ we can obtain $(\mathbb{I}'/\mathfrak{P}_i')[\eta]$ from $\mathbb{I}'/\mathfrak{P}_i'$ by adjoining a 2-power root of unity. (Write $\eta$ as the product of a 2-power order character and an odd order character and note that any odd order root of unity is automatically a square in any ring in which it is an element.)

**Lemma 3.4.7.** *We have $(\mathbb{I}'/\mathfrak{P}_i')[\eta] = \mathbb{I}'/\mathfrak{P}_i'$ for $i = 1, 2$. Therefore $\tilde{\sigma} : \mathbb{I}'/\mathfrak{P}_1' \cong \mathbb{I}'/\mathfrak{P}_2'$ and $\eta$ takes values in $\mathbb{Z}_p[\chi]$.*

*Proof.* Suppose first that $\mathbb{I}'/\mathfrak{P}_2' = (\mathbb{I}'/\mathfrak{P}_2')[\eta]$ but $[(\mathbb{I}'/\mathfrak{P}_1')[\eta] : \mathbb{I}'/\mathfrak{P}_1'] = 2$. Then we have that

$\tilde{\sigma} : (\mathbb{I}'/\mathfrak{P}'_1)[\eta] \cong \mathbb{I}'/\mathfrak{P}'_2$. Note that $(\mathbb{I}'/\mathfrak{P}'_1)[\eta]$ is unramified over $\mathbb{I}'/\mathfrak{P}'_1$ since it is obtained by adjoining a prime-to-$p$ root of unity (namely a 2-power root of unity). Thus the residue field of $(\mathbb{I}'/\mathfrak{P}'_1)[\eta]$ must be a quadratic extension of the residue field $\mathbb{F}$ of $\mathbb{I}'/\mathfrak{P}'_1$. But $\mathbb{F}$ is also the residue field of $\mathbb{I}'/\mathfrak{P}'_2$ and since $(\mathbb{I}'/\mathfrak{P}'_1)[\eta] \cong \mathbb{I}'/\mathfrak{P}'_2$ they must have the same residue field, a contradiction. Therefore we must have $(\mathbb{I}'/\mathfrak{P}'_1)[\eta] = \mathbb{I}'/\mathfrak{P}'_1$.

It remains to deal with the case when $[(\mathbb{I}'/\mathfrak{P}'_1)[\eta] : \mathbb{I}'/\mathfrak{P}'_1] = [(\mathbb{I}'/\mathfrak{P}'_2)[\eta] : \mathbb{I}'/\mathfrak{P}'_2] = 2$. As noted above, these extensions must be unramified and hence the residue field of $(\mathbb{I}'/\mathfrak{P}'_i)[\eta]$ must be the unique quadratic extension $\mathbb{E} = \mathbb{F}[\bar{\eta}]$ of $\mathbb{F}$. Note that $\tilde{\sigma}$ induces an automorphism $\hat{\sigma}$ of $\mathbb{E}$ that necessarily restricts to an automorphism of $\mathbb{F}$. From $\chi^{\tilde{\sigma}} = \eta^2\chi$ we find that

$$\bar{\chi}^{\hat{\sigma}} = \bar{\eta}^2\bar{\chi}.$$

On the other hand $\hat{\sigma}$ is an automorphism of $\mathbb{F}$ and hence is equal to some power of Frobenius. So we see that for some $s \in \mathbb{Z}$ we have $\bar{\eta}^2 = \bar{\chi}^{p^s-1}$. Since $p$ is odd, $p^s - 1$ is even and hence $\bar{\eta}^2$ takes values in $\mathbb{F}_p[\bar{\chi}^2]$. Thus $\bar{\eta}$ takes values in $\mathbb{F}_p[\bar{\chi}] \subseteq \mathbb{F}$, a contradiction to the assumption that $[\mathbb{F}[\bar{\eta}] : \mathbb{F}] = 2$.

Since $\eta^2$ takes values in $\mathbb{Z}_p[\chi]$ and $\mathbb{F}_p[\bar{\eta}] \subseteq \mathbb{F}_p[\bar{\chi}]$, it follows that in fact $\eta$ must take values in $\mathbb{Z}_p[\chi]$. Hence we may take $L_i = K_i$ and $\tilde{\sigma} : \mathbb{I}'/\mathfrak{P}'_1 \cong \mathbb{I}'/\mathfrak{P}'_2$. □

Finally, we summarize how the results in this section fit together to prove Proposition 3.4.1.

*Proof of Proposition 3.4.1.* By Lemma 3.4.2 it suffices to show that, for any two primes $\mathcal{Q}_1 \neq \mathcal{Q}_2$ of $\mathbb{I}_0$ lying over $P_{k,1}$, the image of $\operatorname{Im}\rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ is open. Proposition 3.4.3 says that if that is not the case, then there is an isomorphism $\sigma : \mathbb{I}_0/\mathcal{Q}_1 \cong \mathbb{I}_0/\mathcal{Q}_2$ and a character $\varphi : H \to Q(\mathbb{I}_0/\mathcal{Q}_2)^\times$ such that $\operatorname{tr}\rho_{f_{\mathfrak{P}_1}}|_H^\sigma = \operatorname{tr}\rho_{f_{\mathfrak{P}_2}}|_H \otimes \varphi$. The obstruction theory arguments allow us to lift $\sigma$ and $\varphi$ to $\tilde{\sigma} : \mathbb{I}'/\mathfrak{P}'_1 \cong \mathbb{I}'/\mathfrak{P}'_2$ and $\tilde{\varphi} : G_{\mathbb{Q}} \to Q(\mathbb{I}/\mathfrak{P}_2)^\times$ such that $\operatorname{tr}\rho_{f_{\mathfrak{P}_1}}^{\tilde{\sigma}} = \operatorname{tr}\rho_{f_{\mathfrak{P}_2}} \otimes \tilde{\varphi}$. Theorem 3.2.1 allows us to lift $\tilde{\sigma}$ to an element of $\Gamma$ that sends $\mathfrak{P}'_1$ to $\mathfrak{P}'_2$. But $\mathfrak{P}'_1$ and $\mathfrak{P}'_2$ lie over different primes of $\mathbb{I}_0$ and $\Gamma$ fixes $\mathbb{I}_0$, so we reach a contradiction. Therefore the image of $\operatorname{Im}\rho$ in the product $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ is open. □

## 3.5 Proof of main theorem

In this section we use the compatibility between the conjugate self-twists of $F$ and those of its classical specializations established in section 3.2 to relate $\mathbb{I}_0/\mathcal{Q}$ to the ring appearing in the work of Ribet [38, 39] and Momose [34]. This allows us to use their results to finish the proof of Theorem 3.1.4.

We begin by recalling the work of Ribet and Momose. We follow Ribet's exposition in [39] closely. Let $f = \sum_{n=1}^{\infty} a(n,f)q^n$ be a classical eigenform of weight $k$. Let $K = \mathbb{Q}(\{a(n,f) : n \in \mathbb{Z}^+\})$ with ring of integers $\mathcal{O}$. Denote by $\Gamma_f$ the group of conjugate self-twists of $f$. Let $E = K^{\Gamma_f}$ and $H_f = \cap_{\sigma\in\Gamma_f} \ker \eta_\sigma$. For any character $\psi$, let $G(\psi)$ denote the Gauss sum of the primitive character of $\psi$. For $\sigma, \tau \in \Gamma_f$ Ribet defined

$$c(\sigma,\tau) := \frac{G(\eta_\sigma^{-1})G(\eta_\tau^{-\sigma})}{G(\eta_{\sigma\tau}^{-1})}.$$

One shows that $c$ is a 2-cocycle on $\Gamma_f$ with values in $K^\times$.

Let $\mathfrak{X}$ be the central simple $E$-algebra associated to $c$. Then $K$ is the maximal commutative semisimple subalgebra of $\mathfrak{X}$. It can be shown that $\mathfrak{X}$ has order two in the Brauer group of $E$, and hence there is a 4-dimensional $E$-algebra $D$ that represents the same element as $\mathfrak{X}$ in the Brauer group of $E$. Namely, if $\mathfrak{X}$ has order one then $D = M_2(E)$ and otherwise $D$ is a quaternion division algebra over $E$.

For a prime $p$, recall that we have a Galois representation

$$\rho_{f,p} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p)$$

associated to $f$. The following theorem is due to Ribet in the case when $f$ has weight 2 [38].

**Theorem 3.5.1** (Momose [34]). *We may view $\rho_{f,p}|_{H_f}$ as a representation valued in $(D \otimes_\mathbb{Q} \mathbb{Q}_p)^\times$. Furthermore, letting $\mathfrak{n}$ denote the reduced norm map on $D$, the image of $\rho_{f,p}|_{H_f}$ is open in*

$$\{x \in (D \otimes_\mathbb{Q} \mathbb{Q}_p)^\times : \mathfrak{n}x \in \mathbb{Q}_p^\times\}.$$

In particular, when $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a matrix algebra, the above theorem tell us that $\operatorname{Im} \rho_{f,p}|_{H_f}$ is open in

$$\{\mathbf{x} \in \mathrm{GL}_2(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p) : \det \mathbf{x} \in (\mathbb{Z}_p^\times)^{k-1}\}.$$

Let $\mathfrak{p}$ be a prime of $\mathcal{O}_E$ lying over $p$, and let $\rho_{f,\mathfrak{p}}$ be the representation obtained by projecting $\rho_{f,p}|_{H_f}$ to the $\mathcal{O}_{E_\mathfrak{p}}$-component. Under the assumption that $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a matrix algebra Theorem 3.5.1 implies that $\rho_{f,\mathfrak{p}}$ is $\mathcal{O}_{E_\mathfrak{p}}$-full. Finally, Brown and Ghate proved that if $f$ is ordinary at $p$, then $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a matrix algebra [2, Theorem 3.3.1].

Thus, the Galois representation associated to each classical specialization of our $\mathbb{I}$-adic form $F$ is $\mathcal{O}_{E_\mathfrak{p}}$-full with respect to the appropriate ring $\mathcal{O}_{E_\mathfrak{p}}$. We must show that $E_\mathfrak{p}$ is equal to $Q(\mathbb{I}_0/\mathcal{Q})$, where $\mathcal{Q}$ corresponds to $\mathfrak{p}$ in a way we will make precise below.

Recall that we have a fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathfrak{P} \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ be an arithmetic prime of $\mathbb{I}$, and let $\mathcal{Q}$ be the prime of $\mathbb{I}_0$ lying under $\mathfrak{P}$. As usual, let $\mathfrak{P}' = \mathfrak{P} \cap \mathbb{I}'$. Let $D(\mathfrak{P}'|\mathcal{Q}) \subseteq \Gamma$ be the decomposition group of $\mathfrak{P}'$ over $\mathcal{Q}$. Let

$$K_\mathfrak{P} = \mathbb{Q}(\{\iota_p^{-1}(a(n, f_\mathfrak{P})) : n \in \mathbb{Z}^+\}) \subset \overline{\mathbb{Q}},$$

and let $\Gamma_\mathfrak{P}$ be the group of all conjugate self-twists of the classical modular form $f_\mathfrak{P}$. Set $E_\mathfrak{P} = K_\mathfrak{P}^{\Gamma_\mathfrak{P}}$. Let $\mathfrak{q}_\mathfrak{P}$ be the prime of $K_\mathfrak{P}$ corresponding to the embedding $\iota_p|_{K_\mathfrak{P}}$, and set $\mathfrak{p}_\mathfrak{P} = \mathfrak{q}_\mathfrak{P} \cap E_\mathfrak{P}$. Let $D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P}) \subseteq \Gamma_\mathfrak{P}$ be the decomposition group of $\mathfrak{q}_\mathfrak{P}$ over $\mathfrak{p}_\mathfrak{P}$. Thus we have that the completion $K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}$ of $K_\mathfrak{P}$ at $\mathfrak{q}_\mathfrak{P}$ is equal to $Q(\mathbb{I}/\mathfrak{P})$ and $\mathrm{Gal}(K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}/E_{\mathfrak{P},\mathfrak{p}_\mathfrak{P}}) = D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$. Thus we may view $D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$ as the set of all automorphisms of $K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}$ that are conjugate self-twists of $f_\mathfrak{P}$.

With this in mind, we see that there is a natural group homomorphism

$$\Phi : D(\mathfrak{P}'|\mathcal{Q}) \to D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$$

since any element of $D(\mathfrak{P}'|\mathcal{Q})$ stabilizes $\mathfrak{P}'$ and hence induces an automorphism of $Q(\mathbb{I}'/\mathfrak{P}') = Q(\mathbb{I}/\mathfrak{P}) = K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}$. The induced automorphism will necessarily be a conjugate self-twist of $f_\mathfrak{P}$ since we started with a conjugate self-twist of $F$. Thus we get an element of $D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$. The main compatibility result is that $\Phi$ is an isomorphism.

**Proposition 3.5.2.** *The natural group homomorphism* $\Phi$ *is an isomorphism. Therefore* $Q(\mathbb{I}_0/\mathcal{Q}) = E_{\mathfrak{P},\mathfrak{p}_{\mathfrak{P}}}$.

*Proof.* The fact that $\Phi$ is injective is easy. Namely, if $\sigma \in D(\mathfrak{P}'|\mathcal{Q})$ acts trivially on $K_{\mathfrak{P},\mathfrak{q}_{\mathfrak{P}}}$ then for almost all $\ell$ we have

$$a(\ell, f_{\mathfrak{P}}) = a(\ell, f_{\mathfrak{P}})^{\sigma} = \eta_{\sigma}(\ell) a(\ell, f_{\mathfrak{P}}).$$

Since $F$ (and hence its arithmetic specialization $f_{\mathfrak{P}}$) does not have CM it follows that $\eta_{\sigma} = 1$. Hence $\sigma = 1$ and $\Phi$ is injective.

To see that $\Phi$ is surjective, let $\sigma \in D(\mathfrak{q}_{\mathfrak{P}}|\mathfrak{p}_{\mathfrak{P}})$. By Theorem 3.2.1 we see that there is $\tilde{\sigma} \in \operatorname{Aut} \mathbb{I}'$ that is a conjugate self-twist of $F$ and $\sigma \circ \mathfrak{P} = \mathfrak{P} \circ \tilde{\sigma}$. That is, $\tilde{\sigma} \in D(\mathfrak{P}'|\mathcal{Q})$ and $\Phi(\tilde{\sigma}) = \sigma$. We have

$$E_{\mathfrak{P},\mathfrak{p}_{\mathfrak{P}}} = K_{\mathfrak{P},\mathfrak{q}_{\mathfrak{P}}}^{D(\mathfrak{q}_{\mathfrak{P}}|\mathfrak{p}_{\mathfrak{P}})} = Q(\mathbb{I}'/\mathfrak{P}')^{D(\mathfrak{P}|\mathcal{Q})}.$$

A general fact from commutative algebra [1, Theorem V.2.2.2] tells us that

$$Q(\mathbb{I}'/\mathfrak{P}')^{D(\mathfrak{P}|\mathcal{Q})} = Q(\mathbb{I}_0/\mathcal{Q}),$$

as desired. □

**Corollary 3.5.3.** *Let* $\mathcal{Q}$ *be a prime of* $\mathbb{I}_0$ *lying over an arithmetic prime of* $\Lambda$. *There is a nonzero* $\mathbb{I}_0/\mathcal{Q}$*-ideal* $\bar{\mathfrak{a}}_{\mathcal{Q}}$ *such that*

$$\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}}) \subseteq \operatorname{Im}(\rho_F \bmod \mathcal{Q}\mathbb{I}') \subseteq \prod_{\mathfrak{P}'|\mathcal{Q}} \mathrm{GL}_2(\mathbb{I}'/\mathfrak{P}'),$$

*where the inclusion of* $\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}})$ *in the product is via the diagonal embedding* $\mathrm{GL}_2(\mathbb{I}_0/\mathcal{Q}) \hookrightarrow \prod_{\mathfrak{P}'|\mathcal{Q}} \mathrm{GL}_2(\mathbb{I}'/\mathfrak{P}')$. *Hence the image of* $\operatorname{Im}\rho$ *in* $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ *is open.*

*Proof.* For a prime $\mathfrak{P}$ of $\mathbb{I}$, write $\mathcal{O}_{\mathfrak{P}}$ for the ring of integers of $E_{\mathfrak{P},\mathfrak{p}_{\mathfrak{P}}}$. By Theorem 3.5.1 and the remarks following it, for each prime $\mathfrak{P}$ of $\mathbb{I}$ lying over $\mathcal{Q}$ we have $\operatorname{Im}\rho_{f_{\mathfrak{P}}}$ contains $\Gamma_{\mathcal{O}_{\mathfrak{P}}}(\bar{\mathfrak{a}}_{\mathfrak{P}})$ for some nonzero $\mathcal{O}_{\mathfrak{P}}$-ideal $\bar{\mathfrak{a}}_{\mathfrak{P}}$. While $\mathbb{I}_0/\mathcal{Q}$ need not be integrally closed, by Proposition 3.5.2 we see that $\bar{\mathfrak{a}}_{\mathfrak{P}} \cap (\mathbb{I}_0/\mathcal{Q})$ is a nonzero $\mathbb{I}_0/\mathcal{Q}$-ideal.

Thus we have

$$\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathfrak{P}} \cap \mathbb{I}_0/\mathcal{Q}) \subseteq \Gamma_{\mathcal{O}_{\mathfrak{P}}}(\bar{\mathfrak{a}}_{\mathfrak{P}}) \subseteq \operatorname{Im} \rho_{f_{\mathfrak{P}}} = \operatorname{Im} \rho_F \bmod \mathfrak{P} \subseteq \mathrm{GL}_2(\mathbb{I}'/\mathfrak{P}').$$

Let $\bar{\mathfrak{a}}_{\mathcal{Q}} = \cap_{\mathfrak{P}|\mathcal{Q}} \bar{\mathfrak{a}}_{\mathfrak{P}} \cap \mathbb{I}_0/\mathcal{Q}$. This is a finite intersection of nonzero $\mathbb{I}_0/\mathcal{Q}$-ideals and hence is nonzero. The first statement follows from the above inclusions.

For the statement about $\rho$, recall that $\rho_F|_{H_0}$ is valued in $\mathrm{GL}_2(\mathbb{I}_0)$ and hence $\operatorname{Im} \rho_F|_{H_0}$ mod $\mathcal{Q}$ lies in the diagonal copy of $\mathrm{GL}_2(\mathbb{I}_0/\mathcal{Q})$ in $\prod_{\mathfrak{P}'|\mathcal{Q}} \mathrm{GL}_2(\mathbb{I}'/\mathfrak{P}')$. Since $H$ is open in $G_{\mathbb{Q}}$ by replacing $\bar{\mathfrak{a}}_{\mathcal{Q}}$ with a smaller $\mathbb{I}_0/\mathcal{Q}$-ideal if necessary, we may assume that $\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}})$ is contained in the image of $\rho_F|_H$ in $\mathrm{GL}_2(\mathbb{I}_0/\mathcal{Q})$. Since $\rho$ and $\rho_F$ are equal on elements of determinant 1 and $\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}}) \subseteq \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$, it follows that $\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}})$ is contained in the image of $\operatorname{Im} \rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$. That is, the image of $\operatorname{Im} \rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open. □

*Summary of Proof of Theorem 3.1.4.* Theorem 3.3.1, which will be proved in the next section, allows us to create a representation $\rho : H \to \mathrm{SL}_2(\mathbb{I}_0)$ with the property that if $\rho$ is $\mathbb{I}_0$-full then so is $\rho_F$. This is important for the use of Pink's theory in section 3.3 as well as for the techniques of section 3.4. Proposition 3.3.2 shows that it is sufficient to prove that the image of $\operatorname{Im} \rho$ in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open for some arithmetic prime $P$ of $\Lambda$. Proposition 3.4.1 further reduces the problem to showing that the image of $\rho$ modulo $\mathcal{Q}$ is open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ for all primes $\mathcal{Q}$ of $\mathbb{I}_0$ lying over a fixed arithmetic prime $P$ of $\Lambda$.

This reduces the problem to studying the image of a Galois representation attached to one of the classical specializations of $F$ (twisted by the inverse square root of the determinant). Hence we can apply the work of Ribet and Momose, but only after we show that $Q(\mathbb{I}_0/\mathcal{Q})$ is the same field that occurs in their work. This is done in Proposition 3.5.2, though the main input is Theorem 3.2.1. □

## 3.6 Obtaining an $\mathrm{SL}_2(\mathbb{I}_0)$-valued representation

In this section we prove Theorem 3.3.1.

**Theorem 3.3.1.** *Assume that $\bar{\rho}_F$ is absolutely irreducible and $H_0$-regular. If $V = \mathbb{I}'^2$ is the module on which $G_{\mathbb{Q}}$ acts via $\rho_F$, then there is a basis for $V$ such that all of the following happen simultaneously:*

1. *$\rho_F$ is valued in $\mathrm{GL}_2(\mathbb{I}')$;*
2. *$\rho_F|_{D_p}$ is upper triangular;*
3. *$\rho_F|_{H_0}$ is valued in $\mathrm{GL}_2(\mathbb{I}_0)$;*
4. *There is a matrix $\boldsymbol{j} = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$, where $\zeta$ and $\zeta'$ are roots of unity, such that $\boldsymbol{j}$ normalizes the image of $\rho_F$ and $\zeta \not\equiv \zeta' \bmod p$.*

It is well known that so long as $\bar{\rho}_F$ is absolutely irreducible we may assume that $\rho_F$ has values in $\mathrm{GL}_2(\mathbb{I}')$ and the local representation $\rho_F|_{D_p}$ is upper triangular [17, Theorem 4.3.2]. To show that $\rho_F|_{H_0}$ has values in $\mathrm{GL}_2(\mathbb{I}_0)$ we begin by investigating the structure of $\Gamma$.

**Proposition 3.6.1.** *The group $\Gamma$ is a finite abelian 2-group.*

*Proof.* Let $S$ be the set of primes $\ell$ for which $a(\ell, F)^\sigma = \eta_\sigma(\ell)a(\ell, F)$ for all $\sigma \in \Gamma$, so $S$ excludes only finitely many primes. For $\ell \in S$, let

$$b_\ell := \frac{a(\ell, F)^2}{\det \rho_F(\mathrm{Frob}_\ell)}.$$

It turns out that $b_\ell \in \mathbb{I}_0$. To see this, note that since $\bar{\rho}_F$ is absolutely irreducible, for any $\sigma \in \Gamma$ we have $\rho_F^\sigma \cong \eta_\sigma \otimes \rho_F$ over $\mathbb{I}'$. Taking determinants we find that $\det \rho_F^{\sigma-1} = \eta_\sigma^2$. Thus we have

$$(a(\ell, F)^\sigma)^2 = \eta_\sigma(\ell)^2 a(\ell, F)^2 = \det \rho_F(\mathrm{Frob}_\ell)^{\sigma-1} a(\ell, F)^2,$$

from which it follows that $b_\ell^\sigma = b_\ell$. Solving for $a(\ell, F)$ in the definition of $b_\ell$ we find that

$$Q(\mathbb{I}') = Q(\mathbb{I}_0)[\sqrt{b_\ell \det \rho_F(\mathrm{Frob}_\ell)} : \ell \in S].$$

Recall that for $\ell \in S$ we have $\det \rho_F(\mathrm{Frob}_\ell) = \chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1}$, where $\kappa(\langle \ell \rangle) \in 1 + \mathfrak{m}_\Lambda$. (Currently all that matters is that $\kappa$ is valued in $1 + \mathfrak{m}_\Lambda$. For a precise definition of $\kappa$, see

(2.6).) In particular, $\sqrt{\kappa(\langle\ell\rangle)} \in \Lambda$. Similarly, we can write $\ell = \langle\ell\rangle\omega(\ell)$ with $\langle\ell\rangle \in 1 + p\mathbb{Z}_p$ and $\omega(\ell) \in \mu_{p-1}$. So $\sqrt{\langle\ell\rangle} \in \Lambda$ as well.

Let

$$\mathcal{K} = Q(\mathbb{I}_0)[\sqrt{b_\ell}, \sqrt{\det \rho_F(\mathrm{Frob}_\ell)} : \ell \in S],$$

which is an abelian extension of $Q(\mathbb{I}_0)$ since it is obtained by adjoining square roots. The above argument shows that in fact $\mathcal{K}$ is obtained from $Q(\mathbb{I}_0)[\sqrt{b_\ell} : \ell \in S]$ by adjoining finitely many roots of unity, namely the square roots of the values of $\chi$ and the square roots of $\mu_{p-1}$. As odd order roots of unity are automatically squares, we can write $\mathcal{K} = Q(\mathbb{I}_0)[\sqrt{b_\ell} : \ell \in S][\mu_{2^s}]$ for some $s \in \mathbb{Z}^+$. Thus we have

$$\mathrm{Gal}(\mathcal{K}/Q(\mathbb{I}_0)) \cong \mathrm{Gal}(Q(\mathbb{I}_0)[\sqrt{b_\ell} : \ell \in S]/Q(\mathbb{I}_0)) \times \mathrm{Gal}(Q(\mathbb{I}_0)[\mu_{2^s}]/Q(\mathbb{I}_0)).$$

By Kummer theory the first group is an elementary abelian 2-group. The second group is isomorphic to $(\mathbb{Z}/2^s\mathbb{Z})^\times$ and hence is a 2-group. As $\Gamma$ is a quotient of $\mathrm{Gal}(\mathcal{K}/Q(\mathbb{I}_0))$ it follows that $\Gamma$ is a finite abelian 2-group, as claimed. $\square$

For ease of notation let $\pi = \bar{\rho}_F|_{H_0} : H_0 \to \mathrm{GL}_2(\mathbb{F})$. Let $D$ be a non-square in $\mathbb{F}$, and let $\mathbb{E} = \mathbb{F}[\sqrt{D}]$ be the unique quadratic extension of $\mathbb{F}$.

**Lemma 3.6.2.** *Let $K$ be a field and $\mathcal{S} \subset \mathrm{GL}_n(K)$ a set of nonconstant semisimple operators that can be simultaneously diagonalized over $\overline{K}$. If $\boldsymbol{y} \in \mathrm{GL}_n(\overline{K})$ such that $\boldsymbol{y}\mathcal{S}\boldsymbol{y}^{-1} \subset \mathrm{GL}_n(K)$, then there is a matrix $\boldsymbol{z} \in \mathrm{GL}_n(K)$ such that $\boldsymbol{z}\mathcal{S}\boldsymbol{z}^{-1} = \boldsymbol{y}\mathcal{S}\boldsymbol{y}^{-1}$. In particular, if $\pi$ is irreducible over $\mathbb{F}$ but not absolutely irreducible, then $\mathbb{E}$ is the splitting field for $\pi$.*

*Proof.* Let $\sigma \in G_K := \mathrm{Gal}(\overline{K}/K)$. Then for any $\mathbf{x} \in \mathcal{S}$ we have $\mathbf{y}^\sigma\mathbf{x}\mathbf{y}^{-\sigma} = (\mathbf{y}\mathbf{x}\mathbf{y}^{-1})^\sigma = \mathbf{y}\mathbf{x}\mathbf{y}^{-1}$, so $\mathbf{y}^{-1}\mathbf{y}^\sigma$ centralizes $\mathbf{x}$. As elements in $\mathcal{S}$ are simultaneously diagonalizable, they have the same centralizer in $\mathrm{GL}_n(\overline{K})$. Since elements of $\mathcal{S}$ are semisimple, their centralizer is a torus and hence isomorphic to $(\overline{K})^{\oplus n}$. It's not hard to show that $a : G_K \to (\overline{K}^\times)^{\oplus n}$ given by $\sigma \mapsto \mathbf{y}^{-1}\mathbf{y}^\sigma$ is a 1-cocycle. (Here we view $(\overline{K}^\times)^{\oplus n}$ as a $G_K$-module by letting elements of $G_K$ act component-wise.) By Hilbert's Theorem 90 we have $H^1(G_K, (\overline{K}^\times)^{\oplus n}) =$

$H^1(G_K, \overline{K}^{\times})^{\oplus n} = 0$. Hence $a$ is a coboundary. That is, there is some $\alpha \in (\overline{K}^{\times})^{\oplus n}$ such that

$$a_\sigma = \mathbf{y}^{-1}\mathbf{y}^\sigma = \alpha^{-1}\alpha^\sigma$$

for all $\sigma \in G_K$. Thus $(\mathbf{y}\alpha^{-1})^\sigma = \mathbf{y}\alpha^{-1}$ for all $\sigma \in G_K$, so $\mathbf{z} := \mathbf{y}\alpha^{-1} \in \mathrm{GL}_n(K)$. But $\alpha$ commutes with $\mathcal{S}$ and so $\mathbf{z}\mathcal{S}\mathbf{z}^{-1} = \mathbf{y}\mathcal{S}\mathbf{y}^{-1}$, as claimed.

To deduce the claim about $\pi$, let $\mathcal{S} = \operatorname{Im}\pi$. The fact that $\mathcal{S}$ is semisimple follows from Clifford's Theorem since $\bar{\rho}_F$ is absolutely irreducible [22, Theorem 6.5, Corollary 6.6]. If $\pi$ is not absolutely irreducible then there is a matrix $\mathbf{y} \in \mathrm{GL}_2(\overline{\mathbb{F}})$ that simultaneously diagonalizes $\mathcal{S}$. Note that every matrix in $\operatorname{Im}\pi$ has eigenvalues in $\mathbb{E}$. Indeed every matrix has a quadratic characteristic polynomial and $\mathbb{E}$ is the unique quadratic extension of $\mathbb{F}$. Thus, taking $K = \mathbb{E}$ we see that $\mathbf{y}\mathcal{S}\mathbf{y}^{-1} \subset \mathrm{GL}_2(K)$. The first statement of the lemma tells us that $\operatorname{Im}\pi$ is diagonalizable over $\mathbb{E}$. Since $\pi$ is irreducible over $\mathbb{F}$ and $[\mathbb{E} : \mathbb{F}] = 2$, it follows that $\mathbb{E}$ is the smallest extension of $\mathbb{F}$ over which $\operatorname{Im}\pi$ is diagonalizable. $\square$

Let $Z$ be the centralizer of $\operatorname{Im}\pi$ in $M_2(\mathbb{F})$. Since $\bar{\rho}_F$ is $H_0$-regular, exactly one of the following three cases must occur:

1. The representation $\pi$ is absolutely irreducible. In this case $Z$ consists of scalar matrices over $\mathbb{F}$.

2. The representation $\pi$ is not absolutely irreducible, but $\pi$ is irreducible over $\mathbb{F}$. In this case we may assume
$$Z = \left\{ \begin{pmatrix} \alpha & \beta D \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{F} \right\} \cong \mathbb{E}.$$

3. The representation $\pi$ is reducible over $\mathbb{F}$. In this case we may assume that $Z$ consists of diagonal matrices over $\mathbb{F}$.

Recall that since $\bar{\rho}_F$ is absolutely irreducible, for any $\sigma \in \Gamma$ we have $\rho_F^\sigma \cong \eta_\sigma \otimes \rho_F$. That is, there is some $\mathbf{t}_\sigma \in \mathrm{GL}_2(\mathbb{I}')$ such that

$$\rho_F(g)^\sigma = \eta_\sigma(g)\mathbf{t}_\sigma\rho_F(g)\mathbf{t}_\sigma^{-1}$$

for all $g \in G_{\mathbb{Q}}$. Then for all $\sigma, \tau \in \Gamma, g \in G_{\mathbb{Q}}$ we have

$$\eta_{\sigma\tau}(g)\mathbf{t}_{\sigma\tau}\rho_F(g)\mathbf{t}_{\sigma\tau}^{-1} = \rho(g)^{\sigma\tau} = \eta_\sigma^\tau(g)\eta_\tau(g)\mathbf{t}_\sigma^\tau\mathbf{t}_\tau\rho_F(g)\mathbf{t}_\tau^{-1}\mathbf{t}_\sigma^{-\tau}.$$

Using the fact that $\eta_{\sigma\tau} = \eta_\sigma^\tau\eta_\tau$ we see that $c(\sigma,\tau) := \mathbf{t}_{\sigma\tau}^{-1}\mathbf{t}_\sigma^\tau\mathbf{t}_\tau$ commutes with the image of $\rho_F$. As $\rho_F$ is absolutely irreducible, $c(\sigma,\tau)$ must be a scalar. Hence $c$ represents a 2-cocycle of $\Gamma$ with values in $\mathbb{I}'^\times$.

We will need to treat case 2 ($\pi$ is irreducible over $\mathbb{F}$ but not absolutely irreducible) a bit differently, so we establish notation that will unify the proofs that follow. For a finite extension $M$ of $\mathbb{Q}_p$, let $\mathcal{O}_M$ denote the ring of integers of $M$. Let $K$ be the largest finite extension of $\mathbb{Q}_p$ for which $\mathcal{O}_K[[T]]$ is contained in $\mathbb{I}'$. So $K$ has residue field $\mathbb{F}$. Let $L$ be the unique unramified quadratic extension of $K$. Write $\mathbb{J} = \Lambda_{\mathcal{O}_L}[\{a(\ell,F) : \ell \nmid N\}]$. Note that the residue field of $\mathbb{J}$ is the unique quadratic extension of $\mathbb{F}$. Let

$$A = \begin{cases} \mathbb{J} & \text{in case 2} \\ \mathbb{I}' & \text{else.} \end{cases}$$

Let $\kappa$ be the residue field of $A$, so $\kappa = \mathbb{E}$ in case 2 and $\kappa = \mathbb{F}$ otherwise.

Since $L$ is obtained from $K$ by adjoining some prime-to-$p$ root of unity, in case 2 it follows that $Q(A)$ is Galois over $Q(\mathbb{I}_0)$ with Galois group isomorphic to $\Gamma \times \mathbb{Z}/2\mathbb{Z}$. In particular, we have an action of $\Gamma$ on $A$ in all cases. Let $B = A^\Gamma$. In case 2, $A$ is a quadratic extension of $B$ and $B \cap \mathbb{I}' = \mathbb{I}_0$. Otherwise $B = \mathbb{I}_0$. We may consider the 2-cocycle $c$ in $H^2(\Gamma, A^\times)$.

**Lemma 3.6.3.** *With notation as above,* $[c] = 0 \in H^2(\Gamma, A^\times)$. *Thus there is a function* $\zeta : \Gamma \to A^\times$ *such that* $c(\sigma,\tau) = \zeta(\sigma\tau)^{-1}\zeta(\sigma)^\tau\zeta(\tau)$ *for all* $\sigma,\tau \in \Gamma$.

*Proof.* Consider the exact sequence $1 \to 1 + \mathfrak{m}_A \to A^\times \to \kappa^\times \to 1$. Note that for $j > 0$ we have $H^j(\Gamma, 1 + \mathfrak{m}_A) = 0$ since $1 + \mathfrak{m}_A$ is a $p$-profinite group for $p > 2$ and $\Gamma$ is a 2-group by Lemma 3.6.1. Thus the long exact sequence in cohomology gives isomorphisms

$$H^j(\Gamma, A^\times) \cong H^j(\Gamma, \kappa^\times)$$

for all $j > 0$. Hence it suffices to prove that $[\bar{c}] = 0 \in H^2(\Gamma, \kappa^\times)$.

Let $\sigma \in \Gamma$ and $h \in H_0$. Recall that $\Gamma$ acts trivially on $\mathbb{F}$ by Proposition 3.2.4. Since $\rho_F^\sigma(h) = \eta_\sigma(h)\mathbf{t}_\sigma\rho_F(h)\mathbf{t}_\sigma^{-1}$ and $\eta_\sigma(h) = 1$ it follows that $\bar{\mathbf{t}}_\sigma \in Z$.

We now split into the three cases depending on the irreducibility of $\pi$. Suppose we are in case 1, so $\pi$ is absolutely irreducible and $\kappa = \mathbb{F}$. Then $\bar{\mathbf{t}}_\sigma$ must be a scalar in $\mathbb{F}^\times$. Call it $\bar{\zeta}(\sigma)$. Then $\bar{c}(\sigma,\tau) = \bar{\zeta}(\sigma\tau)^{-1}\bar{\zeta}(\sigma)^\tau\bar{\zeta}(\tau)$, and so $[\bar{c}] = 0 \in H^2(\Gamma,\mathbb{F}^\times)$.

In case 2, using the description of $Z$ above we see that $\bar{\mathbf{t}}_\sigma = \left(\begin{smallmatrix}\alpha_\sigma & \beta_\sigma D\\ \beta_\sigma & \alpha_\sigma\end{smallmatrix}\right)$ for some $\alpha_\sigma, \beta_\sigma \in \mathbb{F}$. This becomes a scalar, say $\bar{\zeta}(\sigma) = \alpha_\sigma + \beta_\sigma\sqrt{D}$, over $\mathbb{E} = \kappa$. Thus $\bar{\mathbf{t}}_\sigma = \bar{\zeta}(\sigma)$. As above $\bar{c}(\sigma,\tau) = \bar{\zeta}(\sigma\tau)^{-1}\bar{\zeta}(\sigma)^\tau\bar{\zeta}(\tau)$, and thus $[\bar{c}] = 0 \in H^2(\Gamma,\kappa^\times)$.

Finally, in case 3 we have that $\bar{\mathbf{t}}_\sigma$ is a diagonal matrix. The diagonal map $\mathbb{F} \hookrightarrow \mathbb{F}\oplus\mathbb{F}$ induces an injection $H^2(\Gamma,\mathbb{F}^\times) \hookrightarrow H^2(\Gamma,\mathbb{F}^\times\oplus\mathbb{F}^\times)$. The fact that $\bar{\mathbf{t}}_\sigma$ is a diagonal matrix allows us to calculate that the image of $[\bar{c}]$ in $H^2(\Gamma,\mathbb{F}^\times\oplus\mathbb{F}^\times)$ is 0. Since the map is an injection, it follows that $[\bar{c}] = 0 \in H^2(\Gamma,\mathbb{F}^\times)$, as desired. □

Replace $\mathbf{t}_\sigma \in \mathrm{GL}_2(\mathbb{I}')$ by $\mathbf{t}_\sigma\zeta(\sigma)^{-1} \in \mathrm{GL}_2(A)$. Then we still have $\rho_F^\sigma = \eta_\sigma\mathbf{t}_\sigma\rho_F\mathbf{t}_\sigma^{-1}$, and now $\mathbf{t}_{\sigma\tau} = \mathbf{t}_\sigma^\tau\mathbf{t}_\tau$. That is, $\sigma \mapsto \mathbf{t}_\sigma$ is a nonabelian 1-cocycle with values in $\mathrm{GL}_2(A)$. Since $F$ is primitive we have $Q(\mathbb{I}) = Q(\mathbb{I}')$. Thus by [17, Theorem 4.3.2] we see that $\rho_F|_{D_p}$ is isomorphic to an upper triangular representation over $Q(\mathbb{I}')$. Under the assumptions that $\bar{\rho}_F$ is absolutely irreducible and $H_0$-regular, the proof of [17, Theorem 4.3.2] goes through with $\mathbb{I}'$ in place of $\mathbb{I}$. That is, $\rho_F|_{D_p}$ is isomorphic to an upper triangular representation over $\mathbb{I}'$. Let $V = \mathbb{I}'^2$ be the representation space for $\rho_F$ with basis chosen such that

$$\rho_F|_{D_p} = \begin{pmatrix}\varepsilon & u\\ 0 & \delta\end{pmatrix},$$

and assume $\bar{\varepsilon} \neq \bar{\delta}$. Let $V[\varepsilon] \subset V$ be the free direct summand of $V$ on which $D_p$ acts by $\varepsilon$ and $V[\delta]$ be the quotient of $V$ on which $D_p$ acts by $\delta$. Let $V_A = V \otimes_{\mathbb{I}'} A$. Similarly for $\lambda \in \{\varepsilon,\delta\}$ let $V_A[\lambda] := V[\lambda]\otimes_{\mathbb{I}'} A$. For $\mathbf{v} \in V_A$, define

$$\mathbf{v}^{[\sigma]} := \mathbf{t}_\sigma^{-1}\mathbf{v}^\sigma, \tag{3.7}$$

where $\sigma$ acts on $\mathbf{v}$ component-wise. Note that in case 2 we are using the action of $\Gamma$ on $A$ described prior to Lemma 3.6.3.

**Lemma 3.6.4.** *For all $\sigma, \tau \in \Gamma$ we have $(\boldsymbol{v}^{[\sigma]})^{[\tau]} = \boldsymbol{v}^{[\sigma\tau]}$, so this defines an action of $\Gamma$ on $V_A$. Furthermore, this action stabilizes $V_A[\varepsilon]$ and $V_A[\delta]$.*

*Proof.* The formula (3.7) defines an action since $\sigma \mapsto \mathbf{t}_\sigma$ is a nonabelian 1-cocycle. Let $\lambda$ be either $\delta$ or $\varepsilon$. Let $\mathbf{v} \in V_A[\lambda]$ and $\sigma \in \Gamma$. We must show that $\mathbf{v}^{[\sigma]} \in V_A[\lambda]$. Let $d \in D_p$. Using the fact that $\mathbf{v} \in V_A[\lambda]$ and $\rho_F^\sigma = \eta_\sigma \mathbf{t}_\sigma \rho_F \mathbf{t}_\sigma^{-1}$ we find that

$$\rho_F(d)\mathbf{v}^{[\sigma]} = \eta_\sigma^{-1}(d)\lambda^\sigma(d)\mathbf{v}^{[\sigma]}.$$

Note that for all $d \in D_p$

$$\begin{pmatrix} \varepsilon^\sigma(d) & u^\sigma(d) \\ 0 & \delta^\sigma(d) \end{pmatrix} = \rho_F^\sigma(d) = \eta_\sigma(d)\mathbf{t}_\sigma \rho_F(d)\mathbf{t}_\sigma^{-1} = \eta_\sigma(d)\mathbf{t}_\sigma \begin{pmatrix} \varepsilon(d) & u(d) \\ 0 & \delta(d) \end{pmatrix} \mathbf{t}_\sigma^{-1}. \tag{3.8}$$

Using the fact that $\varepsilon \neq \delta$ and that $\rho_F|_{D_p}$ is indecomposable [9, 49] we see that $u/(\varepsilon - \delta)$ cannot be a constant. (If $u/(\varepsilon - \delta) = \alpha$ is a constant, then conjugating by $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ makes $\rho_F|_{D_p}$ diagonal.) Hence $\mathbf{t}_\sigma$ must be upper triangular. Therefore (3.8) implies that $\lambda^\sigma(d) = \eta_\sigma(d)\lambda(d)$, and thus

$$\rho_F(d)\mathbf{v}^{[\sigma]} = \eta_\sigma^{-1}(d)\lambda^\sigma(d)\mathbf{v}^{[\sigma]} = \lambda(d)\mathbf{v}^{[\sigma]}.$$

□

We are now ready to show that $\rho_F|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0)$.

**Theorem 3.6.5.** *Let $\rho_F : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_F|_{D_p}$ is upper triangular. Assume that $\bar{\rho}_F$ is absolutely irreducible and $H_0$-regular. Then $\rho_F|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0)$.*

*Proof.* We have an exact sequence of $A[D_p]$-modules

$$0 \to V_A[\varepsilon] \to V_A \to V_A[\delta] \to 0 \tag{3.9}$$

that is stable under the new action of $\Gamma$ defined in Lemma 3.6.4. Tensoring with $\kappa$ over $A$ we get an exact sequence of $\kappa$-vector spaces

$$V_\kappa[\bar{\varepsilon}] \to V_\kappa \to V_\kappa[\bar{\delta}] \to 0. \tag{3.10}$$

Since $V_A[\varepsilon]$ is a direct summand of $V_A$, the first arrow is injective. Since $V_A[\varepsilon]$ and $V_A$ are free $A$-modules, it follows that $\dim_\kappa V_\kappa[\bar{\varepsilon}] = 1$ and $\dim_\kappa V_\kappa = 2$. Counting dimensions in (3.10) now tells us that $\dim_\kappa V_\kappa[\bar{\delta}] = 1$.

Going back to the exact sequence (3.9) we can take $\Gamma$-invariants since all of the modules are stable under the new action of $\Gamma$. This gives an exact sequence of $B[D_p \cap H_0]$-modules

$$0 \to V_A[\varepsilon]^\Gamma \to V_A^\Gamma \to V_A[\delta]^\Gamma \to H^1(\Gamma, V_A[\varepsilon]).$$

Since $\Gamma$ is a 2-group by Lemma 3.6.1 and $V_A[\varepsilon] \cong A$ is $p$-profinite, we find that $H^1(\Gamma, V_A[\varepsilon]) = 0$. Tensoring with $\kappa^\Gamma$ over $B$ we get an exact sequence

$$V_A[\varepsilon]^\Gamma \otimes_B \kappa^\Gamma \to V_A^\Gamma \otimes_B \kappa^\Gamma \to V_A[\delta]^\Gamma \otimes_B \kappa^\Gamma \to 0.$$

If $\dim_{\kappa^\Gamma} V_A[\lambda]^\Gamma \otimes_B \kappa^\Gamma = 1$ for $\lambda \in \{\varepsilon, \delta\}$, then it follows from Nakayama's Lemma that $V_A[\lambda]^\Gamma$ is a free $B$-module of rank 1. Hence $V_A^\Gamma$ is a free $B$-module of rank 2. In all the cases except case 2, this completes the proof. In case 2 the above argument tells us that if we view $\rho_F$ as a $\mathrm{GL}_2(A)$-valued representation, then $\rho_F|_{H_0}$ takes values in $\mathrm{GL}_2(B)$. We know that $\rho_F$ actually has values in $\mathrm{GL}_2(\mathbb{I}')$ and hence $\rho_F|_{H_0}$ has values in $\mathrm{GL}_2(B \cap \mathbb{I}') = \mathrm{GL}_2(\mathbb{I}_0)$.

Thus we must show that for $\lambda \in \{\varepsilon, \delta\}$ we have $\dim_{\kappa^\Gamma} V_A[\lambda]^\Gamma \otimes_B \kappa^\Gamma = 1$. Note that $V_A[\lambda]^\Gamma \otimes_B \kappa^\Gamma = V_\kappa[\bar{\lambda}]^\Gamma$. When we are not in case 2, $\Gamma$ acts trivially on $\kappa$ and hence

$$\dim_{\mathbb{F}} V_{\mathbb{F}}[\bar{\lambda}]^\Gamma = \dim_{\mathbb{F}} V_{\mathbb{F}}[\bar{\lambda}] = 1.$$

Now assume we are in case 2, so $\kappa = \mathbb{E}$. Since $\dim_{\mathbb{E}} V_{\mathbb{E}}[\bar{\lambda}] = 1$ we can choose some nonzero $\mathbf{v} \in V_{\mathbb{E}}[\bar{\lambda}]$. We would like to show that

$$\sum_{\sigma \in \Gamma} \mathbf{v}^{[\sigma]} \neq 0$$

since the right hand side is $\Gamma$-invariant.

Since $V_{\mathbb{E}}[\bar{\lambda}]$ is 1-dimensional, for each $\sigma \in \Gamma$ there is some $\alpha_\sigma \in \mathbb{E}^\times$ such that $\mathbf{v}^{[\sigma]} = \alpha_\sigma \mathbf{v}$. Thus

$$\sum_{\sigma \in \Gamma} \mathbf{v}^{[\sigma]} = \sum_{\sigma \in \Gamma} \alpha_\sigma \mathbf{v}.$$

If $\sum_{\sigma\in\Gamma} \alpha_\sigma \neq 0$ then we are done. Otherwise we can change $\mathbf{v}$ to $a\mathbf{v}$ for any $a \in \mathbb{E}^\times$. It is easy to see that $(a\mathbf{v})^{[\sigma]} = a^\sigma \alpha_\sigma a^{-1}(a\mathbf{v})$ and thus changing $\mathbf{v}$ to $a\mathbf{v}$ changes $\alpha_\sigma$ to $a^\sigma a^{-1}\alpha_\sigma$. So we need to show that there is some $a \in \mathbb{E}^\times$ such that $\sum_{\sigma\in\Gamma} a^{\sigma-1} \neq 0$. In other words, we are interested in the zeros of the function

$$f(x) = \sum_{\sigma\in\Gamma} \alpha_\sigma x^{\sigma-1}$$

on $\mathbb{E}$. By Artin's Theorem on characters [26, Theorem VI.4.1], $f$ is not identically zero on $\mathbb{E}$. Therefore $\dim_{\mathbb{F}} V_{\mathbb{E}}[\bar{\lambda}]^\Gamma \geq 1$.

To get equality, let $0 \neq \mathbf{w} \in V_{\mathbb{E}}[\bar{\lambda}]^\Gamma$. Since $V_{\mathbb{E}}[\bar{\lambda}]^\Gamma \subseteq V_{\mathbb{E}}[\bar{\lambda}]$ and $\dim_{\mathbb{E}} V_{\mathbb{E}}[\bar{\lambda}] = 1$, any element of $V_{\mathbb{E}}[\bar{\lambda}]^\Gamma$ is an $\mathbb{E}$-multiple of $\mathbf{w}$. If $\beta \in \mathbb{E} \setminus \mathbb{F}$ then $\sigma$ does not fix $\beta$. Thus

$$(\beta\mathbf{w})^{[\sigma]} = \beta^\sigma \mathbf{w}^{[\sigma]} = \beta^\sigma \mathbf{w} \neq \beta\mathbf{w}.$$

Hence $V_{\mathbb{E}}[\bar{\lambda}]^\Gamma = \mathbb{F}\mathbf{w}$ and $\dim_{\mathbb{F}} V_{\mathbb{E}}[\bar{\lambda}]^\Gamma = 1$, as desired. □

Finally, we modify $\rho_F$ to obtain the normalizing matrix $\mathbf{j}$ in the last part of Theorem 3.3.1.

**Lemma 3.6.6.** *Suppose $\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_F|_{D_p}$ is upper triangular and $\rho_F|_{H_0}$ is valued in $\mathrm{GL}_2(\mathbb{I}_0)$. Assume $\bar{\rho}_F$ is absolutely irreducible and $H_0$-regular. Then there is an upper triangular matrix $\boldsymbol{x} \in \mathrm{GL}_2(\mathbb{I}_0)$ and roots of unity $\zeta$ and $\zeta'$ such that $\boldsymbol{j} := \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$ normalizes the image of $\boldsymbol{x}\rho_F\boldsymbol{x}^{-1}$ and $\zeta \not\equiv \zeta' \bmod p$.*

*Proof.* This argument is due to Hida [17, Lemma 4.3.20]. As $\bar{\rho}_F$ is $H_0$-regular there is an $h \in H_0$ such that $\bar{\varepsilon}(h) \neq \bar{\delta}(h)$. Let $\zeta$ and $\zeta'$ be the roots of unity in $\mathbb{I}_0$ satisfying $\zeta \equiv \varepsilon(h) \bmod \mathfrak{m}_0$ and $\zeta' \equiv \delta(h) \bmod \mathfrak{m}_0$. By our choice of $h$ we have $\zeta \not\equiv \zeta' \bmod p$.

Let $q = |\mathbb{F}|$. Then for some $u \in \mathbb{I}_0$

$$\lim_{n\to\infty} \rho_F(h)^{q^n} = \begin{pmatrix} \zeta & u \\ 0 & \zeta' \end{pmatrix}.$$

Conjugating $\rho_F$ by $\left(\begin{smallmatrix} 1 & u/(\zeta-\zeta') \\ 0 & 1 \end{smallmatrix}\right)$ preserves all three of the desired properties, and the image of the resulting representation is normalized by $\mathbf{j} = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$. □

# CHAPTER 4

# Lifting without Deformation Theory

As we saw in Chapter 3, Theorem 3.2.1 was a key component in the proof of Theorem 3.1.4. Although the statement of Theorem 3.1.4 is purely automorphic, deformation theory played a key role in the proof we gave in Section 3.2. In this section we give a purely automorphic proof of a version of Theorem 3.2.1.

## 4.1 Twists as endomorphisms of a Hecke algebra

In this section we seek to reformulate the existence of conjugate self-twists in terms of commutative diagrams involving certain Hecke algebras. We use Wiles' interpretation of Hida families presented in section 2.2.3 and the notation $M(\psi, \eta)$ introduced prior to Lemma 3.2.6.

For the rest of this section, fix a Dirichlet character $\eta$ with values in $\mathbb{Z}[\chi]$. Let $M$ be a positive integer multiple of $M(\chi, \eta)$. We wish to unify the classical and $\mathbb{J}$-adic cases in what follows. Let $A$ be either a ring of integers $\mathcal{O}$ in a number field containing $\mathbb{Z}[\chi]$ or an integral domain $\mathbb{J}$ that is finite flat over $\Lambda$ and contains $\mathbb{Z}[\chi]$. We shall write $S(M, \chi; A)$ for either $S_k(\Gamma_0(M), \chi; \mathcal{O})$ when $A = \mathcal{O}$ or $\mathbb{S}(M, \chi; \mathbb{J})$ when $A = \mathbb{J}$. Let

$$r_{\chi,\eta}(M) = \begin{cases} R_{\chi,\eta}(M) & \text{when } A = \mathcal{O} \\ \mathbb{R}_{\chi,\eta}(M) & \text{when } A = \mathbb{J}. \end{cases}$$

Denote by ${}^{M}T(n)$ the $n$-th Hecke operator on either $S(M, \chi; A)$ or $S(M, \eta^2\chi; A)$. Note that we use this notation ${}^{M}T(n)$ even when $(n, M) > 1$. The Hecke operators are compatible with $r_{\chi,\eta}(M)$ in the following sense.

**Lemma 4.1.1.** *For all $n \in \mathbb{Z}^+$ we have*

$$^{M}T(n) \circ r_{\chi,\eta}(M) = \eta(n) r_{\chi,\eta}(M) \circ {}^{M}T(n).$$

*In particular, both maps are zero when $(n, M) > 1$.*

*Proof.* The classical case follows from the $\mathbb{J}$-adic case by specialization, so we give the proof in the $\mathbb{J}$-adic case. (Incidentally, the classical case can be proved by exactly the same argument.)

It suffices to prove the lemma when $n = \ell$ is prime. Let $G \in \mathbb{S}(M, \chi; \mathbb{J})$ and recall that by definition of $^{M}T(\ell)$ we have

$$a(m, G|^{M}T(\ell)) = a(m\ell, G) + \kappa(\langle \ell \rangle)\chi(\ell)\ell^{-1} a(m/\ell, G),$$

where $\kappa : 1 + p\mathbb{Z}_p \to \Lambda^{\times}$ was defined in (2.6) and $a(m/\ell, G) = 0$ if $\ell \nmid m$. Applying this formula to $\mathbb{R}_{\chi,\eta}(M)(G) \in \mathbb{S}(M, \eta^2\chi; \mathbb{J})$ we calculate that for all $m \in \mathbb{Z}^+$

$$a(m, \mathbb{R}_{\chi,\eta}(M)(G)|^{M}T(\ell)) = \eta(\ell) a(m, \mathbb{R}_{\chi,\eta}(M)(G|^{M}T(\ell))).$$

This implies that

$$^{M}T(\ell) \circ \mathbb{R}_{\chi,\eta}(M)(G) = \eta(\ell)\mathbb{R}_{\chi,\eta}(M) \circ {}^{M}T(\ell)(G),$$

as desired. ∎

For the rest of the chapter assume further that $\eta$ is a quadratic character, so $r_{\chi,\eta}(M)$ is an endomorphism of $S(M, \chi; A)$. Let $h(M, \chi; A)$ be the Hecke algebra of $S(M, \chi; A)$. Recall the duality between them (Theorems 2.2.1 and 2.2.3). Let $\theta_{\chi,\eta}(M)$ be the $A$-algebra endomorphism of $h(M, \chi; A)$ induced by $r_{\chi,\eta}(M)$ via dualtiy. By Lemma 3.2.6 if $p \nmid c(\eta)$ then $\theta_{\chi,\eta}(M)$ restricts to an endomorphism of $\mathbf{h}^{\mathrm{ord}}(M, \chi; \mathbb{J})$.

**Lemma 4.1.2.** *For all $n \in \mathbb{Z}^+$ we have*

$$\theta_{\chi,\eta}(M)(^{M}T(n)) = \eta(n)^{M}T(n).$$

*Proof.* By definition $\theta_{\chi,\eta}(M)$ is the map that makes the following diagram commute.

$$\begin{array}{ccc} h(M,\chi;A) & \longleftrightarrow & \mathrm{Hom}_A(S(M,\chi;A),A) \\ {\scriptstyle \theta_{\chi,\eta}(M)}\downarrow & & \downarrow{\scriptstyle r_{\chi,\eta}(M)^*} \\ h(M,\chi;A) & \longleftrightarrow & \mathrm{Hom}_A(S(M,\chi;A),A) \end{array}$$

Note that ${}^{M}T(n)$ corresponds to $({}^{M}T(n),-)_A$ under duality, and $r_{\chi,\eta}(M)^*(({}^{M}T(n),-)_A) = ({}^{M}T(n),-)_A \circ r_{\chi,\eta}(M)$. Using the formula for the action of ${}^{M}T(n)$ on $q$-expansions as in Lemma 4.1.1 together with the definition of $r_{\chi,\eta}(M)$ yields

$$({}^{M}T(n),-)_A \circ r_{\chi,\eta}(M)(f) = \eta(n)({}^{M}T(n),f)_A$$

for all $f \in S(M,\chi)$. Thus $({}^{M}T(n),-)_A \circ r_{\chi,\eta}(M) = \eta(n)({}^{M}T(n),-)_A$ which corresponds to $\eta(n){}^{M}T(n)$ under duality. Thus $\theta_{\chi,\eta}(M)({}^{M}T(n)) = \eta(n){}^{M}T(n)$, as claimed. □

**Lemma 4.1.3.** *Let $f \in S(N,\chi;A)$ be an eigenform and $M$ a positive integer multiple of $N$. There is an eigenform $f_M \in S(M,\chi;A)$ such that $f_M|{}^{M}T(n) = 0$ for all $n$ such that $(n, M/N) > 1$ and $f_M$ has the same eigenvalues as $f$ for all ${}^{M}T(n)$ with $(n, M/N) = 1$.*

*Proof.* Write $M/N = \ell_1 \ldots \ell_t$ for not necessarily distinct primes $\ell_i$. By induction on $t$ it suffices to show that we can construct an eigenform $f_{N\ell_1} \in S(N\ell_1,\chi)$ with $f_{N\ell_1}|{}^{N\ell_1}T(\ell_1) = 0$ and $f_{N\ell_1}$ having the same eigenvalues as $f$ for all primes $\ell \neq \ell_1$.

Let $\lambda_1$ be the eigenvalue of $f$ under ${}^{N}T(\ell_1)$, so

$$f|{}^{N}T(\ell_1) = \lambda_1 f.$$

If $\lambda_1 = 0$ then just viewing $f \in S(N\ell_1,\chi;A)$ has all the desired properties and we may take $f_{N\ell_1} = f$. Otherwise, define $f_{N\ell_1} = f - \lambda_1 f|[\ell_1]$ where $(f|[\ell_1])(z) := f(\ell_1 z)$. It is well known (and can be checked by a calculation with $q$-expansions) that $f|[\ell_1]|{}^{N\ell_1}T(\ell_1) = f|{}^{N}T(\ell_1)$. This implies that $f_{N\ell_1}|{}^{N\ell_1}T(\ell_1) = 0$. For $\ell \neq \ell_1$ one can check that

$${}^{N\ell_1}T(\ell) \circ [\ell_1] = [\ell_1] \circ {}^{N}T(\ell).$$

From this it follows that $f_{N\ell_1}$ and $f$ have the same eigenvalues for ${}^{N\ell_1}T(\ell)$ for all primes $\ell \neq \ell_1$, as desired. □

We are interested in describing conjugate self-twists of an eigenform $f \in S(N, \chi; A)$. Let $A'$ be the subalgebra of $A$ generated by $\{a(\ell, f) : \ell \nmid N\}$ over either $\mathbb{Z}[\chi]$ if $A = \mathcal{O}$, or over $\Lambda_\chi$ when $A = \mathbb{I}$. Note that if $f$ is a newform then $Q(A) = Q(A')$. If $N^2|M$ then the eigenform $f_M$ from Lemma 4.1.3 is an element of $S(M, \chi; A')$. Write $\lambda_{f_M} : h(M, \chi; A') \to A'$ for the $A'$-algebra homomorphism corresponding to $f_M$. That is, $\lambda_{f_M}({}^M T(n)) = a(n, f_M)$ for all $n \in \mathbb{Z}^+$.

**Proposition 4.1.4.** *Let $f \in S(N, \chi; A)$ be primitive and let $\eta$ be a primitive quadratic character. Let $M = c(\eta)N^2$. Then $f$ has a conjugate self-twist with character $\eta$ if and only if there is an automorphism $\sigma$ of $A'$ making the following diagram commute.*

$$\begin{array}{ccc} h(M,\chi;A') & \xrightarrow{\theta_{\chi,\eta}(M)} & h(M,\chi;A') \\ \downarrow{\scriptstyle \lambda_{f_M}} & & \downarrow{\scriptstyle \lambda_{f_M}} \\ A' & \overset{\exists\sigma}{\dashrightarrow} & A' \end{array}$$

*Proof.* Let $f \in S(N, \chi; A)$ be an eigenform.

First suppose that we are given the above diagram for some $\sigma \in \operatorname{Aut} A'$. Let $\ell$ be a prime not dividing $M$. Then from the diagram and the definition of $f_M$ we have $\sigma(a(\ell, f)) = \lambda_{f_M} \circ \theta_{\chi,\eta}(M)({}^M T(\ell))$. From the description of $\theta_{\chi,\eta}(M)$ in Lemma 4.1.2 and the fact that $\eta$ takes values in $A'$ and $\lambda_{f_M}$ is an $A'$-algebra homomorphism, we see that

$$\sigma(a(\ell, f)) = \eta(\ell)\lambda_{f_M}({}^M T(\ell)) = \eta(\ell)a(\ell, f).$$

Thus $\sigma$ is a conjugate self-twist of $f$ with character $\eta$.

Conversely assume that there is a conjugate self-twist $\sigma$ of $f$ with character $\eta$. Then we have that $\rho_f^\sigma \cong \rho_f \otimes \eta$. Since $\rho_f$ is unramified away from $N$ it follows that the only primes $\ell$ for which

$$\sigma(a(\ell, f)) \neq \eta(\ell)a(\ell, f)$$

are those dividing $N$. We need only check that the diagram commutes for ${}^M T(\ell)$ for all primes $\ell$. If $\ell | M/N$ then both compositions are zero. If $\ell \nmid M/N = c(\eta)N$ then using the definition of $\theta_{\chi,\eta}(M)$ and $\lambda_{f_M}$ we see that

$$\sigma \circ \lambda_{f_M} \left( {}^{M}T(\ell) \right) = \lambda_{f_M} \circ \theta_{\chi,\eta}(M) \left( {}^{M}T(\ell) \right),$$

as desired. □

As the previous proposition shows, we will want $\eta$ to be a twist character of $F$ or one of its specializations. We had to impose the condition that $\eta$ be quadratic. By Lemma 3.2.11, this can be achieved by assuming that the Nebentypus $\chi$ is quadratic. By Proposition 3.2.9, for applications to fullness we need only assume that the order of $\chi$ is not divisible by four. In order to use the automorphic lifting techniques developed in the previous section, we must further assume that

$$\chi \text{ has order two.} \tag{4.1}$$

This assumption will be in place for the rest of the chapter.

## 4.2 Reinterpreting $\mathbb{I}$-adic conjugate self-twists

Fix an arithmetic prime $\mathcal{Q}$ of $\mathbb{I}_0$ lying over $P_{k,\varepsilon}$. The total ring of fractions $Q(\mathbb{I}'/\mathcal{Q}\mathbb{I}')$ of $\mathbb{I}'/\mathcal{Q}\mathbb{I}'$ breaks up as a finite product of fields indexed by the primes of $\mathbb{I}'$ lying over $\mathcal{Q}$. Namely

$$Q(\mathbb{I}'/\mathcal{Q}\mathbb{I}') \cong \prod_{\mathfrak{P}'|\mathcal{Q}} Q(\mathbb{I}'/\mathfrak{P}').$$

(This relies two facts. First $\mathcal{Q}$ is an arithmetic prime and hence unramified in $\mathbb{I}'$. Secondly the cokernel of $\mathbb{I}'/\mathcal{Q}\mathbb{I}' \hookrightarrow \prod_{\mathfrak{P}'|\mathcal{Q}} \mathbb{I}'/\mathfrak{P}'$ is finite since $\dim \mathbb{I}' = 2$.) Let $\Gamma_{\mathcal{Q}}$ be the group of all automorphisms $\sigma$ of $Q(\mathbb{I}'/\mathcal{Q}\mathbb{I}')$ for which there is a Dirichlet character $\eta_\sigma$ such that

$$\sigma(a(\ell, F) + \mathcal{Q}\mathbb{I}') = \eta_\sigma(\ell) a(\ell, F) + \mathcal{Q}\mathbb{I}'$$

for all but finitely many primes $\ell$. Since $\mathcal{Q} \subseteq \mathbb{I}_0$ and $\mathbb{I}_0$ is fixed by $\Gamma$, elements of $\Gamma$ preserve $\mathcal{Q}\mathbb{I}'$. Hence there is a natural group homomorphism

$$\Psi : \Gamma \to \Gamma_{\mathcal{Q}}$$

by letting $\sigma \in \Gamma$ act on $Q(\mathbb{I}'/\mathcal{Q}\mathbb{I}')$ via $\sigma(a(\ell, F) + \mathcal{Q}\mathbb{I}') := \sigma(a(\ell, F)) + \mathcal{Q}\mathbb{I}'$. While we expect that $\Psi$ is an isomorphism in general, the purely automorphic techniques only allow us to lift certain elements of $\Gamma_{\mathcal{Q}}$ to $\Gamma$. Let

$$\Gamma^{2,p} = \{\sigma \in \Gamma : \eta_\sigma^2 = 1 \text{ and } p \nmid c(\eta_\sigma)\}$$
$$\Gamma_{\mathcal{Q}}^{2,p} = \{\sigma \in \Gamma_{\mathcal{Q}} : \eta_\sigma^2 = 1 \text{ and } p \nmid c(\eta_\sigma)\}.$$

It is easy to check that $\Gamma^{2,p}$ is a subgroup of $\Gamma$ and $\Gamma_{\mathcal{Q}}^{2,p}$ is a subgroup of $\Gamma_{\mathcal{Q}}$. Note that under assumption (4.1) the condition $\eta_\sigma^2 = 1$ is automatic. Furthermore, by Lemma 3.2.5, $p \nmid c(\eta_\sigma)$. (The proof of Lemma 3.2.5 does not use deformation theory.) We shall show that $\Psi|_{\Gamma^{2,p}} : \Gamma^{2,p} \to \Gamma_{\mathcal{Q}}^{2,p}$ is an isomorphism.

**Proposition 4.2.1.** *The homomorphism $\Psi : \Gamma \to \Gamma_{\mathcal{Q}}$ is injective. Furthermore, $\Psi(\Gamma^{2,p}) = \Gamma_{\mathcal{Q}}^{2,p}$.*

*Proof.* Suppose $\sigma \in \Gamma$ such that $\Psi(\sigma)$ is trivial. Thus for almost all primes $\ell$ we have

$$a(\ell, F) + \mathcal{Q}\mathbb{I}' = \sigma(a(\ell, F)) + \mathcal{Q}\mathbb{I}' = \eta_\sigma(\ell)a(\ell, F) + \mathcal{Q}\mathbb{I}'.$$

Recall that $\mathcal{Q}\mathbb{I}' = \cap_{\mathfrak{P}'|\mathcal{Q}}\mathfrak{P}'$, so for all primes $\mathfrak{P}$ of $\mathbb{I}$ lying over $\mathcal{Q}$ and almost all rational primes $\ell$ we have

$$a(\ell, f_{\mathfrak{P}}) = \eta_\sigma(\ell)a(\ell, f_{\mathfrak{P}}).$$

Since $f_{\mathfrak{P}}$ is a non-CM form it follows that $\eta_\sigma$ must be the trivial character. Therefore $\sigma = 1$ and $\Psi$ is injective.

Now we show that we can lift elements of $\Gamma_{\mathcal{Q}}^{2,p}$. Let $\sigma \in \Gamma_{\mathcal{Q}}^{2,p}$, so for almost all primes $\ell$,

$$\sigma(a(\ell, F) + \mathcal{Q}\mathbb{I}') = \eta_\sigma(\ell)a(\ell, F) + \mathcal{Q}\mathbb{I}'.$$

Let $M = c(\eta_\sigma)N^2$ and consider the map $\theta_{\chi,\eta_\sigma}(M)$ defined before Lemma 4.1.2 with $A = \mathbb{I}$. Since $p \nmid c(\eta_\sigma)$ we know by Lemma 3.2.6 that $\theta_{\chi,\eta_\sigma}(M)$ restricts to an endomorphism:

$$\theta_{\chi,\eta_\sigma}(M) : \mathbf{h}^{\text{ord}}(M, \chi; \mathbb{I}) \to \mathbf{h}^{\text{ord}}(M, \chi; \mathbb{I})$$
$${}^{M}T(n) \mapsto \eta_\sigma(n)\,{}^{M}T(n).$$

Note that $\theta_{\chi,\eta_\sigma}(M)$ restricts to an endomorphism of $\mathbf{h}^{\mathrm{ord}}(M,\chi;\mathbb{I}')$. By Proposition 4.1.4 it suffices to show that $\theta_{\chi,\eta_\sigma}(M)$ preserves the $\mathbb{I}'$-component of the Hecke algebra $\mathbf{h}^{\mathrm{ord}}(M,\chi;\mathbb{I}')$. The induced map $\theta_{\chi,\eta_\sigma}(M)^*$ on spectra must send irreducible components to irreducible components. Furthermore since $\sigma \in \Gamma_{\mathcal{Q}}$ we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathbf{h}^{\mathrm{ord}}(M,\chi;\mathbb{I}') & \xrightarrow{\theta_{\chi,\eta_\sigma}(M)} & \mathbf{h}^{\mathrm{ord}}(M,\chi;\mathbb{I}') \\
\Big\downarrow {\scriptstyle \lambda_{F_M} \bmod \mathcal{Q}\mathbb{I}'} & & \Big\downarrow {\scriptstyle \lambda_{F_M} \bmod \mathcal{Q}\mathbb{I}'} \\
\mathbb{I}'/\mathcal{Q}\mathbb{I}' & \xrightarrow{\quad\sigma\quad} & \mathbb{I}'/\mathcal{Q}\mathbb{I}'
\end{array}
$$

That is, $\theta_{\chi,\eta_\sigma}(M)^*$ maps set of points of $\operatorname{Spec}\mathbb{I}'$ lying over $\mathcal{Q}$ to itself. Hence the two irreducible components $\operatorname{Spec}\mathbb{I}'$ and $\theta_{\chi,\eta_\sigma}(M)^*(\operatorname{Spec}\mathbb{I}')$ of $\operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(M,\chi;\mathbb{I}')$ have nonempty intersection. (Namely, they intersect in some points of $\mathbb{I}'$ lying over $\mathcal{Q}$.) Since $\operatorname{Spec}\mathbf{h}^{\mathrm{ord}}(M,\chi;\mathbb{I}')$ is étale over $\operatorname{Spec}\Lambda$ at arithmetic points [16, Proposition 3.78] and $\mathcal{Q}$ is arithmetic, we must have $\theta_{\chi,\eta_\sigma}(M)^*(\operatorname{Spec}\mathbb{I}') = \operatorname{Spec}\mathbb{I}'$. That is, there is an automorphism $\tilde{\sigma} : \mathbb{I}' \to \mathbb{I}'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbf{h}^{\mathrm{ord}}(M,\chi;\mathbb{I}') & \xrightarrow{\theta_{\chi,\eta_\sigma}(M)} & \mathbf{h}^{\mathrm{ord}}(M,\chi;\mathbb{I}') \\
{\scriptstyle \lambda_{F_M}} \Big\downarrow & & {\scriptstyle \lambda_{F_M}} \Big\downarrow \\
\mathbb{I}' & \xrightarrow{\quad\tilde{\sigma}\quad} & \mathbb{I}' \\
\Big\downarrow & & \Big\downarrow \\
\mathbb{I}'/\mathcal{Q}\mathbb{I}' & \xrightarrow{\quad\sigma\quad} & \mathbb{I}'/\mathcal{Q}\mathbb{I}'
\end{array}
$$

By Lemma 4.1.2 and the definition of $\lambda_{F_M}$ we see that $\tilde{\sigma} \in \Gamma$. As the lower square of the above diagram commutes, it follows that $\Psi(\tilde{\sigma}) = \sigma$, as desired. □

## 4.3 Identifying $\mathbb{I}$-adic and classical decomposition groups

We briefly recall the notation introduced in section 3.5. We have a fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathfrak{P}_0 \in \operatorname{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ be an arithmetic prime of $\mathbb{I}$, and let $\mathcal{Q}$ be the prime of $\mathbb{I}_0$ lying under $\mathfrak{P}_0$. Let $D(\mathfrak{P}_0'|\mathcal{Q}) \subseteq \Gamma$ be the decomposition group of $\mathfrak{P}_0'$ over $\mathcal{Q}$. Let

$$K_{\mathfrak{P}_0} = \mathbb{Q}(\{\iota_p^{-1}(a(n, f_{\mathfrak{P}_0})) : n \in \mathbb{Z}^+\}) \subset \overline{\mathbb{Q}},$$

and let $\Gamma_{\mathfrak{P}_0}$ be the group of all conjugate self-twists of the classical modular form $f_{\mathfrak{P}_0}$. As in the previous section, define

$$\Gamma_{\mathfrak{P}_0}^{2,p} = \{\sigma \in \Gamma_{\mathfrak{P}_0} : \eta_\sigma^2 = 1 \text{ and } p \nmid c(\eta_\sigma)\}.$$

Set $E_{\mathfrak{P}_0} = K_{\mathfrak{P}_0}^{\Gamma_{\mathfrak{P}_0}}$. Let $\mathfrak{q}_{\mathfrak{P}_0}$ be the prime of $K_{\mathfrak{P}_0}$ corresponding to the embedding $\iota_p|_{K_{\mathfrak{P}_0}}$, and set $\mathfrak{p}_{\mathfrak{P}_0} = \mathfrak{q}_{\mathfrak{P}_0}| \cap E_{\mathfrak{P}_0}$. Let $D(\mathfrak{q}_{\mathfrak{P}_0}|\mathfrak{p}_{\mathfrak{P}_0}) \subseteq \Gamma_{\mathfrak{P}}$ be the decomposition group of $\mathfrak{q}_{\mathfrak{P}_0}$ over $\mathfrak{p}_{\mathfrak{P}_0}$. Thus we have that the completion $K_{\mathfrak{P}_0,\mathfrak{q}_{\mathfrak{P}_0}}$ of $K_{\mathfrak{P}_0}$ at $\mathfrak{q}_{\mathfrak{P}_0}$ is equal to $Q(\mathbb{I}/\mathfrak{P}_0)$ and $\mathrm{Gal}(K_{\mathfrak{P}_0,\mathfrak{q}_{\mathfrak{P}_0}}/E_{\mathfrak{P}_0,\mathfrak{p}_{\mathfrak{P}_0}}) = D(\mathfrak{q}_{\mathfrak{P}_0}|\mathfrak{p}_{\mathfrak{P}_0})$. Thus we may view $D(\mathfrak{q}_{\mathfrak{P}_0}|\mathfrak{p}_{\mathfrak{P}_0})$ as the set of all automorphisms of $K_{\mathfrak{P}_0,\mathfrak{q}_{\mathfrak{P}_0}}$ that are conjugate self-twists of $f_{\mathfrak{P}_0}$.

Let

$$\Phi : D(\mathfrak{P}_0'|\mathcal{Q}) \to D(\mathfrak{q}_{\mathfrak{P}_0}|\mathfrak{p}_{\mathfrak{P}_0})$$

be the natural homomorphism defined in section 3.5. We saw that $\Phi$ is an isomorphism in Proposition 3.5.2. In this section we give a second proof that

$$D(\mathfrak{q}_{\mathfrak{P}_0}|\mathfrak{p}_{\mathfrak{P}_0}) \cap \Gamma_{\mathfrak{P}_0}^{2,p} \subseteq \mathrm{Im}\,\Phi.$$

Under assumption (4.1), this shows that $\Phi$ is an isomorphism as in Proposition 3.5.2.

**Theorem 4.3.1.** *We have* $D(\mathfrak{q}_{\mathfrak{P}_0}|\mathfrak{p}_{\mathfrak{P}_0}) \cap \Gamma_{\mathfrak{P}_0}^{2,p} \subseteq \mathrm{Im}\,\Phi$.

*Proof.* Let $\sigma \in D(\mathfrak{q}_{\mathfrak{P}_0}|\mathfrak{p}_{\mathfrak{P}_0}) \cap \Gamma_{\mathfrak{P}_0}^{2,p}$. For any prime$\mathfrak{P}'$ of $\mathbb{I}'$ lying over $\mathcal{Q}$, there is some $\gamma_{\mathfrak{P}'} \in \Gamma$ such that $\gamma_{\mathfrak{P}'}(\mathfrak{P}_0') = \mathfrak{P}'$. Then $\gamma_{\mathfrak{P}'}$ induces an automorphism $\bar{\gamma}_{\mathfrak{P}'}$ of $\overline{\mathbb{Q}}_p$ such that $\bar{\gamma}_{\mathfrak{P}'}(a(\ell, f_{\mathfrak{P}})) = \eta_{\gamma_{\mathfrak{P}'}}(\ell)a(\ell, f_{\mathfrak{P}_0})$ for almost all primes $\ell$. Then

$$\bar{\gamma}_{\mathfrak{P}'}^{-1} \circ \sigma \circ \bar{\gamma}_{\mathfrak{P}'} \in D(\mathfrak{q}_{\mathfrak{P}'}|\mathfrak{p}_{\mathfrak{P}'}).$$

In fact, we can compute the action of this element explicitly. This computation makes use of the fact that all twist characters are quadratic and hence their values are either $\pm 1$. In particular, they are fixed by all automorphisms in question. For almost all primes $\ell$ we have

$$\bar{\gamma}_{\mathfrak{P}'}^{-1} \circ \sigma \circ \bar{\gamma}_{\mathfrak{P}'}(a(\ell, f_{\mathfrak{P}})) = \eta_\sigma(\ell)a(\ell, f_{\mathfrak{P}}). \tag{4.2}$$

This shows that the automorphism $\bar{\gamma}_{\mathfrak{P}'}^{-1} \circ \sigma \circ \bar{\gamma}_{\mathfrak{P}'}$ is independent of the choice of $\gamma_{\mathfrak{P}'}$ sending $\mathfrak{P}'_0$ to $\mathfrak{P}'$.

We can now put all of these automorphisms $\bar{\gamma}_{\mathfrak{P}'}^{-1} \circ \sigma \circ \bar{\gamma}_{\mathfrak{P}'}$ together to obtain an automorphism

$$\pi := \prod_{\mathfrak{P}'|\mathcal{Q}} \bar{\gamma}_{\mathfrak{P}'}^{-1} \circ \sigma \circ \bar{\gamma}_{\mathfrak{P}'}$$

of $Q(\mathbb{I}'/\mathcal{Q}\mathbb{I}') \cong \prod_{\mathfrak{P}'|\mathcal{Q}} Q(\mathbb{I}'/\mathfrak{P}') = \prod_{\mathfrak{P}'|\mathcal{Q}} K_{\mathfrak{P},\mathfrak{q}_{\mathfrak{P}}}$ by simply letting each $\bar{\gamma}_{\mathfrak{P}'}^{-1} \circ \sigma \circ \bar{\gamma}_{\mathfrak{P}'}$ act on $K_{Pp,\mathfrak{q}_{\mathfrak{P}}}$. By equation (4.2) we see that $\pi$ is in fact an element of $\Gamma_{\mathcal{Q}}^{2,p}$. Thus by Proposition 4.2.1 it follows that $\pi$, and hence $\sigma$, can be lifted to an element $\tilde{\sigma} \in \Gamma$. It is clear from the definition of the action of $\sigma$ on $Q(\mathbb{I}'/\mathcal{Q}\mathbb{I}')$ that $\tilde{\sigma} \in D(\mathfrak{P}'_0|\mathcal{Q})$ and $\Phi(\tilde{\sigma}) = \sigma$. □

*Remark* 4. Suppose that $\mathcal{Q}$ lies over $P_{k,1}$ with $k$ divisible by $p-1$. Then $f_{\mathfrak{P}} \in S_k(\Gamma_0(N), \chi)$. Under assumption (4.1) it follows from Lemma 3.2.10 that all twist characters of $f_{\mathfrak{P}}$ are quadratic.

# CHAPTER 5

# Other results

This section contains a collection of small results that I have proved related to the ideas presented in the previous two chapters. The subsections are nearly independent of each other. The results in this section have not, to my knowledge, been published anywhere else.

## 5.1 Another description of $Q(\mathbb{I}_0)$

We can give a more explicit description of $Q(\mathbb{I}_0)$. In fact, the result holds for general fields and more generally than just $\mathrm{GL}_2$-representations. Therefore we shall work with in a more general setting for this section. This description of $Q(\mathbb{I}_0)$ suggests an appropriate analogue when proving big image theorems for larger groups.

Let $K$ be a field of characteristic zero. Let $\mathbb{G}$ be a connected linear algebraic group defined over $K$. Fix an algebraic closure $\overline{K}$ of $K$, and let $L$ be an extension $K$ in $\overline{K}$. Let $G$ be any group and

$$\rho : G \to \mathbb{G}(L)$$

be a representation.

**Definition 5.1.1.** An automorphism $\sigma$ of $L$ fixing $K$ is a *conjugate self-twist* of $\rho$ if there is a character $\eta : G \to L^\times$ such that

$$\rho^\sigma \cong \eta \otimes \rho.$$

Let $\Gamma$ denote the group of all conjugate self-twists of $\rho$, and

$$L_0 = L^\Gamma = \{x \in L : \sigma(x) = x, \forall \sigma \in \Gamma\}.$$

Let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$ and $\mathrm{Ad} : \mathbb{G} \to \mathrm{End}\,\mathfrak{g}$ the adjoint representation. Let $Z(\mathbb{G})$ denote the center of $\mathbb{G}$ and $P\mathbb{G} = \mathbb{G}/Z(\mathbb{G})$.

**Proposition 5.1.2.** *Assume that* $\mathrm{Ad}\,\rho$ *is absolutely irreducible. Assume further that there is an embedding* $\iota : \mathbb{G} \hookrightarrow \mathrm{GL}_n$ *over* $K$ *such that* $Z(\mathbb{G})$ *is contained in the diagonal matrices* $\mathbb{G}_m \subseteq \mathrm{GL}_n$*. Then* $L_0$ *is generated over* $K$ *by the values of the trace of* $\mathrm{Ad}\,\rho$*.*

*Proof.* First take $\sigma \in \Gamma$ and $g \in G$. By definition of $\Gamma$ there is a character $\eta : G \to L^\times$ such that $\rho^\sigma \cong \eta \otimes \rho$. Thus

$$\mathrm{Ad}\,\rho^\sigma \cong \mathrm{Ad}(\eta \otimes \rho) = \mathrm{Ad}\,\rho.$$

The last equality follows from the fact that, via $\iota$, we can view the adjoint representation simply as matrix conjugation. In particular it is unaffected by the scalar values of $\eta$. Applying both sides to $g$ and taking traces shows that $\mathrm{tr}\,\mathrm{Ad}\,\rho(g) \in L_0$.

Now assume that $\sigma \in \mathrm{Gal}(\overline{K}/K(\mathrm{tr}\,\mathrm{Ad}\,\rho))$. We will show that $\sigma|_L$ is a conjugate self-twist of $\rho$ and thus fixes $L_0$ pointwise. Since

$$\mathrm{tr}\,\mathrm{Ad}\,\rho^\sigma(g) = \mathrm{tr}\,\mathrm{Ad}\,\rho(g)$$

for all $g \in G$ and since $\mathrm{Ad}\,\rho$ is absolutely irreducible by assumption, it follows that

$$\mathrm{Ad}\,\rho^\sigma \cong \mathrm{Ad}\,\rho. \tag{5.1}$$

Since $\mathbb{G}$ is connected and $\mathrm{char}\,K = 0$, the kernel of the natural map $\mathrm{Ad} : \mathbb{G} \to \mathrm{End}\,\mathfrak{g}$ is the center of $\mathbb{G}$. (See Lemma 5.1.3 below for this fact.) Thus we may view $\mathrm{Ad}\,\rho : G \to \mathrm{Ad}\,\mathbb{G}(L) \cong P\mathbb{G}(L)$. Indeed, an element $g \in G$ is mapped to its class $[\rho(g)] \in P\mathbb{G}(L)$, and similarly for $\mathrm{Ad}\,\rho^\sigma$. By (5.1) it follows that

$$\rho^\sigma \cong \rho \mod Z(\mathbb{G}).$$

Define $\eta : G \to L^\times$ by

$$\eta(g) = \iota(\rho^\sigma(g)\rho(g)^{-1}) \in \iota(Z(\mathbb{G})) \subseteq \mathbb{G}_m(L) = L^\times.$$

Since $\rho^\sigma(g)\rho(g)^{-1} \in Z(\mathbb{G})$ for all $g \in G$ it follows easily that $\eta$ is a character and $\rho^\sigma = \rho \otimes \eta$, as desired. □

**Lemma 5.1.3.** *If $\mathbb{G}$ is a connected algebraic group over a field of characteristic zero, then $Z(\mathbb{G})$ is the kernel of* $\mathrm{Ad} : \mathbb{G} \to \mathrm{End}\, \mathfrak{g}$.

*Proof.* Embed $\mathbb{G} \hookrightarrow \mathrm{GL}_n$ so that we may view everything as matrices. As $\mathrm{char}\, K = 0$ we have the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ given by

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

If $g \in \ker \mathrm{Ad}$ then $g$ commutes with all elements of $\mathfrak{g}$. Thus $g$ also commutes with all elements of $\exp(\mathfrak{g})$. Since $\mathbb{G}$ is connected, $\exp(\mathfrak{g})$ generates $\mathbb{G}$. Thus $\ker \mathrm{Ad}$ is contained in the center of $\mathbb{G}$. The reverse containment is evident. □

## 5.2 Identifying $G/H$ with $\Gamma^*$

Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation. For now, we will let $\mathcal{O}$ be either a finite extension of $\mathbb{Z}_p$ or a finite extension of $\Lambda = \mathbb{Z}_p[[T]]$. Assume $\mathcal{O}$ is the integral closure of either $\mathbb{Z}_p$ or $\Lambda$ in the field generated by the traces of $\rho$.

Let $N$ be the conductor of $\rho$ and $\chi$ its Nebentypus. Let $\Gamma$ be the group of all conjugate self twists of $\rho$ in the sense of Definition 5.1.1. Assume $\rho$ does not have CM, so there is no nontrivial character $\eta$ for which $\rho \cong \eta \otimes \rho$. Let

$$H = \bigcap_{\sigma \in \Gamma} \ker \eta_\sigma.$$

For any finite abelian group $A$ we'll write $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ for the Pontryagin dual.

The main point of this section is to identify $G/H$ with $\Gamma^*$. In order to do this, we need a few facts about conjugate self twists that were proved in Momose's paper, namely Lemma 3.2.10 and for every $\sigma \in \Gamma$ the character $\eta_\sigma$ has conductor dividing $N$ [34]. That is, $\eta_\sigma$ can be viewed as a map $(\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}$.

**Proposition 5.2.1.** *There is a natural isomorphism $G_{\mathbb{Q}}/H \cong \Gamma^*$.*

*Proof.* Since $\rho$ does not have CM there is a natural inclusion

$$\Gamma \hookrightarrow \mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}), \mathbb{Q}/\mathbb{Z})$$

given by $\sigma \mapsto \eta_\sigma$. Recall that the natural isomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ is given by $a \leftrightarrow \sigma_a$, where $\sigma_a$ acts on a primitive $N$-th root of unity $\zeta_N$ by $\zeta_N^{\sigma_a} = \zeta_N^a$. Therefore Pontryagin duality yields a surjection

$$\Phi : \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \twoheadrightarrow \Gamma^*$$

given explicitly by $\Phi(\sigma_a)(\gamma) = \eta_\gamma(a)$ (where we regard $\mathbb{Q}/\mathbb{Z}$ as the group of all finite order roots of unity).

Note that

$$\ker \Phi = \{\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) : \eta_\gamma(a) = 1, \forall \gamma \in \Gamma\}.$$

Thus under the isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong G_{\mathbb{Q}}/\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_N))$ we have $\ker \Phi$ corresponds to $H \cdot \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_N))/\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_N))$. Therefore we have

$$\Gamma^* \cong \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})/\ker \Phi \cong G_{\mathbb{Q}}/H \cdot \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_N)) \cong \mathrm{Gal}(\overline{\mathbb{Q}}^H \cap \mathbb{Q}(\mu_N)/\mathbb{Q}).$$

Hence it suffices to show that $\overline{\mathbb{Q}}^H \subseteq \mathbb{Q}(\mu_N)$ since $\mathrm{Gal}(\overline{\mathbb{Q}}^H/\mathbb{Q}) \cong \mathbb{G}_{\mathbb{Q}}/H$.

For each $\sigma \in \Gamma$, let $K_\sigma = \overline{\mathbb{Q}}^{\ker \eta_\sigma}$, so each $K_\sigma$ is a finite extension of $\mathbb{Q}$. By Galois theory we know that $\overline{\mathbb{Q}}^H = \prod_{\sigma \in \Gamma} K_\sigma$, the compositum of all $K_\sigma$'s. Thus it suffices to show that $\mathbb{Q}(\mu_N) \supseteq K_\sigma$ for all $\sigma \in \Gamma$.

Since each $\eta_\sigma$ may be viewed as a character of $(\mathbb{Z}/N\mathbb{Z})^\times$ [34], we have

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_N)) \subseteq \ker \eta_\sigma.$$

Thus $\mathbb{Q}(\mu_N) \supseteq K_\sigma$, as desired. □

## 5.3 Residual conjugate self-twists when $\mathrm{Im}\,\bar{\rho}_F \supseteq \mathrm{SL}_2(\mathbb{F}_p)$

In this section we investigate what we can say about conjugate self-twists in the case when $\mathrm{Im}\,\bar{\rho} \supseteq \mathrm{SL}_2(\mathbb{F}_p)$. Let $p \geq 5$ throughout this section. Let $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous odd representation such that $\operatorname{Im}\bar{\rho} \supseteq \mathrm{SL}_2(\mathbb{F}_p)$. Assume that $\mathbb{F}$ is generated over $\mathbb{F}_p$ by the trace of $\rho$, which is possible since $\bar{\rho}$ is absolutely irreducible.

By Serre's Conjecture we know that $\bar{\rho}$ is the reduction of $\rho_f$ for some modular form $f$ [23, 24]. Let $\Gamma$ be the group of conjugate self twists of $f$. We will prove that all conjugate self-twists act trivially on the residue field and thus are contained in the inertia group.

**Lemma 5.3.1.** *Let $p \geq 5$ and let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$. Let $G$ be a subgroup of $\mathrm{GL}_2(\mathbb{F})$ such that $G \supseteq \mathrm{SL}_2(\mathbb{F}_p)$. Assume that $\mathbb{F}$ is generated by $\{\operatorname{tr} g : g \in G\}$ over $\mathbb{F}_p$. Then $G \supseteq \mathrm{SL}_2(\mathbb{F})$. Furthermore, writing $D = \{\det g : g \in G\}$ we have*

$$G = \{x \in \mathrm{GL}_2(\mathbb{F}) : \det x \in D\}.$$

*Proof.* The second claim follows immediately from claim that $G \supseteq \mathrm{SL}_2(\mathbb{F})$. Let $\mathbb{G} = G \cdot \mathbb{F}^\times$ and $\bar{\mathbb{G}} = \mathbb{G}/\mathbb{F}^\times \subseteq \mathrm{PGL}_2(\mathbb{F})$. By Dickson's classification of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}})$ and the facts that $G \supseteq \mathrm{SL}_2(\mathbb{F}_p)$ and $p \geq 5$ it follows that, for some $x \in \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, we have $x\overline{\mathbb{G}}x^{-1} = \mathrm{PSL}_2(\mathbb{E})$ or $x\overline{\mathbb{G}}x^{-1} = \mathrm{PGL}_2(\mathbb{E})$ for some finite extension $\mathbb{E}/\mathbb{F}_p$. Therefore $xGx^{-1} \cdot \mathbb{F}^\times$ is either $\mathrm{GL}_2(\mathbb{E})$ or $\mathrm{SL}_2(\mathbb{E}) \cdot \mathbb{E}^\times$. In either case $\mathbb{F} \subseteq \mathbb{E}$.

Note that intersecting either $\mathrm{GL}_2(\mathbb{E})$ or $\mathrm{SL}_2(\mathbb{E}) \cdot \mathbb{E}^\times$ with $\mathrm{SL}_2(\mathbb{E})$ gives $\mathrm{SL}_2(\mathbb{E})$. On the other hand,

$$xGx^{-1} \cdot \mathbb{F}^\times \cap \mathrm{SL}_2(\mathbb{E}) = \bigcup_{\alpha \in \mathbb{F}^\times} x(\alpha^{-1} G_{\alpha^2} \cap \mathrm{SL}_2(\mathbb{F}))x^{-1} \subseteq x\,\mathrm{SL}_2(\mathbb{F})x^{-1},$$

where $G_\alpha = \{g \in G : \det g = \alpha\}$. Therefore

$$|\,\mathrm{SL}_2(\mathbb{E})| \leq |x\,\mathrm{SL}_2(\mathbb{F})x^{-1}| = |\,\mathrm{SL}_2(\mathbb{F})|.$$

Since $\mathbb{F} \subseteq \mathbb{E}$, this is only possible when $\mathbb{F} = \mathbb{E}, G \supseteq \mathrm{SL}_2(\mathbb{F})$, and $x \in \mathrm{GL}_2(\mathbb{F})$, as desired. □

**Proposition 5.3.2.** *Suppose $\sigma$ is an automorphism of $\mathbb{F}$ and $\eta : G_{\mathbb{Q}} \to \mathbb{F}^\times$ such that $\bar{\rho}^\sigma \cong \eta \otimes \sigma$. Then $\sigma = 1$ and $\eta = 1$.*

*Proof.* Let us write $G = \operatorname{Im}\bar{\rho}$ and $H = \bar{\rho}(\cap_{\gamma \in \Gamma} \ker \eta_\gamma)$. We claim that $H \supseteq \mathrm{SL}_2(\mathbb{F}_p)$. Recall that $G_{\mathbb{Q}}/\cap_{\gamma \in \Gamma} \ker \eta_\gamma \cong \Gamma^*$ (Proposition 5.2.1) and $\Gamma$ is a finite abelian 2-group (Proposition

3.6.1). Since $\mathrm{SL}_2(\mathbb{F}_p) \subseteq G$ it follows that $H \cap \mathrm{SL}_2(\mathbb{F}_p)$ is a normal subgroup in $\mathrm{SL}_2(\mathbb{F}_p)$ with index a power of 2. Since $\mathrm{PSL}_2(\mathbb{F}_p)$ is simple for $p \geq 5$ and the sequence

$$1 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PSL}_2(\mathbb{F}_p)$$

is non-split, it follows that we must have $H \cap \mathrm{SL}_2(\mathbb{F}_p) = \mathrm{SL}_2(\mathbb{F}_p)$.

Let $\mathbb{F}_0 = \mathbb{F}^\Gamma$. By Theorem 3.3.1 we may assume $H \subseteq \mathrm{GL}_2(\mathbb{F}_0)$. By Lemma 5.3.1 and the fact that $H \supseteq \mathrm{SL}_2(\mathbb{F}_p)$ we have $H \supseteq \mathrm{SL}_2(\mathbb{F}_0)$ and $G \supseteq \mathrm{SL}_2(\mathbb{F})$. Let $q = |\mathbb{F}_0|$ and $n = [\mathbb{F} : \mathbb{F}_0]$. Let $D_G = \{\det g : g \in G\}$ and $D_H = \{\det h : h \in H\}$. Then we have

$$|G| = |\mathrm{SL}_2(\mathbb{F})||D_G| = q^n(q^{2n} - 1)|D_G|$$
$$|H| = |\mathrm{SL}_2(\mathbb{F}_0)||D_H| = q(q^2 - 1)|D_H|.$$

Thus if $n > 1$ then $q|[G : H]$. But $[G : H]|[G_{\mathbb{Q}} : \cap_{\gamma \in \Gamma} \ker \eta_\gamma|] = |\Gamma^*|$, and $\Gamma$ is a 2-group. Since $p \geq 5$ this is impossible. Therefore we must have $n = 1$ and $\Gamma$ acts trivially on $\mathbb{F}$. □

## 5.4 Containing $\mathrm{SL}_2$

One is often interested in determining when the image of given representation contains $\mathrm{SL}_2$ of some appropriate ring. Certainly if this is the case, then the image of the residual representation will contain $\mathrm{SL}_2(\mathbb{F}_p)$. In this section we discuss sufficient conditions for a group of matrices to contain $\mathrm{SL}_2$.

Let $R$ be a local $p$-profinite ring with residue field $\mathbb{F}$ of characteristic $p > 2$. Let $\mathfrak{m}$ denote the maximal ideal of $R$. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(R)$ be a continuous representation (often assumed to be ordinary and modular). Fix a closed subgroup $G \leq \mathrm{GL}_2(R)$.

Write $\overline{G}$ for the image of $G$ under the natural projection to $\mathrm{GL}_2(\mathbb{F})$, so $\overline{G} \cong G \cdot \Gamma_R(\mathfrak{m})/\Gamma_R(\mathfrak{m})$. For each $k \geq 1$, write $G(k)$ for the image of $G$ in $\mathrm{GL}_2(R/\mathfrak{m}^k)$, so $G(1) = \overline{G}$.

**Lemma 5.4.1.** *Let $k \geq 1$. Assume $\Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1}) \subseteq G(k+1)$ and $G(k) \supseteq \mathrm{SL}_2(R/\mathfrak{m}^k)$. Then $G(k+1) \supseteq \mathrm{SL}_2(R/\mathfrak{m}^{k+1})$.*

*Proof.* Let $M_n(\mathfrak{m}^k/\mathfrak{m}^{k+1})$ denote the set of all $n \times n$-matrices with all entries in $\mathfrak{m}^k/\mathfrak{m}^{k+1}$. Note that the determinant map induces an isomorphism

$$(1 + M_n(\mathfrak{m}^k/\mathfrak{m}^{k+1}))/\Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1}) \cong 1 + \mathfrak{m}^k/\mathfrak{m}^{k+1}.$$

Since $\Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1}) \subseteq G(k+1)$ we have that

$$(G(k+1) \cap (1 + M_n(\mathfrak{m}^k/\mathfrak{m}^{k+1})))/\Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1})$$

is a subgroup of $(1+M_n(\mathfrak{m}^k/\mathfrak{m}^{k+1}))/\Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1})$. Let $D_{k+1}$ be the image of $G(k+1) \cap (1+M_n(\mathfrak{m}^k/\mathfrak{m}^{k+1}))$ under the determinant map. Then we see that for any $y \in M_n(\mathfrak{m}^k/\mathfrak{m}^{k+1})$, we have $1 + y \in G(k+1)$ if and only if $\det(1+y) \in D_{k+1}$.

Let $x \in \mathrm{SL}_n(R/\mathfrak{m}^{k+1})$. Since $G(k) \supseteq \mathrm{SL}_n(R/\mathfrak{m}^k)$ there is some $g \in G(k+1)$ such that $x \equiv g \mod (\mathfrak{m}^k/\mathfrak{m}^{k+1})$. That is, $xg^{-1} \in 1 + M_n(\mathfrak{m}^k/\mathfrak{m}^{k+1})$. Write $xg^{-1} = 1 + y$ for some $y \in M_n(\mathfrak{m}^k/\mathfrak{m}^{k+1})$. Note that

$$\det(1+y) = \det(xg^{-1}) = \det g^{-1} \in D_{k+1},$$

and therefore $1 + y \in G(k+1)$ by the conclusion of the first paragraph of this proof. □

There is something more that can be said in the case when $R$ is a DVR.

**Lemma 5.4.2.** *Assume* $\mathrm{char}\,\mathbb{F} \nmid n$. *Let $R$ be a DVR and $k \geq 1$. If $\overline{G} \supseteq \mathrm{SL}_n(\mathbb{F})$ and $G(k+1) \cap \Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1}) \neq \{1\}$, then $G(k+1) \supseteq \Gamma_{R/m^{k+1}(m^k/m^{k+1})}$.*

*Proof.* Let $\pi$ be a uniformizer of $R$. Note that $G(k+1)$ acts on $G(k+1) \cap \Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1})$ by conjugation. Furthermore, the group $\Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1})$ with multiplication is isomorphic to the additive group $\mathfrak{sl}_n(\mathbb{F})$ via $x \mapsto \frac{1}{\pi^k}(x-1)$. From this formula, we see that for any $g \in G(k+1)$ we have $gxg^{-1} \mapsto \bar{g}(\frac{1}{\pi^k}(x-1))\bar{g}^{-1}$. By assumption, $\overline{G} \supseteq \mathrm{SL}_n(\mathbb{F})$. Since $\mathrm{char}\,\mathbb{F} \nmid n$, the conjugation action of $\mathrm{SL}_n(\mathbb{F})$ on $\mathfrak{sl}_2(\mathbb{F})$ is absolutely irreducible. Therefore $G(k+1) \supseteq \Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1})$, as desired. □

Now we give some conditions under which we can remove the first assumption in Lemma 5.4.1 and still retain the conclusion. As above, $R$ will be a DVR with uniformizer $\pi$.

**Lemma 5.4.3.** *Assume $R$ is a DVR, and let $k \geq 2$. If $G(k) \supseteq \mathrm{SL}_2(R/\mathfrak{m}^k)$ then $G(k+1) \supseteq \mathrm{SL}_2(R/\mathfrak{m}^{k+1})$.*

*Proof.* By Lemma 5.4.1 it suffices to show that $G(k+1) \supseteq \Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1})$. Choose integers $i, j \geq 1$ such that $i+j = k$ (which is possible as $k \geq 2$). As $G(k) \supset \mathrm{SL}_2(R/\mathfrak{m}^k)$ we can find $x, y \in G$ such that $x \equiv \left(\begin{smallmatrix} 1 & \pi^i \\ 0 & 1 \end{smallmatrix}\right) \mod \mathfrak{m}^k$ and $y \equiv \left(\begin{smallmatrix} 1 & 0 \\ \pi^j & 1 \end{smallmatrix}\right) \mod \mathfrak{m}^k$. A direct computation with matrices shows that there is some $u \in (R/\mathfrak{m}^{k+1})^\times$ for which $xyx^{-1}y^{-1} \equiv \left(\begin{smallmatrix} 1+u\pi^k & 0 \\ 0 & 1-u\pi^k \end{smallmatrix}\right)$ mod $\mathfrak{m}^{k+1}$. Since $x, y \in G$ we have $xyx^{-1}y^{-1} \in G$ and hence $G(k+1) \cap \Gamma_{R/\mathfrak{m}^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1}) \neq \{1\}$. Therefore, by Lemma 5.4.2, it follows that $G(k+1) \supseteq \Gamma_{R/m^{k+1}}(\mathfrak{m}^k/\mathfrak{m}^{k+1})$, as desired. □

**Corollary 5.4.4.** *Let $R$ be a DVR and $G \leq \mathrm{GL}_2(R)$ such that $\overline{G} \supseteq \mathrm{SL}_2(\mathbb{F})$ and $G(2) \supseteq \Gamma_{R/\mathfrak{m}^2}(\mathfrak{m}/\mathfrak{m}^2)$. Then $G \supseteq \mathrm{SL}_2(R)$.*

*Proof.* Direct application of Lemma 5.4.1 and Lemma 5.4.3. □

Now we study what happens going from mod $\mathfrak{m}$ to mod $\mathfrak{m}^2$.

**Proposition 5.4.5.** *Let $R$ be a DVR with finite residue field $\mathbb{F}$ with $\mathrm{char}\,\mathbb{F} = p > 2$. Let $H$ be a subgroup of $\mathrm{GL}_2(R/\mathfrak{m}^2)$ such that the projection of $H$ to $\mathrm{GL}_2(\mathbb{F})$ contains $\mathrm{SL}_2(\mathbb{F})$.*

1. *If $0 \neq p \in R/\mathfrak{m}^2$ then $H \supseteq \mathrm{SL}_2(R/\mathfrak{m}^2)$.*

2. *If $0 = p \in R/\mathfrak{m}^2$ then $H \supseteq \mathrm{SL}_2(R/\mathfrak{m}^2)$ if and only if $H \cap \Gamma_{R/\mathfrak{m}^2}(\mathfrak{m}/\mathfrak{m}^2) \neq \{1\}$.*

*Proof.* First assume that $0 \neq p \in R/\mathfrak{m}^2$. Let $w \in \mathbb{F}$ and choose $x \in H$ such that $x \equiv \left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)$ mod $\mathfrak{m}^2$. We claim that if $w \neq 0$ then $x$ has order $p^2$. To see this, one shows by induction on $n$ that, for $a, b, c, d \in \mathfrak{m}/\mathfrak{m}^2$ and $\tilde{w}$ a lift of $w$,

$$\begin{pmatrix} 1+a & \tilde{w}+b \\ c & 1+d \end{pmatrix}^n = \begin{pmatrix} 1+na+\alpha_n\tilde{w}c & n\tilde{w}+nb+\tilde{w}^2\beta_n c+\alpha_n\tilde{w}(a+d) \\ nc & 1+nd+\alpha_n\tilde{w}c \end{pmatrix},$$

where $\alpha_{n+1} = \alpha_n + n, \beta_{n+1} = \alpha_n + \beta_n, \alpha_1 = 0, \beta_1 = 0$. One calculates that $\alpha_p, \beta_p$ are both divisible by $p$ and hence taking $n = p$ in the above formula proves that $x^p \neq 1$. Since

$x^p \in G \cap \Gamma_{R/\mathfrak{m}^2}(\mathfrak{m}/\mathfrak{m}^2)$ it follows from Lemma 5.4.2 that $H \supseteq \Gamma_{R/\mathfrak{m}^2}(\mathfrak{m}/\mathfrak{m}^2)$ and then Lemma 5.4.1 gives the desired result.

For the case when $0 = p \in R/\mathfrak{m}^2$, one direction is obvious. On the other hand, if $H \cap \Gamma_{R/\mathfrak{m}^2}(\mathfrak{m}/\mathfrak{m}^2) \neq \{1\}$ then Lemma 5.4.2 implies that $H \supseteq \mathrm{SL}_2(\mathcal{O}/\mathfrak{m}^2)$. □

In an appendix of [30], Boston proves the following general proposition.

**Proposition 5.4.6.** *Let $R$ be a complete noetherian local ring with maximal ideal $\mathfrak{m}$. Assume $R/\mathfrak{m}$ is a finite field with characteristic different from 2. If $H$ is a closed subgroup of $\mathrm{SL}_2(R)$ whose image modulo $\mathfrak{m}^2$ is equal to $\mathrm{SL}_2(R/\mathfrak{m}^2)$, then $H = \mathrm{SL}_2(R)$.*

I was interested in whether the following generalization of Boston's theorem might hold.

**Question.** *Let $R, \mathfrak{m}$ be as in Proposition 5.4.6. Let $H$ be a subgroup of $\mathrm{GL}_2(R)$ such that the image of $H$ in $\mathrm{GL}_2(R/\mathfrak{m}^2)$ contains $\mathrm{SL}_2(R/\mathfrak{m}^2)$. Does $H$ contain $\mathrm{SL}_2(R)$?*

I suspect the answer to the above question is 'no'. Here is my proposed counterexample. Let $R = \Lambda = \mathbb{Z}_p[[T]]$ and $H$ be the subgroup (topologically) generated by $\mathrm{SL}_2(\mathbb{Z}_p) \subset \mathrm{SL}_2(\Lambda)$ and $\tau = \left(\begin{smallmatrix} 1+T^2 & T \\ 0 & 1 \end{smallmatrix}\right)$. Then the image of $H$ in $\mathrm{SL}_2(\Lambda/\mathfrak{m}_\Lambda^2)$ is contained in $\mathrm{SL}_2(\Lambda/\mathfrak{m}_\Lambda^2)$, and I believe $\mathrm{SL}_2(\Lambda/\mathfrak{m}_\Lambda^2)$ should be generated by $\mathrm{SL}_2(\mathbb{Z}/p^2\mathbb{Z})$ and $\left(\begin{smallmatrix} 1 & T \\ 0 & 1 \end{smallmatrix}\right)$. However, $H \not\subseteq \mathrm{SL}_2(\Lambda)$ since $\det \tau = 1 + T^2$. It seems that $H$ should be too small to contain $\mathrm{SL}_2(\Lambda)$. Indeed, it is hard to imagine how one would get $\left(\begin{smallmatrix} 1 & T \\ 0 & 1 \end{smallmatrix}\right)$ in $H$, but I cannot prove this is the case. Part of the difficulty lies in the fact that neither $\mathrm{SL}_2(\mathbb{Z}_p)$ nor the cyclic subgroup (topologically) generated by $\tau$ normalize the other. Therefore, there are relatively few tools from group theory available to study the group they generate.

However, this is a purely group theoretic counterexample. It is not obvious how one would construct an element like $\tau$ in the image of a Galois representation. Therefore, one could refine the question by requiring that $H$ be the image of a Galois representation. Perhaps that refinement has a positive answer.

# CHAPTER 6

# Conjectures and Future Work

## 6.1 Determining the $\mathbb{I}_0$-level and relation to $p$-adic $L$-functions

Theorem 3.1.4 guarantees that there is a non-zero $\mathbb{I}_0$-ideal $\mathfrak{a}_0$ such that the image of $\rho_F$ contains $\Gamma_{\mathbb{I}_0}(\mathfrak{a}_0)$. The largest such ideal is called the $\mathbb{I}_0$-*level* of $\rho_F$ and is denoted $\mathfrak{c}_{\mathbb{I}_0,F}$. A natural question is to determine the $\mathbb{I}_0$-level. More generally, as discussed in Remark 2 one knows that for *any* $\Lambda$-order $R$ in $Q(\mathbb{I}_0)$, there is a non-zero $R$-ideal $\mathfrak{a}_R$ such that $\operatorname{Im}\rho_F \supseteq \Gamma_R(\mathfrak{a}_R)$. We may define the largest such $R$-ideal to be the $R$-*level* of $\rho_F$, denoted $\mathfrak{c}_{R,F}$. Thus, one might ask for a natural choice of $R^{\mathrm{can}}$ such that $\Gamma_{R^{\mathrm{can}}}(\mathfrak{c}_{R^{\mathrm{can}},F})$ is maximal among $\Gamma_R(\mathfrak{c}_{R,F})$, as $R$ varies over $\Lambda$-orders of $Q(\mathbb{I}_0)$.

**Conjecture 6.1.1.** *We can describe* $R^{\mathrm{can}}$ *explicitly as* $R^{\mathrm{can}} = \Lambda[\operatorname{tr}\operatorname{ad}\rho_F]$.

The plausibility of this conjecture, comes from Proposition 5.1.2, namely that $Q(\mathbb{I}_0) = Q(\Lambda)(\operatorname{tr}\operatorname{ad}\rho_F)$. That is, the field fixed by conjugate self-twists is equal to the field generated by the trace of the adjoint representation. Conjecture 6.1.1 can be seen as an integral refinement of this result.

Recall the heuristic from the introduction: the image of $\rho_F$ should be as large as possible, subject to the symmetries of $F$. Theorem 3.1.4 shows that $Q(\mathbb{I}_0)$ is the largest field such that $\rho_F$ is $R$-full for every $\Lambda$-order $R$ of $Q(\mathbb{I}_0)$. In this sense, the conjugate self-twists of $F$ account for all of its symmetries. Therefore, Proposition 5.1.2 shows that, on the level of fields, $\operatorname{tr}\operatorname{ad}\rho_F$ captures the same information as the symmetries of $F$. The heuristic therefore suggests that if Conjecture 6.1.1 is false, then there is a new subtle type of symmetry of $F$ that is only visible at the integral level.

I have the following outline for a strategy to prove Conjecture 6.1.1 in the case when $\operatorname{Im}\bar{\rho}_F \supseteq \mathrm{SL}_2(\mathbb{F}_p)$. The idea is to identify $R^{\mathrm{can}}$ as an appropriate universal deformation ring and the prove an $R = \mathbb{T}$ theorem, where the Hecke side can be identified as $\Lambda[\operatorname{tr}\operatorname{ad}\rho_F]$. Indeed, $\operatorname{ad}\rho_F$ is essentially an orthogonal representation, and so one can look at the universal deformation ring $R$ parametrizing orthogonal deformations of $\operatorname{ad}\bar{\rho}_F$, as in Tilouine's book [46]. Under the assumption that $\operatorname{Im}\rho_F \supseteq \mathrm{SL}_2(\mathbb{F}_p)$, the adjoint representation $\operatorname{ad}\bar{\rho}_F$ will be absolutely irreducible and hence $R$ will exist [46]. By an analogue of Carayol's Theorem we may assume $\operatorname{ad}\rho_F$ takes values in the orthogonal group over $\Lambda[\operatorname{tr}\operatorname{ad}\rho_F]$. By universality, there is an algebra homomorphism $\alpha : R \to \mathbb{I}^\Gamma$, where we define $\mathbb{I}^\Gamma = Q(\mathbb{I})^\Gamma \cap \mathbb{I}$ in the case when $\mathbb{I}$ is not normal. Thus, if we can prove that $\alpha$ is surjective and $R = \Lambda[\operatorname{tr}\operatorname{ad}\rho^{\mathrm{univ}}]$, then it will follow that $\mathbb{I}^\Gamma = \Lambda[\operatorname{tr}\operatorname{ad}\rho_F]$ and we'll have the desired integral statement.

The $R^{\mathrm{can}}$-level of $\rho_F$ is expected to depend on the shape of the image of the residual representation. The following is expected to be the correct generalization of Hida's Theorem II in [19].

**Conjecture 6.1.2.** *Let $p > 2$ be prime and $F$ a Hida family that is satisfies the hypotheses of Theorem 3.1.4.*

1. *If $\operatorname{Im}\rho_F \supseteq \mathrm{SL}_2(\mathbb{F}_p)$, then $\mathfrak{c}_{R^{\mathrm{can}},F} = R^{\mathrm{can}}$. That is, $\operatorname{Im}\rho_F \supseteq \mathrm{SL}_2(R^{\mathrm{can}})$.*

2. *Suppose that $\bar{\rho}_F \cong \operatorname{Ind}_M^{\mathbb{Q}}\bar{\psi}$ for an imaginary quadratic field $M$ in which $p$ splits and a character $\bar{\psi} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{F}}_p^\times$. Assume $M$ is the only quadratic field for which $\bar{\rho}_F$ is induced from a Hecke character of that field. Under minor conditions on the tame level of $F$, there is a product $\mathcal{L}_0$ of anticyclotomic Katz $p$-adic $L$-functions such that $\mathfrak{c}_{R^{\mathrm{can}},F}$ is a factor of $\mathcal{L}_0$. Furthermore, every prime factor of $\mathcal{L}_0$ is a factor of $\mathfrak{c}_{R^{\mathrm{can}},F}$ for some $F$.*

3. *If $\bar{\rho} \cong \operatorname{Ind}_M^{\mathbb{Q}}\bar{\psi}$ for a real quadratic field $M$ and character $\bar{\psi} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{F}}_p^\times$ and $\bar{\rho}$ is not induced from any imaginary quadratic field, then $\mathfrak{c}_{R^{\mathrm{can}},F} = (1+T)^m - 1$ for some integer $m \geq 0$.*

4. *If the image of $\bar{\rho}_F$ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ is tetrahedral, octahedral, or icosahedral, then $\mathfrak{c}_{R^{\mathrm{can}},F} = T^n \cdot R^{\mathrm{can}}$ for some integer $n \geq 1$.*

Note that even when $\mathbb{I}_0 = \Lambda$, part (1) of Conjecture 6.1.2 is stronger than Hida's Theorem II [19]. Indeed, under the assumption that $\mathrm{Im}\,\bar{\rho}_F \supseteq \mathrm{SL}_2(\mathbb{F}_p)$, Hida's theorem only guarantees that $\mathfrak{c}_{\Lambda,F} \supseteq \mathfrak{m}_\Lambda^r$ for some $r \geq 0$. In proving case (1) of the conjecture, I will make use of Manoharmayum's recent work that shows $\mathrm{Im}\,\rho_F \supseteq \mathrm{SL}_2(W)$ for a finite unramified extension $W$ of $\mathbb{Z}_p$ [29]. By combining this with the $\Lambda$-module structure on the Pink Lie algebra associated to $\mathrm{Im}\,\rho_F$ that was used in the proof of Theorem 3.1.4, I hope to show that $\mathrm{Im}\,\rho_F \supseteq \mathrm{SL}_2(R^{\mathrm{can}})$.

Parts (2) and (3) of Conjecture 6.1.2 are also stronger than Hida's Theorem II [19]. When $\bar{\rho}_F$ is induced from an imaginary quadratic field (and $\mathbb{I} = \Lambda$), Hida's theorem shows that $\mathfrak{c}_{\Lambda,F}$ is a factor of $\mathcal{L}_0^2$ and every prime factor of $\mathcal{L}_0$ is a factor of $\mathfrak{c}_{\Lambda,F}$ for some $F$. Furthermore, Conjecture 6.1.2.1 is a natural extension of the work of Mazur-Wiles [30] and Fischman [8]. The strategy for proving case (2) is as follows. I will relate the $\mathbb{I}_0$-level to the congruence ideal of $F$ as in Hida's proof of Theorem II [19]. The connection to Katz $p$-adic $L$-functions is then obtained by relating the congruence ideal to the $p$-adic $L$-function through known cases of the Main Conjecture of Iwasawa Theory. This should yield that $\mathfrak{c}_{R^{\mathrm{can}},F} | \mathcal{L}_0^2$.

Next, suppose for simplicity that $|\Gamma| = 2$ and that the non-trivial element of $\Gamma$ is the involution defined in [20]. Then under the hypothesis that the class number of $M$ is prime to $p$ and assuming the semi-simplicity conjecture, by [20, Theorem 8.1] we have that $\mathbb{I} = \mathbb{I}_0[\sqrt{\mathcal{L}_0}]$ with $\mathcal{L}_0$ square-free. In particular, $\mathbb{I}_0 = \Lambda$ in this case. The idea is that by using $\mathbb{I} = \mathbb{I}_0[\mathcal{L}_0]$, we may be able to remove the square and conclude that $\mathfrak{c}_{R^{\mathrm{can}},F} = \mathcal{L}_0$ in this case.

Note that part (3) of the above conjecture differs from [19, Theorem II] by a square, just as in the second part of Conjecture 6.1.2. The rationale for these two conjectures is the same. Namely, in [20] Hida shows that in the case when $\mathbb{I}_0 = \Lambda$ and $|\Gamma_F| = 2$, we have $\mathbb{I} = \Lambda[\sqrt{f}]$, where $f$ is $(1+T)^m - 1$ for some integer $m$ in the real quadratic case, and $f$ is the relevant product of anticyclotomic Katz $p$-adic $L$-functions in the imaginary quadratic case.

Proving Conjecture 6.1.2 would yield refined information about the images of Galois representations attached to Hida families. It is the first step in completely determining the images of such representations.

## 6.2 Computing $\mathcal{O}_0$-levels of classical Galois representations

The goal of this project is to compute the level of Galois representations coming from classical modular forms and thus completely determine the image of such a representation. Let $f$ be a non-CM classical Hecke eigenform, $\mathfrak{p}$ a prime of the ring of integers of the field generated by the Fourier coefficients of $f$, and $\rho_{f,\mathfrak{p}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ the associated $p$-adic representation. Let $\pi$ be a uniformizer of the subring $\mathcal{O}_0$ of $\mathcal{O}$ fixed by all conjugate self-twists. By the work of Ribet [39] and Momose [34], there is a minimal non-negative integer $c(f,\mathfrak{p})$ such that $\mathrm{Im}\,\rho_f$ contains $\Gamma_{\mathcal{O}_0}(\pi^{c(f,\mathfrak{p})})$. Their work shows that $c(f,\mathfrak{p}) = 0$ for all but finitely many primes $\mathfrak{p}$. However, relatively little is known about the case when $c(f,\mathfrak{p})$ is positive and the weight of $f$ is greater than 2. I plan to study how $c(f,\mathfrak{p})$ changes as $f$ varies over the classical specializations of a non-CM Hida family that is congruent to a CM family.

This project will have both theoretical and computational components. First, I will establish a relationship between $c(f,\mathfrak{p})$ and the congruence number of $f$, which should also be related to values of the Katz $p$-adic $L$-function, as suggested by the proof of Theorem II in [19]. Once this is established, I will create a method in Sage [45] to compute $c(f,\mathfrak{p})$ by computing the congruence number of $f$. Using the new functionality, I will create a large data set of levels of classical Galois representations in Hida families, which will likely lead to new conjectures to be studied theoretically.

## 6.3 Analogue of the Mumford-Tate Conjecture in $p$-adic families

Another way to describe the work of Ribet and Momose is as follows. Given a modular form $f$, there is an algebraic group $G$, defined over $\mathbb{Q}$, such that $G \times_{\mathbb{Q}} \mathbb{Q}_p$ is equal to the connected component of the identity of the Zariski closure of the image of the $p$-adic Galois representation associated to $f$. Conjecturally, $G$ is the Mumford-Tate group of the motive attached to $f$. Thus, one can say that the images of the $\ell$-adic Galois representations in the compatible system coming from $f$ are essentially independent of the prime $\ell$. We can view this as a "horizontal independence" in families.

Hida has proposed an analogue of the Mumford-Tate Conjecture for $p$-adic families of Galois representations, which can be seen as a "vertical independence" in $p$-adic families. For an arithmetic prime $\mathfrak{P}$ of $\mathbb{I}$, write $\mathrm{MT}_{\mathfrak{P}}$ for the Mumford-Tate group of the compatible system containing $\rho_{f_{\mathfrak{P}}}$, so $\mathrm{MT}_{\mathfrak{P}}$ is an algebraic group over $\mathbb{Q}$. Let $\kappa(\mathfrak{P}) = \mathbb{I}_{\mathfrak{P}}/\mathfrak{P}_{\mathfrak{P}}$, and write $G_{\mathfrak{P}}$ for the Zariski closure of $\operatorname{Im}\rho_{f_{\mathfrak{P}}}$ in $\mathrm{GL}_2(\kappa(\mathfrak{P}))$. Let $G_{\mathfrak{P}}^{\circ}$ be the connected component of the identity of $G_{\mathfrak{P}}$ and $G'_{\mathfrak{P}}$ the (closed) derived subgroup of $G_{\mathfrak{P}}$. Finally, let $\Gamma_F$ denote the group generated by the conjugate self-twists of a non-CM Hida family $F$.

**Conjecture 6.3.1** (Hida). *Assume $F$ is non-CM. There is a simple algebraic group $G'$, defined over $\mathbb{Q}_p$, such that for all arithmetic primes $\mathfrak{P}$ of $\mathbb{I}$ one has $G'_{\mathfrak{P}} \cong G' \times_{\mathbb{Q}_p} \kappa(\mathfrak{P})$ and $\mathrm{Res}_{\mathbb{Q}_p}^{\kappa(\mathfrak{P})} G_{\mathfrak{P}}$ is (the ordinary factor of) $\mathrm{MT}_{\mathfrak{P}} \times_{\mathbb{Q}} \mathbb{Q}_p$. Furthermore, the component group $G_{\mathfrak{P}}/G_{\mathfrak{P}}^0$ is canonically isomorphic to the Pontryagin dual of the decomposition group of $\mathfrak{P}$ in $\Gamma_F$.*

By obtaining a sufficiently precise understanding of images of Galois representations attached to Hida families through the first project, I plan to prove results along the lines of Conjecture 6.3.1. The relationship to the Pontryagin dual of $\Gamma_F$ can be seen in Proposition 5.2.1.

Conjecture 6.3.1 is significant because it suggests that the images of classical specializations of the Galois representation attached to a Hida family are even more related to one another than previously thought. Not only do they arise as specializations of some group in $\mathrm{GL}_2(\mathbb{I})$, they can all be found simply by base change from a single group, at least up to abelian error.

## 6.4 Other settings

The above projects can be studied in more general settings than Hida families of elliptic modular forms.

### 6.4.1 Hilbert modular forms

Let $F$ be a totally real number field and $f$ a primitive Hilbert modular form defined over $F$ with level $\mathfrak{n}$, an ideal of the ring of integers $\mathcal{O}_F$ of $F$. For any prime $\mathfrak{l}$ of $\mathcal{O}_F$, write $a(\mathfrak{l}, f)$ for the eigenvalue of $f$ under the $\mathfrak{l}$-th Hecke operator. Let $K$ be the number field generated by all $a(\mathfrak{l}, f)$'s and $\mathcal{O}$ its ring of integers. Then for any prime $\mathfrak{p}$ of $\mathcal{O}$, there is a continuous Galois representation [16, Theorem 2.43]

$$\rho_{f,\mathfrak{p}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$$

that is unramified outside of $\mathfrak{n}p$ such that for every prime $\mathfrak{l} \nmid \mathfrak{n}p$ we have

$$\operatorname{tr} \rho_{f,\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{l}}) = a(\mathfrak{l}, f).$$

The images of the residual representations of such $\rho_{f,\mathfrak{p}}$ have been studied by Dimitrov and Dieulefait [7, 6]. However, the analogue of Ribet and Momose's work has not been completed for Galois representations coming from Hilbert modular forms. Since these are $\mathrm{GL}_2$-representations, one expects that the relevant "symmetries" in the heuristic from the introduction should still be conjugate self-twists. Unlike in bigger groups, $\mathrm{GL}_2$-representations do not have enough room for more complicated symmetries.

Furthermore, ordinary Hilbert modular forms can be put into $p$-adic families as well. However, in this case, the base ring $\Lambda$ may have more variables. Indeed, let $r = [F : \mathbb{Q}] + 1 + \delta_F$, where $\delta_F$ is the defect of Leopoldt's Conjecture for $F$. Thus, $\delta_F = 0$ if Leopoldt's Conjecture is true for $F$. Then Hida's big Hecke algebra will be finite over $\Lambda \cong \mathbb{Z}_p[[T_1, \ldots, T_r]]$. As usual, one has a big Galois representation attached to such a family, say

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{I}),$$

where $\mathbb{I}$ is an irreducible component of a Hida Hecke algebra for Hilbert modular forms over $F$.

In principle, many of the techniques presented in Chapter 3 of this thesis should apply to the study of the images of such representations as well. However, one would first need to study the images of the classical Galois representations and obtain results analogous to those of Ribet and Momose in the elliptic modular setting. Once that is done, one of the main challenges in studying Galois representations $\rho$ attached to $p$-adic families of Hilbert modular forms would be turning the Pink-Lie algebra of the image of $\rho$ into a $\Lambda$-algebra when $F$ has more than one prime lying over $p$. Indeed, this would require that one be able to find a basis in which for each prime $\mathfrak{p}|p$, the local representation $\rho|_{D_\mathfrak{p}}$ is upper triangular. While it is known that $\rho|_{D_\mathfrak{p}}$ can be made upper triangular for each $\mathfrak{p}|p$, it is not known (and may not be true) that this can be done simultaneously. However, one can first treat the case where there is only one prime of $F$ lying over $p$. In this case, it should be possible to prove a result analogous to Theorem 3.1.4 for $\rho$. Once this is accomplished, the projects in the first three subsections of this chapter could be studied in the analogous setting of representations coming from Hilbert modular forms and $p$-adic families thereof.

### 6.4.2 Non-ordinary elliptic modular forms

Tilouine and his collaborators proved an analogue of Theorem 3.1.4 in the non-ordinary $\mathrm{GL}_2$-setting [3] by building on the ideas in this thesis. They study the image of the Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated to a finite slope, non-ordinary Coleman family $\mathbf{f}$ of elliptic modular forms, where $\mathbb{I}^\circ$ is an irreducible component of an appropriate Hecke algebra. Their analogue of Theorem 3.1.4 is as follows: they identify a certain subring $\mathbb{I}_0^\circ$ of $\mathbb{I}^\circ[p^{-1}]$, an $\mathbb{I}_0^\circ$-algebra $\mathcal{B}_r$, and a Lie algebra $\mathcal{H}_r \subseteq \mathfrak{sl}_2(\mathcal{B}_r)$ associated to the image of $\rho$ such that

$$\mathcal{H}_r \supseteq \mathfrak{l} \cdot \mathfrak{sl}_2(\mathcal{B}_r)$$

for a nonzero $\mathbb{I}_0^\circ$-ideal $\mathfrak{l}$ [3, Theorem 6.2]. The largest such $\mathbb{I}_0^\circ$-ideal is called the *Galois level* of $\rho$. Loosely speaking, this says that the image of $\rho$ is "large with respect to $\mathbb{I}_0^\circ$" in the sense that it is not contained in a smaller algebraic group like a Borel or a unipotent subgroup. However, their methods do not allow them to actually identify a large subgroup of the image of $\rho$; they are only able to measure the size of the Lie algebra $\mathcal{H}_r$. Thus their work does not recover Theorem 3.1.4

Having proved the existence of the Galois level $\mathfrak{l}$, they proceed to relate it to a certain (fortuitous) congruence ideal $\mathfrak{c}$. The ideal $\mathfrak{c}$ measures congruences between the given family $\mathbf{f}$ and CM forms of slope at most that of $\mathbf{f}$. In particular, in Theorem 7.1 they show that $\mathfrak{l}$ and $\mathfrak{c}$ have the same support whenever $\overline{\rho}$ is not induced from a Hecke character of a real quadratic field. Finally, Theorem 7.4 shows that whenever $\overline{\rho}$ is not induced from a Hecke character of a real quadratic field, one has $\mathfrak{c}^2 \subseteq \mathfrak{l} \subseteq \mathfrak{c}$. This is the analogue of [19, Theorem 8.6], and the proof follows that of Hida. One would hope that by solving Conjecture 6.1.2, similar techniques could be used to show that $\mathfrak{l} = \mathfrak{c}$ in the non-ordinary setting.

The key new idea in [3] is to introduce the relative Sen operator in order to show that the Lie algebra $\mathcal{H}_r$ is an algebra over a sufficiently large ring. In [19] and in Chapter 3 above, the analogous step crucially depended on the representation being ordinary. Another important contribution of this paper is their definition (in Section 3) of the fortuitous congruence ideal $\mathfrak{c}$. Since there are no CM components of positive slope, the congruence ideal cannot be defined as the (scheme-theoretic) intersection between the given family and CM components as in the ordinary case. The fact that the fortuitous congruence ideal is related to the Galois level in the same way as in the ordinary case indicates that the fortuitous congruence ideal is the appropriate generalization to the non-ordinary setting.

The computational questions posed in Section 6.2 are also valid for non-ordinary forms. However the division algebra $D$ appearing in Momose's Theorem 3.5.1 may not be a matrix algebra when base changed to $\mathbb{Q}_p$. Thus, one works with congruence subgroups of $(D\otimes_{\mathbb{Q}}\mathbb{Q}_p)^\times$, or rather their Lie algebras.

### 6.4.3 Siegel modular forms

Hida and Tilouine proved an analogue of Hida's Theorem II [19] for representations associated to Hida families of Siegel modular forms, which take values in $\mathrm{GSp}_4$ [21]. There are two main difficulties they overcome in their work: the types of symmetries are much more complicated than CM versus non-CM, and Pink's theory of Lie algebras is only valid for $\mathrm{SL}_2$. The images of the classical representations associated to Siegel modular forms were studied by Dieulefait [5].

One of the ideas they use to overcome the limitations of Pink's theory of Lie algebras is that for an algebraic group $G$ and a root $\alpha$ of $G$, one has an embedding $i_\alpha : \mathrm{SL}_2 \hookrightarrow G$. Furthermore, $G$ is generated by the images of all such embeddings. Thus, they can apply Pink's theory of Lie algebras to each $\mathrm{SL}_2$ and then combine them to study all of $G$.

One of the challenges in studying representations for larger groups is that there are more types of symmetries that can contribute to the size of the image. In the $\mathrm{GL}_2$-case, the only way to get a $\mathrm{GL}_2$-representation from a smaller dimension is to induce a Hecke character. This is what gives rise to the CM versus non-CM dichotomy Theorem 3.1.4. For larger groups, there are more ways to create representations from smaller dimensional ones. More precisely, let $\rho : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathcal{O}_{\mathfrak{p}})$ be a $p$-adic Galois representation associated to a Siegel modular form. Write $G$ for the connected component of the identity of the Zariski closure of the image of $\rho$ and $G'$ for the derived subgroup of $G$. Write $\rho^{\mathrm{conn}}$ for $\rho$ restricted to the preimage of $G$. Then $G'$ takes one of the following forms [21]:

1. $G' \sim \mathrm{Sp}_4$
2. $G' \sim \mathrm{SL}_2 \times \mathrm{SL}_2$ with $\rho^{\mathrm{conn}} \cong \rho_1 \oplus \rho_2$ with $\rho_1 \ncong \rho_2 \otimes \chi$ for any character $\chi$
3. $G' \sim \mathrm{SL}_2$, with $\rho^{\mathrm{conn}} \cong \chi \oplus \chi' \oplus \rho_1$, with $\chi, \chi'$ characters and $\rho_1$ an irreducible two-dimensional representation
4. $G' \sim \mathrm{SL}_2$ via the symmetric cube representation of $\mathrm{SL}_2$

5. $G' \sim \mathrm{SL}_2$ with $\rho^{\mathrm{conn}} \cong \rho_1 \oplus (\rho_1 \otimes \chi)$ for a two-dimensional irreducible representation $\rho_1$ and a character $\chi$

6. $G' \sim \{1\}$ with $\rho \cong \mathrm{Ind}_M^{\mathbb{Q}} \chi$ for a character $\chi$ and a degree four CM field $M/\mathbb{Q}$.

In each of these cases, one can formulate what it should mean for the representation $\rho$ to be full. This is done in [21], and the analogous results to Chapter 3 are proven in the case when $\mathbb{I} = \Lambda$. Thus, one can study the first three projects in this setting as well. Indeed, Hida and Tilouine have already proved some results relating the Galois level of such representations to appropriate congruence ideals [21].

## 6.5 The image of $\rho_{\mathbb{T}}$

The ultimate goal of this program of studying images of modular Galois representations would be to completely understand the image of $\rho_{\mathbb{T}}$, where $\mathrm{Spec}\,\mathbb{T}$ is a non-CM connected component of the primitive part of Hida's big Hecke algebra. Indeed, by the $R = \mathbb{T}$ philosophy, $\mathbb{T}$ should essentially be the universal deformation ring for "nice" deformations of $\bar{\rho}_{\mathbb{T}}$. Thus, by understanding the image of $\rho_{\mathbb{T}}$, we would have a good description of the image of any deformation of $\bar{\rho}_{\mathbb{T}}$.

The description of $\mathrm{Im}\,\rho_{\mathbb{T}}$ is expected to depend heavily on the geometry of $\mathrm{Spec}\,\mathbb{T}$, which is still not well understood. As a toy example, we will describe what we think would be true in the case when $\mathrm{Spec}\,\mathbb{T} = \mathrm{Spec}\,\mathbb{I}_1 \cup \mathrm{Spec}\,\mathbb{I}_2$, with $\mathrm{Spec}\,\mathbb{I}_i$ irreducible components of $\mathrm{Spec}\,\mathbb{T}$. (We continue to assume $\mathrm{Spec}\,\mathbb{T}$ is a non-CM connected component of the primitive part of Hida's big Hecke algebra and that $\bar{\rho}_{\mathbb{T}}$ is absolutely irreducible.) Let $p_i : \mathbb{T} \to \mathbb{I}_i$ be the natural projection induced by the inclusion $\mathrm{Spec}\,\mathbb{I}_i \hookrightarrow \mathrm{Spec}\,\mathbb{T}$. Then $\mathbb{T}$ can be viewed as a subring of $\mathbb{I}_1 \times \mathbb{I}_2$ via $p_1 \oplus p_2$. In fact, we can give a precise description of the image of $\mathbb{T}$ inside $\mathbb{I}_1 \times \mathbb{I}_2$ using Hida's theory of congruence modules [10, 11]. Let $\mathfrak{a} = \ker p_2$ and $\mathfrak{b} = \ker p_1$, which can be viewed as ideals of $\mathbb{I}_1$ and $\mathbb{I}_2$, respectively. The kernel of the natural map $\mathbb{T} \to \mathbb{I}_2/\mathfrak{b}$ is $\mathfrak{a} \oplus \mathfrak{b}$. Indeed, $\ker(\mathbb{T} \to S/\mathfrak{b}) = \{(x, y) \in \mathbb{T} : y \in \mathfrak{b}\}$. However, if $(x, y) \in \mathbb{T}$

and $y \in \mathfrak{b} = \mathbb{T} \cap (0 \times \mathbb{I}_2)$, then

$$(x, 0) = (x, y) - (0, y) \in \mathbb{T} \cap (\mathbb{I}_1 \times 0) = \mathfrak{a}.$$

Thus $\mathbb{T}/(\mathfrak{a} \oplus \mathfrak{b}) \cong \mathbb{I}_2/\mathfrak{b}$. Similarly, $\mathbb{T}/(\mathfrak{a} \oplus \mathfrak{b}) \cong \mathbb{I}_1/\mathfrak{a}$, so we obtain an isomorphism

$$\mathbb{I}_1/\mathfrak{a} \cong \mathbb{I}_2/\mathfrak{b}.$$

It now follows that

$$\mathbb{T} = \{(x, y) \in \mathbb{I}_1 \times \mathbb{I}_2 : x \bmod \mathfrak{a} = y \bmod \mathfrak{b}\},$$

where we have identified $\mathbb{I}_1/\mathfrak{a} = \mathbb{I}_2/\mathfrak{b}$.

Define the ring

$$\mathbb{I} = \{(x, y) \in \mathbb{I}_1 \oplus \mathbb{I}_2 : x \equiv y \bmod \mathfrak{m}_{\mathbb{T}}\}.$$

It is easy to see that $\mathbb{T}$ lies inside $\mathbb{I}$. Furthermore, $\mathbb{I}$ is a local ring with maximal ideal $\mathfrak{m}_{\mathbb{I}} = \{(x, y) \in \mathbb{I} : x \equiv 0 \bmod \mathfrak{m}_{\mathbb{I}_1}\}$.

Since $\mathbb{I}_1, \mathbb{I}_2$ are non-CM irreducible components of $\mathbb{T}$, we have their corresponding rings $\mathbb{I}_{1,0}, \mathbb{I}_{2,0}$ fixed by conjugate self-twists defined in Chapter 3. Let $\mathbb{T}_0 = \mathbb{T} \cap (\mathbb{I}_{1,0} \times \mathbb{I}_{2,0})$. By Theorem 3.3.1 it follows that for an appropriate finite index subgroup $H$ of $G_{\mathbb{Q}}$, the representation $\rho_{\mathbb{T}}|_H$ takes values in $\mathrm{GL}_2(\mathbb{T}_0)$. Let $\mathfrak{a}_0 = \mathfrak{a} \cap \mathbb{I}_{1,0}$ and $\mathfrak{b}_0 = \mathfrak{b} \cap \mathbb{I}_{2,0}$.

**Conjecture 6.5.1.** *Let* $\operatorname{Spec} \mathbb{T}$ *be a non-CM primitive connected component of Hida's big Hecke algebra that has exactly two irreducible components,* $\operatorname{Spec} \mathbb{I}_1$ *and* $\operatorname{Spec} \mathbb{I}_2$*.*

1. *The* $\mathbb{I}_{1,0}$*-level of* $\rho_{\mathbb{I}_1}$ *is equal to* $\mathfrak{a}_0$*, and the* $\mathbb{I}_{2,0}$*-level of* $\rho_{\mathbb{I}_2}$ *is equal to* $\mathfrak{b}_0$*.*

2. *The representation* $\rho_{\mathbb{T}}$ *is* $\mathbb{T}_0$*-full, and the* $\mathbb{T}_0$*-level of* $\rho_{\mathbb{T}}$ *is* $\mathfrak{a}_0 \oplus \mathfrak{b}_0 \subset \mathbb{T}_0$*.*

Note that the first part of the conjecture is essentially a rephrasing of Conjecture 6.1.2, since $L$-functions correspond to congruence ideals by the Main Conjecture. In terms of the heuristic from the introduction, the conjecture amounts to saying that there should not be any extra symmetries of the geometric object $\operatorname{Spec} \mathbb{T}$ that are not accounted for by the symmetries of its irreducible components. The justification for this expectation is that in

Section 3.2, we essentially showed that a conjugate self-twist of a connected component of the Hecke algebra does not permute its irreducible components. However, in order to turn this idea into a proof of the conjecture, one must think carefully about the difference between $\mathbb{I}$ and $\mathbb{I}'$, since it was really $\operatorname{Spec}\mathbb{I}'$ that was shown to be stable under conjugate self-twists in Section 3.2.

Note that the picture becomes significantly more complicated when there are more than two irreducible components of $\operatorname{Spec}\mathbb{T}$. Indeed, Hida's theory of congruence modules works well for comparing one irreducible component to the rest of the space, but if there are more than two irreducible components, it is somewhat less clear how they all interact. The geometry of their intersection (which is what the congruence module captures) can be significantly more complicated. Hida has pointed out that there may not be any known examples of $\operatorname{Spec}\mathbb{T}$ that have more than two components. Perhaps one could hope to prove this is indeed the case in some generality by studying the image of $\rho_{\mathbb{T}}$.

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