Images of Galois representations of Hida families

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Introduction and Heuristic

Let $G_{\mathbb{Q}}$ be the absolute Galois group of the rational numbers. A *Galois representation* is a continuous homomorphism

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(A),$$

for a topological ring A. In practice, Galois representations arise from the action of $G_{\mathbb{Q}}$ on the cohomology of varieties defined over $\mathbb{Q}$.

Heuristic. The image of a Galois representation should be as large as possible subject to the symmetries (cf. Definition 1) of the geometric object from which it arose.

Definitions

Fix a prime $p > 2$ and embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Definition 1. [6] Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a cuspidal Hecke eigenform, and let K be the number field generated by $\{a_n : n \in \mathbb{Z}^+\}$. An automorphism σ of K is a *conjugate self-twist* of f if there is a nontrivial Dirichlet character η_σ such that

$$a_\ell^\sigma = \eta_\sigma(\ell) a_\ell$$

for almost all primes ℓ. If the identity automorphism is a conjugate self-twist of f, then we say f has *complex multiplication (CM)*.

Let $\Lambda = \mathbb{Z}_p[[T]]$. For an integer $k \geq 2$ and a p-power root of unity ζ, let $P_{k,\zeta} = (1 + T - \zeta(1+p)^k)\Lambda$. Such primes (and primes lying over them) are called *arithmetic*.

Definition 2. [2] Let $\mathbb{I}$ be an integral domain that is finite flat over Λ. A formal power series $F = \sum_{n=1}^{\infty} A_n q^n \in \mathbb{I}[[q]]$ is a *Hida family* if $A_p \in \mathbb{I}^\times$ and, for every prime ideal $\mathfrak{P}$ of $\mathbb{I}$ lying over some $P_{k,\zeta}$, we have

- $A_n \bmod \mathfrak{P} \in \overline{\mathbb{Q}}$ (rather than just $\overline{\mathbb{Q}}_p$)
- $f_{\mathfrak{P}} := \sum_{n=1}^{\infty} (A_n \bmod \mathfrak{P}) q^n$ is the q-expansion of a classical modular form of weight k and the appropriate level and nebentypus.

Main Theorem

Let $F = \sum_{n=1}^{\infty} A_n q^n \in \mathbb{I}[[q]]$ be a Hida familly. Hida showed [1] that there is a Galois representation $\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I})$ that is unramified outside a finite set of primes and such that

$$\mathrm{tr}\, \rho_F(\mathrm{Frob}_\ell) = A_\ell$$

for all primes ℓ at which ρ_F is unramified.
We can define *conjugate self-twists* of F and the notion of *CM* following Definition 1 but replacing K with the field of fractions of $\mathbb{I}$. For simplicity assume that $\mathbb{I}$ is normal, and let $\mathbb{I}_0$ be the subring of $\mathbb{I}$ fixed by all conjugate self-twists of F.

Theorem 1. *(L., [4]) Let F be a non-CM Hida family. Assume that the residual representation $\bar{\rho}_F$ is absolutely irreducible and satisfies a minor $\mathbb{Z}_p$-regularity condition. Then there is a non-zero $\mathbb{I}_0$-ideal $\mathfrak{a}_0$ such that, in an appropriate basis, the image of ρ_F contains*

$$\ker(\mathrm{SL}_2(\mathbb{I}_0) \to \mathrm{SL}_2(\mathbb{I}_0/\mathfrak{a}_0)).$$

Proof: Lifting twists

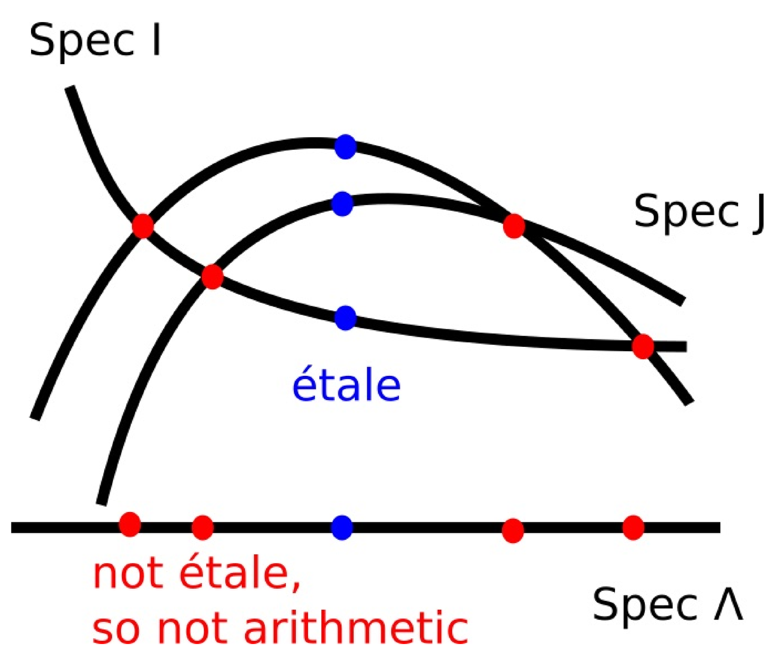
We keep the notation from Theorem 1 above. The following is a key input to the proof of Theorem 1.

Theorem 2. *(L., [4]) Let $\mathfrak{P}$ be an arithmetic prime of $\mathbb{I}$ and σ a conjugate self-twist of $f_{\mathfrak{P}}$. If σ preserves the local field generated by the Fourier coefficients of $f_{\mathfrak{P}}$, then σ can be lifted to a conjugate self-twist $\tilde{\sigma}$ of F.*

Step 1. Show that σ can be lifted to a conjugate self-twist Σ of the (unrestricted) universal deformation of $\bar{\rho}_F$ using abstract deformation theory.

Step 2. Show that Σ preserves an appropriate Hida Hecke algebra. Thus $\mathrm{Spec}\, \mathbb{I}$ and $\Sigma^* \mathrm{Spec}\, \mathbb{I}$ are both irreducible components of the Hecke algebra. They intersect at the arithmetic point $\mathfrak{P}$.

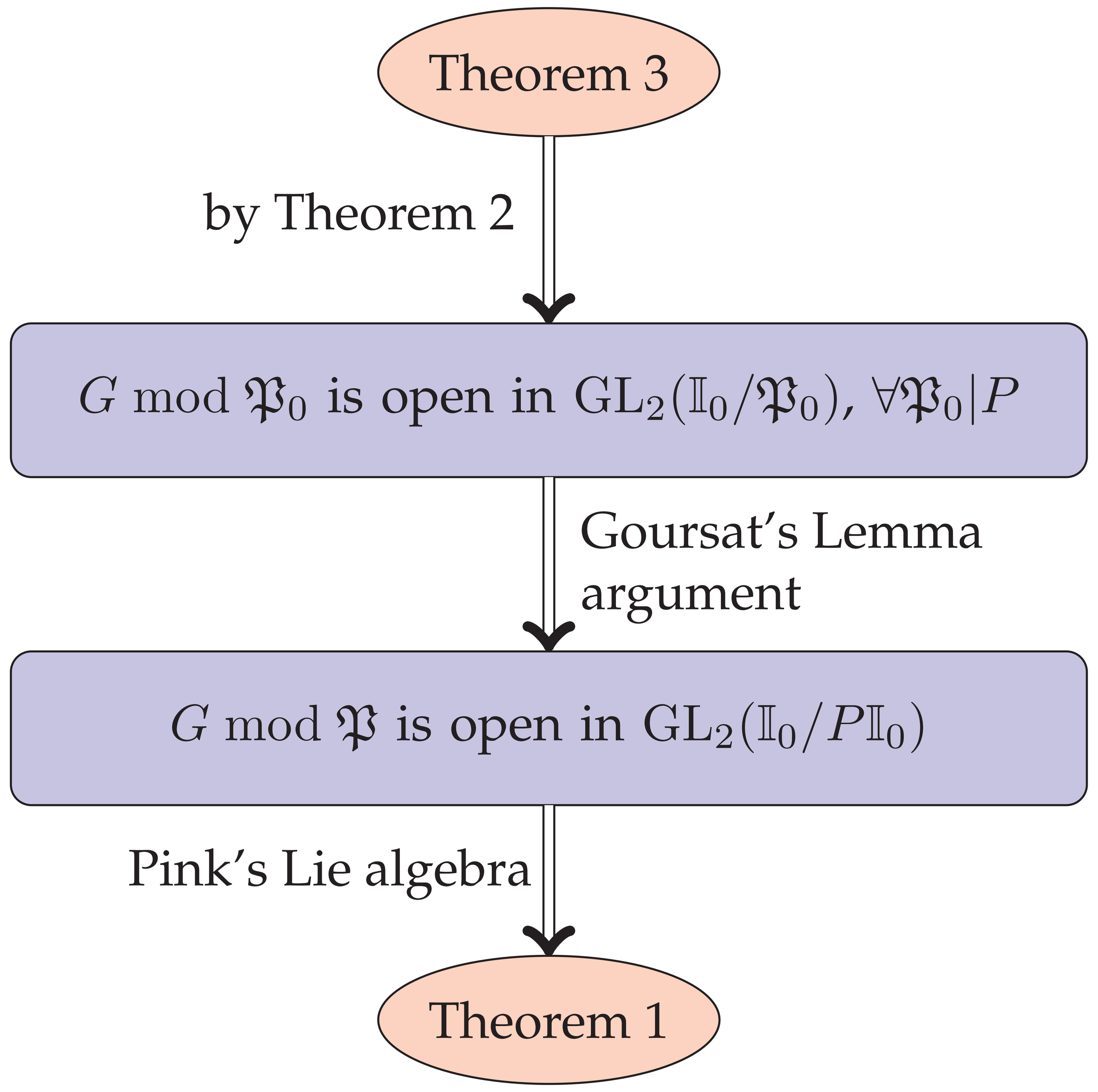
Step 3. Use the fact that the Hecke algebra is étale over Λ at arithmetic points [3] to conclude that Σ descends to the desired automorphism $\tilde{\sigma}$ of $\mathbb{I}$.



Proof: Reduction steps

Theorem 3. *[7, 5] Let f be a classical non-CM cuspidal eigenform. Let $\mathcal{O}_0$ be the ring fixed by all conjugate self-twists of f. Then for any prime $\mathfrak{p}$, the image of $\rho_{f,\mathfrak{p}}$ contains an open subgroup of $\mathcal{O}_{0,\mathfrak{p}}$.*

Let $G = \mathrm{Im}\, \rho_F$ and P be an arithmetic prime of Λ.



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Future Work

- What is the largest $\mathbb{I}_0$-ideal $\mathfrak{a}_0$ that satisfies Theorem 1? We expect the answer is related to the congruence ideal of F, which in some cases can be related to p-adic L-functions.
- To what extent can we completely determine the image of ρ_F?
- Is there an analogue of the Mumford-Tate Conjecture for p-adic families of Galois representations?

References

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