Images of $\mathbf{GL_2}$-type Galois representations

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(joint work with Andrea Conti, Anna Medvedovsky)

1. Introduction

Throughout this talk, let p denote a prime and A a complete, local, noetherian, pro-p ring. Write $\mathfrak{m}_A$ for the maximal ideal of A and $\mathbb{F} = A/\mathfrak{m}_A$. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ be a continuous representation unramified outside a finite set of primes. Write $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ for ρ modulo $\mathfrak{m}_A$. We will study the following question for such representations ρ under mild conditions.

Question 1.1. *If ρ is irreducible, not induced from a character on an index 2 subgroup, and $\mathrm{Im}\,\rho$ is infinite, what is $\mathrm{Im}\,\rho$?*

Let us begin with a brief history about answers to this question in certain cases. Serre showed that if ρ comes from the p-adic Tate module of a non-CM elliptic curve $E/\mathbb{Q}$, then $\mathrm{Im}\,\rho$ is open in $\mathrm{GL}_2(\mathbb{Z}_p)$ [10]. There are Galois representations attached to classical elliptic cuspidal Hecke eigenforms as well.

Theorem 1.2. [9], [7] *If ρ arises from a non-CM cuspidal eigenform of weight at least 2, then there is a subring $\mathcal{O} \subseteq A$ finite over $\mathbb{Z}_p$ such that either:*

1. *$\mathrm{Im}\,\rho \cap \mathrm{SL}_2(A)$ contains (with finite index) an open subgroup of $\mathrm{SL}_2(\mathcal{O})$.*
2. *$\mathrm{Im}\,\rho \cap \mathrm{SL}_2(A)$ contains (with finite index) an open subgroup of the norm 1 elements in a (non-split) division algebra over the fraction field of $\mathcal{O}$.*

Note that for modular forms, $\det \rho$ need not surject onto $A^\times$. This is the reason for intersecting $\mathrm{Im}\,\rho$ with $\mathrm{SL}_2(A)$ in the above (and following) results. The ring $\mathcal{O}$ in Theorem 1.2 is the subring of A fixed by the conjugate self-twists of ρ (see Definition 3.3).

When A is a ring of Krull dimension larger than 1, openness is no longer a good measure of the size of the image of ρ. Thus we introduce, for any $0 \neq \mathfrak{a}$ ideal of A,

$$\Gamma_A(\mathfrak{a}) := \ker(\mathrm{SL}_2(A) \to \mathrm{SL}_2(A/\mathfrak{a})) = \left\{ \left(\begin{smallmatrix} 1+a & b \\ c & 1+d \end{smallmatrix}\right) \in \mathrm{SL}_2(A) : a, b, c, d \in \mathfrak{a} \right\}.$$

This allows us to state the following theorem, which was proved under various assumptions by a variety of people.

Theorem 1.3. [6], [3], [4], [5] *If $p \neq 2$, ρ arises from a non-CM Hida family, and $\bar{\rho}$ is absolutely irreducible and "regular", then there exists a ring $\mathbb{I}$ finite over $\mathbb{Z}_p[\![T]\!]$ such that $\mathrm{Im}\,\rho \supseteq \Gamma_{\mathbb{I}}(\mathfrak{a})$ for some $0 \neq \mathfrak{a}$ ideal of $\mathbb{I}$.*

We will not define the slightly technical "regular" condition in the above theorem. Instead, see Theorem 3.4 below for an idea of what this assumption entails. Finally, some work has been done to generalize the above theorem to the positive finite slope case.

Theorem 1.4. [2] *If $p \neq 2$, ρ arises from a Coleman family, and $\bar{\rho}$ is absolutely irreducible and "regular", then a certain Lie algebra attached to $\mathrm{Im}\,\rho$ is "big".*

This theorem gives a hint about how all these big image theorems are proved. Indeed, the key tool is to attach a p-adic Lie algebra to $\operatorname{Im}\rho$ and show that this Lie algebra is big. If the theory of Lie algebras is sufficiently strong, one can translate this information back into information about the group $\operatorname{Im}\rho$.

There are some problems with classical p-adic Lie algebras. For example, $\log_p$ does not behave well if A has characteristic p. Furthermore, in characteristic zero, one often needs to invert p, thus losing information at p. Most importantly for us, classical Lie algebras are not well-behaved over large dimensional rings.

2. Pink's Lie algebras

Henceforth, assume $p \neq 2$ and A is as above. Let Γ be a closed pro-p subgroup of $\mathrm{SL}_2(A)$. Pink [8] defines a function

$$\Theta : \mathrm{SL}_2(A) \to \mathfrak{sl}_2(A)$$
$$x \mapsto x - \frac{1}{2}\operatorname{tr} x.$$

Following Pink, we define $L_1(\Gamma) = L_1$ to be the closed $\mathbb{Z}_p$-submodule of $\mathfrak{sl}_2(A)$ topologically generated by $\Theta(\Gamma)$. Let $L_{n+1}(\Gamma) = L_{n+1}$ be the closed $\mathbb{Z}_p$-submodule of $\mathfrak{sl}_2(A)$ generated by $[L_1, L_n]$.

For example, if $\Gamma = \Gamma_A(\mathfrak{a})$ for some A-ideal $\mathfrak{a} \neq 0$ then one can show that

$$L_n(\Gamma) = \mathfrak{sl}_2(\mathfrak{a}^n) := \left\{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) : a, b, c \in \mathfrak{a} \right\}.$$

Let $H_n = \Theta^{-1}(L_n) \cap \mathrm{SL}_2(A)$ and $\Gamma^{(n+1)}$ be the closed subgroup of Γ topologically generated by commutators $(\Gamma, \Gamma^{(n)})$, where $\Gamma^{(1)} := \Gamma$.

Theorem 2.1. [8]

- *$L_{n+1} \subseteq L_n$ for all $n \geq 1$;*
- *Γ is a normal subgroup of H_1 and H_1/Γ is abelian;*
- *$\Gamma^{(n)} = H_n$ for all $n \geq 2$.*

These Lie algebras work well in characteristic p and can be defined over large dimensional rings, but a major drawback is that L_n is only a module over $\mathbb{Z}_p$. Furthermore, it is unclear how to generalize the theory to groups other than SL_2.

3. Bellaïche's theory and our results

Let $\rho : \Pi \to \mathrm{GL}_2(A)$ be a continuous representation where:

(1) Every open subgroup $\Pi' < \Pi$ satisfies $\dim_{\mathbb{F}_p} \operatorname{Hom}(\Pi', \mathbb{F}_p) < \infty$;
(2) $\bar{\rho}$ is absolutely irreducible and projectively non-abelian;
(3) $\det \rho$ maps isomorphically to $\det \bar{\rho}$;
(4) A is generated by $\operatorname{tr}\rho(\Pi)$ as a $W(\mathbb{F})$-algebra ($\iff$ as a $W(\mathbb{F})$-module).

Theorem 3.1. [1] *If ρ is not induced from a character and $\operatorname{Im}\rho$ is infinite, then there exists $\mathbb{F}_p \subseteq \mathbb{F}_q \subseteq \mathbb{F}$ and closed $W(\mathbb{F}_q)$-submodules $I, B \subseteq \mathfrak{m}_A$ such that*

(1) *$W(\mathbb{F}_q)L_1 = \left(\begin{smallmatrix} I & B \\ B & I \end{smallmatrix}\right)^0 := \left\{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) : a \in I, b, c \in B \right\}$;*
(2) *A is generated by $W(\mathbb{F}) + I + I^2 + B$ as a $W(\mathbb{F})$-module;*

(3) *if $\bar\rho$ is not induced from a character, then $I = B$ and $I^2 \subseteq I$.*

Corollary 3.2. [1] *Assume further that A is a domain. If there is some $\left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \mu_0\end{smallmatrix}\right) \in \operatorname{Im}\bar\rho$ with $\lambda_0, \mu_0 \in \mathbb{F}_p^\times$ and $\lambda_0 \neq \pm\mu_0$, then $\operatorname{Im}\rho \supseteq \Gamma_{\mathbb{Z}_p[\![I]\!]}(\mathfrak{a})$ for some ideal $\mathfrak{a} \neq 0$ of $\mathbb{Z}_p[\![I]\!]$.*

Definition 3.3. *A conjugate self-twist (CST) of ρ is a pair (σ, η_σ) such that σ is a ring automorphism of A and $\eta_\sigma : \Pi \to W(\mathbb{F})^\times$ is a homomorphism such that for all $g \in \Pi$,*

$$\sigma(\operatorname{tr}\rho(g)) = \eta_\sigma(g)\operatorname{tr}\rho(g).$$

Let Σ_ρ be the subgroup of automorphisms of A consisting of all σ such that (σ, η_σ) is a conjugate self-twist of ρ for some $\eta_\sigma : \Pi \to W(\mathbb{F})^\times$. Let $\mathbb{E} = \mathbb{F}^{\Sigma_{\bar\rho}}$, the subfield of $\mathbb{F}$ fixed by every $\sigma \in \Sigma_{\bar\rho}$.

Theorem 3.4 (Conti-L.-Medvedovsky)**.**

- *L_2 is a module over $W(\mathbb{E})[\![I^2]\!]$.*
- *If there is some $\left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \mu_0\end{smallmatrix}\right) \in \operatorname{Im}\bar\rho$ such that $\lambda_0\mu_0^{-1} \in \mathbb{E}^\times \setminus \{\pm 1\}$ (we will call such a $\bar\rho$ **regular**), then L_3 is a module over $W(\mathbb{E})[\![I]\!]$.*
- *If $\operatorname{Im}\bar\rho \supseteq \mathrm{SL}_2(\mathbb{F}_p)$ and $p \geq 7$, then L_1 is a $W(\mathbb{E})[\![I]\!]$-module.*

Theorem 3.5 (Conti-L.-Medvedovsky)**.** *Let $\Pi_0 = \cap_{\sigma\in\Sigma_\rho} \ker\eta_\sigma$. Assume that the projective image of $\bar\rho$ is not abelian, $\bar\rho$ is regular, and $\bar\rho|_{\Pi_0}$ is multiplicity free. If A is a domain, then $W(\mathbb{E})[\![I]\!]$ and A^{Σ_ρ} have the same field of fractions.*

Corollary 3.6 (Conti-L.-Medvedovsky)**.** *Under the conditions of Theorem 3.5, there is some ideal $0 \neq \mathfrak{a} \subset A^{\Sigma_\rho}$ such that $\operatorname{Im}\rho \supseteq \Gamma_{A^{\Sigma_\rho}}(\mathfrak{a})$.*

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