LONG TIME DYNAMICS OF NERNST-PLANCK-NAVIER-STOKES SYSTEMS

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ABSTRACT. We consider the Nernst-Planck-Navier-Stokes system describing the electrodiffusion of ions in a viscous Newtonian fluid. We prove the exponential nonlinear stability of constant steady states in the case of periodic boundary conditions in any dimension of space without constraints on the number of species, valences and diffusivities. We consider also the case of two spatial dimensions, and we prove the exponential stability from arbitrary large data.

1. INTRODUCTION

Electrodiffusion is the motion of ions interacting with a fluid through electrical forces, and among themselves due to molecular diffusion and electrostatic forces. Electrodiffusion has a wide variety of applications in computational physics, electrochemisty, biophysics, electrophysiology, and neurophysiology (see [9] and references therein).

We consider an electrodiffusion model describing the evolution of n ionic species in a d-dimensional fluid. The ionic concentrations $c_i(x,t)$'s evolve according to the Nernst-Planck equations

$$\partial_t c_i + u \cdot \nabla c_i = D_i \Delta c_i + D_i \nabla \cdot (z_i c_i \nabla \Phi) \tag{1}$$

for $i = 1, \dots, n$, where $z_1, \dots, z_n$ and $D_1, \dots, D_n$ are respectively the valences and diffusivities of the species. The potential $\Phi(x,t)$ obeys the Poisson equation

$$-\epsilon \Delta \Phi = \rho \tag{2}$$

where

$$\rho = z_1 c_1 + \dots + z_n c_n \tag{3}$$

and ϵ is a positive constant proportional to the square of the Debye length. The velocity of the fluid $u(x,t)$ satisfies the Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - \rho \nabla \Phi \tag{4}$$

and the divergence-free condition

$$\nabla \cdot u = 0. \tag{5}$$

Here $p(x,t)$ represents the pressure of the fluid, and ν is the kinematic viscosity.

In this paper we consider the Nernst-Planck-Navier-Stokes (NPNS) system given in (1)–(5) on the d-dimensional torus $\mathbb{T}^d = [0, 2\pi]^d$ with periodic boundary conditions. We prove the exponential nonlinear stability of constant steady states in the case of periodic boundary conditions in any dimension of space without constraints on the number of species, valences and diffusivities. We consider also the case of two spatial dimensions, and we prove the exponential stability from arbitrary large data.

The NPNS system has been intensely studied in different situations and dimensions. In [12] and [13], it has been shown that the system has global weak solutions in the two and three dimensional cases for homogeneous Dirichlet boundary conditions and Neumann boundary conditions respectively. In [6], the NPNS system was considered in a two-dimensional bounded domain with different types of boundary conditions. Blocking boundary conditions, which are conditions imposing the vanishing of the normal flux of ions at the

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boundary model the situations where ions do not cross the boundary of the domain. Other boundary conditions were also studied, where some ions may cross some parts of the boundary while being blocked from crossing others. These kind of boundary conditions are called selective. A special case of selective boundary conditions, where the chemical potentials are constant on the boundary were singled out as uniform selective. In all three cases (blocking, uniform selective, general selective), the existence of global smooth solutions has been shown in [6] and the global convergence to steady states was shown for blocking and uniform selective boundary conditions. In [7], existence of global regular solutions on a three-dimensional bounded domain has been established for general selective boundary conditions in the cases of two ionic species and many ionic species having equal diffusivities.

The different types of boundary conditions described above give rise to different dynamical consequences. In the cases of a 2D or a 3D bounded domain with blocking or uniform selective boundary conditions, nonlinear stability of Boltzmann states has been obtained in [8]. Instabilities have been studied numerically [11, 15], and observed physically [10] for general selective boundary conditions.

In [2], we considered the NPNS system on the two-dimensional torus with periodic boundary conditions for two ionic species with valences 1 and -1 and same diffusivities, and we proved that the velocity of the fluid converges exponentially in time to zero and the ionic concentrations converge exponentially in time to their initial average. In the presence of body forces in the fluid and/or some added charge density, we showed that the system has a finite dimensional global attractor. In [4], we addressed the forced NPNS system for n ionic species with different valences and diffusivities and we proved the existence of a global analytic solution on the two-dimensional torus and a local analytic solution in the three-dimensional case.

In this paper, we first investigate the existence of global regular solutions of the NPNS system for n ionic species in higher dimensions without imposing any restrictions on the diffusivities and the valences of the species but rather on the size of the initial data. We consider natural spaces in which we measure the size of initial data and prove a nonlinear global stability result, namely that small initial data yield global solutions (Theorem 1) and these global solutions converge exponentially to steady states, which in the periodic setting are constant concentrations and zero velocity. Thus we generalize our previous result from [2] to arbitrary dimension, arbitrary number of species and arbitrary valences and diffusivities. Secondly, we consider the case of two spatial dimensions, with large regular initial data, different diffusivities, and different valences. We obtain global exponential decay (Theorem 5) meaning that the solutions converge exponentially fast to steady states. This result is based on a new application of a logarithmic Sobolev inequality for the basic energy principle of the NPNS equations.

2. Main Results

2.1. **Functional setting.**

Let $\mathcal{D}(\mathbb{T}^d)$ be the space of $C^\infty(\mathbb{R}^d)$ functions that are 2π periodic, and $\mathcal{D}_0(\mathbb{T}^d)$ be the subspace of mean-free functions in $\mathcal{D}(\mathbb{T}^d)$. We denote by $\mathcal{D}'(\mathbb{T}^d)$ and $\mathcal{D}_0'(\mathbb{T}^d)$ their dual spaces respectively. For $f \in \mathcal{D}(\mathbb{T}^d)$, we define the Fourier transform of f by

$$k \in \mathbb{Z}^d \mapsto \mathcal{F}f(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(x) e^{-ik\cdot x} dx \tag{6}$$

and we denote its inverse by $\mathcal{F}^{-1}$.

Let Φ be a nonnegative, decreasing, infinitely differentiable, radial function such that $\Phi(r) = 1$ for $r \in \left[0, \frac{1}{2}\right]$ and $\Phi(r) = 0$ for $r \in \left[\frac{5}{8}, \infty\right]$. For each $j \in \mathbb{Z}$, we let $\Psi_j(r) = \Phi\left(2^{-j-1}r\right) - \Phi(2^{-j}r)$, and we define the homogeneous blocks

$$\Delta_j f(x) = (\mathcal{F}^{-1}\Psi_j(|\cdot|) * f)(x) \tag{7}$$

and the lower frequency cutoff functions

$$S_j f(x) = \sum_{k \le j-1} \Delta_k f(x). \tag{8}$$

for $f \in \mathcal{D}_0'(\mathbb{T}^d)$.

Let $L^p(\mathbb{T}^d)$ be the space of 2π-periodic functions with the norm

$$\|f\|_{L^p(\mathbb{T}^d)} = \left(\int_{\mathbb{T}^d} |f(x)|^p dx\right)^{1/p} \tag{9}$$

for $p \in [1, \infty)$ with the usual convention when $p = \infty$.

For $s \in \mathbb{R}, 1 \le p, q \le \infty$, we define the homogeneous Besov space

$$\dot{B}^s_{p,q}(\mathbb{T}^d) = \left\{ f \in \mathcal{D}_0'(\mathbb{T}^d) : \|f\|_{\dot{B}^s_{p,q}(\mathbb{T}^d)} = \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j f\|^q_{L^p(\mathbb{T}^d)}\right)^{1/q} < \infty \right\} \tag{10}$$

and the time-dependent homogeneous Besov spaces

$$\tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{T}^d)) = \left\{ f \in \mathcal{D}_0'(\mathbb{T}^d) : \|f\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{T}^d))} = \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j f\|^q_{L^r(0,T;L^p(\mathbb{T}^d))}\right)^{1/q} < \infty \right\}. \tag{11}$$

For $s > 0$, we denote by $H^s(\mathbb{T}^d)$ the Sobolev spaces of measurable periodic functions f

$$f = \sum_{k\in\mathbb{Z}^d} f_k e^{ik\cdot x}. \tag{12}$$

obeying

$$\|f\|^2_{H^s} = \sum_{k\in\mathbb{Z}^d} (1 + |k|^s)^2 |f_k|^2 < \infty. \tag{13}$$

2.2. **Results.** In this paper, we address the global well-posedness and long-time behavior of solutions to the Nernst-Planck-Navier-Stokes system described by (1)–(5). The conservation of the spatial averages of the velocity u and ionic concentrations c_i in time is a key property of the model that is frequently used in the paper:

Remark 1. *Let $(u, c_1, \dots, c_n)$ be a solution to* (1)–(5). *Integrating the ionic concentration equation* (1) *in the spatial variable over the d-dimensional torus, and using the divergence-free property satisfied by u, we infer that*

$$\frac{d}{dt}\int_{\mathbb{T}^d} c_i(x,t)dx = 0$$

for any $t \ge 0$ and $i \in \{1, \cdots, n\}$, and consequently

$$\int_{\mathbb{T}^d} c_i(x,t)dx = \int_{\mathbb{T}^d} c_i(x,0)dx$$

for any $t \ge 0$ and $i \in \{1, \cdots, n\}$. Similarly, we integrate the velocity equation (4) *over $\mathbb{T}^d$, use the vanishing of the nonlinear term in ρ that follows from the identity*

$$\int_{\mathbb{T}^d} \rho\nabla\Phi dx = -\epsilon\int_{\mathbb{T}^d} \Delta\Phi\nabla\Phi dx = \epsilon\int_{\mathbb{T}^d} \nabla\Phi\nabla\cdot(\nabla\Phi)dx = -\int_{\mathbb{T}^d} \nabla\Phi\rho dx, \tag{14}$$

integrate in time, and deduce that

$$\int_{\mathbb{T}^d} u(x,t)dx = \int_{\mathbb{T}^d} u(x,0)dx \tag{15}$$

for any $t \ge 0$.

We prove first the global existence and uniqueness of regular solutions of the d-dimensional NPNS system (1)–(5) with small initial data:

Theorem 1. *Let $1 \le p < \infty$. Let $u_0 \in \dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{T}^d)$ be divergence-free with a zero spatial average. Let $c_1(0), \dots, c_n(0) \in \dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{T}^d)$. For $1 \le p < \infty$, let E_p be the functional space defined by*

$$E_p = \left\{ F \in \mathcal{D}_0'(\mathbb{T}^d) : \|F\|_{E_p} = \|F\|_{\tilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} + \|F\|_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}} < \infty \right\}. \tag{16}$$

There exists an $\epsilon > 0$ such that for any $\epsilon_0 \in (0, \epsilon)$, if

$$\|u_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \sum_{i=1}^n \|c_i(0)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} < \epsilon_0 \tag{17}$$

then the the system (1)–(5) has a unique global-in-time solution $(u, c_1, \dots, c_n)$ in $(E_p)^{n+1}$ obeying

$$\|u\|_{E_p} + \sum_{i=1}^n \|c_i\|_{E_p} < 2\epsilon_0. \tag{18}$$

The proof of Theorem 1 is based on a fixed point iteration introduced in [5]. Namely, we let $S^0 = 0$ and $S^{(n)}$ be the solution of the linear parabolic system approximating (1)–(5), forced by matching nonlinear terms depending only on $S^{(n-1)}$. We show that the iterative inequality

$$\|S^{(n)}\|_{E_p} \le C_0 + C_1 \|S^{(n-1)}\|_{E_p}^2 \tag{19}$$

holds for all $n \in \mathbb{N}$, where C_0 is a positive constant depending only on the size of the initial data and C_1 is a positive universal constant. This estimate yields global unique regular solutions when C_0 is sufficiently small. The main challenges arise from estimating the nonlinearities in the functional space E_p, which is based on decomposing the nonlinear terms using the paraproduct decomposition and estimating using the uniform boundedness of the dyadic blocks in L^p spaces, Bernstein's inequality, the localization of the heat kernel, and the boundedness of the Riesz transform in Besov spaces.

Preserved for all positive times, the smallness of the solution of (1)–(5) in E_p can be used to show that the L^2 norm of the velocity decays exponentially in time to zero and the L^2 norm of the ionic concentrations decay exponentially in time to their initial spatial averages:

Theorem 2. *Let $u_0 \in \dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$ be divergence-free with a zero spatial average. Let $c_1(0), \dots, c_n(0) \in \dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$. Suppose the initial data $(u_0, c_1(0), \dots, c_n(0))$ is sufficiently small in $\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$, and the initial concentrations $(c_1(0), \dots, c_n(0))$ are sufficiently small in $L^2(\mathbb{T}^d)$. Then the unique solution of $(u, c_1, \dots, c_n)$ of (1)–(5) obeys*

$$\|u(t)\|_{L^2}^2 + \|c_i(t) - \bar{c}_i(0)\|_{L^2}^2 \le \left(\|u_0\|_{L^2}^2 + \sum_{i=1}^n \|c_i(0) - \bar{c}_i(0)\|_{L^2}^2 \right) e^{-ct} \tag{20}$$

where c is a positive constant depending only on the parameters of the problem, and $\bar{c}_i(0)$ is the initial spatial average of the ionic concentration c_i.

It is shown in [4] that the two-dimensional electrodiffusion model (1)–(5) has a unique global regular solution for arbitrary initial data, that is if the initial velocity is $H^1(\mathbb{T}^2)$ regular and the initial concentrations are nonnegative and $H^1(\mathbb{T}^2)$ regular, then the system (1)–(5) has a unique solution $(u, c_1, \dots, c_n)$ on $[0, \infty)$ satisfying

$$u \in L^\infty(0, T; H^1(\mathbb{T}^2)) \cap L^2(0, T; H^2(\mathbb{T}^2)) \tag{21}$$

and

$$c_i \in L^\infty(0, T; H^1(\mathbb{T}^2)) \cap L^2(0, T; H^2(\mathbb{T}^2)) \tag{22}$$

for $i = 1, \dots, n$ and for any $T > 0$. Moreover, the ionic concentrations are nonnegative for all positive times $t > 0$.

In fact, the solution to the model (1)–(5) is smooth provided that the initial data is smooth:

Remark 2. *Let $k \geq 1$. Suppose $u_0 \in H^k(\mathbb{T}^2)$ is divergence-free and mean-free, and $c_i(0) \in H^k(\mathbb{T}^2)$ is nonnegative. Then the solution to the Nernst-Planck-Navier-Stokes system* (1)–(5) *belongs to $L^\infty(0,T;H^k(\mathbb{T}^2))$ and $L^2(0,T;H^{k+1}(\mathbb{T}^2))$. Indeed, a Galerkin approximation scheme can be adapted to show that the solution lies in the aforementioned Lebesgue spaces on a short time interval $[0,T_k]$, and this regularity propagates to the whole interval $[0,T]$ by a continuity criterion that follows from the uniform-in-time boundedness of the solution in Sobolev spaces (see Theorem 4).*

We prove in this paper the exponential decay to steady state of the unique solution of (1)–(5) for any regular large initial data for the case where the ionic species have equal diffusivities:

Theorem 3. *Let $d = 2$ and suppose $D_1 = \cdots = D_n$. Let $u_0 \in H^1(\mathbb{T}^2)$ be divergence-free and $c_i(0) \in H^1(\mathbb{T}^2)$ be nonnegative. Then there is a positive constant C_0 depending exponentially on the initial data and the parameters of the problem, and a positive constant γ_0 depending only on the parameters of the problem such that the unique solution $(u, c_1, \dots, c_n)$ of* (1)–(5) *obeys*

$$\|\nabla u(t)\|_{L^2}^2 + \sum_{i=1}^n \|\nabla c_i(t)\|_{L^2}^2 + \int_t^{t+1} \left[\|\Delta u(s)\|_{L^2}^2 + \sum_{i=1}^n \|\Delta c_i(s)\|_{L^2}^2\right] ds \leq C_0 e^{-\frac{\gamma_0}{2}t} \tag{23}$$

for any $t \geq 0$.

We note that Theorem 3 generalizes the time decay obtained in [2] for two ionic species with valences 1 and -1 and same diffusivities. We show furthermore that the exponential decay in time holds in all Sobolev spaces:

Theorem 4. *Let $d = 2$ and suppose $D_1 = \cdots = D_n$. Let $k \geq 2$. Let $u_0 \in H^k(\mathbb{T}^2)$ be divergence-free and $c_i(0) \in H^k(\mathbb{T}^2)$ be nonnegative. Suppose there is a positive constant C_{k-1} depending only on the initial data and the parameters of the problem, and a positive constant γ depending only on the parameters of the problem such that the solution $(u, c_1, \dots, c_n)$ of* (1)–(5) *obeys*

$$\begin{aligned}&\|(-\Delta)^{\frac{k-1}{2}} u(t)\|_{L^2}^2 + \sum_{i=1}^n \|(-\Delta)^{\frac{k-1}{2}} c_i(t)\|_{L^2}^2 \\ &+ \int_t^{t+1} \left[\|(-\Delta)^{\frac{k}{2}} u(s)\|_{L^2}^2 + \sum_{i=1}^n \|(-\Delta)^{\frac{k}{2}} c_i(s)\|_{L^2}^2\right] ds \leq C_{k-1} e^{-\frac{\gamma}{2^k}t}\end{aligned} \tag{24}$$

for any $t \geq 0$. Then there is a positive constant C_k depending only on the initial data and the parameters of the problem such that the solution $(u, c_1, \dots, c_n)$ of (1)–(5) *obeys*

$$\begin{aligned}&\|(-\Delta)^{\frac{k}{2}} u(t)\|_{L^2}^2 + \sum_{i=1}^n \|(-\Delta)^{\frac{k}{2}} c_i(t)\|_{L^2}^2 \\ &+ \int_t^{t+1} \left[\|(-\Delta)^{\frac{k+1}{2}} u(s)\|_{L^2}^2 + \sum_{i=1}^n \|(-\Delta)^{\frac{k+1}{2}} c_i(s)\|_{L^2}^2\right] ds \leq C_k e^{-\frac{\gamma}{2^{k+1}}t}\end{aligned} \tag{25}$$

for any $t \geq 0$.

Our main result for the two-dimensional NPNS system is the global exponential decay to steady state for n ionic species with different valences and diffusivities:

Theorem 5. *Let $d = 2$ and consider the Nernst-Planck-Navier-Stokes system* (1)–(5) *on $\mathbb{T}^2$ with different valences and diffusivities. Let $k \geq 1$. Let $u_0 \in H^k(\mathbb{T}^2)$ be divergence-free and $c_i(0) \in H^k(\mathbb{T}^2)$ be nonnegative. Then there exist a positive constant $C_{1,k}$ depending only on k, the H^k norm of the initial data and the parameters of the problem, and a positive constant Γ_0 depending only on the parameters of the problem, such that the solution $(u, c_1, \dots, c_n)$ of* (1)–(5) *obeys for all $k \geq 0$*

$$\|(-\Delta)^{\frac{k}{2}} u(t)\|_{L^2}^2 + \sum_{i=1}^n \|(-\Delta)^{\frac{k}{2}} c_i(t)\|_{L^2}^2 \leq C_k e^{-\frac{\Gamma_0}{2^{k+1}}t} \tag{26}$$

for any $t \geq 0$.

The proof of Theorem 5 exploits the dissipative structure of the NPNS system. A logarithmic Sobolev inequality (182) is used to show that the dissipation dominates the energy (187). The proof of decay of higher norms is based on the proofs of Theorems 3 and 4.

Remark 3. *We point out that the tools required to prove Theorem 5 in the case of n ionic species with arbitrary diffusivities are different from those employed in Theorem 3 when the ions have equal diffusivities. Indeed, this latter case relies only on the boundedness of the potential Φ in $L^{\infty}(0,T;H^1(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$ and yields the time decay of the ionic concentrations in $L^2(\mathbb{T}^2)$ to their initial spatial averages. In contrast, the case of different diffusivities requires $c_i \ln c_i$ bounds in $L^1(\mathbb{T}^2)$ for the concentrations and a novel logarithmic Sobolev inequality to obtain the desired decay in $L^2(\mathbb{T}^2)$. We shed light on these differences in the proofs of Theorems 3 and 5.*

3. Preliminaries

We recall some estimates from the Littlewood-Paley theory:

Proposition 1. [3, 14] *Let $f \in \mathcal{D}_0'(\mathbb{T}^d)$.*

(1) *Let $1 \le p \le \infty$. Let k be a nonnegative integer. For all $j \in \mathbb{Z}$, we have*

$$\sup_{|\alpha|=k} \|\partial^{\alpha}\Delta_j f\|_{L^p(\mathbb{T}^d)} \le C_k 2^{jk}\|\Delta_j f\|_{L^p(\mathbb{T}^d)}. \tag{27}$$

(2) *Let $1 \le p \le q \le \infty$. For all $j \in \mathbb{Z}$, we have*

$$\|\Delta_j f\|_{L^q(\mathbb{T}^d)} \le C 2^{dj\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Delta_j f\|_{L^p(\mathbb{T}^d)} \tag{28}$$

Moreover, the continuous Besov embedding

$$\dot{B}^{s}_{p_1,q_1}(\mathbb{T}^d) \hookrightarrow \dot{B}^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,q_2}(\mathbb{T}^d) \tag{29}$$

holds for $1 \le p_1 \le p_2 \le \infty, 1 \le q_1 \le q_2 \le \infty$ and $s \in \mathbb{R}$.

(3) *Let $1 \le p \le \infty, t \ge 0, \alpha > 0$. Then*

$$\|e^{-t\Lambda^{\alpha}}\Delta_j f\|_{L^p(\mathbb{T}^d)} \le Ce^{-C^{-1}t2^{j\alpha}}\|\Delta_j f\|_{L^p(\mathbb{T}^d)} \tag{30}$$

holds for all $j \in \mathbb{Z}$. Here Λ^{α} is the Fourier multiplier with symbol $|k|^{\alpha}$.

(4) *For $k \in \{1,\dots,d\}$, let $R_k = \partial_k\Lambda^{-1}$. Let $R = (R_1,\dots,R_d)$ be the periodic Riesz transform. For each $p \in [1,\infty]$, there is a positive constant $C > 0$ depending only on p and d (independent of j) such that*

$$\|\Delta_j Rf\|_{L^p(\mathbb{T}^d)} \le C\|\Delta_j f\|_{L^p(\mathbb{T}^d)} \tag{31}$$

holds for all $j \in \mathbb{Z}$. Hence, for $s \in \mathbb{R}$ and $1 \le p,q \le \infty$, R is bounded from $\dot{B}^s_{p,q}(\mathbb{T}^d)$ onto itself.

We note that there exists a negative integer j_0 such that $\Delta_j f$ vanishes for $j \le j_0$, a fact that follows from the definitions.

The following proposition is used to decompose the dyadic blocks of the product of two functions:

Proposition 2. [3] *Let $f, g \in \mathcal{D}_0'(\mathbb{T}^d)$. Then*

$$\begin{aligned}\Delta_j(fg) &= \sum_{k\ge j-2}\Delta_j(S_{k+1}f\Delta_k g) + \sum_{k\ge j-2}\Delta_j(S_k g\Delta_k f)\\ &= \sum_{k\ge j-2}\Delta_j(S_{k+1}g\Delta_k f) + \sum_{k\ge j-2}\Delta_j(S_k f\Delta_k g)\end{aligned} \tag{32}$$

holds for any $j \in \mathbb{Z}$.

We recall the uniform Gronwall lemma for decay:

Lemma 1. [1] *Let $y(t) \geq 0$ obey a differential inequality*

$$\frac{d}{dt}y + c_1 y \leq F_1 + F(t)$$

with initial datum $y(0) = y_0$ with F_1 a nonnegative constant, and $F(t) \geq 0$ obeying

$$\int_t^{t+1} F(s)ds \leq g_0 e^{-c_2 t} + F_2$$

where c_1, c_2, g_0 are positive constants and F_2 is a nonnegative constant. Then

$$y(t) \leq y_0 e^{-c_1 t} + g_0 e^{c_1 + c}(t+1)e^{-ct} + \frac{1}{c_1}F_1 + \frac{e^{c_1}}{1 - e^{-c_1}}F_2$$

holds with $c = \min\{c_1, c_2\}$.

Finally, we state and prove the following fractional product estimate:

Lemma 2. *Let $k \geq 1$. Let $f, g \in H^k(\mathbb{T}^2)$ be mean-zero functions. Then there is a positive constant C_k depending on k such that*

$$\begin{aligned}\|(-\Delta)^{\frac{k}{2}}(fg)\|_{L^2} \leq C_k \|f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{L^2}^{\frac{1}{2}} \|(-\Delta)^{\frac{k}{2}} g\|_{L^2}^{\frac{1}{2}} \|(-\Delta)^{\frac{k+1}{2}} g\|_{L^2}^{\frac{1}{2}} \\ + C_k \|(-\Delta)^{\frac{k}{2}} f\|_{L^2}^{\frac{1}{2}} \|(-\Delta)^{\frac{k+1}{2}} f\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\nabla g\|_{L^2}^{\frac{1}{2}}\end{aligned} \tag{33}$$

holds.

Proof. The bound (33) follows from the fractional inequality

$$\|(-\Delta)^{\frac{k}{2}}(fg)\|_{L^2} \leq C_k \left[\|f\|_{L^4} \|(-\Delta)^{\frac{k}{2}} g\|_{L^4} + \|(-\Delta)^{\frac{k}{2}} f\|_{L^4} \|g\|_{L^4} \right] \tag{34}$$

followed by applications of the Ladyzhenskaya's interpolation inequality. We omit further details.

In this paper, the letter C (or $C_i, i = 1, 2, \ldots$) will be frequently used to denote a positive constant depending only on universal constants and the parameters of the problems, and this constant may change from line to line along the proofs.

4. Proof of Theorem 1

In the proof of Theorem 1 below, we use the following auxiliary propositions.

Proposition 3. *Suppose $F, G \in \mathcal{D}_0'(\mathbb{T}^d)$. Let*

$$\mathcal{B}(F, G) = \int_0^t e^{(t-s)\Delta} \nabla \cdot (FG)(s) ds. \tag{35}$$

and

$$\mathcal{B}'(F, G) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (FG)(s) ds \tag{36}$$

where $\mathbb{P}$ is the Leray-Hodge projector on divergence-free vector fields. Then

$$\|\mathcal{B}(F, G)\|_{E_p} \leq C \|F\|_{E_p} \|G\|_{E_p} \tag{37}$$

and

$$\|\mathcal{B}'(F, G)\|_{E_p} \leq C \|F\|_{E_p} \|G\|_{E_p}. \tag{38}$$

Proof. For $j \in \mathbb{Z}$, we apply Δ_j to $\mathcal{B}(F, G)$. Since Δ_j and ∇ commutes, we have

$$\Delta_j \mathcal{B}(F, G) = \int_0^t e^{(t-s)\Delta} \nabla \cdot \Delta_j (FG)(s) ds \tag{39}$$

and hence

$$\|\Delta_j \mathcal{B}(F, G)\|_{L^p} \leq C \int_0^t e^{-(t-s)2^{2j}} 2^j \|\Delta_j (FG)(s)\|_{L^p} ds \tag{40}$$

in view of Bernstein's inequality (27) and the localization of the heat kernel (30). Now we use Proposition 2 to decompose $\Delta_j(FG)$ as

$$\Delta_j(FG) = \sum_{k\ge j-2} \Delta_j(S_{k+1}F\Delta_k G) + \sum_{k\ge j-2}\Delta_j(S_kG\Delta_kF). \tag{41}$$

Taking the L^p norm and using the uniform-in-j boundedness of Δ_j on L^p, we obtain

$$\|\Delta_j(FG)\|_{L^p} \le C\sum_{k\ge j-2}\|S_{k+1}F\|_{L^\infty}\|\Delta_kG\|_{L^p} + C\sum_{k\ge j-2}\|S_kG\|_{L^\infty}\|\Delta_kF\|_{L^p} \tag{42}$$

and consequently

$$\begin{aligned}\|\Delta_j\mathcal{B}(F,G)\|_{L^p} &\le C\int_0^t e^{-(t-s)2^{2j}}2^j\sum_{k\ge j-2}\|S_{k+1}F(s)\|_{L^\infty}\|\Delta_kG(s)\|_{L^p}ds\\ &\quad + C\int_0^t e^{-(t-s)2^{2j}}2^j\sum_{k\ge j-2}\|S_kG(s)\|_{L^\infty}\|\Delta_kF(s)\|_{L^p}ds\\ &= \mathcal{B}_{1,j}(F,G) + \mathcal{B}_{2,j}(F,G)\end{aligned} \tag{43}$$

where

$$\mathcal{B}_{1,j}(F,G) = C\int_0^t e^{-(t-s)2^{2j}}2^j\sum_{k\ge j-2}\|S_{k+1}F(s)\|_{L^\infty}\|\Delta_kG(s)\|_{L^p}ds \tag{44}$$

and

$$\mathcal{B}_{2,j}(F,G) = C\int_0^t e^{-(t-s)2^{2j}}2^j\sum_{k\ge j-2}\|S_kG(s)\|_{L^\infty}\|\Delta_kF(s)\|_{L^p}ds. \tag{45}$$

We start by estimating $\mathcal{B}_{1,j}(F,G)$ in L_t^∞ and L_t^1. In view of Bernstein's inequality (28), we estimate

$$\|S_{k+1}F(s)\|_{L^\infty} \le \sum_{l\le k}\|\Delta_lF(s)\|_{L^\infty} \le C\sum_{l\le k}2^{l\frac{d}{p}}\|\Delta_lF(s)\|_{L^p} \le C\|F\|_{\widetilde{L}_t^\infty\dot{B}_{p,1}^{\frac{d}{p}}}, \tag{46}$$

hence

$$\begin{aligned}\|\mathcal{B}_{1,j}(F,G)\|_{L_t^\infty} &\le C\|F\|_{\widetilde{L}_t^\infty\dot{B}_{p,1}^{\frac{d}{p}}}\left\|\int_0^t 2^je^{-(t-s)2^{2j}}\sum_{k\ge j-2}\|\Delta_kG(s)\|_{L^p}ds\right\|_{L_t^\infty}\\ &\le C\|F\|_{\widetilde{L}_t^\infty\dot{B}_{p,1}^{\frac{d}{p}}}\sum_{k\ge j-2}2^j\|\Delta_kG\|_{L_t^1L^p}.\end{aligned} \tag{47}$$

Since Δ_kG has a zero spatial average for any $k\in\mathbb{Z}$, we bound

$$\|\Delta_kG\|_{L_t^1L^p} \le C\|\nabla\Delta_kG\|_{L_t^1L^p} \le C2^k\|\Delta_kG\|_{L_t^1L^p} \tag{48}$$

in view of Poincaré's inequality and Bernstein's inequality. Multiplying $\|\mathcal{B}_{1,j}(F,G)\|_{L_t^\infty}$ by $2^{j\frac{d}{p}}$, we obtain

$$\begin{aligned}2^{j\frac{d}{p}}\|\mathcal{B}_{1,j}(F,G)\|_{L_t^\infty} &\le C\|F\|_{\widetilde{L}_t^\infty\dot{B}_{p,1}^{\frac{d}{p}}}\sum_{k\ge j-2}2^{(j-k)\left(\frac{d}{p}+1\right)}2^{k\left(\frac{d}{p}+1\right)}\|\Delta_kG\|_{L_t^1L^p}\\ &\le C\|F\|_{\widetilde{L}_t^\infty\dot{B}_{p,1}^{\frac{d}{p}}}\sum_{k\ge j-2}2^{(j-k)\left(\frac{d}{p}+1\right)}2^{k\left(\frac{d}{p}+2\right)}\|\Delta_kG\|_{L_t^1L^p}.\end{aligned} \tag{49}$$

We apply the ℓ^1 norm in j and we use Young's convolution inequality to conclude that

$$\left\|2^{j\frac{d}{p}}\|\mathcal{B}_{1,j}(F,G)\|_{L_t^\infty}\right\|_{\ell^1} \le C\|F\|_{\widetilde{L}_t^\infty\dot{B}_{p,1}^{\frac{d}{p}}}\|G\|_{\widetilde{L}_t^1\dot{B}_{p,1}^{\frac{d}{p}+2}}. \tag{50}$$

On the other hand, taking the L^1_t norm of $\mathcal{B}_{1,j}(F,G)$ yields

$$\begin{aligned}\|\mathcal{B}_{1,j}(F,G)\|_{L^1_t} &\le C\|F\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \left\| \int_0^t 2^j e^{-(t-s)2^{2j}} \sum_{k\ge j-2} \|\Delta_k G(s)\|_{L^p} ds \right\|_{L^1_t} \\ &\le C\|F\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \sum_{k\ge j-2} 2^{-j} \|\Delta_k G\|_{L^1_t L^p}. \end{aligned} \tag{51}$$

We multiply both sides by $2^{j\left(\frac{d}{p}+2\right)}$, and we obtain

$$\begin{aligned} 2^{j\left(\frac{d}{p}+2\right)} \|\mathcal{B}_{1,j}(F,G)\|_{L^1_t} &\le C\|F\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \sum_{k\ge j-2} 2^{(j-k)\left(\frac{d}{p}+1\right)} 2^{k\left(\frac{d}{p}+1\right)} \|\Delta_k G\|_{L^1_t L^p} \\ &\le C\|F\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \sum_{k\ge j-2} 2^{(j-k)\left(\frac{d}{p}+1\right)} 2^{k\left(\frac{d}{p}+2\right)} \|\Delta_k G\|_{L^1_t L^p}, \end{aligned} \tag{52}$$

hence

$$\left\| 2^{j\left(\frac{d}{p}+2\right)} \|\mathcal{B}_{1,j}(F,G)\|_{L^1_t} \right\|_{\ell^1} \le C\|F\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \|G\|_{\widetilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}}. \tag{53}$$

Now, we estimate $\mathcal{B}_{2,j}(FG)$ in L^∞_t and L^1_t. In view of Bernstein's inequality (28) and Poincaré's inequality, we estimate

$$\|S_k G(s)\|_{L^\infty} \le \sum_{l\le k-1} \|\Delta_l G(s)\|_{L^\infty} \le C \sum_{l\le k-1} 2^{l\frac{d}{p}} 2^l \|\Delta_l G(s)\|_{L^p} \le C 2^k \|G\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}}, \tag{54}$$

hence

$$2^{j\frac{d}{p}} \|\mathcal{B}_{2,j}(F,G)\|_{L^\infty_t} \le C\|G\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \sum_{k\ge j-2} 2^{(j-k)\left(\frac{d}{p}+1\right)} 2^{k\left(\frac{d}{p}+2\right)} \|\Delta_k F\|_{L^1_t L^p}. \tag{55}$$

We take the ℓ^1 norm in j and we obtain

$$\left\| 2^{j\frac{d}{p}} \|\mathcal{B}_{2,j}(F,G)\|_{L^\infty_t} \right\|_{\ell^1} \le C\|G\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \|F\|_{\widetilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}}. \tag{56}$$

Taking the L^1_t norm of $\mathcal{B}_{2,j}(F,G)$, multiplying both sides by $2^{j\left(\frac{d}{p}+2\right)}$ and then taking the ℓ^1 norm in j yield

$$\left\| 2^{j\left(\frac{d}{p}+2\right)} \|\mathcal{B}_{2,j}(F,G)\|_{L^1_t} \right\|_{\ell^1} \le C\|G\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \|F\|_{\widetilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}}. \tag{57}$$

Putting (50), (53), (56) and (57) together, we obtain (37). The proof of (38) is similar to that of (37) and is based on the fact that the Leray projector $\mathbb{P}$ is bounded on Besov spaces. We omit further details.

Remark 4. *The product estimates* (37) *and* (38) *hold in the whole space setting and were used in* [5] *to estimate the transport nonlinear term driving the Navier-Stokes equations in* $\mathbb{R}^d$*. The proof in the periodic setting provided above is somewhat simpler due to the Poincaré inequality* (48)*, which is not available in* $\mathbb{R}^d$*.*

Proposition 4. *Let* $g \in \mathcal{D}'_0(\mathbb{T}^d)$*, and let* v *be the solution of the d-dimensional Laplace equation*

$$\Delta v = g \tag{58}$$

with periodic boundary conditions. Let

$$\mathcal{S}(g,v) = \int_0^t e^{(t-s)\Delta} \mathbb{P}\left[g\nabla v\right](s) ds. \tag{59}$$

Then

$$\|\mathcal{S}(g,v)\|_{E_p} \le C\|g\|^2_{E_p} \tag{60}$$

Proof. For $j \in \mathbb{Z}$, we have

$$\|\Delta_j \mathcal{S}(g,v)\|_{L^p} \leq C \int_0^t e^{-(t-s)2^{2j}} \|\Delta_j (g\nabla v)(s)\|_{L^p} ds \tag{61}$$

in view of (30) and the boundedness of the Leray projector on L^p spaces. Decomposing $\Delta_j(g\nabla v)$ as

$$\Delta_j(g\nabla v) = \sum_{k\geq j-2} \Delta_j(S_{k+1}g\Delta_k \nabla v) + \sum_{k \geq j-2} \Delta_j(S_k \nabla v \Delta_k g), \tag{62}$$

we estimate

$$\begin{aligned}\|\Delta_j \mathcal{S}(g,v)\|_{L^p} &\leq C \int_0^t e^{-(t-s)2^{2j}} \sum_{k\geq j-2} \|S_{k+1}g(s)\|_{L^\infty} \|\Delta_k \nabla v(s)\|_{L^p} ds \\ &+ C \int_0^t e^{-(t-s)2^{2j}} \sum_{k\geq j-2} \|S_k \nabla v(s)\|_{L^\infty} \|\Delta_k g(s)\|_{L^p} ds \\ &= \mathcal{S}_{1,j}(g,v) + \mathcal{S}_{2,j}(g,v)\end{aligned} \tag{63}$$

where

$$\mathcal{S}_{1,j}(g,v) = C \int_0^t e^{-(t-s)2^{2j}} \sum_{k\geq j-2} \|S_{k+1}g(s)\|_{L^\infty} \|\Delta_k \nabla v(s)\|_{L^p} ds \tag{64}$$

and

$$\mathcal{S}_{2,j}(g,v) = C \int_0^t e^{-(t-s)2^{2j}} \sum_{k\geq j-2} \|S_k \nabla v(s)\|_{L^\infty} \|\Delta_k g(s)\|_{L^p} ds. \tag{65}$$

Now we estimate $\mathcal{S}_{1,j}(g,v)$ and $\mathcal{S}_{2,j}(g,v)$ in L_t^∞ and L_t^1.

In view of Bernstein's inequality (28), we have

$$\|S_{k+1}g(s)\|_{L^\infty} \leq C\|g\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}}}, \tag{66}$$

and consequently

$$\|\mathcal{S}_{1,j}(g,v)\|_{L_t^\infty} \leq C\|g\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \sum_{k\geq j-2} \|\Delta_k \nabla v\|_{L_t^1 L^p}. \tag{67}$$

In view of Poincaré's inequality and Bernstein's inequality, we have

$$\|\Delta_k \nabla v\|_{L_t^1 L^p} \leq C\|\nabla\nabla\Delta_k \Delta v\|_{L_t^1 L^p} \leq C 2^{2k} \|\Delta_k g\|_{L_t^1 L^p} \tag{68}$$

for all $k \in \mathbb{Z}$. We multiply $\|\mathcal{S}_{1,j}(g,v)\|_{L_t^\infty}$ by $2^{j\frac{d}{p}}$ and we obtain

$$\begin{aligned} 2^{j\frac{d}{p}} \|\mathcal{S}_{1,j}(g,v)\|_{L_t^\infty} &\leq C\|g\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \sum_{k\geq j-2} 2^{(j-k)\frac{d}{p}} 2^{k\frac{d}{p}} \|\Delta_k \nabla v\|_{L_t^1 L^p} \\ &\leq C\|g\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \sum_{k\geq j-2} 2^{(j-k)\frac{d}{p}} 2^{k\left(\frac{d}{p}+2\right)} \|\Delta_k g\|_{L_t^1 L^p}. \end{aligned} \tag{69}$$

We apply the ℓ^1 norm in j and we use Young's convolution inequality to conclude that

$$\left\| 2^{j\frac{d}{p}} \|\mathcal{S}_{1,j}(g,v)\|_{L_t^\infty} \right\|_{\ell^1} \leq C\|g\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \|g\|_{\widetilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}+2}}. \tag{70}$$

Now we take the L_t^1 norm of $\mathcal{S}_{1,j}(F,G)$, and we obtain

$$\begin{aligned} \|\mathcal{S}_{1,j}(g,v)\|_{L_t^1} &\leq C\|g\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \left\| \int_0^t e^{-(t-s)2^{2j}} \sum_{k\geq j-2} \|\Delta_k \nabla v(s)\|_{L^p} ds \right\|_{L_t^1} \\ &\leq C\|g\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \sum_{k\geq j-2} 2^{-2j} \|\Delta_k \nabla v\|_{L_t^1 L^p}. \end{aligned} \tag{71}$$

We multiply both sides by $2^{j\left(\frac{d}{p}+2\right)}$, and we obtain

$$
\begin{aligned}
2^{j\left(\frac{d}{p}+2\right)}\|\mathcal{S}_{1,j}(g,v)\|_{L^1_t} &\le C\|g\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \sum_{k\ge j-2} 2^{(j-k)\frac{d}{p}} 2^{k\frac{d}{p}} \|\Delta_k \nabla v\|_{L^1_t L^p} \\
&\le C\|g\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \sum_{k\ge j-2} 2^{(j-k)\frac{d}{p}} 2^{k\left(\frac{d}{p}+2\right)} \|\Delta_k g\|_{L^1_t L^p},
\end{aligned}
\tag{72}
$$

where we have used the Poincaré inequality. Hence

$$
\left\| 2^{j\left(\frac{d}{p}+2\right)} \|\mathcal{S}_{1,j}(g,v)\|_{L^1_t} \right\|_{\ell^1} \le C\|g\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \|g\|_{\widetilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}}. \tag{73}
$$

Similarly, we estimate $\mathcal{S}_{2,j}(g,v)$ in L^∞_t and L^1_t. In view of Poincaré's inequality and Bernstein's inequality, we have

$$
\|S_k \nabla v\|_{L^\infty_t L^\infty} \le C \sum_{l\le k-1} 2^{l\frac{d}{p}} \|\Delta_l \nabla v\|_{L^\infty_t L^p} \le C \sum_{l\le k-1} 2^{l\frac{d}{p}} \|\Delta_l g\|_{L^\infty_t L^p} \le C\|g\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \tag{74}
$$

for all $k \in \mathbb{Z}$. Multiplying $\|\mathcal{S}_{2,j}(g,v)\|_{L^\infty_t}$ by $2^{j\frac{d}{p}}$, applying the Poincaré inequality twice, taking the ℓ^1 norm in j, and finally using Young's convolution inequality, we obtain

$$
\left\| 2^{j\frac{d}{p}} \|\mathcal{S}_{2,j}(g,v)\|_{L^\infty_t} \right\|_{\ell^1} \le C\|g\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \|g\|_{\widetilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}}. \tag{75}
$$

For the L^1_t norm of $\mathcal{S}_{2,j}(g,v)$, we have

$$
\|\mathcal{S}_{2,j}(g,v)\|_{L^1_t} \le C\|g\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \sum_{k\ge j-2} 2^{-2j} \|\Delta_k g\|_{L^1_t L^p}. \tag{76}
$$

We multiply both sides by $2^{j\left(\frac{d}{p}+2\right)}$ and we estimate. We obtain

$$
\left\| 2^{j\left(\frac{d}{p}+2\right)} \|\mathcal{S}_{2,j}(g,v)\|_{L^1_t} \right\|_{\ell^1} \le C\|g\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} \|g\|_{\widetilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}}. \tag{77}
$$

Putting (70), (73), (75) and (77) together, we obtain (60).

Proof of Theorem 1. Let $u^{(0)} = c_1^{(0)} = \cdots = c_n^{(0)} = 0$. For each positive integer m, let $(u^{(m)}, c_1^{(m)}, \dots, c_n^{(m)})$ be the solution of

$$
\begin{cases}
\partial_t u^{(m)} - \nu\Delta u^{(m)} = -\mathbb{P}(u^{(m-1)}\cdot\nabla u^{(m-1)}) - \mathbb{P}(\rho^{(m-1)}\nabla\Phi^{(m-1)}) \\
\nabla\cdot u^{(m)} = 0 \\
\rho^{(m)} = z_1 c_1^{(m)} + \cdots + z_n c_n^{(m)} \\
-\epsilon\Delta\Phi^{(m)} = \rho^{(m)} \\
\partial_t c_i^{(m)} - D_i\Delta c_i^{(m)} = -u^{(m-1)}\cdot\nabla c_i^{(m-1)} + D_i\nabla\cdot(z_i c_i^{(m-1)}\nabla\Phi^{(m-1)}), i = 1, \dots, n
\end{cases}
\tag{78}
$$

posed on $\mathbb{T}^d \times [0,\infty)$. For simplicity, suppose $\nu = D_1 = \cdots = D_n = 1$. For each m, the smooth solution $(u^{(m)}, c_1^{(m)}, \dots, c_n^{(m)})$ of (78) can be written in the form

$$
u^{(m)}(t) = e^{t\Delta}u_0 - \mathcal{B}'(u^{m-1}, u^{m-1}) - \mathcal{S}(\rho^{(m-1)}, \Phi^{(m-1)}) \tag{79}
$$

and

$$
c_i^{(m)}(t) = e^{t\Delta}c_i(0) - \mathcal{B}(u^{m-1}, c_i^{m-1}) + z_i\mathcal{B}(c_i^{(m-1)}, \nabla\Phi^{(m-1)}) \tag{80}
$$

for $i = 1, \dots, n$, where the operators $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{S}$ are defined in the previous two propositions.

For each integer m, we let

$$
a_m = \|u^{(m)}\|_{E_p} + \sum_{i=1}^n \|c_i^{(m)}\|_{E_p} \tag{81}
$$

and we show that

$$\|a_m\|_{E_p} \le C_1 \|a_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + C_2 \|a_{m-1}\|_{E_p}^2. \tag{82}$$

First, we note that

$$\|e^{t\Delta}\Delta_j u_0\|_{L_t^\infty L^p} \le C\|e^{-t2^{2j}}\Delta_j u_0\|_{L_t^\infty L^p} \le C\|\Delta_j u_0\|_{L^p} \tag{83}$$

and

$$\|e^{t\Delta}\Delta_j u_0\|_{L_t^1 L^p} \le C\|e^{-t2^{2j}}\Delta_j u_0\|_{L_t^1 L^p} \le C2^{-2j}\|\Delta_j u_0\|_{L^p} \tag{84}$$

for each $j \in \mathbb{Z}$. Thus

$$\|e^{t\Delta}u_0\|_{E_p} = \|e^{t\Delta}u_0\|_{\widetilde{L}_t^\infty \dot{B}^{\frac{d}{p}}_{p,1}} + \|e^{t\Delta}u_0\|_{\widetilde{L}_t^1 \dot{B}^{\frac{d}{p}+2}_{p,1}} \le C\|u_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}}. \tag{85}$$

Similarly, we have

$$\sum_{i=1}^n \|e^{t\Delta}c_i(0)\|_{E_p} \le C\sum_{i=1}^n \|c_i(0)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}. \tag{86}$$

In view of Proposition 3, we have

$$\mathcal{B}(u^{m-1}, c_i^{m-1}) \le C\|u^{(m-1)}\|_{E_p}\|c_i^{(m-1)}\|_{E_p}, \tag{87}$$

and

$$\mathcal{B}'(u^{m-1}, u^{m-1}) \le C\|u^{(m-1)}\|_{E_p}^2 \tag{88}$$

and in view of Proposition 4, we have

$$\mathcal{S}(\rho^{(m-1)}, \Phi^{(m-1)}) \le C\|\rho^{(m-1)}\|_{E_p}^2 \le C\sum_{i=1}^n \|c_i^{(m-1)}\|_{E_p}^2 \tag{89}$$

where we have bounded the valences in absolute value by their maximum value which is absorbed by the constant C. In view of Proposition 3 and the Poincaré inequality, we have

$$\begin{aligned}\mathcal{B}(c_i^{(m-1)}, \nabla\Phi^{(m-1)}) &\le C\|c_i^{(m-1)}\|_{E_p}\|\nabla\Phi^{(m-1)}\|_{E_p} \\ &\le C\|c_i^{(m-1)}\|_{E_p}\|\rho^{(m-1)}\|_{E_p} \\ &\le C\sum_{j=1}^n \|c_i^{(m-1)}\|_{E_p}\|c_j^{(m-1)}\|_{E_p}.\end{aligned} \tag{90}$$

Putting (85)–(90) together and applying Young's inequality, we obtain (82).

Remark 5. *The unique solution* $(u, c_1, \dots, c_n)$ *obtained in Theorem 1 when* $p = 2$ *obeys*

$$u \in L^\infty(0,\infty; H^{\frac{d}{2}}) \cap L^1(0,\infty; H^{\frac{d}{2}+2}) \tag{91}$$

and

$$c_i \in L^\infty(0,\infty; H^{\frac{d}{2}}) \cap L^1_{loc}(0,\infty; H^{\frac{d}{2}+2}) \tag{92}$$

for $i = 1, \dots, n$*, provided that the initial concentrations are in* L^2*. Indeed, in view of the continuous Besov embedding* (29)*, we have*

$$\|u(t)\|_{\dot{B}^{\frac{d}{2}}_{2,2}} \le C\|u(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} = C\sum_{j\in\mathbb{Z}} 2^{\frac{dj}{2}}\|\Delta_j u(t)\|_{L^2} \le C\sum_{j\in\mathbb{Z}} 2^{\frac{dj}{2}}\|\Delta_j u(t)\|_{L_t^\infty L^2} = C\|u\|_{\tilde{L}_t^\infty \dot{B}^{\frac{d}{2}}_{2,1}} \tag{93}$$

and similarly

$$\|c_i(t)\|_{\dot{B}^{\frac{d}{2}}_{2,2}} \le C\|c_i\|_{\tilde{L}_t^\infty \dot{B}^{\frac{d}{2}}_{2,1}} \tag{94}$$

for $i = 1, \dots, n$. But $B^{\frac{d}{2}}_{2,2} = \dot{B}^{\frac{d}{2}}_{2,2} \cap L^2$ coincides with the Sobolev space $H^{\frac{d}{2}}$, hence the velocity of the fluid and the ionic concentrations are in $L^{\infty}(0, \infty; H^{\frac{d}{2}})$. On the other hand,

$$\int_0^{\infty} \|u(t)\|_{\dot{B}^{\frac{d}{p}+2}_{2,2}} dt \le C \int_0^{\infty} \|u(t)\|_{\dot{B}^{\frac{d}{p}+2}_{2,1}} dt = C \int_0^{\infty} \sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{p}+2)} \|\Delta_j u(t)\|_{L^2} dt$$

$$= C \sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{p}+2)} \|\Delta_j u(t)\|_{L^1_t L^2} = C \|u\|_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{2,1}} \tag{95}$$

and

$$\int_0^{\infty} \|c_i(t)\|_{\dot{B}^{\frac{d}{p}+2}_{2,2}} \le C \|c_i\|_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{2,1}} \tag{96}$$

for $i = 1, \dots, n$. But $B^{\frac{d}{p}+2}_{2,2} = \dot{B}^{\frac{d}{p}+2}_{2,2} \cap L^2$ coincides with the Sobolev space $H^{\frac{d}{p}+2}$, yielding the $L^1(0, \infty; H^{\frac{d}{p}+2})$ regularity of the velocity in view of the Poincaré inequality and the the $L^1_{loc}(0, \infty; H^{\frac{d}{p}+2})$ regularity of the concentrations .

Remark 6. *The unique solution $(u, c_1, \dots, c_n)$ is sufficiently small in the Sobolev space $H^{\frac{d}{2}}$ provided that the initial ionic concentrations is sufficiently small in $L^2 \cap \dot{B}^{\frac{d}{2}}_{2,1}$ and the initial velocity is sufficiently small in $\dot{B}^{\frac{d}{2}}_{2,1}$.*

5. Proof of Theorem 2

Proof of Theorem 2. We take the L^2 inner product of the velocity equation with u. In view of the divergence-free condition obeyed by u, the nonlinear term vanishes, that is

$$\int_{\mathbb{T}^d} (u \cdot \nabla u) \cdot u dx = 0 \tag{97}$$

and we obtain the differential equation

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 = - \int_{\mathbb{T}^d} \rho \nabla \Phi \cdot u. \tag{98}$$

Elliptic regularity together with the Gagliardo-Nirenberg interpolation inequality applied to the mean zero function ρ yield the bound

$$\|\nabla \Phi\|_{L^{\infty}} \le C \|\rho\|_{L^{d+1}} \le C \|\rho\|_{H^{\frac{d}{2}}}^{\frac{d-1}{d+1}} \|\rho\|_{L^2}^{\frac{2}{d+1}} \le C \|\rho\|_{H^{\frac{d}{2}}}, \tag{99}$$

hence

$$\left| \int_{\mathbb{T}^d} \rho \nabla \Phi \cdot u \right| \le \|\nabla \Phi\|_{L^{\infty}} \|\rho\|_{L^2} \|u\|_{L^2} \le C \|\rho\|_{H^{\frac{d}{2}}} \|\nabla \rho\|_{L^2} \|u\|_{L^2}$$

$$\le C \left(\max_{1 \le i \le n} z_i \right) \left(\sum_{i=1}^n \|\nabla c_i\|_{L^2} \right) \|u\|_{L^2} \|\rho\|_{H^{\frac{d}{2}}}$$

$$\le C_1 \left(\sum_{i=1}^n \frac{D_i}{4} \|\nabla c_i\|_{L^2}^2 \right) \|\rho\|_{H^{\frac{d}{2}}}^2 + \frac{\nu}{2} \|\nabla u\|_{L^2}^2 \tag{100}$$

in view of the Poincaré inequality applied to the mean zero functions u and ρ. Here C_1 depends on n, the valences z_i's, and the diffusivities D_i's. Referring to Remark 6, we can choose the initial ionic concentrations to be small enough in $H^{\frac{d}{2}}$ so that

$$C_1 \|\rho\|_{H^{\frac{d}{2}}} \le C_1 \sum_{i=1}^n |z_i| \|c_i\|_{H^{\frac{d}{2}}} \le 1 \tag{101}$$

yielding the bound

$$\left|\int_{\mathbb{T}^d} \rho \nabla \Phi \cdot u\right| \le \sum_{i=1}^n \frac{D_i}{4} \|\nabla c_i\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u\|_{L^2}^2. \tag{102}$$

Now we take the L^2 inner product of the ith ionic concentration equation with c_i. We obtain the differential equation

$$\frac{1}{2}\frac{d}{dt}\|c_i - \bar{c}_i\|_{L^2}^2 + D_i\|\nabla c_i\|_{L^2}^2 = -D_i z_i \int_{\mathbb{T}^d} (c_i - \bar{c}_i)\nabla\Phi \cdot \nabla c_i - D_i z_i \bar{c}_i \int_{\mathbb{T}^d} \nabla \Phi \cdot \nabla c_i \tag{103}$$

where $\bar{c}_i = \bar{c}_i(0)$ is the time-independent spatial average of the ionic concentration c_i. In view of Hölder's inequality, the Poincaré inequality applied to the mean zero function $c_i - \bar{c}_i$, elliptic regularity and the Gagliardo-Nirenberg inequality, we estimate

$$\begin{aligned}\left|D_i z_i \int_{\mathbb{T}^d} (c_i - \bar{c}_i)\nabla\Phi \cdot \nabla c_i\right| &\le D_i|z_i|\|c_i - \bar{c}_i\|_{L^2}\|\nabla\Phi\|_{L^\infty}\|\nabla c_i\|_{L^2} \\ &\le C D_i|z_i|\|\nabla c_i\|_{L^2}^2\|\nabla\Phi\|_{L^\infty} \\ &\le C_2\left(\frac{D_i}{8}\|\nabla c_i\|_{L^2}^2\right)\|\rho\|_{H^{\frac{d}{2}}} \end{aligned}\tag{104}$$

where C_2 depends on z_i. We choose the initial concentrations to be small enough in $H^{\frac{d}{2}}$ so that

$$C_2\|\rho\|_{H^{\frac{d}{2}}} \le 1 \tag{105}$$

which gives the bound

$$\sum_{i=1}^n \left|D_i z_i \int_{\mathbb{T}^d} (c_i - \bar{c}_i)\nabla\Phi\cdot\nabla c_i\right| \le \sum_{i=1}^n \frac{D_i}{8}\|\nabla c_i\|_{L^2}^2. \tag{106}$$

Using the Poisson equation obeyed by the potential Φ, the boundedness of the Riesz transforms $R = (R_1, \dots, R_d)$ on $L^2(\mathbb{T}^d)$, and the Poincaré inequality, we have

$$\|\nabla\Phi\|_{L^2} = \frac{1}{\epsilon}\|R\Lambda^{-1}\rho\|_{L^2} \le C\|\Lambda^{-1}\rho\|_{L^2} \le C\|\nabla\rho\|_{L^2} \tag{107}$$

and thus

$$\left|-D_i z_i \bar{c}_i \int_{\mathbb{T}^d} \nabla\Phi\cdot\nabla c_i\right| \le D_i|z_i||\bar{c}_i|\|\nabla\Phi\|_{L^2}\|\nabla c_i\|_{L^2} \tag{108}$$

$$\le C D_i|z_i||\bar{c}_i|\left(\sum_{j=1}^n |z_i|\|\nabla c_j\|_{L^2}\right)\|\nabla c_i\|_{L^2} \tag{109}$$

$$\le C_3|\bar{c}_i|\sum_{j=1}^n \frac{D_j}{8}\|\nabla c_j\|_{L^2}^2. \tag{110}$$

Here C_3 is a positive constant that depends on D_i, z_i and n. We choose the initial concentrations to be small enough in L^2 so that

$$C_3\sum_{i=1}^n |\bar{c}_i| \le 1 \tag{111}$$

and we obtain the bound

$$\sum_{i=1}^n \left|-D_i z_i \bar{c}_i \int_{\mathbb{T}^d} \nabla\Phi\cdot\nabla c_i\right| \le \sum_{i=1}^n \frac{D_i}{8}\|\nabla c_i\|_{L^2}^2 \tag{112}$$

Adding the equations of the velocity and the ionic concentrations, we obtain the differential inequality,

$$\frac{d}{dt}\left(\|u\|_{L^2}^2 + \sum_{i=1}^n \|c_i - \bar{c}_i\|_{L^2}^2\right) + \nu\|\nabla u\|_{L^2}^2 + \sum_{i=1}^n D_i\|\nabla c_i\|_{L^2}^2 \le 0 \tag{113}$$

Let

$$c = \min\{\nu, D_1, \dots, D_n\} \tag{114}$$

In view of the Poincaré inequality, we have

$$\frac{d}{dt}\left(\|u\|_{L^2}^2+\sum_{i=1}^n\|c_i-\bar{c}_i(0)\|_{L^2}^2\right)+c\left(\|u\|_{L^2}^2+\sum_{i=1}^n\|c_i-\bar{c}_i(0)\|_{L^2}^2\right)\le 0. \tag{115}$$

Integrating in time from 0 to t gives the exponentially decaying bound (20).

6. PROOF OF THEOREM 3

Proof of Theorem 3. The proof will be divided into three main steps. For simplicity, we assume that $\epsilon = 1$.

Step 1. Bounds for the L^2 norm of the velocity u and the H^{-1} norm of the charge density ρ. The ionic concentrations evolve according to the equations

$$\partial_t(z_ic_i)+u\cdot\nabla(z_ic_i)-D\Delta(z_ic_i)=D\nabla\cdot(z_i^2c_i\nabla\Phi) \tag{116}$$

for all $i\in\{1,\dots,n\}$. Consequently, the charge density ρ obeys

$$\partial_t\rho+u\cdot\nabla\rho-D\Delta\rho=D\sum_{i=1}^n\nabla\cdot(z_i^2c_i\nabla\Phi). \tag{117}$$

Now we take the L^2 inner product of this latter equation with $\Lambda^{-2}\rho$. We obtain the equation

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-1}\rho\|_{L^2}^2+D\|\rho\|_{L^2}^2=-\int_{\mathbb{T}^2}(u\cdot\nabla\rho)\Lambda^{-2}\rho-D\int_{\mathbb{T}^2}\sum_{i=1}^n z_i^2c_i\nabla\Phi\cdot\nabla\Lambda^{-2}\rho \tag{118}$$

The L^2 norm of the velocity u satisfies

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\nu\|\nabla u\|_{L^2}^2=-\int_{\mathbb{T}^2}\rho\nabla\Phi\cdot u \tag{119}$$

We add the equations (118) and (119), and we estimate. Integrating by parts, using the divergence-free condition obeyed by the velocity, and using the Poisson equation obeyed by the potential Φ, we obtain the cancellation

$$\int_{\mathbb{T}^2}(u\cdot\nabla\rho)\Lambda^{-2}\rho+\int_{\mathbb{T}^2}\rho\nabla\Phi\cdot u=0. \tag{120}$$

The nonnegativity of the ionic concentrations implies

$$-D\int_{\mathbb{T}^2}\sum_{i=1}^n z_i^2c_i\nabla\Phi\cdot\nabla\Lambda^{-2}\rho=-D\int_{\mathbb{T}^2}\sum_{i=1}^n z_i^2c_i\nabla\Phi\cdot\nabla\Phi=-D\int_{\mathbb{T}^2}\sum_{i=1}^n z_i^2c_i|\nabla\Phi|^2\le 0. \tag{121}$$

Putting (118)–(121) together, we get the differential inequality

$$\frac{d}{dt}\left\{\|u\|_{L^2}^2+\|\Lambda^{-1}\rho\|_{L^2}^2\right\}+2\nu\|\nabla u\|_{L^2}^2+2D\|\rho\|_{L^2}^2\le 0 \tag{122}$$

Letting

$$\gamma=\min\{2\nu,2D\}, \tag{123}$$

we obtain the bounds

$$\|u(t)\|_{L^2}^2+\|\Lambda^{-1}\rho(t)\|_{L^2}^2\le(\|u_0\|_{L^2}^2+\|\Lambda^{-1}\rho_0\|_{L^2}^2)e^{-\gamma t}, \tag{124}$$

$$\int_0^t\left\{\nu\|\nabla u(s)\|_{L^2}^2+D\|\rho(s)\|_{L^2}^2\right\}ds\le\frac{1}{2}\left(\|u_0\|_{L^2}^2+\|\Lambda^{-1}\rho_0\|_{L^2}^2\right), \tag{125}$$

and

$$\int_t^{t+1}\left\{\nu\|\nabla u(s)\|_{L^2}^2+D\|\rho(s)\|_{L^2}^2\right\}ds\le\frac{1}{2}\left(\|u_0\|_{L^2}^2+\|\Lambda^{-1}\rho_0\|_{L^2}^2\right)e^{-\gamma t} \tag{126}$$

for all $t\ge 0$.

Step 2. Bounds for the L^2 norm of $c_i-\bar{c}_i(0)$. For each $i\in\{1,\dots,n\}$, we take the L^2 inner product of the equation (1) obeyed by c_i with c_i and we obtain

$$\frac{1}{2}\frac{d}{dt}\|c_i-\bar{c}_i(0)\|_{L^2}^2+D\|\nabla c_i\|_{L^2}^2=-D\int_{\mathbb{T}^2}z_i(c_i-\bar{c}_i(0))\nabla\Phi\cdot\nabla c_i-D\int_{\mathbb{T}^2}z_i\bar{c}_i(0)\nabla\Phi\cdot\nabla c_i. \tag{127}$$

In view of the Ladyzhenskaya's interpolation inequality, we have

$$\|\nabla\Phi\|_{L^4} \le C\|\Lambda^{-1}\rho\|_{L^2}^{\frac{1}{2}}\|\rho\|_{L^2}^{\frac{1}{2}} \tag{128}$$

and

$$\|c_i - \bar{c}_i(0)\|_{L^4} \le C\|c_i - \bar{c}_i\|_{L^2}^{\frac{1}{2}}\|\nabla c_i\|_{L^2}^{\frac{1}{2}} \tag{129}$$

and hence

$$\left|D\int_{\mathbb{T}^2} z_i(c_i - \bar{c}_i(0))\nabla\Phi\cdot\nabla c_i\right| \le CD|z_i|\|\Lambda^{-1}\rho\|_{L^2}^{\frac{1}{2}}\|\rho\|_{L^2}^{\frac{1}{2}}\|c_i - \bar{c}_i\|_{L^2}^{\frac{1}{2}}\|\nabla c_i\|_{L^2}^{\frac{3}{2}} \tag{130}$$

In view of the boundedness of the Riesz transform on L^2, we bound

$$\left|D\int_{\mathbb{T}^2} z_i\bar{c}_i(0)\nabla\Phi\cdot\nabla c_i\right| \le D\bar{c}_i(0)\|\nabla\Phi\|_{L^2}\|\nabla c_i\|_{L^2} \le CD\bar{c}_i(0)\|\Lambda^{-1}\rho\|_{L^2}\|\nabla c_i\|_{L^2}. \tag{131}$$

This yields the differential inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|c_i - \bar{c}_i(0)\|_{L^2}^2 + \frac{D}{2}\|\nabla c_i\|_{L^2}^2\\ &\le C\|\Lambda^{-1}\rho\|_{L^2}^2\|\rho\|_{L^2}^2\|c_i - \bar{c}_i(0)\|_{L^2}^2 + C\bar{c}_i(0)^2\|\Lambda^{-1}\rho\|_{L^2}^2\end{aligned} \tag{132}$$

after an application of Young's inequality. Hence

$$\begin{aligned}&\frac{d}{dt}\|c_i - \bar{c}_i(0)\|_{L^2}^2 + D\|c_i - \bar{c}_i(0)\|_{L^2}^2\\ &\le 2C\|\Lambda^{-1}\rho\|_{L^2}^2\|\rho\|_{L^2}^2\|c_i - \bar{c}_i(0)\|_{L^2}^2 + 2C\bar{c}_i(0)^2\|\Lambda^{-1}\rho\|_{L^2}^2\end{aligned} \tag{133}$$

in view of the Poincaré inequality applied to the mean-free function $c_i - \bar{c}_i$, and so

$$\frac{d}{dt}\|c_i - \bar{c}_i(0)\|_{L^2}^2 + \left(\frac{\gamma}{2} - 2C\|\Lambda^{-1}\rho\|_{L^2}^2\|\rho\|_{L^2}^2\right)\|c_i - \bar{c}_i(0)\|_{L^2}^2 \le 2C\bar{c}_i(0)^2\|\Lambda^{-1}\rho\|_{L^2}^2 \tag{134}$$

since $\gamma/2 \le D$. Let

$$r(t) = \int_0^t\left(\frac{\gamma}{2} - 2C\|\Lambda^{-1}\rho(s)\|_{L^2}^2\|\rho(s)\|_{L^2}^2\right)ds. \tag{135}$$

Multiplying by the factor $e^{r(t)}$, we obtain

$$\frac{d}{dt}\left(e^{r(t)}\|c_i(t) - \bar{c}_i(0)\|_{L^2}^2\right) \le 2C\bar{c}_i(0)^2e^{r(t)}\|\Lambda^{-1}\rho(t)\|_{L^2}^2 \le 2C\bar{c}_i(0)^2e^{\frac{\gamma}{2}t}\|\Lambda^{-1}\rho(t)\|_{L^2}^2. \tag{136}$$

Integrating in time from 0 to t and using (124), we obtain

$$\begin{aligned}&e^{r(t)}\|c_i(t) - \bar{c}_i(0)\|_{L^2}^2 - \|c_i(0) - \bar{c}_i(0)\|_{L^2}^2\\ &\le \int_0^t 2C\bar{c}_i(0)^2e^{\frac{\gamma}{2}s}e^{-\gamma s}(\|u_0\|_{L^2}^2 + \|\Lambda^{-1}\rho_0\|_{L^2}^2)ds\\ &\le C_\gamma\bar{c}_i(0)^2(\|u_0\|_{L^2}^2 + \|\Lambda^{-1}\rho_0\|_{L^2}^2)\end{aligned} \tag{137}$$

for any $t \ge 0$. In view of (124) and (125),

$$r(t) \ge \frac{\gamma}{2}t - 2C\int_0^t \Gamma_0\|\rho(s)\|_{L^2}^2ds \ge \frac{\gamma}{2}t - \Gamma_0' \tag{138}$$

for any $t \ge 0$. Here Γ_0 and Γ_0' are constants depending only on the initial data and the parameters of the problem. Therefore, we have

$$\|c_i(t) - \bar{c}_i(0)\|_{L^2}^2 \le \Gamma e^{-\frac{\gamma}{2}t} \tag{139}$$

for any $t \ge 0$ and all $i \in \{1, \dots, n\}$. Integrating (132) in time from t to $t+1$, we obtain

$$\int_t^{t+1}\|\nabla c_i(s)\|_{L^2}^2ds \le \Gamma'e^{-\frac{\gamma}{2}t} \tag{140}$$

for any $t \ge 0$ and all $i \in \{1, \dots, n\}$. Here Γ and Γ' are positive constants depending exponentially on the initial data.

Step 3. L^2 gradient bounds. The L^2 norm of the gradient of u evolves according to the energy equality

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 = \int_{\mathbb{T}^2} \rho\nabla\Phi\cdot\Delta u. \tag{141}$$

We bound

$$\left|\int_{\mathbb{T}^2}\rho\nabla\Phi\cdot\Delta u\right| \le \|\Delta u\|_{L^2}\|\rho\|_{L^4}\|\nabla\Phi\|_{L^4} \le C\|\Delta u\|_{L^2}\|\rho\|_{L^2}^{\frac12}\|\nabla\rho\|_{L^2}^{\frac12}\|\Lambda^{-1}\rho\|_{L^2}^{\frac12}\|\rho\|_{L^2}^{\frac12} \tag{142}$$

using Hölder's inequality with exponents $2, 4, 4$ and Ladyzhenskaya's inequality. This yields the differential inequality

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 \le C\|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + C\|\rho\|_{L^2}^2\|\Lambda^{-1}\rho\|_{L^2}^2. \tag{143}$$

In view of the Gronwall lemma 1, and the exponentially decaying estimates (124), (139) and (140), we conclude that

$$\|\nabla u(t)\|_{L^2}^2 \le \Gamma_1^1 e^{-\frac{\gamma}{2}t} \tag{144}$$

and

$$\int_t^{t+1}\|\Delta u(s)\|_{L^2}^2 ds \le \Gamma_1^2 e^{-\frac{\gamma}{2}t} \tag{145}$$

for all $t \ge 0$. Now we take the L^2 inner product of the equation (1) obeyed by c_i with $-\Delta c_i$ and we estimate. We obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla c_i\|_{L^2}^2 + D\|\Delta c_i\|_{L^2}^2 &\le D|z_i|\left|\int_{\mathbb{T}^2}\nabla\cdot(c_i\nabla\Phi)\Delta c_i\right| + \left|\int_{\mathbb{T}^2}(u\cdot\nabla c_i)\Delta c_i\right| \\ &\le C\left|\int_{\mathbb{T}^2}(c_i-\bar{c}_i)\Delta\Phi\Delta c_i\right| + C\left|\int_{\mathbb{T}^2}(\nabla c_i\cdot\nabla\Phi)\Delta c_i\right| + C\left|\int_{\mathbb{T}^2}\bar{c}_i\Delta\Phi\Delta c_i\right| + \left|\int_{\mathbb{T}^2}(u\cdot\nabla c_i)\Delta c_i\right|\end{aligned} \tag{146}$$

We bound the three terms in (146) using interpolation inequalities, and we get

$$\begin{aligned}\left|\int_{\mathbb{T}^2}(c_i-\bar{c}_i)\Delta\Phi\Delta c_i\right| &\le \|c_i-\bar{c}_i\|_{L^4}\|\Delta\Phi\|_{L^4}\|\Delta c_i\|_{L^2} \le C\|c_i-\bar{c}_i\|_{L^2}^{\frac12}\|\nabla c_i\|_{L^2}^{\frac12}\|\rho\|_{L^2}^{\frac12}\|\nabla\rho\|_{L^2}^{\frac12}\|\Delta c_i\|_{L^2} \\ &\le \frac{D}{8}\|\Delta c_i\|_{L^2}^2 + C\|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + C\|c_i-\bar{c}_i\|_{L^2}^2\|\nabla c_i\|_{L^2}^2\end{aligned} \tag{147}$$

for the first term,

$$\begin{aligned}\left|\int_{\mathbb{T}^2}(\nabla c_i\cdot\nabla\Phi)\Delta c_i\right| &\le \|\nabla c_i\|_{L^4}\|\nabla\Phi\|_{L^4}\|\Delta c_i\|_{L^2} \le C\|\nabla c_i\|_{L^2}^{\frac12}\|\Delta c_i\|_{L^2}^{\frac32}\|\Lambda^{-1}\rho\|_{L^2}^{\frac12}\|\rho\|_{L^2}^{\frac12} \\ &\le \frac{D}{8}\|\Delta c_i\|_{L^2}^2 + C\|\Lambda^{-1}\rho\|_{L^2}^2\|\rho\|_{L^2}^2\|\nabla c_i\|_{L^2}^2\end{aligned} \tag{148}$$

for the second term, and

$$\left|\int_{\mathbb{T}^2}\bar{c}_i\Delta\Phi\Delta c_i\right| \le |\bar{c}_i(0)|\|\Delta\Phi\|_{L^2}\|\Delta c_i\|_{L^2} \le \frac{D}{8}\|\Delta c_i\|_{L^2}^2 + C|\bar{c}_i(0)|^2\|\rho\|_{L^2}^2 \tag{149}$$

for the third term. As for the nonlinear term in u, we integrate by parts and we estimate to get

$$\left|\int_{\mathbb{T}^2}(u\cdot\nabla c_i)\Delta c_i\right| \le \|\nabla u\|_{L^2}\|\nabla c_i\|_{L^4}^2 \le C\|\nabla u\|_{L^2}\|\nabla c_i\|_{L^2}\|\Delta c_i\|_{L^2} \le \frac{D}{8}\|\Delta c_i\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla c_i\|_{L^2}^2. \tag{150}$$

Putting (146)–(150) together, we end up with the energy inequality

$$\begin{aligned}\frac{d}{dt}\|\nabla c_i\|_{L^2}^2 + D\|\Delta c_i\|_{L^2}^2 &\le C\|\nabla u\|_{L^2}^2\|\nabla c_i\|_{L^2}^2 + C\|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 \\ &+ C\|c_i-\bar{c}_i\|_{L^2}^2\|\nabla c_i\|_{L^2}^2 + C\|\Lambda^{-1}\rho\|_{L^2}^2\|\rho\|_{L^2}^2\|\nabla c_i\|_{L^2}^2 + C|\bar{c}_i(0)|^2\|\rho\|_{L^2}^2\end{aligned} \tag{151}$$

The decaying-in-time estimates (124), (126), (139), (140) and (144) give the bounds

$$\|\nabla c_i(t)\|_{L^2}^2 \le \Gamma_1^3 e^{-\frac{\gamma}{2}t} \tag{152}$$

and

$$\int_t^{t+1} \|\Delta c_i(s)\|_{L^2}^2 ds \le \Gamma_1^4 e^{-\frac{\gamma}{2}t} \tag{153}$$

for all $t \ge 0$, after an application of the uniform Gronwall lemma 1. This ends the proof of Theorem 3.

7. Proof of Theorem 4

Proof of Theorem 4. We take the L^2 inner product of the equation satisfied by the velocity u in (1)–(5) with $(-\Delta)^k u$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^2 + \nu\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^2 = -\int_{\mathbb{T}^2}(u\cdot\nabla u)\cdot(-\Delta)^k u - \int_{\mathbb{T}^2}(\rho\nabla\Phi)\cdot(-\Delta)^k u. \tag{154}$$

The nonlinear term in u is estimated as

$$\left|\int_{\mathbb{T}^2}(u\cdot\nabla u)\cdot(-\Delta)^k u\right| \le \|(-\Delta)^{\frac{k-1}{2}}(u\cdot\nabla u)\|_{L^2}\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2} \tag{155}$$

via integration by parts followed by an application of the Cauchy-Schwarz inequality. In view of the divergence-free condition obeyed by u and the product estimate given by Lemma 2, we bound

$$\begin{aligned}\|(-\Delta)^{\frac{k-1}{2}}(u\cdot\nabla u)\|_{L^2} &= \|(-\Delta)^{\frac{k-1}{2}}\nabla(u\cdot u)\|_{L^2} \le C\|(-\Delta)^{\frac{k}{2}}(u\cdot u)\|_{L^2}\\ &\le C_k\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^{\frac12}\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^{\frac12}\end{aligned} \tag{156}$$

and hence

$$\begin{aligned}\left|\int_{\mathbb{T}^2}(u\cdot\nabla u)\cdot(-\Delta)^k u\right| &\le C_k\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^{\frac12}\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^{\frac32}\\ &\le \frac{\nu}{4}\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^2 + C_k\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^2\\ &\le \frac{\nu}{4}\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^2 + C_k\|\nabla u\|_{L^2}^4\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^2\end{aligned} \tag{157}$$

by Young's inequality followed by an application of the Poincaré inequality to the mean-free function u. As for the nonlinear term in ρ, we integrate by parts and we estimate

$$\begin{aligned}\left|\int_{\mathbb{T}^2}(\rho\nabla\Phi)\cdot(-\Delta)^k u\right| &\le \|(-\Delta)^{\frac{k-1}{2}}(\rho\nabla\Phi)\|_{L^2}\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}\\ &\le \frac{\nu}{4}\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^2 + C\|(-\Delta)^{\frac{k-1}{2}}(\rho\nabla\Phi)\|_{L^2}^2.\end{aligned} \tag{158}$$

We bound

$$\begin{aligned}\|(-\Delta)^{\frac{k-1}{2}}(\rho\nabla\Phi)\|_{L^2}^2 &\le C_k\|\rho\|_{L^2}\|\nabla\rho\|_{L^2}\|(-\Delta)^{\frac{k-1}{2}}\nabla\Phi\|_{L^2}\|(-\Delta)^{\frac{k}{2}}\nabla\Phi\|_{L^2}\\ &\quad + C_k\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}\|(-\Delta)^{\frac{k}{2}}\rho\|_{L^2}\|\nabla\Phi\|_{L^2}\|\nabla\nabla\Phi\|_{L^2}\end{aligned} \tag{159}$$

in view of Lemma 2. The potential Φ obeys the Poisson equation $-\Delta\Phi = \rho$, hence

$$\|(-\Delta)^{\frac{k-1}{2}}\nabla\Phi\|_{L^2} \le C\|(-\Delta)^{\frac{k-2}{2}}\rho\|_{L^2}, \tag{160}$$

$$\|(-\Delta)^{\frac{k}{2}}\nabla\Phi\|_{L^2} \le C\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}, \tag{161}$$

$$\|\nabla\Phi\|_{L^2} \le C\|\Lambda^{-1}\rho\|_{L^2} \tag{162}$$

and

$$\|\nabla\nabla\Phi\|_{L^2} \le C\|\rho\|_{L^2}. \tag{163}$$

Putting (159)–(163) together and applying the Poincaré inequality, we end up with

$$\|(-\Delta)^{\frac{k-1}{2}}(\rho\nabla\Phi)\|_{L^2}^2 \le C_k\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^2\|(-\Delta)^{\frac{k}{2}}\rho\|_{L^2}^2. \tag{164}$$

This yields the differential inequality

$$\frac{d}{dt}\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^2 + \nu\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^2 \le C_k\|\nabla u\|_{L^2}^4\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^2 + C_k\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^2\|(-\Delta)^{\frac{k}{2}}\rho\|_{L^2}^2. \quad (165)$$

Recalling that

$$\rho = \sum_{i=1}^n z_i c_i, \quad (166)$$

using the hypothesis of Theorem 4 given by (24), and applying the uniform Gronwall Lemma 1, we obtain

$$\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^2 \le \Gamma_k^1 e^{-\frac{\gamma}{2^{k+1}}t} \quad (167)$$

and

$$\int_t^{t+1}\|(-\Delta)^{\frac{k+1}{2}}u(s)\|_{L^2}^2 ds \le \Gamma_k^2 e^{-\frac{\gamma}{2^{k+1}}t} \quad (168)$$

for any $t \ge 0$.

Now we take the L^2 inner product of the equation obeyed by the ionic concentration c_i in (1)–(5) with $(-\Delta)^k c_i$ and we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^2 + D\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}^2 &= -\int_{\mathbb{T}^2}(u\cdot\nabla c_i)(-\Delta)^k c_i + Dz_i\int_{\mathbb{T}^2}\nabla\cdot(c_i\nabla\Phi)(-\Delta)^k c_i \\ &= -\int_{\mathbb{T}^2}(u\cdot\nabla c_i)(-\Delta)^k c_i + Dz_i\int_{\mathbb{T}^2}(\nabla c_i\cdot\nabla\Phi)(-\Delta)^k c_i \\ &\quad + Dz_i\int_{\mathbb{T}^2}((c_i-\bar{c}_i)\Delta\Phi)(-\Delta)^k c_i + Dz_i\bar{c}_i(0)\int_{\mathbb{T}^2}\Delta\Phi(-\Delta)^k c_i. \end{aligned} \quad (169)$$

Integrating by parts, using the divergence-free condition obeyed by u, and applying Lemma 2, we estimate

$$\begin{aligned} &\left|\int_{\mathbb{T}^2}(u\cdot\nabla c_i)(-\Delta)^k c_i\right| = \left|\int_{\mathbb{T}^2}(u\cdot\nabla(c_i-\bar{c}_i))(-\Delta)^k c_i\right| \\ &\le C\|(-\Delta)^{\frac{k}{2}}(u(c_i-\bar{c}_i))\|_{L^2}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2} \\ &\le C_k\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}^{\frac{3}{2}} \\ &\quad + C_k\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^{\frac{1}{2}}\|c_i-\bar{c}_i\|_{L^2}^{\frac{1}{2}}\|\nabla c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2} \\ &\le \frac{D}{8}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}^2 + C_k\|\nabla u\|_{L^2}^4\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^2 \\ &\quad + C_k\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^2\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^2 + C_k\|\nabla c_i\|_{L^2}^4. \end{aligned} \quad (170)$$

As for the terms involving the potential Φ, we integrate by parts, apply the fractional product inequality given by Lemma 2, use the Poisson equation $-\Delta\Phi = \rho$ and estimate. The first term in Φ is bounded as

$$\begin{aligned} &\left|Dz_i\int_{\mathbb{T}^2}(\nabla c_i\cdot\nabla\Phi)(-\Delta)^k c_i\right| \le C\|(-\Delta)^{\frac{k-1}{2}}(\nabla c_i\cdot\nabla\Phi)\|_{L^2}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2} \\ &\le C_k\|\nabla c_i\|_{L^2}^{\frac{1}{2}}\|\Delta c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k-1}{2}}\nabla\Phi\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k}{2}}\nabla\Phi\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2} \\ &\quad + C_k\|(-\Delta)^{\frac{k-1}{2}}\nabla c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k}{2}}\nabla c_i\|_{L^2}^{\frac{1}{2}}\|\nabla\Phi\|_{L^2}^{\frac{1}{2}}\|\Delta\Phi\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2} \\ &\le C_k\|\nabla c_i\|_{L^2}^{\frac{1}{2}}\|\Delta c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k-2}{2}}\rho\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2} \\ &\quad + C_k\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}^{\frac{1}{2}}\|\rho\|_{L^2}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2} \\ &\le \frac{D}{8}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}^2 + C_k\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^2\|\rho\|_{L^2}^4 \\ &\quad + C_k\|\nabla c_i\|_{L^2}^2\|\Delta c_i\|_{L^2}^2 + C_k\|(-\Delta)^{\frac{k-2}{2}}\rho\|_{L^2}^2\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^2. \end{aligned} \quad (171)$$

The second term in Φ is bounded as follows

$$\begin{aligned}
&\left|Dz_i \int_{\mathbb{T}^2} ((c_i - \bar{c}_i)\Delta\Phi)(-\Delta)^k c_i\right| \le C\|(-\Delta)^{\frac{k-1}{2}}((c_i-\bar{c}_i)\Delta\Phi)\|_{L^2}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}\\
&\le C_k\|c_i-\bar{c}_i\|_{L^2}^{\frac{1}{2}}\|\nabla c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k-1}{2}}\Delta\Phi\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k}{2}}\Delta\Phi\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}\\
&\quad + C_k\|(-\Delta)^{\frac{k-1}{2}}c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^{\frac{1}{2}}\|\Delta\Phi\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta\Phi\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}\\
&\le C_k\|\nabla c_i\|_{L^2}\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k}{2}}\rho\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}\\
&\quad + C_k\|(-\Delta)^{\frac{k-1}{2}}c_i\|_{L^2}^{\frac{1}{2}}\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^{\frac{1}{2}}\|\nabla\rho\|_{L^2}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}\\
&\le \frac{D}{8}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}^2 + C_k\|\nabla c_i\|_{L^2}^4 + C_k\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^2\|(-\Delta)^{\frac{k}{2}}\rho\|_{L^2}^2\\
&\quad + C_k\|\nabla\rho\|_{L^2}^4 + C_k\|(-\Delta)^{\frac{k-1}{2}}c_i\|_{L^2}^2\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^2. \qquad (172)
\end{aligned}$$

The estimation of the last term in Φ is given by

$$\left|Dz_i\bar{c}_i(0)\int_{\mathbb{T}^2}\Delta\Phi(-\Delta)^k c_i\right| \le \frac{D}{8}\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}^2 + C\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^2. \qquad (173)$$

Putting (169)–(173) together and applying the Poincaré inequality, we obtain the differential inequality

$$\begin{aligned}
&\frac{d}{dt}\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^2 + D\|(-\Delta)^{\frac{k+1}{2}}c_i\|_{L^2}^2\\
&\le C_k\|\nabla u\|_{L^2}^4\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^2 + C_k\|(-\Delta)^{\frac{k}{2}}u\|_{L^2}^2\|(-\Delta)^{\frac{k+1}{2}}u\|_{L^2}^2 + C_k\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^2\|\rho\|_{L^2}^4\\
&\quad + C_k\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^2\|(-\Delta)^{\frac{k}{2}}\rho\|_{L^2}^2 + C_k\|(-\Delta)^{\frac{k-1}{2}}c_i\|_{L^2}^2\|(-\Delta)^{\frac{k}{2}}c_i\|_{L^2}^2 + C_k\|(-\Delta)^{\frac{k-1}{2}}\rho\|_{L^2}^2. \qquad (174)
\end{aligned}$$

In view of the exponentially decaying bounds for the velocity (167) and (168), the decaying assumptions imposed on the ionic concentrations in the hypothesis of Theorem 4, and the Gronwall Lemma 1, we obtain

$$\|(-\Delta)^{\frac{k}{2}}c_i(t)\|_{L^2}^2 \le \Gamma_k^3 e^{-\frac{\gamma}{2^{k+1}}t} \qquad (175)$$

and

$$\int_t^{t+1}\|(-\Delta)^{\frac{k+1}{2}}c_i(s)\|_{L^2}^2 ds \le \Gamma_k^4 e^{-\frac{\gamma}{2^{k+1}}t} \qquad (176)$$

for any $t \ge 0$ and $i \in \{1,\dots,n\}$. This completes the proof of Theorem 4.

Remark 7. *As a consequence of theorems 3 and 4, the solution* $(u, c_1, \dots, c_n)$ *to the periodic two-dimensional NPNS system* (1)–(5) *decays exponentially in time in all Sobolev spaces* H^k *provided that the initial data is smooth. Due to the 2D continuous embedding of the Sobolev spaces* H^k *in the Hölder spaces* $C^{k-1-\alpha,\alpha}$ *for* $k > 1$ *and* $\alpha \in (0,1)$, *we conclude that the solution of* (1)–(5) *decays exponentially in time in all Hölder spaces* $C^{k-1-\alpha,\alpha}$, *and so does its time derivatives .*

Remark 8. *The ionic species are assumed to have equal diffusivities. This assumption is needed to obtain the diffusion term* $-D\Delta\rho$ *in* (117) *when summing the equations* (116) *obeyed by the ionic concentrations, and consequently obtain the boundedness of the charge density* ρ *in the space* $L^2(0,T;L^2(\mathbb{T}^2))$ *which is crucial to prove the base step decay.*

Remark 9. *The decaying bounds established for the solution of* (1)–(5) *depend exponentially on the initial data due to* (139) *and* (140). *In the case of two ionic species with valences 1 and -1 and equal diffusivities, the dependence on the initial data is at most polynomial (see* [2]*).*

8. Proof of Theorem 5

In this section, we study the long-time behavior of solutions to the two-dimensional periodic NPNS system for n ionic species with different valences and diffusivities on the periodic box $\mathbb{T}^2$. The main challenges arise from the fact that the charge density ρ does not obey the differential equation (117) that holds in the case of equal diffusivities.

Proof of Theorem 5. The following quantities are the basic relative energy and its dissipation of the NPNS system,

$$\mathcal{E} = \int_{\mathbb{T}^2} \left[\sum_{i=1}^n \bar{c}_i \left(\frac{c_i}{\bar{c}_i} \ln\left(\frac{c_i}{\bar{c}_i}\right) - \frac{c_i}{\bar{c}_i} + 1\right) + \frac{1}{2}\rho\Phi\right] dx, \tag{177}$$

$$\mathcal{D} = \int_{\mathbb{T}^2} \sum_{i=1}^n D_i c_i \left|\nabla \ln c_i + z_i \nabla\Phi\right|^2 dx, \tag{178}$$

and

$$C_i = \int_{\mathbb{T}^2} c_i(x) dx = 4\pi^2 \bar{c}_i, \tag{179}$$

introduced in [6] and used to obtain the global regularity for the NPNS system in two spatial dimensions on bounded domains. We note that the constant C_i is time-independent, a fact that follows from the conservation of the spatial integral of the i-th ionic concentration for all positive times. Without loss of generality, we assume that C_i is nonzero. For simplicity, we assume that $\epsilon = 1$.

Step 1. Nonnegativity of the energy $\mathcal{E}$. In view of the Poisson equation (2) obeyed by the potential Φ, we have

$$\int_{\mathbb{T}^2} \rho\Phi dx = \int_{\mathbb{T}^2} \rho\Lambda^{-2}\rho dx = \|\Lambda^{-1}\rho\|_{L^2}^2 \geq 0 \tag{180}$$

at any time $t \geq 0$. Moreover, the inequality $x\ln x - x + 1 \geq 0$ that holds for any $x \geq 0$ implies that

$$\int_{\mathbb{T}^2} \left[\sum_{i=1}^n \bar{c}_i \left(\frac{c_i}{\bar{c}_i} \ln\left(\frac{c_i}{\bar{c}_i}\right) - \frac{c_i}{\bar{c}_i} + 1\right)\right] dx \geq 0 \tag{181}$$

at any time $t \geq 0$. Putting (180) and (181) together, we conclude that the energy $\mathcal{E}$ is nonnegative.

Step 2. Logarithmic Sobolev estimate. For each $i \in \{1, \dots, n\}$, we show that the logarithmic estimate

$$\int_{\mathbb{T}^2} \bar{c}_i \left(\frac{c_i}{\bar{c}_i} \ln\left(\frac{c_i}{\bar{c}_i}\right) - \frac{c_i}{\bar{c}_i} + 1\right) dx \leq 4\pi^2 \bar{c}_i \ln\left(1 + \frac{C}{2\pi\bar{c}_i} \|\nabla\sqrt{c_i}\|_{L^2}^2\right) \tag{182}$$

holds, where C is a positive universal constant. Indeed, we apply Jensen's inequality to the natural logarithmic concave function and the probability measure $\frac{c_i}{C_i}dx$ to bound

$$\begin{aligned}
&\int_{\mathbb{T}^2} \bar{c}_i \left(\frac{c_i}{\bar{c}_i} \ln\left(\frac{c_i}{\bar{c}_i}\right) - \frac{c_i}{\bar{c}_i} + 1\right) dx = C_i \int_{\mathbb{T}^2} \frac{c_i}{C_i} \ln\left(\frac{c_i}{\bar{c}_i}\right) dx \\
&\leq C_i \ln\left(\int_{\mathbb{T}^2} \frac{c_i^2}{C_i\bar{c}_i} dx\right) = C_i \ln\left(\int_{\mathbb{T}^2} \frac{c_i^2}{(2\pi)^2\bar{c}_i^2} dx\right) \\
&= C_i \ln\left(\left\|\sqrt{\frac{c_i}{2\pi\bar{c}_i}}\right\|_{L^4}^4\right).
\end{aligned} \tag{183}$$

We let $q := \frac{\sqrt{c_i}}{\sqrt{(2\pi)\bar{c}_i}}$ and denote its spatial average over $\mathbb{T}^2$ by $\bar{q}$. In view of Ladyzhenskaya's interpolation inequality, we have

$$\|q\|_{L^4}^4 \leq \|\bar{q}\|_{L^4}^4 + C\|q - \bar{q}\|_{L^2}^2 \|\nabla q\|_{L^2}^2 \leq \|\bar{q}\|_{L^4}^4 + C\|q\|_{L^2}^2 \|\nabla q\|_{L^2}^2 \tag{184}$$

where the last inequality follows from the boundedness of $\bar{q}$ by the L^2 norm of q. Since $\|q\|_{L^2}^2 = 2\pi$, we have

$$\|\bar{q}\|_{L^4}^4 = (2\pi)^2 \left(\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} q(x) dx\right)^4 \leq \frac{1}{(2\pi)^2} \left(\int_{\mathbb{T}^2} q(x)^2 dx\right)^2 = \frac{1}{(2\pi)^2}(2\pi)^2 = 1 \tag{185}$$

by Hölder's inequality. Hence

$$\|q\|_{L^4}^4 \le 1 + C\|\nabla q\|_{L^2}^2, \tag{186}$$

Putting (183) and (186) together gives the desired bound (182), finishing the proof of Step 2.

Step 3. Dissipation controls energy inequality. We show that

$$\mathcal{E} \le \Gamma_1 \mathcal{D} \tag{187}$$

where Γ_1 is a positive constant depending only on the diffusivities. Indeed, we have

$$\begin{aligned}
\mathcal{D} &\ge \left(\min_{1\le i\le n} D_i\right) \sum_{i=1}^n \int_{\mathbb{T}^2} c_i |\nabla \ln c_i + z_i \nabla \Phi|^2 dx \\
&= \left(\min_{1\le i\le n} D_i\right) \sum_{i=1}^n \int_{\mathbb{T}^2} \left(c_i |\nabla \ln c_i|^2 + c_i |z_i|^2 |\nabla \Phi|^2 + 2 z_i c_i (\nabla \ln c_i) \cdot \nabla \Phi\right) dx \\
&= \left(\min_{1\le i\le n} D_i\right) \sum_{i=1}^n \int_{\mathbb{T}^2} \left(\left|\frac{\nabla c_i}{\sqrt{c_i}}\right|^2 + c_i |z_i|^2 |\nabla \Phi|^2 + 2 z_i \nabla c_i \cdot \nabla \Phi\right) dx \\
&\ge \left(\min_{1\le i\le n} D_i\right) \sum_{i=1}^n \int_{\mathbb{T}^2} |2\nabla \sqrt{c_i}|^2 dx + \left(\min_{1\le i\le n} D_i\right) \int_{\mathbb{T}^2} 2\nabla \rho \cdot \nabla \Phi dx \\
&= 4\left(\min_{1\le i\le n} D_i\right) \left(\sum_{i=1}^n \|\nabla \sqrt{c_i}\|_{L^2}^2 + \frac{1}{2}\|\rho\|_{L^2}^2\right).
\end{aligned} \tag{188}$$

Since ρ is mean-free, we have the Poincaré estimate

$$\frac{1}{2}\|\Lambda^{-1}\rho\|_{L^2} \le \frac{1}{2}\|\rho\|_{L^2}. \tag{189}$$

In view of the logarithmic Sobolev estimate derived in Step 2, and the inequality $\ln(1+x) \le x$ that holds for any $x \ge 0$, we bound

$$\int_{\mathbb{T}^2} \bar{c}_i \left(\frac{c_i}{\bar{c}_i} \ln\left(\frac{c_i}{\bar{c}_i}\right) - \frac{c_i}{\bar{c}_i} + 1\right) dx \le C \,\|\nabla \sqrt{c_i}\|_{L^2}^2 \tag{190}$$

Adding (189) and (190), we conclude that

$$\mathcal{E} \le C \sum_{i=1}^n \|\nabla \sqrt{c_i}\|_{L^2}^2 + \frac{1}{2}\|\rho\|_{L^2}. \tag{191}$$

Therefore, we obtain the bound

$$\mathcal{D} \ge C\left(\min_{1\le i\le n} D_i\right)\mathcal{E}, \tag{192}$$

from which we conclude that

$$\mathcal{E} \le \frac{C}{\left(\min_{1\le i\le n} D_i\right)} \mathcal{D} \tag{193}$$

for some positive universal constant C. This finishes the proof of Step 3.

Step 4. Energy equality. The following energy equality

$$\frac{d}{dt}\left\{\frac{1}{2}\|u\|_{L^2}^2 + \mathcal{E}\right\} + \nu\|\nabla u\|_{L^2}^2 + \mathcal{D} = 0 \tag{194}$$

holds for all $t \ge 0$. We refer the reader to the proof of Proposition 2 in [4] for details.

Step 5. Decaying bounds up to uniform constants. The equality (194) implies that

$$\int_0^T \left(\|\mathcal{D} + \nu\|\nabla u\|_{L^2}^2\right) dx \le \frac{1}{2}\|u_0\|_{L^2}^2 + \mathcal{E}_0 \tag{195}$$

for any $T > 0$. But

$$\mathcal{D} \ge 2\left(\min_{1\le i\le n} D_i\right)\|\rho\|_{L^2}^2 \tag{196}$$

and so

$$\int_0^T \left(2\left(\min_{1\le i\le n} D_i\right)\|\rho\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2\right) dt \le \left(\frac{1}{2}\|u_0\|_{L^2}^2 + \mathcal{E}_0\right). \tag{197}$$

Moreover, the energy equality (194) and the Poincaré inequality (187) yield the differential inequality

$$\frac{d}{dt}\left\{\frac{1}{2}\|u\|_{L^2}^2 + \mathcal{E}\right\} + \nu\|u\|_{L^2}^2 + \frac{1}{\Gamma_1}\mathcal{E} \le 0 \tag{198}$$

and so

$$\frac{d}{dt}\left\{\frac{1}{2}\|u\|_{L^2}^2 + \mathcal{E}\right\} + \Gamma_2\left\{\frac{1}{2}\|u\|_{L^2}^2 + \mathcal{E}\right\} \le 0 \tag{199}$$

where $\Gamma_2 > 0$ depends on the diffusivities $D_1, \dots, D_n$ and the kinematic viscosity ν. Therefore, we conclude that

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \mathcal{E}(t) \le \left(\frac{1}{2}\|u_0\|_{L^2}^2 + \mathcal{E}_0\right) e^{-\Gamma_2 t} \tag{200}$$

for any $t \ge 0$. Since

$$\mathcal{E}(t) \ge \frac{1}{2}\|\Lambda^{-1}\rho(t)\|_{L^2}^2 \tag{201}$$

for any $t \ge 0$, we infer that

$$\|u(t)\|_{L^2}^2 + \|\Lambda^{-1}\rho(t)\|_{L^2}^2 \le \left(\|u_0\|_{L^2}^2 + 2\mathcal{E}_0\right) e^{-\Gamma_2 t}. \tag{202}$$

The bounds (26) can be obtained by following the proof of Steps 2 and 3 of Theorem 3 and then the proof of Theorem 4. We omit further details.

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