SCHAUDER ESTIMATES FOR SUB ELLIPTIC EQUATIONS

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Contents

1. Introduction

In this paper we are concerned with Schauder estimates for a class of sub-elliptic equations in $\mathbb{R}^N$ of the kind

$$\mathcal{L} = \sum_{i,j=1}^{m} a_{ij}(x) X_i X_j + X_0, \tag{1.1}$$

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where the X_j's are smooth vector fields in $\mathbb{R}^N$, that is, $X_j = \sum_{k=1}^N b_j^k(x)\partial_{x_k}$, $b_j^k \in C^\infty(\mathbb{R}^N)$, satisfying the following conditions:

(I) There exists a group of dilations $\{\delta_r\}_{r>0}$ in $\mathbb{R}^N$ such that $X_1, \cdots, X_m$ are δ_r-homogeneous of degree one and X_0 is δ_r-homogeneous of degree two (see next section for the definition of δ_r).

(II) If $\mathfrak{a} = \text{Lie}\,\{X_1, \cdots, X_m, X_0\}$, then $\dim \mathfrak{a} = \text{rank}\,\mathfrak{a}(x) = N$ at each $x \in \mathbb{R}^N$.

It is well known, by a theorem of Hörmander, that the rank condition in (II) implies the hypoellipticity of $\sum_{j=1}^m X_j^2 + X_0$. Moreover, from a recent result in [BL], conditions (I) and (II) imply the existence of composition law $\circ$ in $\mathbb{R}^N$ making the triplet

$$\mathbb{G} = (\mathbb{R}^N, \circ, \delta_r)$$

an homogeneous Lie group on which the X_j's are left translation invariant (see Section 6). The homogeneous Lie group $\mathbb{G}$ allows one to introduce a notion of Hölder spaces appropriate for the operator $\mathcal{L}$. Indeed, let $\|\cdot\|$ be the δ_r-homogeneous norm defined in Section 2. Given a function $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ open, and $0 < \theta < 1$, then for each compact $K \subset \Omega$ define

$$[f]_{\theta,K} = \sup_{x,y\in K, x\neq y} \frac{|f(x) - f(y)|}{\|y^{-1} \circ x\|^\theta},$$

and

$$\|f\|_{2+\theta,K} = \sum_{|\alpha|_d \le 2} \sup_K |X^\alpha f| + \sum_{|\alpha|_d = 2} [X^\alpha f]_{\theta,K},$$

where $|\cdot|_d$ and X^α are defined in Section 2.

Finally, let us introduce our last hypothesis on $\mathcal{L}$:

(III) there exists a positive constant λ such that $\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2$ for each $x \in \mathbb{R}^N$ and each $\xi \in \mathbb{R}^m$. Moreover, $a_{ij} = a_{ji}$, and $[a_{ij}]_{\theta,K} < \infty$ for each compact $K \subset \mathbb{R}^N$.

Our main result is the following theorem.

Theorem 1.1 (Main Theorem). *Let $\Omega \subset \mathbb{R}^N$ be open and $K_1, K_2 \Subset \Omega$, $K_1 \subset K_2^\circ$. Then, for each smooth function $u : \Omega \to \mathbb{R}$ we have*

$$\|u\|_{2+\theta,K_1} \le C \left(\|\mathcal{L}u\|_{L^\infty(K_2)} + \|\mathcal{L}u\|_{\theta,K_2} \right). \tag{1.2}$$

The positive constant C depends only on K_1, K_2, the Hölder norms of the a_{ij}'s and the vector fields X_j.

The crucial point in the proof of this theorem is to establish the estimate (1.2) for the prototypical operator $\sum_{j=1}^{m} X_j^2 + X_0$. To obtain this estimate we use a simple technique inspired in the work of Caffarelli, see [CC95]. It only uses

(1) Picone's maximum principle,
(2) the hypoellipticity of $\mathcal{L}$, and
(3) the left translation invariance of $\mathcal{L}$ on $\mathbb{G}$ and its δ_r-homogeneity of degree two.

Our main tools are the Taylor formula for smooth functions on $\mathbb{G}$, some properties of the corresponding Taylor polynomial, and an orthogonality theorem that extends to homogeneous Lie groups a classical theorem of Calderón and Zygmund in Euclidean setting. We would like to stress that this extended theorem seems to have independent interest in its own right and it might be useful in studying differentiability properties of functions in homogeneous Lie groups.

Schauder estimates have been obtained for several operators that are less general than the ones considered in this paper and with different methods, some containing ideas partially similar to ours. For constant coefficients, k-homogeneous and hypoelliptic operators these estimates are proved by L. Simon [Sim97]. L. Capogna and Q. Han [CH01] considered operators of the type (1.1) with $X_0 = 0$ and the X_j's, $j = 1, \cdots, m$, span the first layer of the Lie algebra of a Carnot group in $\mathbb{R}^N$. M. Bramanti and L. Brandolini [BB07] proved Schauder estimates for heat-type operators of the form $a_{ij}X_iX_j - \partial_t$, with X_j smooth satisfying Hörmander's rank condition. For Kolmogorov type operators $a_{ij}\partial_{ij} + X_0 - \partial_t$, where X_0 is an homogeneous vector field with first order polynomial coefficients, Schauder estimates were proved by M. Manfredini [Man97] and A. Lunardi [Lun97]. Polidoro and Di Francesco [PF06] later improved Manfredini results removing the homogeneity hypothesis for X_0. We would like to stress that the techniques used in [BB07], [Man97],[Lun97] and [PF06] are completely different to ours and they are based on fine properties of the fundamental solution of the frozen operator.

The plan of the paper is as follows. In Section 2 we show some properties of polynomials in $\mathbb{G}$, we recall the Taylor formula in homogeneous groups, and we prove our Calderón and Zygmund type orthogonality result, Theorem 2.3. In Section 3, we first show Schauder estimates at the origin for homogeneous hypoelliptic operators in general form. Then, assuming the left invariance translation of the operator on a Lie group, we obtain the previous estimates out of the origin.

In Section 5, we briefly show how to extend to the general case of the operator in (1.1) the estimates proved in the previous section.

Finally, in Section 6 we present some explicit examples of operators, already present in the literature and arising in different contexts, to which our results apply.

2. Preliminaries

Let $\mathbf{d} = (\delta_r)_{r>0}$ be the group of dilations defined by $\delta_r : \mathbb{R}^N \to \mathbb{R}^N$, $\delta_r(x_1, \cdots, x_N) = (r^{\sigma_1}x_1, \cdots, r^{\sigma_N}x_N)$, $1 \le \sigma_1 \le \cdots \le \sigma_N$. Define $\|x\| := \sum_{j=1}^N |x_j|^{1/\sigma_j}$, $x \in \mathbb{R}^n$. Then $\|\delta_r(x)\| = r\|x\|$. If $\alpha = (\alpha_1, \cdots, \alpha_N)$ is a multi-index, $\alpha_j \in \mathbb{Z}, \alpha_j \ge 0$, we let

$$|\alpha|_d := \sum_{j=1}^N \sigma_j \alpha_j,$$

and we denote as usual $|\alpha| = \sum_{j=1}^N \alpha_j$. Notice that if $\delta_r = rx$ is the usual Euclidean dilation, then $|\cdot|_d = |\cdot|$. We remark that $\sigma_1|\alpha| \le |\alpha|_d \le \sigma_N|\alpha|$. A polynomial $p(x)$ in the variables $x_1, \cdots, x_N$ has $\mathbf{d}$-degree k if $p(x) = \sum_{|\alpha|_d \le k} a_\alpha x^\alpha$. We denote by $\mathbb{P}_{m,d}$ the class of all polynomials with $\mathbf{d}$-degree less than or equal than m. Let D^* be any fixed bounded open neighborhood of the origin and define

$$D_\mu := \delta_\mu D^*, \qquad \text{for } \mu > 0.$$

For $p \in \mathbb{P}_{m,d}$ we put $\|p\|_\infty = \sup_{x\in D^*} |p(x)|$, and given $0 < \mu \le 1$, we set $\|p\|_\mu = \mu^{-m} \sup_{x\in D_\mu} |p(x)|$. It is easy to see that there exists a positive constant c^* such that

$$\{x \in \mathbb{R}^N : \|x\| \le 1/c^*\} \subset D \subset \{x \in \mathbb{R}^N : \|x\| \le c^*\},$$

and as a consequence

$$\{x \in \mathbb{R}^N : \|x\| \le r/c^*\} \subset D_r \subset \{x \in \mathbb{R}^N : \|x\| \le c^* r\}.$$

Proposition 2.1. *There exists a positive constant C_1 such that*

$$\|p\|_\infty \le C_1 \|p\|_\mu,$$

for all $p \in \mathbb{P}_{m,d}$ and for all $0 < \mu \le 1$.

Proof. We first observe that if $p \in \mathbb{P}_{m,d}$, $p(x) = \sum_{|\alpha|_d \le m} a_\alpha x^\alpha$, then $|a_\alpha| \le C\|p\|_\infty$ for all $|\alpha|_d \le m$ with C independent of p. Indeed, since $|||p||| = \max_{|\alpha|_d \le m} |a_\alpha|$ is a norm on $P_{m,d}$ and it has finite dimension, then all norms are equivalent.

Next, let $p \in \mathbb{P}_{m,d}$, and set $q_\mu(y) := p(\delta_\mu y)$. We have $\|q_\mu\|_\infty = \mu^m \|p\|_\mu$, and $q_\mu(y) = \sum_{|\alpha|_d \le m} a_\alpha \mu^{|\alpha|_d} y^\alpha$. Then by the first observation,

$$|a_\alpha \mu^{|\alpha|_d}| \le C\mu^m \|p\|_\mu \tag{2.1}$$

and as a consequence, if $z \in D^*$, we then have $|p(z)| \le \sum_{|\alpha|_d \le m} (c^*)^{|\alpha|_d} |a_\alpha| \le C \sum_{|\alpha|_d \le m} \mu^{m-|\alpha|_d} \|p\|_\mu \le C' \|p\|_\mu$. This completes the proof.

□

Proposition 2.2. *Let $0 < \lambda, \theta < 1$ and $p_k \in \mathbb{P}_{m,d}$, $k = 1, 2, \cdots$. Assume that there exists a constant $C_0 > 0$ such that*

$$|p_k(x) - p_{k+1}(x)| \le C_0 \lambda^{(m+\theta)k}, \qquad \text{for } x \in D_{\lambda^k} \text{ and, } k = 1, 2, \cdots. \tag{2.2}$$

Then there exists $p \in \mathbb{P}_{m,d}$ such that

$$|p(x) - p_k(x)| \le C\, \lambda^{k(m+\theta)}, \quad \text{if } x \in D_{\lambda^k}, k = 1, 2, \cdots, \tag{2.3}$$

where the constant C only depends on C_0, λ and θ.

Proof. From (2.2), we get

$$\|p_k - p_{k+1}\|_{\lambda^k} \le C_0\, \lambda^{\theta k}, \tag{2.4}$$

for $k = 1, 2, \cdots$. Hence by Proposition 2.1

$$\|p_k - p_{k+1}\|_\infty \le C_1 \lambda^{\theta k},$$

with C_1 a structural constant, that is, p_k is a Cauchy sequence in $P_{m,d}$. Let $p \in P_{m,d}$ be its limit. We write for each $k \in \mathbb{N}$, $p - p_k = \sum_{j=1}^\infty (p_{k+j} - p_{k+j-1})$, so that, if we denote by a_α and $a_\alpha^{(k)}$ the coefficients of p and p_k, respectively, we have

$$a_\alpha - a_\alpha^{(k)} = \sum_{j=1}^\infty \left(a_\alpha^{(k+j)} - a_\alpha^{(k+j-1)} \right)$$

for each $|\alpha|_d \le m$. Then from (2.1) and (2.4) we get

$$|a_\alpha - a_\alpha^{(k)}| \le C_1 \sum_{j=1}^\infty \|p_{k+j} - p_{k+j-1}\|_{\lambda^{k+j-1}} \lambda^{(k+j-1)(m-|\alpha|_d)} \le C_3 \sum_{j=1}^\infty \lambda^{(k+j-1)(\theta+m-|\alpha|_d)},$$

where C_2 and C_3 are structural constants (C_3 depends on θ). Therefore

$$|a_\alpha - a_\alpha^{(k)}| \le C_4 \lambda^{k(\theta+m-|\alpha|_d)},$$

for all $|\alpha|_d \le m$ and with C_4 depending only on C_3 and λ.

Finally, if $x \in D_{\lambda^k}$, we get

$$|p(x) - p_k(x)| \le \sum_{|\alpha|_d \le m} |a_\alpha - a_\alpha^{(k)}| |x^\alpha| \le C_4 \sum_{|\alpha|_d \le m} (c^*)^{|\alpha|_d} \lambda^{k(\theta+m)} = C_5 \lambda^{k(m+\theta)}.$$

Here we used the inequality $|x^\alpha| = |x_1|^{\alpha_1} \cdots |x_N|^{\alpha_N} \le \|x\|^{\sigma_1 \alpha_1} \cdots \|x\|^{\sigma_N \alpha_N} = \|x\|^{|\alpha|_d}$. □

2.1. **A generalization of a lemma of Calderón and Zygmund.** In this section we assume that $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_r)$ is an homogeneous Lie group with δ_r given in Section 2. Our aim here is to prove the following generalization of the Calderón and Zygmund [CZ61, Lemma 2.6] having independent interest.

Theorem 2.3. *For each $m \geq 0$ there exists a function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that*

$$p \star \phi_\epsilon(x) = \int_{\mathbb{R}^N} p(y)\, \phi_\epsilon(y^{-1} \circ x)\, dy = p(x)$$

for each $p \in \mathbb{P}_{m,d}$ and for every $\epsilon > 0$ where $\phi_\epsilon(y) = \epsilon^{-Q}\phi(\delta_{1/\epsilon} y)$ with $Q = \sigma_1 + \cdots + \sigma_N$.

Theorem 2.3 is a consequence of the following proposition.

For each $m \in \mathbb{Z}_+$, let $a(m) = \{\alpha \in \mathbb{Z}_+^N : |\alpha|_d \leq m\}$ and $V_m = \{v : a(m) \to \mathbb{R}\}$. We have that V_m has dimension $N(m)$. For $v \in V_m$ we write $v_\alpha = v(\alpha)$.

Proposition 2.4. *Let $T : C_0^\infty(\mathbb{R}^N) \to V_m$ be the linear map defined by*

$$\Big(T(\phi)\Big)_\alpha := \int_{\mathbb{R}^N} \phi(y^{-1})\, y^\alpha\, dy$$

for $\alpha \in a(m)$. Then $T(C_0^\infty(\mathbb{R}^N)) = V(m)$. Therefore there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \phi(y)\, dy = 1$ and $\int_{\mathbb{R}^N} \phi(y)\, y^\alpha\, dy = 0$ for all $\alpha \in a(m)$.

Proof. Suppose by contradiction that $T(C_0^\infty(\mathbb{R}^N)) \neq V(m)$, then there exists $w \in V(m)$, $w \neq 0$, such that $T(\phi) \perp w$ for all $\phi \in C_0^\infty(\mathbb{R}^N)$. This means that $T(\phi) \cdot w = 0 = \int_{\mathbb{R}^N} \phi(y^{-1}) \left(\sum_{\alpha \in a(m)} w_\alpha y^\alpha\right) dy = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^N)$ which yields $w = 0$, a contradiction. Here we used the fact that the map $\phi \to \hat{\phi}$, $\hat{\phi}(y) = \phi(y^{-1})$ is a bijection of $C_0^\infty(\mathbb{R}^N)$ onto itself. □

Proof of Theorem 2.3. Fix $x \in \mathbb{R}^N$ and $\alpha \in a(m)$. Then $q_x^\alpha(z) := (x \circ z^{-1})^\alpha$ belongs to $\mathbb{P}_{m,d}$, since the composition law $\circ$ has polynomial components (see [BLU07, Theorem 1.3.15]) and since $\lambda^{-|\alpha|_d} q_x^\alpha(\delta_\lambda x) = \lambda^{-|\alpha|_d}(x \circ (\delta_\lambda z)^{-1})^\alpha = \left(\delta_{1/\lambda} x \circ z^{-1}\right)^\alpha \to (z^{-1})^\alpha$ as $\lambda \to \infty$. Let $p \in \mathbb{P}_{m,d}$, $p(y) = \sum_{|\alpha|_d \leq m} c_\alpha y^\alpha$, and let ϕ be the function in Proposition 2.4. Then keeping in mind that the Lebesgue measure is left translation invariant on $\mathbb{G}$, see [BLU07, Theorem 1.3.21], changing variables we get that

$$p \star \phi_\epsilon(x) = \int_{\mathbb{R}^N} p(x \circ z)\, \phi_\epsilon(z^{-1})\, dz = \sum_{|\alpha|_d \leq m} c_\alpha \int_{\mathbb{R}^N} (x \circ z)^\alpha\, \phi_\epsilon(z^{-1})\, dz = \sum_{|\alpha|_d \leq m} c_\alpha \int_{\mathbb{R}^N} q_x^\alpha(z)\, \phi_\epsilon(z^{-1})\, dz.$$

Writing $q_x^\alpha(z) = \sum_{|\beta|_d \le m} b_{x,\beta}^\alpha z^\beta$, replacing this quantity in the integral yields

$$\begin{aligned} p \star \phi_\epsilon(x) &= \sum_{\alpha,\beta} c_\alpha b_{x,\beta}^\alpha \int_{\mathbb{R}^N} z^\beta \phi_\epsilon(z^{-1})\, dz \\ &= \sum_{\alpha,\beta} c_\alpha b_{x,\beta}^\alpha \epsilon^{|\beta|_d} \int_{\mathbb{R}^N} z^\beta \phi(z^{-1})\, dz = \sum_\alpha c_\alpha b_{x,0}^\alpha = \sum_\alpha c_\alpha q_x^\alpha(0) = p(x). \end{aligned}$$

□

2.2. **A Taylor formula at the origin.** Let $1 \le \sigma_1 \le \cdots \le \sigma_N$ be the exponents in the dilations δ_r and define $\Sigma = \{|\alpha|_d : \alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^N, |\alpha|_d > 0\}$. In general, $\Sigma \ne \mathbb{N}$, and one has $\Sigma = \mathbb{N}$ if $\sigma_j \in \mathbb{N}$ for $i = 1, \cdots, N$. Let $u \in C^\infty(\mathbb{R}^N)$. For each $m \in \Sigma$ define

$$p_{m,d}(u)(x) = \sum_{|\alpha|_d \le m} a_\alpha x^\alpha,$$

where $a_\alpha = \dfrac{D^\alpha u(0)}{\alpha!}$.

Lemma 2.5. *Let $m \in \Sigma$. For each $M > 0$ and $\|x\| \le M$ we have that*

$$u(x) = p_{m,d}(x) + \|x\|^{m+\delta} \omega_{m,M}(u)(x), \tag{2.5}$$

where

$$|\omega_{m,M}(u)(x)| \le C_M(m) \sup_{|\alpha| \le m+1} \|D^\alpha u\|_{L^\infty(\|y\| \le M)},$$

and $\delta = \min\{|\alpha|_d - m : |\alpha|_d > m\}$. We remark that if all the σ_j's are natural numbers, then $|\alpha|_d \in \mathbb{N}$ and so $|\alpha|_d - m > 0$ implies $|\alpha|_d - m \ge 1$ and so in this case we can choose $\delta = 1$.

Proof. From the standard Taylor formula and since $|\alpha| \le |\alpha|_d$ we can write

$$u(x) = \sum_{|\alpha| \le [m]} a_\alpha x^\alpha + g_m(x) = p_{m,d}(u)(x) + \sum_{|\alpha| \le [m], |\alpha|_d > m} a_\alpha x^\alpha + g_m(x),$$

with

$$g_m(x) = \sum_{|\alpha| = [m]+1} \frac{D^\alpha u(tx)}{\alpha!} x^\alpha,$$

for some $0 < t < 1$; here $[m]$ denotes the integer part of m. Notice that if $\|x\| \le M$, then $\|tx\| \le M$ and so keeping in mind the definition of a_α, and since $|\alpha|_d \le \sigma_N |\alpha|$, we get for $\|x\| \le M$ that

$$|u(x) - p_{m,d}(x)| \le C(m) \sup_{|\alpha| \le m+1} \|D^\alpha u\|_{L^\infty(\|y\| \le M)} \sum_{m < |\alpha|_d \le \sigma_N(m+1)} |x^\alpha|. \tag{2.6}$$

On the other hand, we have that $|x^\alpha| \le \|x\|^{|\alpha|_d} = \|x\|^m \|x\|^{|\alpha|_d - m}$, which combined with (2.6) completes the proof. □

Remark 2.6. If $p \in \mathbb{P}_{m,d}$ satisfies $u(x) = p(x) + \|x\|^m O(\|x\|^\delta)$, then it follows from Lemma 2.5 that $p = p_{m,d}$. Therefore $p_{m,d}$ is called the Taylor polynomial of d-degree m of u at $x = 0$, and also denoted by $p_{m,d}(x) = p_{m,d}(u, 0)(x)$. Clearly, $D^\alpha u(0) = D^\alpha p_{m,d}(0)$ for $|\alpha|_d \le m$. To prove that $p = p_{m,d}$, let $p(x) = \sum_{|\alpha|_d \le m} b_\alpha\, x^\alpha$ such that $p(x) = o(\|x\|^m)$ as $x \to 0$. We want to conclude that $b_\alpha = 0$. Set $\{m_1, \cdots, m_n\} = \{|\alpha|_d : |\alpha|_d \le m\}$. We can assume $m_1 < m_2 < \cdots < m_n \le m$. We have $p(\delta_r x) = o(\|\delta_r x\|) = o(r^m \|x\|)$ as $r \to 0$, for each fixed $x \in \mathbb{R}^n$. On the other hand, $p(\delta_r x) = \sum_{|\alpha|_d \le m} b_\alpha r^{|\alpha|_d} x^\alpha = \sum_{j=1}^n \left(\sum_{|\alpha|_d = m_j} b_\alpha x^\alpha\right) r^{m_j}$ and therefore, $\sum_{|\alpha|_d = m_j} b_\alpha x^\alpha = 0$ for all $x \in \mathbb{R}^N$ and we are done.

2.3. **Taylor formula at a general point y.** We assume as in Subsection 2.1 that $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_r)$ is an homogeneous Lie group with δ_r given in Section 2. In such context the Taylor formula from Subsection 2.2 can be translated at any point $y \in \mathbb{R}^N$ and it takes the explicit form given in [Bon, Theorem 2] For our purposes, it is enough to recall this result in the following form. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let $m \in \Sigma$. Then there exists a unique polynomial $P_{m,d}(u, y) \equiv P_y \in \mathbb{P}_{m,d}$ such that

$$u(x) = P_y(x) + \|y^{-1} \circ x\|^{m+\delta} \omega_y(x)$$

where ω_y is bounded in a neighborhood of y and δ is a suitable positive constant. Moreover, $X^\alpha P_y(y) = X^\alpha u(y)$ for every $|\alpha|_d \le m$. Here if $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $X_1, \cdots, X_N$ is a Jacobian basis of the Lie algebra of $\mathbb{G}$, then X^α denotes the differential operator $X^\alpha := X_1^{\alpha_1} \cdots X_N^{\alpha_N}$.

2.4. **Application of Theorem 2.3.** In this subsection, as in the previous one, we assume that $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_r)$ is an homogeneous Lie group.

Theorem 2.7. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a smooth function, $0 < \theta < 1$ and $m \in \Sigma$. Suppose there exists a positive constant A such that for each $y \in \mathbb{R}^N$ there exists a polynomial $P_y \in \mathbb{P}_{m,d}$ with*

$$|u(x) - P_y(x)| \le A \|y^{-1} \circ x\|^{m+\theta}, \tag{2.7}$$

for all $x \in \mathbb{R}^N$. Then there exists a constant C depending only on A, θ, m, and the structure such that

$$|X^\alpha u(x_1) - X^\alpha u(x_2)| \le C \|x_1^{-1} \circ x_2\|^\theta$$

for all $|\alpha|_d = m$, $\alpha = (\alpha_1, \cdots, \alpha_N)$, and all $x_1, x_2 \in \mathbb{R}^N$.

Proof. Fix $x_1, x_2 \in \mathbb{R}^N$ and write

$$u(x) = P_{x_i}(x) + (u(x) - P_{x_i}(x)), \qquad i = 1, 2.$$

Let ϕ be the function from Theorem 2.3, and we may assume $\operatorname{supp}\phi \subset \{\|x\| \le 1\}$. Convolving with ϕ_ϵ we get

$$u \star \phi_\epsilon(x) = P_{x_i}(x) + \Big((u - P_{x_i}) \star \phi_\epsilon\Big)(x), \qquad i = 1, 2.$$

Since P_{x_i} is the Taylor polynomial of d-degree m of u at x_i, we have $X^\alpha P_{x_i}(x) = X^\alpha u(x_i)$ for $i = 1, 2$ and for all $|\alpha|_d = m$ we obtain

$$X^\alpha \Big(u \star \phi_\epsilon(x)\Big) = X^\alpha u(x_i) + \Big((u - P_{x_i}) \star X^\alpha \phi_\epsilon\Big)(x), \qquad i = 1, 2.$$

where in the second term on the right hand side we used the left translation invariance of X^α. We then obtain

$$X^\alpha u(x_1) - X^\alpha u(x_2) = \Big((u - P_{x_2}) \star X^\alpha \phi_\epsilon\Big)(x) - \Big((u - P_{x_1}) \star X^\alpha \phi_\epsilon\Big)(x)$$

for each $x \in \mathbb{R}^N$ and $|\alpha|_d = m$. We set $d(x, y) = \|y^{-1} \circ x\|$. Then d is a quasi distance, i.e., there exists a positive constant C such that $d(x', x'') \le C(d(x', z) + d(z, x''))$ and $d(x', x'') \le C d(x'', x')$ for all $x', x'', z \in \mathbb{R}^N$, see e.g. [BLU07, Proposition 5.1.7 and Corollary 5.1.5]. From (2.8)

$$\begin{aligned}
|\Big((u - P_{x_i}) \star X^\alpha \phi_\epsilon\Big)(x)| &\le \int_{\mathbb{R}^N} |u(y) - P_{x_i}(y)|\, |X^\alpha \phi_\epsilon(y^{-1} \circ x)|\, dy \\
&\le A \int_{\mathbb{R}^N} d(x_i, y)^{m+\theta}\, |X^\alpha \phi_\epsilon(y^{-1} \circ x)|\, dy \\
&\le A\, \epsilon^{-|\alpha|_d} \int_{d(x,y)\le\epsilon} d(x_i, y)^{m+\theta}\, |(X^\alpha \phi)_\epsilon(y^{-1} \circ x)|\, dy.
\end{aligned}$$

Let us now pick $x = x_1$ and $\epsilon = d(x_1, x_2)$. With this choice of x if $d(x, y) \le \epsilon$, then $d(x_i, y) \le C\,(d(x_i, x) + d(x, y)) \le 2C\epsilon$, where C is the constant in the quasi-triangle inequality for d. Therefore,

$$\begin{aligned}
&A\, \epsilon^{-|\alpha|_d} \int_{d(x,y)\le\epsilon} d(x_i, y)^{m+\theta}\, |(X^\alpha \phi)_\epsilon(y^{-1} \circ x)|\, dy \\
&\le A\, C^{m+\theta} \epsilon^\theta \int_{\mathbb{R}^N} |(X^\alpha \phi)_\epsilon(y^{-1} \circ x)|\, dy \\
&= A\, C^{m+\theta} \epsilon^\theta \int_{\mathbb{R}^N} |(X^\alpha \phi)_\epsilon(z^{-1})|\, dz \\
&\le A\, C^{m+\theta} \epsilon^\theta \int_{\mathbb{R}^N} |X^\alpha \phi(z)|\, dz = C\, A\, d^*(x_1, x_2)^\theta.
\end{aligned}$$

□

The proof of the previous theorem can be easily adapted to obtain the following local version. For this it is convenient to define $B_r(y) = \{x \in \mathbb{R}^N : \|y^{-1} \circ x\| < r\}$.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^N$ be open, let $u : \Omega \to \mathbb{R}$ be a smooth function, $0 < \theta < 1$ and $m \in \Sigma$. Suppose there exist positive constants A and R such that for each $y \in \Omega$ with $B_r(y) \subset \Omega$ there exists a polynomial $P_y \in \mathbb{P}_{m,d}$ with*

$$|u(x) - P_y(x)| \leq A\|y^{-1} \circ x\|^{m+\theta}, \tag{2.8}$$

for all $x \in B_r(y)$. Then there exist positive constants C and η depending only on A, θ, m and the structure such that, for each ball $B_{2R}(y) \subset \Omega$,

$$|X^\alpha u(x_1) - X^\alpha u(x_2)| \leq C\,\|x_1^{-1} \circ x_2\|^\theta \tag{2.9}$$

for all $|\alpha|_d = m$, $\alpha = (\alpha_1, \cdots, \alpha_N)$, and all $x_1, x_2 \in B_{\eta R}(y)$.

3. Homogeneous hypo-elliptic operators: Schauder estimates at the origin

Let

$$\mathcal{L} := \sum_{i,j=1}^N a_{ij}\partial_{x_i x_j} + \sum_{j=1}^N b_j(x)\partial_{x_j}, \tag{3.1}$$

and we assume

(H1) $a_{ij}, b_j \in C^\infty(\mathbb{R}^N)$;
(H2) there exists $\alpha > 0$ and $j \in \{1, \cdots, N\}$ such that $a_{jj}(x) \geq \alpha$ for all $x \in \mathbb{R}^N$, that is $\mathcal{L}$ is not totally degenerate.
(H3) $q_{\mathcal{L}}(x, \xi) := \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq 0$ for all $x, \xi \in \mathbb{R}^N$;
(H4) $\mathcal{L}$ is hypo-elliptic;
(H5) $\mathcal{L}$ is d-homogeneous of degree two, i.e.

$$\mathcal{L}(u(\delta_r(x))) = r^2\mathcal{L}u(\delta_r(x)),$$

where δ_r was defined in Section 2

We will show in the Appendix that these assumptions imply that the σ_j's in the dilations δ_r are integers and from now on we assume this is so.

We have that (H2) and (H3) imply the following weak maximum principle (due to Picone, see [BLU07, Theorem 5.13.4]): if $u \in C^2(\Omega)$, $\mathcal{L}u \geq 0$ in Ω bounded, and $\limsup_{x\to y} u(x) \leq 0$ for each $y \in \partial\Omega$, then $u \leq 0$ in Ω.

Proposition 3.1. *Let u be a smooth function such that $\mathcal{L}u = 0$ in an open set Ω containing the origin, and let $p \in \mathbb{P}_{m,d}$ be the d-Taylor polynomial of degree m of the function u at 0.*

Then $\mathcal{L}p = 0$ in $\mathbb{R}^N$.

Proof. We can assume $m \geq 2$, otherwise $\mathcal{L}p = 0$ For $p \in \mathbb{P}_{m,d}$, we can write $p = \sum_{j=0}^{m} p_j$, where p_j a d-homogeneous polynomial of d-degree j. Set $u = p + R$. Then $R \in C^\infty$ and from Lemma 2.5, $R(x) = \|x\|^m O(\|x\|^\delta)$. Moreover, since $\mathcal{L}u = 0$, it follows that $\mathcal{L}R = \mathcal{L}p$. As a consequence

$$\int_\Omega R(x)\mathcal{L}^*(\phi_\epsilon(x))\,dx = \int_\Omega \mathcal{L}p(x)\phi_\epsilon(x)\,dx,$$

for every $\phi \in C_0^\infty(\Omega)$, ϵ sufficiently small, where $\phi_\epsilon(x) = \epsilon^{-Q}\phi(\delta_{1/\epsilon}(x))$, $Q = \sigma_1 + \cdots + \sigma_N$, and $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$. Then

$$\begin{aligned}\left|\int_\Omega R(x)\mathcal{L}^*(\phi_\epsilon(x))\,dx\right| &\leq C\int_{\mathbb{R}^N} \|x\|^{m+\delta}|\mathcal{L}^*(\phi_\epsilon(x))|\,dx \\ &= C\int_{\mathbb{R}^N} \|x\|^{m+\delta}|(\mathcal{L}^*(\phi))_\epsilon(x))|\epsilon^{-2}\,dx \\ &\leq C\,\epsilon^{m+\delta-2}.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_\Omega \mathcal{L}p(x)\phi_\epsilon(x)\,dx &= \sum_{j=2}^{m}\int_\Omega \mathcal{L}p_j(x)\phi_\epsilon(x)\,dx \\ &= \sum_{j=2}^{m}\int_\Omega \mathcal{L}p_j(\delta_\epsilon(x))\phi(x)\,dx \\ &= \sum_{j=2}^{m}\int_\Omega \mathcal{L}p_j(x)\epsilon^{j-2}\phi(x)\,dx, \quad \text{since } \mathcal{L}p_j \text{ is } \delta_r\text{-homogeneous of degree } j-2 \\ &:= \sum_{j=2}^{m} c_j\epsilon^{j-2}.\end{aligned}$$

We then obtain that $\sum_{j=2}^{m} c_j\epsilon^{j-2} = O(\epsilon^{m+\delta-2})$ for ϵ small. This implies that $c_j = 0$ for $2 \leq j \leq m$, that is, $\int_\Omega \mathcal{L}p_j(x)\phi(x)\,dx = 0$ for all $\phi \in C_0^\infty$ and $2 \leq j \leq m$. Therefore $\mathcal{L}p_j = 0$ in Ω[1]. Since $\mathcal{L}p_i = 0$ for $i = 0, 1$, the proof is complete. □

Our starting point to establish the Schauder estimates at $x = 0$ is the following lemma which can be found for example in [Bon69, Theoreme 5.2 and Corollaire 5.2], see also [BLU07, pp. 383-387].

[1]Since $\mathcal{L}$ has smooth coefficients and is d-homogeneous, then its coefficients are polynomial functions. Therefore $\mathcal{L}p_j = 0$ in $\mathbb{R}^n$.

Lemma 3.2. *Assume that conditions (H1)-(H4) hold (not necessarily (H5)) . Then there exists an open set $D^* \subset \Omega$ with $0 \in D^*$ and a smooth function $h \geq 0$ in $\bar{D}^*$ such that $h \in C^\infty(D^*) \cap C(\bar{D}^*)$ and*

$$\mathcal{L}h = -1 \text{ in } D^*$$
$$h = 0 \text{ on } \partial D^*.$$

Lemma 3.3. *Assume that conditions (H1)-(H5) hold. Let $D_r(0) = \delta_r D^*$ and $h_r(x) = r^2 h(\delta_{1/r}x)$. If $\mathcal{L}u \geq -f$ in $D_r(0)$, then*

$$u(x) \leq \sup_{\partial D_r(0)} u + h_r(x) \sup_{D_r(0)} f, \quad x \in D_r(0), \text{ for } x \in D_r(0).$$

Proof. Let $g(x) = u(x) - \sup_{\partial D_r(0)} u - h_r(x) \sup_{D_r(0)} f$. We have $\mathcal{L}g = \mathcal{L}u + \sup_{D_r(0)} f \geq -f + \sup_{D_r(0)} f \geq 0$, that is, g is $\mathcal{L}$-subharmonic in $D_r(0)$. Then $\sup_{D_r(0)} g = \sup_{\partial D_r(0)} g = 0$, so $g \leq 0$ in $D_r(0)$ and the lemma follows. □

We remark that when $\mathcal{L} = \Delta$, the Laplacian, then in Lemmas 3.3 and 3.2, D^* is the unit Euclidean ball centered at 0 and $h(x) = \dfrac{1-|x|^2}{2N}$, where $|x|$ is the Euclidean norm, and so $h_r(x) = r^2 h(x/r) = \dfrac{r^2 - |x|^2}{2N}$.

Lemma 3.4. *Assume that conditions (H1)-(H5) hold. If u is a solution to $\mathcal{L}u = f$ in $D_r(0)$ and v solves $\mathcal{L}v = 0$ and $v = u$ on $\partial D_r(0)$, then*

$$h_r(x) \inf_{D_r(0)} f \leq v(x) - u(x) \leq h_r(x) \sup_{D_r(0)} f,$$

in particular,

$$|u(x) - v(x)| \leq h_r(x) \sup_{D_r(0)} |f|,$$

for all $x \in D_r(0)$.

Proof. Since $\mathcal{L}(v - u) = -f$, the inequality immediately follows from Lemma 3.3. □

Lemma 3.5. *Let $0 < \alpha < \delta$, with δ the number in Lemma 2.5* [2]*. There exist positive constants C_0, ϵ_0 and $0 < \lambda < 1$ such that for any f and any solution to $\mathcal{L}u = f$ in D^* with $\|u\|_{L^\infty(D^*)} \leq 1$ and $\|f\|_{L^\infty(D^*)} \leq \epsilon_0$ there exists a polynomial q of d-degree two such that $\mathcal{L}q = 0$,*

$$|u(x) - q(x)| \leq \lambda^{2+\alpha}, \qquad \text{for } x \in D_\lambda, \tag{3.2}$$

[2] $\delta = 1$ in case all σ_j are integers.

and

$$\|q\|_{L^\infty(D^*)} \le C_0 \tag{3.3}$$

Proof. Let v be such that $\mathcal{L}v = 0$ in D^* with $v = 0$ on ∂D^*. By the maximum principle $\sup_{D^*}|v| \le \sup_{D^*}|u| \le 1$. Since $\mathcal{L}$ is hypoelliptic, from [Bon69, Theorem 7.2] we have

$$|D^\beta v(x)| \le C(n,|\beta|) \sup_{D^*}|v| \le C(n,|\beta|), \qquad \text{for } x \in D_{1/2}. \tag{3.4}$$

We remark that to apply [Bon69, Theorem 7.2] we need $v \ge 0$. Since $-1 \le v \le 1$, so $v+1 \ge 0$, and $\mathcal{L}(v+1) = 0$, we can use $v+1$ instead of v. Let q be the Taylor polynomial of d-degree two of v at 0. The coefficients of q are derivatives of v at $x = 0$. Hence from (3.4) all the coefficients of q, and hence its L^∞ norm in D^*, are bounded by a structural constant C_0. By Proposition 3.1

$$\mathcal{L}q = 0$$

To complete the proof of the lemma we write

$$\begin{aligned} |u(x) - q(x)| &\le |u(x) - v(x)| + |v(x) - q(x)| \\ &\le h_1(x) \sup_{D_1}|f| + C\,\|x\|^{2+\delta} = I + II, \end{aligned}$$

from Lemma 2.5. We have for $\|x\| \le D_\lambda$ that $II \le C\,\lambda^{2+\delta} \le \frac{1}{2}\lambda^{2+\alpha}$, if we pick $\lambda \le \left(\dfrac{1}{2C}\right)^{1/(\delta-\alpha)}$. With this value of λ, we next want

$$I \le \frac{1}{2}\lambda^{2+\alpha}.$$

If $\max_{D^*} h(x)\,\epsilon_0 \le \frac{1}{2}\lambda^{2+\alpha}$, we then have

$$I = h(x) \sup_{D^*}|f| \le \max_{D^*} h(x) \sup_{D^*}|f| \le \max_{D^*} h(x)\,\epsilon_0 \le \frac{1}{2}\lambda^{2+\alpha}$$

and we are done. □

Theorem 3.6. *Suppose $u \in C^2(D^*) \cap C(\bar{D}^*)$, $\mathcal{L}u = f$, and f is Hölder continuous at 0, i.e.,*

$$[f]_{\alpha,0} = \sup_{x\in D^*} \frac{|f(x) - f(0)|}{\|x\|^\alpha} < \infty,$$

with $0 < \alpha < \delta$, where δ is from Lemma 2.5. Then there exists polynomial $p(x,0)$ with d-degree two such that

$$|u(x) - p(x,0)| \le C_1\,\|x\|^{2+\alpha}, \qquad \text{for } x \in D^*, \tag{3.5}$$

with

$$C_1 \le C_0 \left([f]_{\alpha,0} + \|f\|_{L^\infty(D^*)} + \|u\|_{L^\infty(D^*)} \right),$$

and all the coefficients of $p(x,0)$ *are bounded by the same quantity.*

Proof. We may assume that

(i) $f(0) = 0$,
(ii) $[f]_{\alpha,0} + \|f\|_{L^\infty(D^*)} \le \epsilon_0$,
(iii) $\|u\|_{L^\infty(D^*)} \le 1$.

Indeed, if we let

$$v(x) = u(x) + h(x)\, f(0), \qquad g(x) = \epsilon_0 \, \frac{f(x) - f(0)}{[f]_{\alpha,0} + 2\|f\|_{L^\infty(D^*)} + \|v\|_{L^\infty(D^*)}}$$

and

$$\bar{u}(x) = \epsilon_0 \, \frac{v(x)}{[f]_{\alpha,0} + 2\|f\|_{L^\infty(D^*)} + \|v\|_{L^\infty(D^*)}},$$

then $\mathcal{L}\bar{u} = g$ in D^*, g satisfies (i) and (ii), and $\bar{u}$ satisfies (iii). Notice that we can assume $\epsilon_0 \le 1$.

Claim: there exists a sequence polynomials p_k with d-degree two with $\mathcal{L}p_k = 0$ and such that

$$|u(x) - p_k(x)| \le \lambda^{(2+\alpha)k}, \qquad \text{for } x \in D_{\lambda^k}(0) = \delta_{\lambda^k} D^* \tag{3.6}$$

and

$$|p_k(x) - p_{k+1}(x)| \le C\lambda^{k(2+\alpha)} \quad \text{if } x \in D_{\lambda^k} \tag{3.7}$$

for $k = 1, \cdots$ where C is a universal constant. In view of (ii) and (iii) above we can apply Lemma 3.5, and we let $p_1(x)$ be the polynomial in that lemma. Suppose $p_k(x)$ is constructed. We will construct p_{k+1}. Let

$$w(x) = \frac{(u - p_k)(\delta_{\lambda^k} x)}{\lambda^{(\alpha+2)k}}.$$

We have

$$\mathcal{L}w(x) = \frac{1}{\lambda^{(\alpha+2)k}} \left[\lambda^{2k} (\mathcal{L}u)(\delta_{\lambda^k} x) - \lambda^{2k} (\mathcal{L}p_k)(\delta_{\lambda^k} x) \right] = \frac{1}{\lambda^{\alpha k}} f(\delta_{\lambda^k} x) = g_k(x).$$

From (3.6), $\|w\|_{L^\infty(D^*)} \le 1$ and from (i) and (ii) above, and the homogeneity of $\|\cdot\|$, $\|g_k\|_{L^\infty(D^*)} \le \epsilon_0$. Hence by application of Lemma 3.5 to w, we get an $\mathcal{L}$-harmonic polynomial $q_k(x)$ -depending on g_k- with d-degree ≤ 2 such that

$$|w(x) - q_k(x)| \le \lambda^{2+\alpha}, \qquad \text{for } x \in D_\lambda(0). \tag{3.8}$$

From the definition of w and (3.8) we then get

$$|u(\delta_{\lambda^k}x) - p_k(\delta_{\lambda^k}x) - \lambda^{(2+\alpha)k}q_k(x)| \le \lambda^{(2+\alpha)(k+1)}, \quad \text{for } x \in D_\lambda(0).$$

Therefore, if we take

$$p_{k+1}(x) = p_k(x) + \lambda^{(2+\alpha)k}q_k(\delta_{\lambda^{-k}}x), \tag{3.9}$$

then p_{k+1} satisfies (3.6) with k replaced by $k+1$, and (3.7) follows from (3.3). Then the claim is proved.

Consequently, from Proposition 2.2 we get that there exists a polynomial p of d-degree two such that

$$|p(x) - p_k(x)| \le C\,\lambda^{k(2+\alpha)} \quad \text{if } x \in D_{\lambda^k}.$$

Given $x \in D^* = D_1$, let k be a positive integer such that $x \in D_{\lambda^k} \setminus D_{\lambda^{k+1}}$. Hence $\lambda^{k+1}/c^* \le \|x\| \le c^*\lambda^k$ and

$$|u(x) - p(x)| \le |u(x) - p_k(x)| + |p_k(x) - p(x)| \le C\,\|x\|^{2+\alpha}$$

and we are done. □

4. Left invariant homogeneous operators: local Schauder estimates in D^*

In this section we assume the operator $\mathcal{L}$ satisfies in addition to (H1)-(H5) the following hypothesis:

(H6) There exists a Lie group structure in $\mathbb{R}^N$, $\mathbb{G} = (\mathbb{R}^n, \circ)$ such that $\mathcal{L}$ is left translation invariant on $\mathbb{G}$ and the dilations $\mathbf{d} = (\delta_r)_{r>0}$ in Section 2 are homeomorphisms of $\mathbb{G}$. In short, $(\mathbb{R}^N, \circ, \delta_r)$ is an homogeneous Lie group. (See Theorem 6.1).

Let $y \in D^*$ and define

$$\operatorname{dist}(y, \partial D^*) := \sup\{\delta > 0 : y \circ \delta_r D^* \subset D^*\}.$$

Let $0 < \alpha < 1$ and $f : D^* \to \mathbb{R}$. We say that $f \in C^\alpha(D^*)$ if

$$[f]_{\alpha,D^*} := \sup_{\|z\| \le \operatorname{dist}(y,\partial D^*)} \frac{|f(y \circ z) - f(y)|}{\|z\|^\alpha} < \infty.$$

Theorem 4.1. *Let $0 < \alpha < 1$ and let $u \in C^2(D^*) \cap C(D^*)$ be a solution to $\mathcal{L}u = f$ in D^* with $f \in C^\alpha(D^*)$. Then there exist structural constants C, ρ, η such that for each $y \in D^*$ with $\|y\| \le \rho$ there exists a polynomial $p_y(x)$ of d-degree two such that*

$$|u(x) - p_y(x)| \le C^*\|y^{-1} \circ x\|^\alpha, \qquad \text{for } \|y^{-1} \circ x\| < \eta$$

and $C^* \le C\Big([f]_{\alpha,D^*} + \|f\|_{L^\infty(D^*)} + \|u\|_{L^\infty(D^*)}\Big)$.

Proof. Let $y \in D^*$ and $0 < r < \operatorname{dist}(y, \partial D^*)$. Define $g(x) = r^2 f(y \circ \delta_r x)$ and $v(x) = u(y \circ \delta_r x)$ for $x \in D^*$. Since $\mathcal{L}$ is left invariant[3] and homogeneous of degree two, $\mathcal{L}v(x) = r^2(\mathcal{L}u)(y \circ \delta_r x) = g(x)$. We also have that

$$[g]_{\alpha,0} = \sup_{D^*} \frac{|g(x) - g(0)|}{\|x\|^\alpha} = r^2 \sup_{D^*} \frac{|f(y \circ \delta_r x) - f(y)|}{\|x\|^\alpha} = r^{2+\alpha} \sup_{z \in \delta_r D^*} \frac{|f(y \circ z) - f(y)|}{\|z\|^\alpha}.$$

From Theorem 3.6 applied to v, there exists a polynomial $p(x, 0)$ with d-degree two such that

$$|v(x) - p(x, 0)| \le C_1 \|x\|^{2+\alpha}, \qquad \text{for } x \in D^*, \tag{4.1}$$

with

$$C_1 \le C_0 \left([g]_{\alpha,0} + \|g\|_{L^\infty(D^*)} + \|v\|_{L^\infty(D^*)} \right). \tag{4.2}$$

From (4.1) and the definition of v we get that

$$|u(z) - p(\delta_{1/r}(y^{-1} \circ z), 0)| \le \frac{C_1}{r^{2+\alpha}} \|y^{-1} \circ z\|^{2+\alpha}, \qquad \text{for } y^{-1} \circ z \in D_r.$$

We have $\|g\|_{L^\infty(D^*)} = r^2 \|f\|_{L^\infty(y \circ \delta_r D^*)}$, and $\|v\|_{L^\infty(D^*)} = \|u\|_{L^\infty(y \circ \delta_r D^*)}$. If we let $q(z, y) = p(\delta_{1/r}(y^{-1} \circ z), 0)$, and $r = \operatorname{dist}(y, \partial D^*)/2$, then

$$|u(x) - q(x, y)| \le C_1^* \|y^{-1} \circ x\|^{2+\alpha}, \qquad \text{for } y^{-1} \circ x \in D_r,$$

with

$$\begin{aligned} C_1^* &= \frac{C_1}{r^{2+\alpha}} \\ &\le C_0 \left(\sup_{z \in D_r} \frac{|f(y \circ z) - f(y)|}{\|z\|^\alpha} + r^{-\alpha} \|f\|_{L^\infty(y \circ \delta_r D^*))} + r^{-2-\alpha} \|u\|_{L^\infty(y \circ \delta_r D^*))} \right). \end{aligned}$$

We claim that there exist two structural constants $\rho, \eta > 0$ such that $\operatorname{dist}(y, \partial D^*) > 2C^*\eta$ if $\|y\| \le \rho$. Suppose for a moment that the claim holds. Then from the above estimates we get that for each $y \in D^*$ with $\|y\| \le \rho$ we have that there exists a polynomial $p(x, y)$ of d-degree two in x such that

$$|u(x) - p(x, y)| \le C^* \|y^{-1} \circ x\|^{2+\alpha}, \qquad \text{for } \|y^{-1} \circ x\| \le 1/4,$$

with

$$C^* \le C_0 \left([f]_{\alpha, D^*} + \|f\|_{L^\infty(D^*)} + \|u\|_{L^\infty(D^*)} \right).$$

We are then left with the proof of the claim. Notice from [BLU07, Proposition 5.1.7] that the quasi-triangle inequality $\|x' \circ x''\| \le C(\|x'\| + \|x''\|)$ holds for all $x', x'' \in \mathbb{R}^N$. Then if choose $\rho = 1/(2CC^*)$ and $\eta = 1/(2CC^{*3})$, then we have for all $z \in D^*$ and $0 < s < 2C^*\eta$ that $\|y \circ \delta_s z\| \le C(\|y\| + s\|z\|) \le C(\rho + 2C^*\eta C^*) < 1/C^*$. This

[3] If $h(x) = u(y \circ x)$, then $\mathcal{L}h(x) = \mathcal{L}u(y \circ x)$.

implies that $y \circ \delta_s D^* \subset D^*$ for $\|y\| \leq \rho$ and $0 < s < 2C^*\eta$. Hence $\mathrm{dist}(y, \partial D^*) \geq 2C^*\eta$ for each $y \in D^*$ with $\|y\| < \rho$. This completes the proof of the theorem.

□

5. The general case

Combining Theorems 3.6 and 2.7 we obtain the Hölder continuity of solutions to $\mathcal{L}u = \sum_{i=1}^r X_i^2 u + Yu = f$, when f is Hölder continuous with respect to $d(x, y) = \|y^{-1} \circ x\|$. This implies by perturbation Schauder estimates for the operator $\sum_{k=1}^r a_{ij} X_i X_j u + Yu$ with a_{ij} Hölder continuous coefficients with respect to the metric d and uniformly elliptic. Indeed, as usual the proof requires interpolation inequalities for Hölder norms defined with respect to the quasi-metric d. These inequalities have been proved by Chao-Jiang Xu in [Xu92, Proposition 2.2]. The general case then follows using Xu's technique as in Lemma 3.8, Theorem 3.9, and Theorem 4.1 of his paper.

6. Examples

The class of the prototype operators in $\mathbb{R}^n$

$$\mathcal{L}u = \sum_{k=1}^m X_k^2 u + X_0 u, \tag{6.1}$$

to which our results apply is quite wide and due to a recent result of one of us with Bonfiglioli [BL] it can be characterized as follows. Suppose the vector fields in (6.1) are smooth, $X_1, \cdots X_m$ are δ_r-homogeneous of degree one, and X_0 is δ_r-homogeneous of degree two. Thus $\mathcal{L}$ is δ_r-homogeneous of degree two. We also assume that $\mathcal{L}$ satisfies the Hörmander rank condition:

$$\text{rank Lie}\{X_0, X_1, \cdots, X_m\}(x) = N \text{ for all } x \in \mathbb{R}^N. \tag{6.2}$$

It is well known that (6.2) implies that $\mathcal{L}$ is hypoelliptic , [Hö68], and actually (6.2) is also a necessary condition for the hypoellipticity of $\mathcal{L}$, see [Der71]. Let $\mathfrak{a}$ denote the Lie algebra generated by $\{X_0, X_1, \cdots, X_m\}$, then the following theorem holds.

Theorem 6.1. *The following conditions are equivalent:*

(1) *dim* $\mathfrak{a} = N$;
(2) *There exists a composition law* $\circ$ *in* $\mathbb{R}^N$ *such that* $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ *is an homogeneous Lie group and* $\mathcal{L}$ *if left-translation invariant on* $\mathbb{G}$.

The composition law of the group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ in the previous theorem can be constructed as follows. Let $\{Z_1, \dots, Z_N\}$ be a basis of the Lie algebra $\mathfrak{a}$ and define

$$\mathrm{Exp} : \mathfrak{a} \to \mathbb{R}^N, \quad \mathrm{Exp}(X) = \exp(tX)(0)|_{t=1}.$$

We denote by $\mathrm{Exp}(tX)(x)$ the solution of the Cauchy problem

$$\dot{\gamma} = X(\gamma), \quad \gamma(0) = x.$$

The map Exp is a bijection of $\mathfrak{a}$ onto $\mathbb{R}^N$. Let $\mathrm{Log} = \mathrm{Exp}^{-1}$. Then, for every $x, y \in \mathbb{R}^N$ we have

$$x \circ y = \mathrm{Exp}(\mathrm{Log}(y))(x).$$

Our results obviously apply to the operator $a_{ij}X_iX_j$ considered by Capogna and Han and to their parabolic counterpart $a_{ij}X_iX_j - \partial_t$, when $X_1, \cdots, X_m$ span the first layer of the Lie algebra of a Carnot group in $\mathbb{R}^N$.

We will next show other examples.

6.1. **Kolmogorov's operator.** We let $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where x stands for velocity, y stands for position, and t is time. This is the operator defined by

$$\mathcal{K} = \Delta_x + x \cdot D_y - \partial_t = \sum_{j=1}^{n} \partial_{x_j}^2 + Y$$

where $Y = \sum_{i=1}^{n} x_i \partial_{y_i} - \partial_t$. The family of dilations

$$\delta_r(x, y, t) = (rx, r^3 y, r^2 t)$$

makes the operator $\mathcal{K}$ homogeneous of degree two. In addition $\mathcal{K}$ is hypoelliptic, and therefore it satisfies conditions (H1)-(H5). A multiplication compatible with these dilations is given by

$$(x, y, t) \circ (x', y', t') = (x + x', y + y' + t'x, t + t'),$$

for $x, x', y, y' \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The operator $\mathcal{K}$ is left invariant with respect to this multiplication. The triple $(\mathbb{R}^N, \circ, \delta_r)$ with $N = 2n + 1$ is an homogeneous Lie group.

For more details, and for a wider class of Kolmgorov type operators we refer to [LP94] and [BLU07, Sect. 4.3.4].

6.2. **Bony's operator.** Let $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$ and $T = \partial_t$, $S = \partial_s + t\partial_{x_1} + \frac{t^2}{2!}\partial_{x_2} + \cdots + \frac{t^n}{n!}\partial_{x_n}$. For $k \in \mathbb{N}$ let $X_k = [T \cdots [T, S] \cdots]$ where T appears k-times. We have $X_k = \partial_{x_k} + t\partial_{x_{k+1}} + \cdots + \frac{t^{n-k}}{(n-k)!}\partial_{x_n}$ for $1 \le k \le n$ and $X_k = 0$ for $k \ge n+1$. The operator $T^2 + S = \partial_t^2 + \sum_{j=1}^{n} \frac{t^2}{2}\partial_{x_j} + \partial_s$ is hypoelliptic and satisfy (H1)-(H5). The family of dilations

$$\delta_r^*(t, s, x) = (rt, r^2 s, r^3 x_1, r^4 x_2, \cdots, r^{n+2} x_n)$$

makes $T^2 + S$ homogeneous of degree two with respect to δ_r^*. A notion of multiplication compatible with these dilations is

$$(t, s, x) \circ (\tau, \sigma, y) = \left(\tau + t, \sigma + s, x_1 + y_1 + \sigma t, \cdots, x_n + \sum_{k=1}^{n} y_k \frac{t^{n-k}}{(n-k)!} + \sigma \frac{t^n}{n!}\right).$$

The operator $T^2 + S$ is left invariant with respect to this multiplication, see [BLU07, Sect. 4.3.3] for more details.

6.3. **An operator from control theory.** The operator

$$\mathcal{L} = \left(\partial_{x_1} + \sum_{j=2}^{N-1} x_j \partial_{x_{j+1}}\right)^2 - \partial_{x_2}$$

satisfies conditions (H1)-(H6). Indeed, it is homogeneous of degree two with respect to the dilations $\delta_r(x_1, \cdots, x_N) = (rx_1, rx_2, r^3 x_3, \cdots, r^{N-1} x_N)$. A composition law compatible with δ_r and making $\mathcal{L}$ left invariant is given by

$$(x_1, \cdots, x_N) \circ (y_1, \cdots, y_N) = \left(y_1 + x_1, y_2 + x_2, y_3 + x_3 + y_1 x_2, \cdots, \sum_{j=2}^{N} \frac{1}{(N-j)!} y_1^{N-j} x_j\right).$$

For more details see [BLU07, Sect. 4.3.5] and the references therein.

7. Appendix

We have an operator $\mathcal{L}$ with the form

$$\mathcal{L} := \sum_{i,j=1}^{N} a_{ij} \partial_{x_i x_j} + \sum_{j=1}^{N} b_j(x) \partial_{x_j},$$

and we assume

(H1) $a_{ij}, b_j \in C^\infty(\mathbb{R}^N)$;

(H2) there exists $\alpha > 0$ and $j \in \{1, \cdots, N\}$ such that $a_{jj}(x) \geq \alpha$ for all $x \in \mathbb{R}^N$, that is $\mathcal{L}$ is not totally degenerate.

(H3) $q_{\mathcal{L}}(x, \xi) := \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq 0$ for all $x, \xi \in \mathbb{R}^N$;

(H4) $\mathcal{L}$ is hypoelliptic;

(H5) There exists a family of dilations $\delta_\lambda(x_1, \cdots, x_N) = (\lambda^{\sigma_1}x_1, \cdots, \lambda^{\sigma_N}x_N)$, $\sigma_j > 0$ for all j, such that $\mathcal{L}$ is homogeneous of degree n with respect to δ_λ, i.e.

$$\mathcal{L}(u(\delta_\lambda(x))) = r^n \mathcal{L}u(\delta_\lambda(x)).$$

We will prove that there exists another family of dilations $\bar{\delta}_\mu(x_1, \cdots, x_N) = (\mu^{\bar{\sigma}_1}x_1, \cdots, \mu^{\bar{\sigma}_N}x_N)$ with $\bar{\sigma}_1 = 1$ and $\bar{\sigma}_j \in \mathbb{N}$ for $j \geq 2$ such that $\mathcal{L}$ is homogeneous of degree two with respect to $\bar{\delta}_\lambda$, and the coefficients a_{ij} and b_j are polynomial functions.

We split the proof in several steps.

Step 1. Let $\phi(x) = x_j^{2m}$ with $1 \leq j \leq N$ and $m \in \mathbb{N}$. We have $\mathcal{L}\phi(x) = 2mx_j^{2m-2}((2m-1)a_{jj}(x) + x_jb_j(x))$ and $\mathcal{L}(\phi(\delta_\lambda x)) = \mathcal{L}(\lambda^{2m\sigma_j}x_j^{2m}) = \lambda^{2m\sigma_j}\mathcal{L}(x_j^{2m})$. Since $\mathcal{L}$ is homogeneous of degree n, $\mathcal{L}(\phi(\delta_\lambda(x))) = \lambda^n(\mathcal{L}\phi)(\delta_\lambda x)$. This yields

$$(2m-1)a_{jj}(x) + x_jb_j(x) = \lambda^{n-2\sigma_j}\Big((2m-1)a_{jj}(\delta_\lambda x) + \lambda^{\sigma_j}x_jb_j(\delta_\lambda x)\Big)$$

for every m which implies that $a_{jj}(\delta_\lambda x) = \lambda^{2\sigma_j-n}a_{jj}(x)$ and $b_j(\delta_\lambda x) = \lambda^{\sigma_j-n}b_j(x)$. This implies that a_{jj} and b_j are polynomials, see [BLU07]. If $a_{11}(0) \neq 0$, then we get $2\sigma_1 = n$. Then replacing the dilations δ_λ by $\bar{\delta}_\lambda x = (\lambda x_1, \lambda^{\sigma_2/\sigma_1}x_2, \cdots, \lambda^{\sigma_N/\sigma_1}x_N)$ we obtain that $\mathcal{L}$ is $\bar{\delta}_\lambda$-homogeneous of degree two.

Step 2. a_{jj} depends only on $x_1, \cdots, x_{j-1}$. Since a_{jj} is homogeneous of degree $2\sigma_j - 2$, we have $a_{jj}(x) = \sum_{|\alpha|_d = 2\sigma_j-2} c_\alpha x^\alpha$. Suppose that for some $i \geq j$ the function a_{jj} depends of x_i. Let m_i be the maximum exponent of x_i appearing in the sum defining a_{jj}. Since $a_{jj}(x) = ... + x_i^{m_i}P(\hat{x}) + ...$ and $a_{jj} \geq 0$, then m_i is even and greater than one. Hence for α in the sum with $\alpha_i = m_i$, we have $2\sigma_j - 2 = |\alpha|_d \geq \sigma_i m_i \geq 2\sigma_i \geq 2\sigma_j$ for $i \geq j$, a contradiction.

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