LONG TIME BEHAVIOR OF SOLUTIONS OF AN ELECTROCONVECTION MODEL IN $\mathbb{R}^2$

ELIE ABDO AND MIHAELA IGNATOVA

ABSTRACT. We consider a two dimensional electroconvection model which consists of a nonlinear and nonlocal system coupling the evolutions of a charge distribution and a fluid. We show that the solutions decay in time in $L^2(\mathbb{R}^2)$ at the same sharp rate as the linear uncoupled system. This is achieved by proving that the difference between the nonlinear and linear evolution decays at a faster rate than the linear evolution. In order to prove the sharp L^2 decay we establish bounds for decay in $H^2(\mathbb{R}^2)$ and a logarithmic growth in time of a quadratic moment of the charge density.

1. INTRODUCTION

We consider the electroconvection model

$$\partial_t q + u \cdot \nabla q + \Lambda q = 0, \tag{1}$$

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = -qRq, \tag{2}$$

$$\nabla \cdot u = 0 \tag{3}$$

in $\mathbb{R}^2$ describing the evolution of a surface charge density q in a two-dimensional incompressible fluid flowing with a velocity u and a pressure p. Here $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the square root of the two-dimensional Laplacian, and $R = \nabla \Lambda^{-1}$ is the two-dimensional Riesz transform. The initial data

$$u(\cdot, 0) = u_0 \tag{4}$$

and

$$q(\cdot, 0) = q_0 \tag{5}$$

are assumed to be regular enough and have good decay properties. The model is motivated by physical and numerical studies of electroconvection [9, 21, 22]. The nonlocal aspect of the evolution of the charge density and the nonlocal forcing on the Navier-Stokes equations in the model are due to the fact that the fluid and charges are confined to a thin two dimensional film. The global well-posedness of the system in bounded domains was obtained in [7] using commutator estimates and nonlocal nonlinear analysis. In [1], we investigated the long time dynamics of the model in two dimensions, with periodic boundary conditions and with applied voltage. When the fluid is forced by time-independent smooth mean zero body forces, we proved that the model (1)–(5) has a finite dimensional global attractor. In the absence of body forces, the charge density q converges exponentially in time to a unique limit due to the applied voltage, and the velocity u converges exponentially in time to zero. The rate of exponential decay depends on the periodic boundary conditions.

In this paper, we consider the time asymptotic behavior of solutions of (1)–(5) in $\mathbb{R}^2$, and adapt the Fourier splitting method [17, 18] of Schonbek to the present system. The method was initially used in [17] to prove decay of Leray weak solutions [14] of Navier-Stokes equations and to further decay studies for Navier-Stokes equations [3, 11, 18, 19, 23] and many other partial differential

Date: today.

equations (see for instance [4, 8, 10, 15, 24, 25]). Different approaches were employed as well to investigate the time decay [16] and space-time decay [2, 12, 13, 20] of higher-order derivatives of solutions to Navier-Stokes equations.

The electroconvection model (1)–(5) couples Navier-Stokes equations to a scalar equation for a surface charge density q, evolving via advection by u and diffusion by Λ. We obtain in Theorem 1 of section 2 the long time L^2 decay of the type

$$\|q\|_{L^2} = O(t^{-1})$$

and

$$\|u\|_{L^2} = O(t^{-\frac{1}{2}}).$$

This rate of decay is sharp for the linear uncoupled system if the initial data have non vanishing finite L^1 norms, because functions of the form $Q(t) = e^{-t\Lambda^\alpha} q_0$ obey

$$\lim_{t\to\infty} t^{\frac{n}{\alpha}} \|Q(t)\|_{L^2(\mathbb{R}^n)}^2 = C_{n,\alpha} \left(\int_{\mathbb{R}^n} q_0 dx \right)^2$$

for any $\alpha > 0$ and $n \geq 1$. The fact that such a decay is sharp for the nonlinear evolution as well is a consequence of Theorem 4 of section 3 where we prove that $u - U$ with $U(t) = e^{t\Delta} u_0$ and $q - Q$ with $Q(t) = e^{-t\Lambda} q_0$ decay faster in L^2 than u and q, respectively. Similar results were proved for critical SQG in [8]. The critical SQG velocity $u = R^\perp q$ decays in L^2 like q, that is at the rate t^{-1}, which helps lower the size of the nonlinear term $u \cdot \nabla q$ in that equation. In our case, the velocity has slower decay in L^2 due to the Navier-Stokes equation, namely of the order $t^{-\frac{1}{2}}$, and the nonlinear term is larger. The influence of the charge density q is felt by the Navier-Stokes velocity via the electric force $-qRq$. In order to obtain a key fast enough decay at low wave numbers for the difference $v = u - U$, we need to control a moment of q, $\int_{\mathbb{R}^2} |x|^2 |q(x,t)|^2 dx = M^2(t)$, in view of the inequality

$$|\widehat{qRq}(\xi)| \leq C|\xi| \|q\|_{L^2} M(t)$$

(see Lemma 3). We prove that

$$M(t) = O(\sqrt{\log t})$$

for long time, by analyzing the evolution of the quantity $a(x)q(x,t)$ with $a(x) = \sqrt{|x|^2 + 1}$. This analysis uses the boundedness of the commutator between Λ and multiplication by $a(x)$, which we establish in Lemma 1. In addition, in order to achieve the necessary sharp L^2 bounds we have to obtain bounds for the decay of higher norms of both u and q. For instance, H^1 norms of q are of the order

$$\|\nabla q\|_{L^2} = O(t^{-1}).$$

These bounds are obtained by somewhat involved nonlinear and nonlocal analysis, and they are no longer sharp compared to the generic t^{-2} linear behavior.

The paper is organized as follows. In section 2, we study the asymptotic behavior of solutions to the electroconvection model (1)–(5): we prove that the L^2 norm of the surface charge density q decays in time to zero with a rate of order t^{-1} whereas the velocity u decays in time to zero with a rate of order $t^{-\frac{1}{2}}$. We also investigate the rate of decay of their higher-order derivatives, and we obtain decaying-in-time bounds in Hölder spaces $C^{0,\frac{1}{2}}$. In section 3, we prove that the differences $q - Q$ and $u - U$ decay to zero in L^2 faster than q and u, with rates of order $t^{-1-\frac{3}{4}}$ and $t^{-\frac{3}{4}}$, respectively. In the Appendix, we present results on the existence and uniqueness of solutions to (1)–(5), based on the Banach fixed point theorem, the Aubin-Lions lemma and commutator estimates.

2. Long Time Behavior of Solutions

In this section, we consider the long-time behavior of solutions of the electroconvection model described by (1)–(5). We show that the charge density q and the velocity u converge to 0 in the H^2 norm, and we investigate the rate of convergence.

For a function $f \in L^1(\mathbb{R}^2)$, we denote its Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-i\xi\cdot x} dx. \tag{6}$$

Theorem 1. *Let $u_0 \in H^1 \cap L^1$ be divergence-free and $q_0 \in L^4 \cap L^1$. There exist positive constants Γ_0 and Γ_0' depending only on the initial data and some universal constants such that the unique global-in-time solution (q, u) of (1)–(5) obeys*

$$\|q(t)\|_{L^2}^2 \le \frac{\Gamma_0}{(t+1)^2} \tag{7}$$

and

$$\|u(t)\|_{L^2}^2 \le \frac{\Gamma_0'}{t+1} \tag{8}$$

for all $t \ge 0$.

Proof: The proof is divided into several steps.

Step 1 *(Basic energy estimates)*. We take the L^2 inner product of equation (1) with $\Lambda^{-1}q$ and the L^2 inner product of equation (2) with u. Then we add the resulting energy equalities. Integrating by parts, we have the cancellations

$$(u \cdot \nabla u, u)_{L^2} = (\nabla p, u)_{L^2} = 0 \tag{9}$$

and

$$\begin{aligned}(u \cdot \nabla q, \Lambda^{-1} q)_{L^2} + (qRq, u)_{L^2} &= -(u \cdot \nabla \Lambda^{-1} q, q)_{L^2} + (qRq, u)_{L^2} \\ &= -(u \cdot Rq, q)_{L^2} + (qRq, u)_{L^2} = 0\end{aligned} \tag{10}$$

due to the divergence-free condition (3). Thus, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{-\frac{1}{2}} q\|_{L^2}^2 + \|u\|_{L^2}^2\right) + \|q\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = 0. \tag{11}$$

We integrate in time from 0 to t and we take the supremum over all positive times $t \ge 0$. We get

$$\sup_{t\ge 0}\left\{\|\Lambda^{-\frac{1}{2}} q(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \int_0^t 2\left(\|q(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2\right) ds\right\} = \|\Lambda^{-\frac{1}{2}} q_0\|_{L^2}^2 + \|u_0\|_{L^2}^2 \tag{12}$$

ending the proof of Step 1.

Step 2 *(Pointwise bounds for the Fourier transform of the charge density q).* The Fourier transform of q evolves according to

$$\partial_t \widehat{q}(\xi, t) + (\widehat{u \cdot \nabla q})(\xi, t) + \widehat{\Lambda q}(\xi, t) = 0. \tag{13}$$

The fractional Laplacian Λ is a Fourier multiplier with symbol $|\xi|$, hence

$$\partial_t \widehat{q} + |\xi| \widehat{q} = -\widehat{u \cdot \nabla q}. \tag{14}$$

We estimate the Fourier transform of the nonlinear term

$$|\widehat{u \cdot \nabla q}| = |\widehat{\nabla \cdot (uq)}| \le C|\xi| \|u\|_{L^2} \|q\|_{L^2} \tag{15}$$

using the divergence-free condition (3), the boundedness of the Fourier transform of a function by its L^1 norm, and the Cauchy-Schwarz inequality. This yields the differential inequality

$$\partial_t \widehat{q} + |\xi| \widehat{q} \le C|\xi| \|u\|_{L^2} \|q\|_{L^2}. \tag{16}$$

We multiply both sides by the integrating factor $e^{|\xi| t}$ and integrate in time from 0 to t. We obtain the bound

$$|\widehat{q}(\xi, t)| \le |\widehat{q}_0(\xi)| + C|\xi| \int_0^t \|u(s)\|_{L^2} \|q(s)\|_{L^2} ds. \tag{17}$$

As a consequence of Step 1 and the Cauchy-Schwarz inequality, we get the pointwise bound

$$|\widehat{q}(\xi, t)| \le \|q_0\|_{L^1} + C_0 |\xi| \sqrt{t} \tag{18}$$

where C_0 is a time-independent constant depending only on $\|u_0\|_{L^2}$ and $\|\Lambda^{-\frac{1}{2}} q_0\|_{L^2}$. This finishes the proof of Step 2.

Step 3 *(Decaying bound for the L^2 norm of the charge density).* The L^2 norm of q evolves according to

$$\frac{1}{2} \frac{d}{dt} \|q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 = 0. \tag{19}$$

In view of Parseval's identity and the fact that $\Lambda^{\frac{1}{2}}$ is a Fourier multiplier with symbol $|\xi|^{\frac{1}{2}}$, we have

$$\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 = \|\widehat{\Lambda^{\frac{1}{2}} q}\|_{L^2}^2 = \int_{\mathbb{R}^2} |\xi| |\widehat{q}(\xi, t)|^2 d\xi. \tag{20}$$

We bound the dissipation from below

$$\int_{\mathbb{R}^2} |\xi| |\widehat{q}(\xi, t)|^2 d\xi \ge \int_{|\xi| > \rho(t)} |\xi| |\widehat{q}(\xi, t)|^2 d\xi \tag{21}$$

where $\rho(t)$ is the function defined on $[0, \infty)$ by

$$\rho(t) = \frac{r}{2(t+1)} \tag{22}$$

for some positive constant r to be determined later. We note that

$$\begin{aligned} \int_{|\xi| > \rho(t)} |\xi| |\widehat{q}(\xi, t)|^2 d\xi &\ge \rho(t) \int_{|\xi| > \rho(t)} |\widehat{q}(\xi, t)|^2 d\xi \\ &= \rho(t) \int_{\mathbb{R}^2} |\widehat{q}(\xi, t)|^2 d\xi - \rho(t) \int_{|\xi| \le \rho(t)} |\widehat{q}(\xi, t)|^2 d\xi \\ &= \rho(t) \|q\|_{L^2}^2 - \rho(t) \int_{|\xi| \le \rho(t)} |\widehat{q}(\xi, t)|^2 d\xi \end{aligned} \tag{23}$$

where we used Parseval's identity. Consequently, we obtain the energy inequality

$$\frac{d}{dt} \|q\|_{L^2}^2 + 2\rho(t) \|q\|_{L^2}^2 \le 2\rho(t) \int_{|\xi| \le \rho(t)} |\widehat{q}(\xi, t)|^2 d\xi. \tag{24}$$

By the pointwise bound (18) and Fubini's theorem for spherical coordinates, we estimate

$$\begin{aligned} \int_{|\xi| \le \rho(t)} |\widehat{q}(\xi, t)|^2 d\xi &\le \int_{|\xi| \le \rho(t)} \left(\|q_0\|_{L^1} + C_0 |\xi| \sqrt{t} \right)^2 d\xi = C \int_0^{\rho(t)} r \left(\|q_0\|_{L^1} + C_0 r \sqrt{t} \right)^2 dr \\ &\le C \int_0^{\rho(t)} r \left(\|q_0\|_{L^1}^2 + C_0^2 r^2 t \right) dr \le \Gamma_1 \left(\rho(t)^2 + t \rho(t)^4 \right) \end{aligned} \tag{25}$$

where Γ_1 depends only on the initial data. We obtain

$$\frac{d}{dt} \|q\|_{L^2}^2 + 2\rho(t) \|q\|_{L^2}^2 \le 2\Gamma_1 (\rho(t)^3 + t\rho(t)^5) \tag{26}$$

for all $t \geq 0$. We multiply both sides of the inequality by the integrating factor

$$e^{2\int_0^t \rho(s)ds} = e^{r\int_0^t \frac{1}{s+1}ds} = e^{r\ln(t+1)} = (t+1)^r \tag{27}$$

and then we integrate in time from 0 to t. We get

$$\begin{aligned}
\|q(t)\|_{L^2}^2 &\leq \frac{\|q_0\|_{L^2}^2}{(t+1)^r} + \frac{\Gamma_2}{(t+1)^r}\int_0^t \left(\frac{1}{(s+1)^3} + \frac{1}{(s+1)^4}\right)(s+1)^r ds \\
&\leq \frac{\|q_0\|_{L^2}^2}{(t+1)^r} + \frac{\Gamma_2}{(t+1)^r}\left(\frac{(t+1)^{r-2}}{r-2} - \frac{1}{r-2} + \frac{(t+1)^{r-3}}{r-3} - \frac{1}{r-3}\right) \\
&\leq \frac{\|q_0\|_{L^2}^2}{(t+1)^r} + \frac{\Gamma_2}{(r-2)(t+1)^2} + \frac{\Gamma_2}{(r-3)(t+1)^3}
\end{aligned} \tag{28}$$

for any $r > 3$. Here Γ_2 depends on r and the initial data. We choose $r = 4$ and we obtain the bound

$$\|q\|_{L^2}^2 \leq \frac{\Gamma_3}{(t+1)^2} \tag{29}$$

where Γ_3 is a positive constant depending only on the initial data. This completes the proof of (7) and Step 3.

Step 4 *(Pointwise bounds for the Fourier transform of the velocity u).* Applying the Leray Projector $\mathbb{P}$ to equation (2), we have

$$\partial_t u + \mathbb{P}(u\cdot\nabla u) - \Delta u = -\mathbb{P}(qRq), \tag{30}$$

where we used the incompressibility condition (3) and the fact that $\mathbb{P}$ and $-\Delta$ are Fourier multipliers so they commute. Hence the Fourier transform of u obeys

$$\partial_t \widehat{u} + \mathbb{P}\widehat{(u\cdot\nabla u)} - \widehat{\Delta u} = -\widehat{\mathbb{P}(qRq)}. \tag{31}$$

We estimate

$$|\widehat{\mathbb{P}(u\cdot\nabla u)}(\xi,t)| \leq C|\xi||\widehat{u}(\xi,t)|^2 \leq C|\xi|\|u(t)\|_{L^2}^2 \tag{32}$$

and

$$|\widehat{\mathbb{P}(qRq)}(\xi,t)| \leq C\|(qRq)(t)\|_{L^1} \leq C\|q(t)\|_{L^2}^2 \tag{33}$$

in view of the boundedness of the Riesz transforms on $L^2(\mathbb{R}^2)$. We obtain

$$\partial_t \widehat{u} + |\xi|^2\widehat{u} \leq C|\xi|\|u\|_{L^2}^2 + C\|q\|_{L^2}^2 \tag{34}$$

and hence

$$|\widehat{u}(\xi,t)| \leq \|u_0\|_{L^1} + C|\xi|\int_0^t \|u(s)\|_{L^2}^2 ds + C\int_0^t \|q(s)\|_{L^2}^2 ds \tag{35}$$

for all $\xi \in \mathbb{R}^2$ and $t \geq 0$. In view of the bound (12), we get

$$|\widehat{u}(\xi,t)| \leq \Gamma_4 + C|\xi|\int_0^t \|u(s)\|_{L^2}^2 ds \tag{36}$$

where Γ_4 is a positive constant depending only on the initial data. This completes the proof of Step 4.

Step 5 *(Decaying bounds for the L^4 norm of q).* The evolution of the L^4 norm of q is described by the energy equality

$$\frac{1}{4}\frac{d}{dt}\|q\|_{L^4}^4 + \int_{\mathbb{R}^2} q^3\Lambda q dx = 0. \tag{37}$$

In view of the Córdoba-Córdoba inequality [6], the dissipation is bounded from below

$$\int_{\mathbb{R}^2} q^3\Lambda q dx \geq \frac{1}{2}\|\Lambda^{\frac{1}{2}}(q^2)\|_{L^2}^2 \tag{38}$$

and thus

$$\int_{\mathbb{R}^2} q^3 \Lambda q dx \geq c\|q\|_{L^8}^4 \tag{39}$$

due to Gagliardo-Nirenberg inequalities. Using interpolation inequalities in L^p spaces and the uniform boundedness of the L^2 norm of the charge density q by $\|q_0\|_{L^2}$, we have the bound

$$\|q\|_{L^4} \leq \|q\|_{L^2}^{\frac{1}{3}} \|q\|_{L^8}^{\frac{2}{3}} \leq \|q_0\|_{L^2}^{\frac{1}{3}} \|q\|_{L^8}^{\frac{2}{3}} \tag{40}$$

from which we conclude that

$$\int_{\mathbb{R}^2} q^3 \Lambda q dx \geq C\|q_0\|_{L^2}^{-2} \|q\|_{L^4}^6 \tag{41}$$

and hence

$$\frac{d}{dt}\|q\|_{L^4}^4 + \frac{C}{\|q_0\|_{L^2}^2}\|q\|_{L^4}^6 \leq 0. \tag{42}$$

Letting $y = \|q\|_{L^4}^4$, we obtain the Bernouilli ordinary differential inequality

$$\frac{dy}{dt} + \frac{C}{\|q_0\|_{L^2}^2} y^{\frac{3}{2}} \leq 0. \tag{43}$$

We apply a change of variable given by $u = y^{-\frac{1}{2}}$ and we get

$$\frac{-2}{u^3}\frac{du}{dt} + \frac{C}{\|q_0\|_{L^2}^2}\frac{1}{u^3} \leq 0 \tag{44}$$

so

$$\frac{du}{dt} \geq \frac{C}{2\|q_0\|_{L^2}^2}. \tag{45}$$

Integrating in time from 0 to t, we arrive at the bound

$$\|q\|_{L^4}^{-2} \geq \|q_0\|_{L^4}^{-2} + \frac{C}{\|q_0\|_{L^2}^2} t \geq \Gamma_5(1+t) \tag{46}$$

where Γ_5 is a constant depending on the initial data. Consequently, we obtain

$$\|q\|_{L^4} \leq \frac{1}{\sqrt{\Gamma_5}} \frac{1}{(1+t)^{\frac{1}{2}}} \tag{47}$$

for all $t \geq 0$.

Step 6 *(Decaying bound for the L^2 norm of the velocity).* The L^2 norm of the velocity evolves according to

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = -\int_{\mathbb{R}^2} qRq \cdot u dx. \tag{48}$$

In view of Hölder's inequality, the boundedness of the Riesz transforms on $L^4(\mathbb{R}^2)$, and Ladyzhenskaya's interpolation inequality, we bound

$$\left| \int_{\mathbb{R}^2} qRq \cdot u \right| \leq C\|q\|_{L^2}\|q\|_{L^4}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} \tag{49}$$

yielding

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}\|u\|_{L^2}^{\frac{2}{3}}. \tag{50}$$

By Parseval's identity, we have

$$\frac{d}{dt}\|u\|_{L^2}^2 + \int_{\mathbb{R}^2} |\xi|^2 |\widehat{u}(\xi,t)|^2 d\xi \leq C\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}\|u\|_{L^2}^{\frac{2}{3}}. \tag{51}$$

For a positive function $\rho_1(t)$ continuous on $[0,\infty)$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}|\xi|^2|\widehat{u}(\xi,t)|^2d\xi &\geq \int_{|\xi|>\rho_1(t)}|\xi|^2|\widehat{u}(\xi,t)|^2d\xi \geq \rho_1(t)^2\int_{|\xi|>\rho_1(t)}|\widehat{u}(\xi,t)|^2d\xi\\ &\geq \rho_1(t)^2\left(\int_{\mathbb{R}^2}|\widehat{u}(\xi,t)|^2d\xi - \int_{|\xi|\leq\rho_1(t)}|\widehat{u}(\xi,t)|^2d\xi\right)\\ &= \rho_1(t)^2\|u\|_{L^2}^2 - \rho_1(t)^2\int_{|\xi|\leq\rho_1(t)}|\widehat{u}(\xi,t)|^2d\xi. \end{aligned} \tag{52}$$

Consequently, we obtain the energy inequality

$$\frac{d}{dt}\|u\|_{L^2}^2 + \rho_1(t)^2\|u\|_{L^2}^2 \leq C\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}\|u\|_{L^2}^{\frac{2}{3}} + \rho_1(t)^2\int_{|\xi|\leq\rho_1(t)}|\widehat{u}(\xi,t)|^2d\xi. \tag{53}$$

Using (36), we have

$$\begin{aligned}&\int_{|\xi|\leq\rho_1(t)}|\widehat{u}(\xi,t)|^2d\xi \leq C\int_0^{\rho_1(t)} r\left(\Gamma_4^2 + Cr^2\left\{\int_0^t\|u(s)\|_{L^2}^2ds\right\}^2\right)dr\\ &\leq \Gamma_6\rho_1(t)^2 + C\rho_1(t)^4\left(\int_0^t\|u(s)\|_{L^2}^2ds\right)^2 \end{aligned} \tag{54}$$

and thus

$$\frac{d}{dt}\|u\|_{L^2}^2 + \rho_1(t)^2\|u\|_{L^2}^2 \leq \Gamma_6\rho_1(t)^4 + C\rho_1(t)^6\left(\int_0^t\|u(s)\|_{L^2}^2ds\right)^2 + C\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}\|u\|_{L^2}^{\frac{2}{3}}. \tag{55}$$

Multiplying by the integrating factor $e^{\int_0^t\rho_1(s)^2ds}$, and integrating in time from 0 to t, we obtain

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\leq \frac{\|u_0\|_{L^2}^2}{e^{\int_0^t\rho_1(s)^2ds}} + \frac{\Gamma_6}{e^{\int_0^t\rho_1(s)^2ds}}\int_0^t e^{\int_0^s\rho_1(\tau)^2d\tau}\rho_1(s)^4ds\\ &+ \frac{C}{e^{\int_0^t\rho_1(s)^2ds}}\int_0^t\left(e^{\int_0^s\rho_1(\tau)^2d\tau}\rho_1(s)^6\right)\left(\int_0^s\|u(\tau)\|_{L^2}^2d\tau\right)^2ds\\ &+ \frac{C}{e^{\int_0^t\rho_1(s)^2ds}}\int_0^t\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}\|u\|_{L^2}^{\frac{2}{3}}e^{\int_0^s\rho_1(\tau)^2d\tau}ds. \end{aligned} \tag{56}$$

In view of (12), (29) and (47), we estimate

$$\int_0^t\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}\|u\|_{L^2}^{\frac{2}{3}}e^{\int_0^s\rho_1(\tau)^2d\tau}ds \leq \Gamma_7\int_0^t\frac{e^{\int_0^s\rho_1(\tau)^2d\tau}}{(s+1)^2}ds \tag{57}$$

for any $t\geq 0$, and so

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\leq \frac{\|u_0\|_{L^2}^2}{e^{\int_0^t\rho_1(s)^2ds}} + \frac{\Gamma_6}{e^{\int_0^t\rho_1(s)^2ds}}\int_0^t e^{\int_0^s\rho_1(\tau)^2d\tau}\rho_1(s)^4ds\\ &+ \frac{C}{e^{\int_0^t\rho_1(s)^2ds}}\int_0^t\left(e^{\int_0^s\rho_1(\tau)^2d\tau}\rho_1(s)^6\right)\left(\int_0^s\|u(\tau)\|_{L^2}^2d\tau\right)^2ds\\ &+ \frac{\Gamma_7}{e^{\int_0^t\rho_1(s)^2ds}}\int_0^t\frac{e^{\int_0^s\rho_1(\tau)^2d\tau}}{(s+1)^2}ds \end{aligned} \tag{58}$$

for any $t\geq 0$.

In order to obtain the sharp decaying bound for the velocity u, we need the following three sub-steps:

Step 6.1 *(Logarithmic decaying bound for the L^2 norm of the velocity).* We take $\rho_1(t) = (e+t)^{-\frac{1}{2}}\left[\ln(e+t)\right]^{-\frac{1}{2}}$. In this case, the integrating factor is given by

$$e^{\int_0^t \rho_1(s)^2 ds} = e^{\int_0^t \frac{1}{(e+s)\ln(e+s)}ds} = e^{\ln[\ln(e+t)]} = \ln(e+t) \tag{59}$$

and so (58) becomes

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\le \frac{\|u_0\|_{L^2}^2}{\ln(e+t)} + \frac{\Gamma_6}{\ln(e+t)}\int_0^t \frac{1}{(e+s)^2\ln(e+s)}ds \\ &+ \frac{C\|u_0\|_{L^2}^2}{\ln(e+t)}\int_0^t \frac{s^2}{(e+s)^3\left[\ln(e+s)\right]^2}ds + \frac{\Gamma_7}{\ln(e+t)}\int_0^t \frac{\ln(e+s)}{(s+1)^2}ds.\end{aligned} \tag{60}$$

in view of the uniform boundedness of $\|u\|_{L^2}$ by $\|u_0\|_{L^2}$. We note that

$$\int_0^t \frac{s^2}{(e+s)^3\left[\ln(e+s)\right]^2}ds \le \int_0^t \frac{1}{(e+s)\left[\ln(e+s)\right]^2}ds = 1 - \frac{1}{\ln(e+t)} \le 1 \tag{61}$$

for any $t \ge 0$. Therefore,

$$\|u(t)\|_{L^2}^2 \le \frac{\Gamma_8}{\ln(e+t)} \tag{62}$$

for all $t \ge 0$, where Γ_8 is a constant depending only on the initial data.

Step 6.2 *(Almost sharp decaying bound for the L^2 norm of the velocity).* In order to improve the logarithmic decay (62), we take $\rho_1(t) = r^{\frac{1}{2}}(t+1)^{-\frac{1}{2}}$ for some r to be chosen later. In this case, the integrating factor is given by

$$e^{\int_0^t \rho_1(s)^2 ds} = e^{r\int_0^t \frac{1}{(s+1)}ds} = e^{r\ln(t+1)} = (t+1)^r \tag{63}$$

and so (58) becomes

$$\|u(t)\|_{L^2}^2 \le \frac{\|u_0\|_{L^2}^2}{(t+1)^r} + \frac{\Gamma_9}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^2}ds + \frac{C}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^3}\left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 ds$$

for all $t \ge 0$. Here Γ_9 is a constant depending only on the initial data and r. We have

$$\frac{\Gamma_9}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^2}ds = \frac{\Gamma_9}{(r-1)(t+1)^r}\left((t+1)^{r-1} - 1\right) \le \frac{\Gamma_9}{(r-1)(t+1)} \tag{64}$$

for any $r > 1$. Moreover, applying the Cauchy-Schwarz inequality in the time variable yields

$$\left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 \le s\int_0^s \|u(\tau)\|_{L^2}^4 d\tau, \tag{65}$$

so that

$$\begin{aligned}\frac{C}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^3}\left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 ds &\le \frac{C}{(t+1)^r}\left(\int_0^t (s+1)^{r-2}ds\right)\left(\int_0^t \|u(s)\|_{L^2}^4 ds\right) \\ &\le \frac{C}{(r-1)(t+1)}\left(\int_0^t \|u(s)\|_{L^2}^4 ds\right)\end{aligned} \tag{66}$$

for any $r > 1$. Taking $r = 2$ and using (62) give

$$\|u(t)\|_{L^2}^2 \le \frac{\Gamma_{10}}{t+1} + \frac{\Gamma_{10}}{t+1}\int_0^t \frac{\|u(s)\|_{L^2}^2}{\ln(e+s)}ds \tag{67}$$

and so

$$(t+1)\|u(t)\|_{L^2}^2 \le \Gamma_{10} + C'\Gamma_{10}\int_0^t \frac{(s+1)\|u(s)\|_{L^2}^2}{(s+e)\ln(e+s)}ds \tag{68}$$

for any $t \geq 0$. By Gronwall's inequality, we obtain

$$(t+1)\|u(t)\|_{L^2}^2 \leq \Gamma_{10} + C'\Gamma_{10}^2 \int_0^t \frac{e^{\int_s^t \frac{1}{(e+\tau)\ln(e+\tau)}d\tau}}{(e+s)\ln(e+s)} ds$$
$$= \Gamma_{10} + C'\Gamma_{10}^2 \int_0^t \frac{\ln(e+t)}{(e+s)\left[\ln(e+s)\right]^2} ds \leq \Gamma_{10} + C'\Gamma_{10}^2 \ln(e+t). \tag{69}$$

Therefore,

$$\|u(t)\|_{L^2}^2 \leq \frac{\Gamma_{11}\ln(t+e)}{t+1} \tag{70}$$

for any $t \geq 0$, where Γ_{11} is a constant depending only on the initial data.

Step 6.3 *(Sharp decaying bound for the L^2 norm of the velocity).* Finally, we prove (8). We take $\rho_1(t) = \sqrt{2}(t+1)^{-\frac{1}{2}}$ as in the previous sub-step, and we obtain the bound

$$\|u(t)\|_{L^2}^2 \leq \frac{\|u_0\|_{L^2}^2}{(t+1)^2} + \frac{\Gamma_{12}}{t+1} + \frac{C}{(t+1)^2} \int_0^t \frac{1}{s+1} \left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau \right)^2 ds \tag{71}$$

for all $t \geq 0$. We note that

$$\int_0^s \|u(\tau)\|_{L^2}^2 d\tau \leq \Gamma_{13} \int_0^s \frac{\ln(\tau+e)}{\tau+1} d\tau \leq C\Gamma_{13} \int_0^s \frac{\ln(\tau+e)}{\tau+e} d\tau \leq \Gamma_{14}\left[\ln(s+e)\right]^2 \tag{72}$$

and so

$$\int_0^t \frac{1}{s+1} \left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau \right)^2 ds \leq \Gamma_{15} \int_0^t \frac{1}{\sqrt{s+1}} ds \tag{73}$$

for all $t \geq 0$. Therefore,

$$\|u(t)\|_{L^2}^2 \leq \frac{\Gamma_{16}}{t+1} \tag{74}$$

for all $t \geq 0$, where Γ_{16} is a positive constant depending only on the initial data. This ends the proof of Theorem 1.

Now we study the rate of convergence of the gradients of the charge density and the velocity.

Theorem 2. *Let $u_0 \in H^1 \cap L^1$ be divergence-free such that $\nabla u_0 \in L^1$. Let $q_0 \in H^1 \cap L^1$ such that $\nabla q_0 \in L^1$. There exist positive constants K_0 and K_0' depending only on the initial data and some universal constants such that the unique global-in-time solution (q, u) of* (1)–(5) *obeys*

$$\|\nabla u(t)\|_{L^2}^2 \leq \frac{K_0}{t+1} \tag{75}$$

and

$$\|\nabla q(t)\|_{L^2}^2 \leq \frac{K_0'}{(t+1)^2} \tag{76}$$

for all $t \geq 0$.

Proof: The proof is divided into 5 steps.

Step 1 *(Pointwise bounds for the Fourier transform of ∇u).* The Fourier transform of the gradient of u satisfies

$$\partial_t \widehat{\nabla u} + \nabla \widehat{\mathbb{P}(u \cdot \nabla u)} - \widehat{\nabla \Delta u} = -\nabla \widehat{\mathbb{P}(qRq)} \tag{77}$$

yielding the differential inequality

$$\partial_t \widehat{\nabla u} + |\xi|^2 \widehat{\nabla u} \leq |\xi|^2 \|u\|_{L^2}^2 + |\xi| \|q\|_{L^2}^2. \tag{78}$$

Integrating in time from 0 to t, and using (12), we obtain

$$|\widehat{\nabla u}(\xi, t)| \leq \|\nabla u_0\|_{L^1} + K_1|\xi| + K_2|\xi|^2 t \tag{79}$$

where K_1 and K_2 are positive constants depending only on the initial data.

We note that the pointwise estimate (79) is not the sharpest. Indeed, one can use the decaying bounds for the L^2 norms of the velocity and the surface charge density derived in Theorem 1 instead of using (12), and this will result in a bound whose growth in t is slower when $t \geq 1$. However, this will not improve the decay in the upcoming step, so we disregard this observation.

Step 2 *(Decaying bound for the L^2 norm of ∇u).* We take the L^2 inner product of equation (2) with $-\Delta u$ and we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int_{\mathbb{R}^2} qRq \cdot \Delta u. \tag{80}$$

The nonlinear term $(u \cdot \nabla u, \Delta u)_{L^2}$ vanishes due to the fact that the matrix $M^t M^2$ has a zero trace where M is the two-by-two traceless matrix whose entries are given by $M_{ij} = \frac{\partial u_i}{\partial x_j}$ and M^t is its transpose. In view of Hölder's inequality with exponents 4,4,2, the boundedness of the Riesz transforms on $L^4(\mathbb{R}^2)$ and Young's inequality, we obtain

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq C\|q\|_{L^4}^4. \tag{81}$$

Using the L^4 estimate (47), we have

$$\|q\|_{L^4}^4 \leq K_3(1+t)^{-2} \tag{82}$$

where K_3 depends on the initial data. We note that the initial charge density is assumed to be in H^1 and so it belongs to L^4 due to the Sobolev embedding of $H^1(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$. Going back to (81), we have

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq CK_3(t+1)^{-2}. \tag{83}$$

For $t \in [0, \infty)$, we let

$$\rho_2(t) = r^{\frac{1}{2}}(t+1)^{-\frac{1}{2}} \tag{84}$$

for some $r > 0$ to be chosen later. By Parseval's identity, we get

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \rho_2(t)^2\|\nabla u\|_{L^2}^2 \leq CK_3(1+t)^{-2} + \rho_2(t)^2 \int_{|\xi| \leq \rho_2(t)} |\widehat{\nabla u}(\xi, t)|^2 d\xi. \tag{85}$$

In view of the pointwise bound (79), we have

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \rho_2(t)^2\|\nabla u\|_{L^2}^2 \leq K_4(t+1)^{-2} + K_5\rho_2(t)^2\left[\rho_2(t)^2 + \rho_2(t)^4 + \rho_2(t)^6 t^2\right] \tag{86}$$

for $t \geq 0$. We multiply by the integrating factor $(t+1)^r$ and then we integrate in time from 0 to t. We obtain

$$\begin{aligned}\|\nabla u\|_{L^2}^2 \leq \frac{\|\nabla u_0\|_{L^2}^2}{(t+1)^r} &+ \frac{K_4}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^2} ds \\ &+ \frac{K_6}{(t+1)^r}\int_0^t (1+s)^r \left(\frac{1}{(1+s)^2} + \frac{1}{(1+s)^3}\right) ds\end{aligned} \tag{87}$$

where K_6 depends only on the initial data. We can take any $r > 2$ and we obtain the bound (75).

Step 3 *(Bounds for $\int_0^t (s+1)^\gamma \|\Delta u(s)\|_{L^2}^2 ds$ where $\gamma \neq 1$ is a real number).* Let $\gamma \neq 1$. The differential inequality (81) yields

$$\frac{d}{dt}((t+1)^\gamma\|\nabla u\|_{L^2}^2) - \gamma(t+1)^{\gamma-1}\|\nabla u\|_{L^2}^2 + (t+1)^\gamma\|\Delta u\|_{L^2}^2 \leq C(t+1)^\gamma\|q\|_{L^4}^4 \tag{88}$$

for all $t \geq 0$. Integrating in time from 0 to t and using (47) and (75), we obtain

$$\int_0^t (s+1)^\gamma \|\Delta u\|_{L^2}^2 ds \leq \|\nabla u_0\|_{L^2}^2 + (\gamma K_0 + K_7) \int_0^t (s+1)^{\gamma-2} ds$$

$$= \|\nabla u_0\|_{L^2}^2 + \frac{\gamma K_0 + K_7}{\gamma - 1} \left[(t+1)^{\gamma-1} - 1\right] \tag{89}$$

for some positive constant K_7 depending on $\|q_0\|_{L^2}$ and $\|q_0\|_{L^4}$.

Step 4 *(Pointwise bounds for the Fourier transform of ∇q).* The Fourier transform of the gradient of q satisfies

$$\partial_t \widehat{\nabla q} + \nabla \widehat{(u \cdot \nabla q)} + \widehat{\nabla \Lambda q} = 0, \tag{90}$$

hence

$$\partial_t \widehat{\nabla q} + |\xi| \widehat{\nabla q} \leq |\xi|^2 \|u\|_{L^2} \|q\|_{L^2}. \tag{91}$$

Using (12), we obtain the pointwise bound

$$|\widehat{\nabla q}(\xi, t)| \leq \|\nabla q_0\|_{L^1} + K_8 |\xi|^2 \sqrt{t} \tag{92}$$

for all $\xi \in \mathbb{R}^2$ and $t \geq 0$. Here K_8 depends only on the initial data.

Step 5 *(Decaying bound for the L^2 norm of ∇q).* The L^2 norm of the gradient of q evolves according to the energy equality

$$\frac{1}{2} \frac{d}{dt} \|\nabla q\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 = (u \cdot \nabla q, \Delta q)_{L^2}. \tag{93}$$

In view of the Ladyzhenskaya interpolation inequality

$$\|\nabla u\|_{L^4} \leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \tag{94}$$

and the interpolation inequality [1]

$$\|\nabla q\|_{L^{\frac{8}{3}}}^2 \leq C \|q\|_{L^4}^{\frac{1}{2}} \|\Lambda^{\frac{3}{2}} q\|_{L^2}^{\frac{3}{2}}, \tag{95}$$

we estimate the nonlinear term

$$|(u \cdot \nabla q, \Delta q)_{L^2}| \leq \|\nabla u\|_{L^4} \|\nabla q\|_{L^{\frac{8}{3}}}^2 \leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \|q\|_{L^4}^{\frac{1}{2}} \|\Lambda^{\frac{3}{2}} q\|_{L^2}^{\frac{3}{2}}. \tag{96}$$

Applying Young's inequality, we obtain

$$\frac{d}{dt} \|\nabla q\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 \|\Delta u\|_{L^2}^2 \|q\|_{L^4}^2. \tag{97}$$

In view of (75) and (82), we have

$$\frac{d}{dt} \|\nabla q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} \nabla q\|_{L^2}^2 \leq \frac{K_9}{(t+1)^2} \|\Delta u\|_{L^2}^2 \tag{98}$$

for all $t \geq 0$. Here K_9 depends only the initial data. Letting

$$\rho_3(t) = r(t+1)^{-1}, \tag{99}$$

we split the dissipation term,

$$\|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 \geq \rho_3(t) \|\nabla q\|_{L^2}^2 - \rho_3(t) \int_{|\xi| \leq \rho_3(t)} |\widehat{\nabla q}(\xi, t)|^2 d\xi \tag{100}$$

yielding the differential inequality

$$\frac{d}{dt} \|\nabla q\|_{L^2}^2 + \rho_3(t) \|\nabla q\|_{L^2}^2 \leq \frac{K_9}{(t+1)^2} \|\Delta u\|_{L^2}^2 + \rho_3(t) \int_{|\xi| \leq \rho_3(t)} |\widehat{\nabla q}(\xi, t)|^2 d\xi \tag{101}$$

In view of the pointwise bound for $\widehat{\nabla q}$ given by (92), we have

$$\int_{|\xi|\le\rho_3(t)} |\widehat{\nabla q}(\xi,t)|^2 d\xi \le K_{10}\left(\rho_3(t)^2 + \rho_3(t)^6 t\right) \tag{102}$$

and so

$$\frac{d}{dt}\|\nabla q\|_{L^2}^2 + \rho_3(t)\|\nabla q\|_{L^2}^2 \le \frac{K_9}{(t+1)^2}\|\Delta u\|_{L^2}^2 + K_{10}\left(\rho_3(t)^3 + \rho_3(t)^7 t\right). \tag{103}$$

We multiply both sides by $(t+1)^r$ and we integrate in time from 0 to t. We obtain

$$\|\nabla q(t)\|_{L^2}^2 \le \frac{\|\nabla q_0\|_{L^2}^2}{(t+1)^r} + \frac{K_{11}}{(t+1)^r}\int_0^t (s+1)^{r-2}\|\Delta u(s)\|_{L^2}^2 ds \tag{104}$$

$$+ \frac{K_{12}}{(t+1)^r}\int_0^t \left[(s+1)^{r-3} - (s+1)^{r-6}\right] ds. \tag{105}$$

In view of (89) applied with $\gamma = r-2$, we have

$$\int_0^t (s+1)^{r-2}\|\Delta u(s)\|_{L^2}^2 ds \le \|\nabla u_0\|_{L^2}^2 + \frac{(r-2)K_0 + K_7}{r-3}\left[(t+1)^{r-3} - 1\right] \tag{106}$$

for any $r \neq 3$, and so

$$\frac{K_{11}}{(t+1)^r}\int_0^t (s+1)^{r-2}\|\Delta u(s)\|_{L^2}^2 ds \le \frac{K_{13}}{(t+1)^3} \tag{107}$$

for any $r > 3$. Here K_{13} depends on the initial data and r. Putting (104) and (107) together and choosing $r = 6$ give the desired decay (76). This completes the proof of Theorem 2.

Now we establish decaying bounds for higher order derivatives. We need the following proposition.

Proposition 1. *Let $u_0 \in H^1 \cap L^1$ be divergence-free such that $\nabla u_0 \in L^1$. Let $q_0 \in H^1 \cap L^1$ such that $\nabla q_0 \in L^1$. Let $\beta > 3$. There exist a positive universal constant C and positive constants c_1, c_2, and c_3 depending only on the initial data such that the solution q of* (1)–(5) *obeys*

$$\int_0^t (s+1)^\beta \|\Lambda^{\frac{3}{2}} q(s)\|_{L^2}^2 ds \le \|\nabla q_0\|_{L^2}^2 + C\|\nabla u_0\|_{L^2}^2 + C\frac{(\beta-2)c_1 + c_2}{\beta-3}(t+1)^{\beta-3} + \frac{\beta c_3}{\beta-2}(t+1)^{\beta-2} \tag{108}$$

for all $t \ge 0$.

Proof: In view of the differential inequality (97), we have

$$\frac{d}{dt}(t+1)^\beta\|\nabla q\|_{L^2}^2 - \beta(t+1)^{\beta-1}\|\nabla q\|_{L^2}^2 + (t+1)^\beta\|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 \le C(t+1)^\beta\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2\|q\|_{L^4}^2. \tag{109}$$

Integrating in time from 0 to t, using the bounds (47) and (75) and applying (89) with $\gamma = \beta - 2$, we obtain (108).

Theorem 3. *Let $u_0 \in H^2 \cap L^1$ be divergence-free such that $\nabla u_0 \in L^1$ and $\Delta u_0 \in L^1$. Let $q_0 \in H^2 \cap L^1$ such that $\nabla q_0 \in L^1$ and $\Delta q_0 \in L^1$. There exist positive constants M_0 and M_0' depending only on the initial data and some universal constants such that the unique global-in-time solution (q,u) of* (1)–(5) *obeys*

$$\|\Delta u(t)\|_{L^2}^2 \le \frac{M_0}{t+1} \tag{110}$$

and

$$\|\Delta q(t)\|_{L^2}^2 \le \frac{M_0'}{(t+1)^2} \tag{111}$$

for all $t \ge 0$.

Proof: The Fourier transform of Δu obeys

$$\partial_t \widehat{\Delta u} + |\xi|^2 \widehat{\Delta u} \le C|\xi|^3 \|u\|_{L^2}^2 + C|\xi|^2 \|q\|_{L^2}^2. \tag{112}$$

Consequently, it satisfies the pointwise bound

$$|\widehat{\Delta u}(\xi,t)| \le \|\Delta u_0\|_{L^1} + M_1|\xi|^3 t + M_2|\xi|^2 \tag{113}$$

for all $\xi \in \mathbb{R}^2$ and all $t \ge 0$. Here M_1 and M_2 are positive constants depending only on the initial data. The L^2 norm of Δu evolves according to the energy equality

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \|\nabla \Delta u\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta(qRq)\cdot \Delta u dx - \int_{\mathbb{R}^2} \Delta(u\cdot\nabla u)\cdot \Delta u dx. \tag{114}$$

Integrating by parts, using (3), and applying Ladyzhenskaya's interpolation inequality, we estimate the second term on the right hand side in (114) as

$$\left|\int_{\mathbb{R}^2} \Delta(u\cdot\nabla u)\cdot\Delta u dx\right| \le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2}. \tag{115}$$

In view of the boundedness of the Riesz transforms on L^4 and the continuous embedding of $\dot{H}^{\frac{1}{2}}$ in L^4, we obtain for the first term on the right hand side in (114)

$$\left|\int_{\mathbb{R}^2} \Delta(qRq)\cdot\Delta u dx\right| \le C\|q\|_{L^4}\|\Lambda^{\frac{3}{2}}q\|_{L^2}\|\nabla\Delta u\|_{L^2}. \tag{116}$$

From (114)–(116) and using Young's inequality, we obtain the energy inequality

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 \le C\|q\|_{L^4}^2\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2. \tag{117}$$

In view of Parseval's identity, we have

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \rho_2(t)^2\|\Delta u\|_{L^2}^2 \le C\|q\|_{L^4}^2\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2 + \rho_2(t)^2\int_{|\xi|\le\rho_2(t)}|\widehat{\Delta u}(\xi,t)|^2 d\xi \tag{118}$$

where ρ_2 is the function defined by (84). The decay bounds (82) and (75) together with the pointwise bound for the Fourier transform of Δu given by (113) yield

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \rho_2(t)^2\|\Delta u\|_{L^2}^2 \le \frac{M_3}{t+1}\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 + \frac{M_4}{t+1}\|\Delta u\|_{L^2}^2 + M_5\rho_2(t)^2\left[\rho_2(t)^2 + \rho_2(t)^8t^2 + \rho_2(t)^6\right] \tag{119}$$

for all $t \ge 0$, where M_3, M_4 and M_5 are positive constants depending only on the initial data. Multiplying by the integrating factor and integrating in time from 0 to t, we obtain

$$\begin{aligned}\|\Delta u\|_{L^2}^2 &\le \frac{\|\Delta u_0\|_{L^2}^2}{(t+1)^r} + \frac{M_3}{(t+1)^r}\int_0^t (s+1)^{r-1}\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 ds \\ &+ \frac{M_4}{(t+1)^r}\int_0^t (s+1)^{r-1}\|\Delta u\|_{L^2}^2 ds + \frac{M_5}{(t+1)^r}\int_0^t \left[(s+1)^{r-2} + (s+1)^{r-3} + (s+1)^{r-4}\right] ds.\end{aligned} \tag{120}$$

We choose $r = 5$. In view of the bound (89) applied with $\gamma = r-1$ and Proposition 1 applied with $\beta = r-1$, we obtain (110).

The Fourier transform of the Laplacian of q satisfies

$$\partial_t \widehat{\Delta q} + |\xi|\widehat{\Delta q} \le |\xi|^3\|u\|_{L^2}\|q\|_{L^2} \tag{121}$$

and hence

$$|\widehat{\Delta q}(\xi,t)| \le \|\Delta q_0\|_{L^1} + M_8|\xi|^3\sqrt{t} \tag{122}$$

for all $\xi \in \mathbb{R}^2$ and $t \geq 0$. Here M_8 depends only on the initial data. Now, we establish decaying estimate for $\|\Delta q\|_{L^2}^2$ which evolves according to

$$\frac{1}{2}\frac{d}{dt}\|\Delta q\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} q\|_{L^2}^2 = 2\int_{\mathbb{R}^2} (\nabla u \cdot \nabla(\nabla q))\Delta q dx + \int_{\mathbb{R}^2} (\Delta u \cdot \nabla q)\Delta q dx. \tag{123}$$

In view of the Gagliardo-Nirenberg interpolation inequality

$$\|\Delta q\|_{L^2} \leq C\|\Lambda^{\frac{5}{2}} q\|_{L^2}^{\frac{4}{5}} \|q\|_{L^2}^{\frac{1}{5}}, \tag{124}$$

the Sobolev embedding inequality

$$\|\Delta q\|_{L^4} \leq C\|\Lambda^{\frac{5}{2}} q\|_{L^2}, \tag{125}$$

and the bound

$$\|\nabla\nabla q\|_{L^2} = \|\nabla\Lambda^{-1}\nabla\Lambda^{-1}\Delta q\|_{L^2} \leq C\|\Delta q\|_{L^2} \tag{126}$$

that follows from the boundedness of the Riesz transforms on L^2, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta q\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} q\|_{L^2}^2 &\leq C\|\nabla u\|_{L^4}\|\Delta q\|_{L^2}\|\Delta q\|_{L^4} + C\|\Delta u\|_{L^2}\|\nabla q\|_{L^4}\|\Delta q\|_{L^4} \\ &\leq C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\Lambda^{\frac{5}{2}} q\|_{L^2}^{\frac{9}{5}}\|q\|_{L^2}^{\frac{1}{5}} + C\|\Delta u\|_{L^2}\|\Lambda^{\frac{3}{2}} q\|_{L^2}\|\Delta q\|_{L^4} \\ &\leq \frac{1}{2}\|\Lambda^{\frac{5}{2}} q\|_{L^2}^2 + C\|\nabla u\|_{L^2}^5\|\Delta u\|_{L^2}^5\|q\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2\|\Lambda^{\frac{3}{2}} q\|_{L^2}^2. \end{aligned} \tag{127}$$

Consequently,

$$\begin{aligned}\frac{d}{dt}\|\Delta q\|_{L^2}^2 + \rho_2(t)^2\|\Delta q\|_{L^2}^2 &\leq C\|\nabla u\|_{L^2}^5\|\Delta u\|_{L^2}^5\|q\|_{L^2}^2 \\ &+ C\|\Delta u\|_{L^2}^2\|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 + \rho_2(t)^2\int_{|\xi|\leq\rho_2(t)^2} |\widehat{\Delta q}(\xi,t)|^2 d\xi\end{aligned} \tag{128}$$

where ρ_2 is defined by (84). In view of the estimates (7), (75) and (110), Proposition 1 applied with $\beta = r - 1$, and the pointwise bound for the Fourier transform of Δq given by (122), we obtain (111). This ends the proof of Theorem 3.

Let $C^{0,\frac{1}{2}}$ be the space of bounded $1/2$-Hölder continuous functions on $\mathbb{R}^2$ with

$$\|f\|_{C^{0,\frac{1}{2}}} = \|f\|_{L^\infty} + \sup_{x,y\in\mathbb{R}^2, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\frac{1}{2}}}. \tag{129}$$

In view of the continuous Sobolev embedding of $W^{1,4}$ into $C^{0,\frac{1}{2}}$, the Ladyzhenskaya interpolation inequality, and Theorems 1, 2, and 3, we obtain the following statement.

Corollary 1. *Let $u_0 \in H^2 \cap L^1$ be divergence-free such that $\nabla u_0 \in L^1$ and $\Delta u_0 \in L^1$. Let $q_0 \in H^2 \cap L^1$ such that $\nabla q_0 \in L^1$ and $\Delta q_0 \in L^1$. There exist positive constants A_0 and A_0' depending only on the initial data and some universal constants such that the unique global-in-time solution (q,u) of* (1)–(5) *obeys*

$$\|u(t)\|_{C^{0,\frac{1}{2}}}^2 \leq \frac{A_0}{t+1} \tag{130}$$

and

$$\|q(t)\|_{C^{0,\frac{1}{2}}}^2 \leq \frac{A_0'}{(t+1)^2} \tag{131}$$

for all $t \geq 0$.

3. Decomposition of the solution

In this section, we decompose the charge density q and the velocity u solutions of (1)–(5) in the sum of solutions Q and U of the linear equations

$$\partial_t Q + \Lambda Q = 0 \tag{132}$$

and

$$\partial_t U - \Delta U = 0 \tag{133}$$

with initial datum $Q(0) = q_0$ and $U(0) = u_0$ and remainders. We study the decays of the remainders $q - Q$ and $u - U$ in L^2 and we show that they are faster than the decays of the L^2 norms of q and u respectively. The solutions of (132) and (133) are given explicitly by

$$Q(t) = \int_{\mathbb{R}^2} K_t^1(x-w)q_0(w)dw \tag{134}$$

and

$$U(t) = \int_{\mathbb{R}^2} K_t^2(x-w)u_0(w)dw \tag{135}$$

where K_t^s is the kernel defined by its Fourier transform

$$\mathcal{F}(K_t^s)(\xi) = e^{-|\xi|^s t}. \tag{136}$$

The following proposition describes the decay of ∇Q and ∇U in L^∞.

Proposition 2. *Suppose $q_0 \in L^1$ such that $\int_{\mathbb{R}^2} |\xi||\widehat{q_0}(\xi)|d\xi < \infty$ and $u_0 \in L^1$ such that $\int_{\mathbb{R}^2} |\xi||\widehat{u_0}(\xi)|d\xi < \infty$. Then there exist positive constants R_0 and R_0' depending only on the initial data such that the solutions Q and U of the linear equations* (132) *and* (133) *satisfy*

$$\|\nabla Q(t)\|_{L^\infty} \le \frac{R_0}{(t+1)^3} \tag{137}$$

and

$$\|\nabla U(t)\|_{L^\infty} \le \frac{R_0'}{(t+1)^{\frac{3}{2}}} \tag{138}$$

for all $t \ge 0$.

Proof: In view of Parseval's identity and the translation property of the Fourier transform, we have

$$\nabla Q(x) = \nabla K_t^1 * q_0(x) = \int_{\mathbb{R}^2} \nabla K_t^1(x-w)q_0(w)dw = \int_{\mathbb{R}^2} e^{-2\pi i x\cdot\xi}\widehat{\nabla K_t^1}(\xi)\widehat{q_0}(\xi)d\xi. \tag{139}$$

On one hand,

$$\|\nabla Q\|_{L^\infty} \le C\|\widehat{q_0}\|_{L^\infty} \int_{\mathbb{R}^2} |\xi||\widehat{K_t^1}(\xi)|d\xi \le C\|q_0\|_{L^1} \int_0^\infty r^2 e^{-rt}dr \le C\|q_0\|_{L^1} t^{-3} \tag{140}$$

and so

$$t^3\|\nabla Q\|_{L^\infty} \le C\|q_0\|_{L^1}. \tag{141}$$

On the other hand,

$$\|\nabla Q\|_{L^\infty} \le C\int_{\mathbb{R}^2} |\xi||\widehat{K_t^1}(\xi)||\widehat{q_0}(\xi)|d\xi \le C\int_{\mathbb{R}^2} |\xi||\widehat{q_0}(\xi)|d\xi \tag{142}$$

for all $t \ge 0$. Hence

$$(1+t)^3\|\nabla Q\|_{L^\infty} \le 4(1+t^3)\|\nabla Q\|_{L^\infty} \le C\left(\|q_0\|_{L^1} + \int_{\mathbb{R}^2} |\xi||\widehat{q_0}(\xi)|d\xi\right) \tag{143}$$

for all $t \ge 0$, yielding (137). Similarly, we have

$$(1+t)^{\frac{3}{2}} \|\nabla U\|_{L^\infty} \le C(1+t^{\frac{3}{2}}) \|\nabla U\|_{L^\infty} \le C\left(\|u_0\|_{L^1} + \int_{\mathbb{R}^2} |\xi| |\widehat{u_0}(\xi)| d\xi \right) \tag{144}$$

for all $t \ge 0$, yielding (138).

Remark 1. *The assumptions $\int_{\mathbb{R}^2} |\xi||\widehat{q_0}(\xi)|d\xi < \infty$ and $\int_{\mathbb{R}^2} |\xi||\widehat{u_0}(\xi)|d\xi < \infty$ are required to obtain the uniform-in-time boundedness of the L^∞ norms ∇Q and ∇U for small times $t \in (0,1)$. This imposed regularity can be dropped since we are interested in studying the long-time behavior of solutions.*

Next, we consider the pointwise behavior of the Fourier transforms of the differences $q - Q$ and $u - U$. We need first the following lemmas.

Lemma 1. *For $f \in L^2(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, we let*

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{\sqrt{|y|^2+1} - \sqrt{|x|^2+1}}{|x-y|^3} f(y) dy. \tag{145}$$

There exists a universal constant $C > 0$ (independent of f) such that

$$\|Tf\|_{L^2} \le C\|f\|_{L^2}. \tag{146}$$

Proof: We write

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (a(y) - a(x)) k(x-y) f(y) dy. \tag{147}$$

where $a(x)$ is the function defined on $\mathbb{R}^2$ by

$$a(x) = \sqrt{|x|^2+1} \tag{148}$$

and $k(x)$ is the function defined on $\mathbb{R}^2 \setminus \{0\}$ by

$$k(x) = \frac{1}{|x|^3}. \tag{149}$$

We note that k is homogeneous of degree -3. Moreover, the gradient of a is given by

$$\nabla a(x) = \left(\frac{x_1}{\sqrt{|x|^2+1}}, \frac{x_2}{\sqrt{|x|^2+1}} \right) \tag{150}$$

and satisfies $\|\nabla a\|_{L^\infty} \le 1$. Therefore, T is a well-defined operator and bounded on L^2 (see page 435 in Section 2 of [5]).

Using Lemma 1, we study the evolution of $(\sqrt{|x|^2+1})q(x)$ in $L^2(\mathbb{R}^2)$.

Lemma 2. *Let $u_0 \in H^1 \cap L^1$ be divergence-free such that $\nabla u_0 \in L^1$. Let $q_0 \in H^1 \cap L^1$ such that $\nabla q_0 \in L^1$. Furthermore, suppose that $\int_{\mathbb{R}^2} |x|^2 q_0(x)^2 dx < \infty$. Then there exists a positive constant $R_1 > 0$ depending only on the initial data such that*

$$\|(\sqrt{|\cdot|^2+1}) q(\cdot, t)\|_{L^2} \le R_1 \ln(t+1) + \|(\sqrt{|\cdot|^2+1}) q_0(\cdot)\|_{L^2} \tag{151}$$

holds for all $t \ge 0$.

Proof: Let $a(x) = \sqrt{|x|^2 + 1}$. The evolution of aq is described by

$$\partial_t(aq) + au \cdot \nabla q + a\Lambda q = 0. \tag{152}$$

Multiplying by aq and integrating in the space variable over $\mathbb{R}^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|aq\|_{L^2}^2 + \int_{\mathbb{R}^2}(a\Lambda q)aq = -\int_{\mathbb{R}^2}(au\cdot\nabla q)aq. \tag{153}$$

The cancellation

$$\int_{\mathbb{R}^2}(u\cdot\nabla(aq))aq = 0 \tag{154}$$

holds due to (3), so we can rewrite the nonlinear term as

$$-\int_{\mathbb{R}^2}(au\cdot\nabla q)aq = \int_{\mathbb{R}^2}(u\cdot\nabla a)q^2 a. \tag{155}$$

By Hölder's inequality, Ladyzhenskaya's interpolation inequality, and the decaying bounds for the L^2 norms of $q, u, \nabla u$ and ∇q given by (7), (8), (75) and (76), respectively, we estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(u\cdot\nabla a)q^2 a\right| &\le \|\nabla a\|_{L^\infty}\|q\|_{L^4}\|u\|_{L^4}\|aq\|_{L^2}\\ &\le C\|q\|_{L^2}^{\frac{1}{2}}\|\nabla q\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|aq\|_{L^2} \le R_2(t+1)^{-\frac{3}{2}}\|aq\|_{L^2}\end{aligned} \tag{156}$$

for some constant R_2 depending only on the initial data. Now we write the linear term as the sum

$$\begin{aligned}\int_{\mathbb{R}^2}(a\Lambda q)aq &= \int_{\mathbb{R}^2} aq\Lambda(aq) + \int_{\mathbb{R}^2}(aq)\left[a\Lambda q - \Lambda(aq)\right]\\ &= \|\Lambda^{\frac{1}{2}}(aq)\|_{L^2}^2 + \int_{\mathbb{R}^2}(aq)\left[a\Lambda q - \Lambda(aq)\right].\end{aligned} \tag{157}$$

By the Cauchy-Schwarz inequality, we bound

$$\left|\int_{\mathbb{R}^2}(aq)\left[a\Lambda q - \Lambda(aq)\right]\right| \le \|aq\|_{L^2}\|a\Lambda q - \Lambda(aq)\|_{L^2}. \tag{158}$$

The pointwise formula for the fractional Laplacian of order 1 yields

$$\begin{aligned}(a\Lambda q - \Lambda(aq))(x) &= C\int_{\mathbb{R}^2}\left[\frac{a(x)q(x) - a(x)q(y)}{|x-y|^3} - \frac{a(x)q(x) - a(y)q(y)}{|x-y|^3}\right]dy\\ &= C\int_{\mathbb{R}^2}\frac{a(y) - a(x)}{|x-y|^3}q(y)dy\end{aligned} \tag{159}$$

where C is positive universal constant. As a consequence of Lemma 1 and (7), we obtain

$$\|a\Lambda q - \Lambda(aq)\|_{L^2} \le C\|q\|_{L^2} \le C(t+1)^{-1}. \tag{160}$$

Therefore, the L^2 norm of aq obeys the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|aq\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}(aq)\|_{L^2} \le \left[R_2(t+1)^{-\frac{3}{2}} + C(t+1)^{-1}\right]\|aq\|_{L^2} \tag{161}$$

so

$$\frac{1}{2}\frac{d}{dt}\|aq\|_{L^2}^2 \le R_3(t+1)^{-1}\|aq\|_{L^2} \tag{162}$$

for some positive constant R_3 depending only on the initial data. Dividing both sides of the inequality by $\|aq\|_{L^2}$, we get

$$\frac{d}{dt}\|aq\|_{L^2} \le R_3(t+1)^{-1}. \tag{163}$$

Integrating in time from 0 to t, we obtain (151).

The following lemma is needed to obtain a growth in $|\xi|$ for the Fourier transform of $\mathbb{P}(qRq)$.

Lemma 3. *Let $f \in L^2(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} |x|^2 f(x)^2 dx < \infty$. Then*

$$|\widehat{\mathbb{P}(fRf)}(\xi)| \le C|\xi| \|f\|_{L^2} \left(\int_{\mathbb{R}^2} |x|^2 |f(x)|^2 dx \right)^{\frac{1}{2}}. \tag{164}$$

where $\mathbb{P}$ is the Leray projector and $R = (R_1, R_2)$ is the Riesz transform vector on $\mathbb{R}^2$.

Proof: The Leray projector is a Fourier multiplier with a symbol denoted by $m(\xi)$. We have

$$\widehat{\mathbb{P}(fRf)}(\xi) = m(\xi)\widehat{fRf}(\xi) \tag{165}$$

for all $\xi \in \mathbb{R}^2$. We note that $m(\xi)$ is bounded uniformly in ξ. Now, the Fourier transform of fRf at ξ is given by

$$\widehat{fRf}(\xi) = \int_{\mathbb{R}^2} f(x) Rf(x) e^{-i\xi \cdot x} dx \tag{166}$$

for $\xi \in \mathbb{R}^2$. Since the Riesz transform is antisymmetric, we have

$$\int_{\mathbb{R}^2} f(x) Rf(x) dx = 0 \tag{167}$$

and so we can write $\widehat{fRf}$ at ξ as

$$\widehat{fRf}(\xi) = \int_{\mathbb{R}^2} f(x) Rf(x) \left(e^{-i\xi \cdot x} - 1 \right) dx. \tag{168}$$

Using the identity

$$|e^{-i\xi \cdot x} - 1| \le |\xi||x| \tag{169}$$

that holds for all $x, \xi \in \mathbb{R}^2$, we estimate

$$|\widehat{fRf}(\xi)| \le |\xi| \int_{\mathbb{R}^2} |x||f(x)||Rf(x)| dx \le |\xi| \|Rf\|_{L^2} \left(\int_{\mathbb{R}^2} |x|^2 |f(x)|^2 dx \right)^{\frac{1}{2}} \tag{170}$$

in view of the Cauchy-Schwarz inequality. This gives the pointwise estimate (164).

As a consequence of lemmas 2 and 3, we obtain the following statement.

Proposition 3. *Let $u_0 \in H^1 \cap L^1$ be divergence-free such that $\nabla u_0 \in L^1$ and $q_0 \in H^1 \cap L^1$ such that $\nabla q_0 \in L^1$. Furthermore, suppose that $\int_{\mathbb{R}^2} |x|^2 q_0(x)^2 dx < \infty$. Let (q, u) be the solution of (1)–(5). Let $\zeta = q - Q$ and $v = u - U$. Then there exist positive constants R_4, R_5 and R_6 depending only on the initial data such that the Fourier transforms of ζ and v satisfy the pointwise bounds*

$$|\widehat{\zeta}(\xi, t)| \le R_4 |\xi| \tag{171}$$

and

$$|\widehat{v}(\xi, t)| \le R_5 |\xi| \ln(t+1) + R_6 |\xi| \ln^2(t+1) \tag{172}$$

for all $\xi \in \mathbb{R}^2$ and $t \ge 0$.

Proof: The Fourier transform of ζ obeys

$$\partial_t \widehat{\zeta} + |\xi| \widehat{\zeta} = -\widehat{u \cdot \nabla q} \le |\xi| \|u\|_{L^2} \|q\|_{L^2}. \tag{173}$$

Consequently,

$$|\widehat{\zeta}(\xi, t)| \le \int_0^t |\xi| \|u\|_{L^2} \|q\|_{L^2} \le R_4 |\xi| \tag{174}$$

in view of the decaying bounds (7) and (8). The Fourier transform of v evolves according to

$$\partial_t \widehat{v} + |\xi|^2 \widehat{v} = -\widehat{\mathbb{P}(u \cdot \nabla u)} - \widehat{\mathbb{P}(qRq)}. \tag{175}$$

Thus

$$|\widehat{v}(\xi,t)| \le C|\xi| \int_0^t \|u\|_{L^2}^2 ds + C|\xi| \int_0^t \|q\|_{L^2} \left(\int_{\mathbb{R}^2} |x|^2 q(x)^2 dx \right)^{\frac{1}{2}} ds \tag{176}$$

by Lemma 3. In view of Lemma 2 and the decaying estimates (7) and (8), we obtain (172).

Theorem 4. *Let $u_0 \in H^1 \cap L^1$ be divergence-free such that $\nabla u_0 \in L^1$. Let $q_0 \in H^1 \cap L^1$ such that $\nabla q_0 \in L^1$. Furthermore, suppose that $\int_{\mathbb{R}^2} |x|^2 q_0(x)^2 dx < \infty$, $\int_{\mathbb{R}^2} |\xi||\widehat{q_0}(\xi)| d\xi < \infty$, and $\int_{\mathbb{R}^2} |\xi||\widehat{u_0}(\xi)| d\xi < \infty$. Let (q,u) be the solution of (1)–(5). Then there exist positive constants R_7 and R_8 depending only on the initial data such that the differences $q - Q$ and $u - U$ satisfy*

$$\|q(t) - Q(t)\|_{L^2}^2 \le \frac{R_7}{(t+1)^{2+\frac{3}{2}}} \tag{177}$$

and

$$\|u(t) - U(t)\|_{L^2}^2 \le \frac{R_8}{(t+1)^{1+\frac{1}{2}}} \tag{178}$$

for all $t \ge 0$.

Proof: Let $\zeta = q - Q$ and $v = u - U$. We have

$$\partial_t \zeta + \Lambda \zeta = -u \cdot \nabla q. \tag{179}$$

Taking the L^2 inner product of equation (179) with ζ and using (3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}\zeta\|_{L^2}^2 = \int_{\mathbb{R}^2} (u \cdot \nabla q) Q dx \le \|u\|_{L^2}\|q\|_{L^2}\|\nabla Q\|_{L^\infty}. \tag{180}$$

As a consequence of Theorem 1 and the bound (137), we obtain the energy inequality

$$\frac{d}{dt}\|\zeta\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}\zeta\|_{L^2}^2 \le \frac{R_9}{(t+1)^{4+\frac{1}{2}}} \tag{181}$$

where R_9 is a positive constant depending only on the initial data. For a fixed r, we let $\rho(t) = r(t+1)^{-1}$. Then

$$\frac{d}{dt}\|\zeta\|_{L^2}^2 + \rho(t)\|\zeta\|_{L^2}^2 \le \frac{R_9}{(t+1)^{4+\frac{1}{2}}} + \rho(t) \int_{|\xi| \le \rho(t)} |\widehat{\zeta}(\xi,t)|^2 d\xi. \tag{182}$$

Using (171), we estimate

$$\int_{|\xi| \le \rho(t)} |\widehat{\zeta}(\xi,t)|^2 d\xi \le R_{10}\rho(t)^4 \tag{183}$$

and we obtain

$$\frac{d}{dt}\|\zeta\|_{L^2}^2 + \rho(t)\|\zeta\|_{L^2}^2 \le \frac{R_9}{(t+1)^{4+\frac{1}{2}}} + R_{10}\rho(t)^5. \tag{184}$$

Multiplying by the factor $(s+1)^r$, integrating in the time variable s from 0 to t, and choosing any $r > 4$, we obtain the desired bound (177). Now, v obeys

$$\partial_t v - \Delta v = -u \cdot \nabla u - qRq - \nabla p. \tag{185}$$

Taking the L^2 inner product of this latter equation with v and using the fact that v is divergence-free, we get the energy equation

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2} + \|\nabla v\|_{L^2}^2 = \int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot U dx - \int_{\mathbb{R}^2} (qRq) \cdot v dx. \tag{186}$$

We estimate

$$\int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot U dx \le \|u\|_{L^2}^2 \|\nabla U\|_{L^\infty} \le \frac{R_{11}}{(t+1)^{1+\frac{3}{2}}} \tag{187}$$

in view of the bounds (8) and (138), and

$$\int_{\mathbb{R}^2} (qRq) \cdot v dx \le C\|q\|_{L^4}^2 \|v\|_{L^2} \le C\|q\|_{L^2} \|\nabla q\|_{L^2} \|v\|_{L^2} \le \frac{R_{12}}{(t+1)^{1+\frac{3}{2}}} \tag{188}$$

in view of the decaying estimate (7), (76) and (130). This yields the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \rho(t)\|v\|_{L^2}^2 \le \frac{R_{13}}{(t+1)^{1+\frac{3}{2}}} + \rho(t)\int_{|\xi|\le\sqrt{\rho(t)}} |\widehat{v}(\xi,t)|^2 d\xi \tag{189}$$

where $\rho(t) = r(t+1)^{-1}$. Using the pointwise bound for the Fourier transform of v given by (172), we have

$$\int_{|\xi|\le\sqrt{\rho(t)}} |\widehat{v}(\xi,t)|^2 d\xi \le R_{14}\left[\ln^2(t+1) + \ln^4(t+1)\right]\rho(t)^2 \le R_{15}\sqrt{t+1}\rho(t)^2, \tag{190}$$

hence

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \rho(t)\|v\|_{L^2}^2 \le \frac{R_{13}}{(t+1)^{1+\frac{3}{2}}} + R_{15}\sqrt{t+1}\rho(t)^3. \tag{191}$$

We multiply both sides by $(s+1)^r$, we integrate from 0 to t, we choose any $r > 3/2$, and we obtain (178).

4. Appendix: Existence and Uniqueness of Solutions

In this appendix, we prove the existence of weak and strong solutions for the electroconvection model (1)–(5).

Definition 1. *A solution (q,u) of (1)–(5) is said to be a weak solution on $[0,T]$ if it solves (1)–(5) in the sense of distributions, u is divergence-free in the sense of distributions,*

$$u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \tag{192}$$

and

$$q \in L^\infty(0,T;L^2) \cap L^2(0,T;H^{1/2}). \tag{193}$$

Theorem 5. *Let $u_0 \in L^2$ be divergence-free, let $q_0 \in L^2$. Let $T > 0$ be arbitrary. There exists a weak solution (q,u) of the system (1)–(5) on $[0,T]$.*

Proof. We briefly sketch the main ideas of the proof. For $0 < \epsilon \le 1$, we consider a viscous approximation of (1)–(5) given by

$$\begin{cases} \partial_t q^\epsilon + J_\epsilon(u^\epsilon \cdot \nabla q^\epsilon) + \Lambda q^\epsilon - \epsilon\Delta q^\epsilon = 0 \\ \partial_t u^\epsilon + J_\epsilon(u^\epsilon \cdot \nabla u^\epsilon) - \Delta u^\epsilon + \nabla p^\epsilon = -J_\epsilon(q^\epsilon R q^\epsilon) \\ \nabla \cdot u^\epsilon = 0 \end{cases} \tag{194}$$

with smoothed out initial data, where J_ϵ is a standard mollifier operator, $u^\epsilon = J_\epsilon u$, $q^\epsilon = J_\epsilon q$ and $p^\epsilon = J_\epsilon p$. For each $\epsilon > 0$, we consider the map

$$(q(t),u(t)) \mapsto \Phi_\epsilon((q,u))(t) = (e^{\epsilon t\Delta}J_\epsilon q_0 - \mathcal{A}_t^\epsilon(q^\epsilon,u^\epsilon), e^{t\Delta}J_\epsilon u_0 - \mathcal{B}_t^\epsilon(q^\epsilon,u^\epsilon)) \tag{195}$$

where

$$\mathcal{A}_t^\epsilon(q^\epsilon,u^\epsilon) = \int_0^t e^{\epsilon(t-s)\Delta}J_\epsilon(u^\epsilon\cdot\nabla q^\epsilon)(s)ds + \int_0^t e^{\epsilon(t-s)\Delta}\Lambda q^\epsilon(s)ds \tag{196}$$

and

$$\mathcal{B}_t^\epsilon(q^\epsilon, u^\epsilon) = \int_0^t e^{(t-s)\Delta} J_\epsilon \mathbb{P}(u^\epsilon \cdot \nabla u^\epsilon)(s) ds + \int_0^t e^{(t-s)\Delta} J_\epsilon \mathbb{P}(q^\epsilon R q^\epsilon)(s) ds. \tag{197}$$

There exists a time $T_\epsilon = T_\epsilon(\epsilon, \|u_0\|_{L^2}, \|q_0\|_{L^2}) > 0$ such that the map Φ_ϵ is a contraction on the Banach space

$$X_T = L^\infty(0,T; \bar{B}_{L^2}(2\|q_0\|_{L^2}) \oplus L^\infty(0,T; \bar{B}_{L^2_\sigma}(2\|u_0\|_{L^2}) \tag{198}$$

where $\bar{B}_{L^2}(r)$ is the closed ball in L^2, and $\bar{B}_{L^2_\sigma}$ is the closed ball in the space of L^2 divergence-free vectors. Consequently, Φ_ϵ has a fixed point $(q^\epsilon, u^\epsilon) \in X_{T_\epsilon}$ solving (194). This solution extends to the time interval $[0,T]$, and this can be obtained by establishing uniform-in-time bounds for (q^ϵ, u^ϵ) on $[0,T]$. Indeed, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{-\frac{1}{2}} q^\epsilon\|_{L^2}^2 + \|u^\epsilon\|_{L^2}^2\right) + \|q^\epsilon\|_{L^2}^2 + \|\nabla u^\epsilon\|_{L^2}^2 + \epsilon\|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}^2 = 0 \tag{199}$$

as shown in (11). Hence the family of mollified velocities $(u^\epsilon)_\epsilon$ is uniformly bounded in $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$. On the other hand, the L^2 norm of q^ϵ evolves according to

$$\frac{1}{2}\frac{d}{dt}\|q^\epsilon\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}^2 + \epsilon\|\nabla q^\epsilon\|_{L^2}^2 = 0, \tag{200}$$

and so the family of mollified charge densities $(q^\epsilon)_\epsilon$ is uniformly bounded in $L^\infty(0,T;L^2) \cap L^2(0,T;H^{\frac{1}{2}})$. The q^ϵ and u^ϵ equations imply that the sequence of time derivatives $(\partial_t q^\epsilon)_\epsilon$ and $(\partial_t u^\epsilon)_\epsilon$ are uniformly bounded in $L^2(0,T;H^{-\frac{3}{2}})$ and $L^2(0,T;H^{-1})$ respectively. By the Aubin-Lions lemma, the sequence $((q^\epsilon, u^\epsilon))_\epsilon$ has a subsequence that converges strongly in $L^2(0,T;L^2)$ to a weak solution (q,u) of (1)–(5). We omit further details.

Definition 2. *A weak solution (q,u) of* (1)–(5) *is said to be a strong solution on $[0,T]$ if*

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2) \tag{201}$$

and

$$q \in L^\infty(0,T;L^4) \cap L^2(0,T;H^{1/2}). \tag{202}$$

Theorem 6. *Let $u_0 \in H^1$ be divergence-free and $q_0 \in L^4$. Let $T > 0$ be arbitrary. There exists a unique strong solution (u,q) of the system* (1)–(5) *on $[0,T]$.*

Proof. We take the L^2 inner product of the equation satisfied by q^ϵ in (194) with $(q^\epsilon)^3$. In view of the divergence-free condition satisfied by u^ϵ, the nonlinear term vanishes, that is

$$\int_{\mathbb{R}^2} u^\epsilon \cdot \nabla q^\epsilon (q^\epsilon)^3 dx = 0. \tag{203}$$

By the Córdoba-Córdoba inequality ([6]), we have

$$\int_{\mathbb{R}^2} (q^\epsilon)^3 \Lambda q^\epsilon dx \geq 0 \tag{204}$$

and

$$-\int_{\mathbb{R}^2} (q^\epsilon)^3 \Delta q^\epsilon dx \geq 0. \tag{205}$$

Consequently, we obtain

$$\frac{1}{4}\frac{d}{dt}\|q^\epsilon\|_{L^4}^4 \leq 0 \tag{206}$$

which yields the boundedness of q in $L^\infty(0,T;L^4(\mathbb{R}^2))$ by the Banach Alaoglu theorem and the lower semi-continuity of the norm. The L^2 norm of ∇u^ϵ obeys the energy inequality

$$\frac{d}{dt}\|\nabla u^\epsilon\|_{L^2}^2 + \|\Delta u^\epsilon\|_{L^2}^2 \le C\|q^\epsilon\|_{L^4}^4 \tag{207}$$

as shown in (80), yielding the boundedness of u in $L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$. Now we prove the uniqueness of strong solutions. Suppose (q_1,u_1) and (q_2,u_2) are strong solutions of (1)–(5) with same initial data. Let $q = q_1 - q_2$, $u = u_1 - u_2$ and $p = p_1 - p_2$. Then q satisfies

$$\partial_t q + \Lambda q = -u_1\cdot\nabla q - u\cdot\nabla q_2 \tag{208}$$

and u satisfies

$$\partial_t u - \Delta u + \nabla p = -qRq_1 - q_2Rq - u_1\cdot\nabla u - u\cdot\nabla u_2. \tag{209}$$

We take the L^2 inner product of (208) with $\Lambda^{-1}q$ and the L^2 inner product of (209) with u. We add the resulting energy equalities. We have a cancellation

$$-\int_{\mathbb{R}^2}(u\cdot\nabla q_2)\Lambda^{-1}q dx - \int_{\mathbb{R}^2}(q_2Rq)\cdot u dx = 0 \tag{210}$$

obtained from integration by parts. In view of the Ladyzhenskaya's interpolation inequality, we estimate

$$\left|\int_{\mathbb{R}^2}(qRq_1)\cdot u dx\right| \le C\|q\|_{L^2}\|q_1\|_{L^4}\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12} \le \frac14\|\nabla u\|_{L^2}^2 + \frac14\|q\|_{L^2}^2 + C\|q_1\|_{L^4}^4\|u\|_{L^2}^2 \tag{211}$$

and

$$\left|\int_{\mathbb{R}^2}(u\cdot\nabla u_2)\cdot u dx\right| \le \|u\|_{L^4}^2\|\nabla u_2\|_{L^2} \le \frac14\|\nabla u\|_{L^2}^2 + C\|\nabla u_2\|_{L^2}^2\|u\|_{L^2}^2. \tag{212}$$

Now we write

$$\int_{\mathbb{R}^2}(u_1\cdot\nabla q)\Lambda^{-1}q dx = \int_{\mathbb{R}^2}\left(\Lambda^{-\frac12}(u_1\cdot\nabla q) - u_1\cdot\nabla\Lambda^{-\frac12}q\right)\Lambda^{-\frac12}q dx \tag{213}$$

via integration by parts, and we show below that

$$\left|\int_{\mathbb{R}^2}\left(\Lambda^{-\frac12}(u_1\cdot\nabla q) - u_1\cdot\nabla\Lambda^{-\frac12}q\right)\Lambda^{-\frac12}q dx\right| \le C\|u_1\|_{H^2}\|q\|_{L^2}\|\Lambda^{-\frac12}q\|_{L^2}. \tag{214}$$

Putting (210)-(214) together, we obtain the energy inequality

$$\frac{d}{dt}\left[\|\Lambda^{-\frac12}q\|_{L^2}^2 + \|u\|_{L^2}^2\right] \le C\left[\|u_1\|_{H^2}^2 + \|\nabla u_2\|_{L^2}^2 + \|q_1\|_{L^4}^4\right]\left[\|\Lambda^{-\frac12}q\|_{L^2}^2 + \|u\|_{L^2}^2\right] \tag{215}$$

from which we obtain uniqueness. Finally, we show that the estimate (214) holds by establishing the commutator estimate

$$\|\Lambda^{-\frac12}(u_1\cdot\nabla q) - u_1\cdot\nabla\Lambda^{-\frac12}q\|_{L^2} \le C\|u_1\|_{H^2}\|q\|_{L^2}. \tag{216}$$

Indeed, let $w\in L^2(\mathbb{R}^2)$. By Parseval's identity, we have

$$\int_{\mathbb{R}^2}(\Lambda^{-\frac12}(u_1\cdot\nabla q)-u_1\cdot\nabla\Lambda^{-\frac12}q)(x)w(x)dx = \int_{\mathbb{R}^2}\mathcal{F}(\Lambda^{-\frac12}(u_1\cdot\nabla q)-u_1\cdot\nabla\Lambda^{-\frac12}q)(\xi)\mathcal{F}w(\xi)d\xi. \tag{217}$$

But

$$\mathcal{F}(\Lambda^{-\frac12}(u_1\cdot\nabla q))(\xi) = \int_{\mathbb{R}^2}|\xi|^{-\frac12}(\xi\cdot\mathcal{F}u_1(\xi-y))\mathcal{F}q(y)dy \tag{218}$$

and

$$\mathcal{F}(u_1\cdot\nabla\Lambda^{-\frac12}q)(\xi) = \int_{\mathbb{R}^2}|y|^{-\frac12}(\xi\cdot\mathcal{F}u_1(\xi-y))\mathcal{F}q(y)dy. \tag{219}$$

Consequently,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}(\Lambda^{-\frac{1}{2}}(u_1\cdot\nabla q)-u_1\cdot\nabla\Lambda^{-\frac{1}{2}}q)(x)w(x)dx\right|\\&\leq\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\min\{|\xi|,|y|\}\left||\xi|^{-\frac{1}{2}}-|y|^{-\frac{1}{2}}\right||\mathcal{F}u_1(\xi-y)||\mathcal{F}q(y)||\mathcal{F}w(\xi)|dyd\xi\end{aligned}\tag{220}$$

where we used

$$|\xi\cdot\mathcal{F}u_1(\xi-y)|\leq\min\{|\xi|,|y|\}\,|\mathcal{F}u_1(\xi-y)|\tag{221}$$

which holds due to the fact that the velocity is divergence-free. We note that

$$\min\{|\xi|,|y|\}\left||\xi|^{-\frac{1}{2}}-|y|^{-\frac{1}{2}}\right|\leq\frac{\min\{|\xi|,|y|\}}{|\xi|^{\frac{1}{2}}|y|^{\frac{1}{2}}}|\xi-y|^{\frac{1}{2}}\leq|\xi-y|^{\frac{1}{2}}\tag{222}$$

for all $\xi,y\in\mathbb{R}^2$. Therefore,

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(\Lambda^{-\frac{1}{2}}(u_1\cdot\nabla q)-u_1\cdot\nabla\Lambda^{-\frac{1}{2}}q)(x)w(x)dx\right|&\leq\||.|^{\frac{1}{2}}\mathcal{F}u_1(.)\|_{L^1}\|q\|_{L^2}\|w\|_{L^2}\\&\leq C\|u_1\|_{H^2}\|q\|_{L^2}\|w\|_{L^2}\end{aligned}\tag{223}$$

by Hölder's inequality and Young's convolution inequality. This gives (216) completing the proof of Theorem 6.

References

[1] E. Abdo, M. Ignatova, *Long time dynamics of a model of electroconvection*, Trans. Amer. Math. Soc. **374**, 5849–5875 (2021).

[2] C. Amrouche, V. Girault, M.E. Schonbek, T.P. Schonbek, *Pointwise decay of solutions and of higher derivatives to Navier-Stokes equations*, SIAM J. Math. Anal. **31**, 740–753 (2000).

[3] C. Bjorland, C.J. Niche, *On the decay of infinite energy solutions to the Navier-Stokes equations in the plane*, Physica D: Nonlinear Phenomena **240** (7), 670–674 (2011).

[4] C. Bjorland , M.E. Schonbek, *On questions of decay and existence for the viscous camassa-holm equations*, Ann. I. H. Poincaré-NA **25**, 907-936 (2008).

[5] A.P. Calderón, *Singular Integrals*, Bull. Amer. Math. Soc. **72**, 427–465 (1966).

[6] A. Córdoba, D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. **249**, 511–528 (2004).

[7] P. Constantin, T. Elgindi, M. Ignatova, V. Vicol, *On some electroconvection models*, Journal of Nonlinear Science **27**, 197–211 (2017).

[8] P. Constantin, J. Wu, *Behavior of solutions of 2D quasi-geostrophic equations*, SIAM J. Math. Anal. **30**, 937–948 (1999).

[9] Z.A. Daya, V.B. Deyirmenjian, S.W. Morris, J.R. de Bruyn, *Annular electroconvection with shear*, Phys. Rev. Lett. **80**, 964–967 (1998).

[10] B. Dong, Y. Li, *Large time behavior to the system of incompressible non-newtonian fluds in* $\mathbb{R}^2$, J. Math. Anal. Appl. **298**,667-676 (2004).

[11] R. Kajikiya, T. Miyakawa, *On* L^2 *decay of weak solutions of the Navier-Stokes equations in* $\mathbb{R}^n$, Math. Z. **192**, 135–148 (1986).

[12] I. Kukavica, *On the weighted decay for solutions of the Navier-Stokes system*, Nonlinear Analysis: Theory, Methods and Applications **70** (6), 2466–2470 (2009).

[13] I. Kukavica, *Space-time decay for Solutions of the Navier-Stokes equations*, Indiana Univ. Math. J. **50**, 205–222(2001).

[14] J. Leray, *Sur le mouvement d'un liquide visquex emplissant l'espace*, Acta Math. **63**, 193–248(1934).

[15] C.J. Niche , M.E. Schonbek , *Decay of weak solutions to the 2d dissipative quasi-geostrophic equation*, Comm. Math. Phys. **276**, 93-115 (2007).

[16] M. Oliver, E.S. Titi, *Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in* $\mathbb{R}^n$, J. Funct. Anal. **172**, 1–18 (2000).

[17] M.E. Schonbek, *L^2 decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **88**, 209–222 (1985).

[18] M.E. Schonbek, *Uniform decay rated for parabolic conservations laws*, Nonlinear Analysis: Theory, Methods and Applications **10**, 943-956 (1986).

[19] M.E. Schonbek, M. Wiegner, *On the decay of higher-order norms of the solutions of Navier-Stokes equations*, Proc. Roy. Soc. Edinburgh Sect. A **126**, 677–685 (1996).

[20] S. Takahashi, *A weigthed equation approach to decay rate estimates for the Navier-Stokes equations*, Nonlinear Anal. **37**, 751–789 (1999).

[21] P. Tsai, Z. Daya, S. Morris, *Charge transport scaling in turbulent electroconvection*, Phys. Rev E **72**, 046311-1-12 (2005).

[22] P. Tsai, Z.A. Daya, V.B. Deyirmenjian, S.W. Morris, *Direct numerical simulation of supercritical annular electroconvection*, Phys. Rev E **76**, 1–11 (2007).

[23] M. Wiegner, *Decay results for weak solutions of the Navier-Stokes equations on $\mathbb{R}^n$*, J. London Math. Soc. **35**, 303–313 (1987).

[24] X. Zhao, *Asymptotic behavior of solutions to a new hall-MHD system*, Acta Applicandae Mathematicae **157**, 205–216 (2018).

[25] C. Zhao, B. Li, *Time decay rate of weak solutions to the generalized MHD equations in $\mathbb{R}^2$*, Appl. Math. Comput. **292**, 1–8 (2017).

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122
E-mail address: abdo@temple.edu

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122
E-mail address: ignatova@temple.edu