REGULARITY FOR THE NEAR FIELD PARALLEL REFRACTOR AND REFLECTOR PROBLEMS

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ABSTRACT. We prove local $C^{1,\alpha}$ estimates of solutions for the parallel refractor and reflector problems under local assumptions on the target set Σ, and no assumptions are made on the smoothness of the densities.

CONTENTS

1. INTRODUCTION

Suppose we have a domain $\Omega \subset \mathbb{R}^n$ and a domain Σ contained in an n dimensional surface in $\mathbb{R}^{n+1}$; Ω is referred as the extended source, and Σ is referred as the target domain, receiver, or screen to be illuminated. Let n_1 and n_2 be the indexes of refraction of two homogeneous and isotropic media I and II, respectively, and suppose that from the extended source Ω surrounded by medium I, radiation emanates in the vertical direction e_{n+1} with intensity $f(x)$ for $x \in \Omega$, and Σ is surrounded by media II. That is, all emanating rays from Ω are collimated. A *parallel refractor* is an optical surface $\mathcal{R}$ interface between media I and II, such that all rays refracted by $\mathcal{R}$ into medium II are received at the surface Σ, and the

prescribed radiation intensity received at each point $p \in \Sigma$ is $\sigma(p)$. Assuming no loss of energy in this process, we have the conservation of energy equation $\int_\Omega f(x)\,dx = \int_\Sigma \sigma(p)\,dp$. Under general conditions on Ω and Σ, and when σ is a Radon measure in D, the existence of parallel refractors is proved in [GT13].

The purpose of this paper is to study local regularity of parallel refractors and reflectors. Indeed, under suitable conditions on the target and the measure σ, we prove local $C^{1,\alpha}$ estimates. More precisely, if u is a parallel refractor in Ω, the target Σ satisfies the local condition (3.3) from $(x^\star, u(x^\star))$, and the measure σ satisfies a local condition at that point, condition (6.1), then $u \in C^{1,\alpha}$ in a neighborhood of the point $x^\star$.

Throughout the paper we assume that media II is denser than media I, that is, $n_1 < n_2$. When $n_1 > n_2$, the geometry of the refractor changes. One needs to use hyperboloids of revolution instead of ellipsoids as indicated in [GH09].

The plan of the paper is as follows. Section 2 contains results concerning estimates of ellipsoids of revolution, and Subsection 2.5 contains basic assumptions on the target. Section 3 contains assumptions on the target modeled on the conditions introduced by Loeper [Loe09, Proposition 5.1]. Indeed, we assume the target satisfies the local condition (3.2). We also introduce the differential condition (3.3), similar in form to condition (A3) of Ma, Trudinger and Wang [MTW05], and show in Theorem 3.2 that (3.3) and (3.2) are equivalent. In Section 4, using the geometry of the ellipsoids, we prove that if an ellipsoid supports a parallel refractor locally, then it supports the refractor globally provided the target satisfies the condition (AW) given in (4.2). The main result in this section is Proposition 4.2 used later in the proof of Theorem 5.3. Section 5 contains the main results, Lemma 5.2 and Theorem 5.3, and also Proposition 5.4 used later for the application of these results to show regularity of parallel refractors constructed in [GT13], Corollary 6.3. Section 6 contains Hölder estimates of gradients of refractors under the assumptions (6.1) and (6.2) on the target Σ and the measure σ on Σ. Section 7 contains examples of targets verifying the assumptions, see condition (7.3). Up to this point in the paper, refractors are defined with ellipsoids supporting the refractor from above. Refractors can also be defined with ellipsoids supporting from below, and in Section 8 we obtain the same regularity results for refractors with this definition. In Section 9 we obtain similar regularity results for the *near field parallel reflector* problem. In this case, the proofs are simpler because the differential condition (9.4) implies the global inequality (9.3).

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2. Preliminaries

2.1. **Refraction.** We briefly review the process of refraction. Our setting is $\mathbb{R}^{n+1}$. Points will be denoted by $X = (x, x_{n+1})$. We consider parallel rays traveling in the unit direction e_{n+1}. Let T be a hyperplane with upper unit normal N and $X \in T$. We assume that the region below T has refractive index n_1 and the region above

T has refractive index n_2 and $\kappa := \dfrac{n_1}{n_2} < 1$, e.g., air to glass. In such case, by Snell's law of refraction, a ray coming from below in direction e_{n+1} that hits T at X is refracted in the unit direction

$$\Lambda = \kappa\, e_{n+1} + \delta N, \quad \text{with} \quad \delta = -\kappa\, e_{n+1} \cdot N + \sqrt{1 + \kappa^2\, ((e_{n+1} \cdot N)^2 - 1)},$$
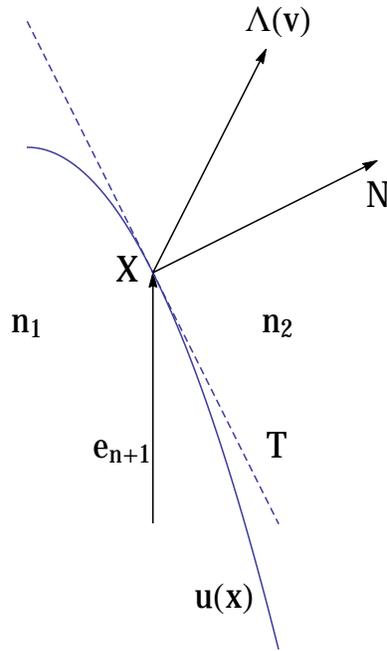
where $\delta > 0$ since $\kappa < 1$.

FIGURE 1. Snell's law, $n_1 < n_2$

In particular, if $v \in \mathbb{R}^n$ and the hyperplane T is so that the unit upper normal $N = \dfrac{(-v, 1)}{\sqrt{1 + |v|^2}}$, then the refracted unit direction is

$$(2.1) \qquad \Lambda(v) := \left(-\frac{\delta}{\sqrt{1 + |v|^2}}\, v, \; \frac{\delta}{\sqrt{1 + |v|^2}} + \kappa \right) := (-Q(v)v, Q(v) + \kappa),$$

with $Q(v)^2 |v|^2 + (Q(v) + \kappa)^2 = 1$ and $Q > 0^*$. The refracted ray is $X + s\Lambda$, for $s > 0$. Here we have in mind that T is the tangent plane to a refractor u at $(x, u(x))$ and so $v = Du(x)$, see Figure 1.

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*We have in this case, $\delta = \dfrac{-\kappa + \sqrt{1 + (1 - \kappa^2)|v|^2}}{\sqrt{1 + |v|^2}}$.

2.2. **Ellipsoids.** Given $b > 0$ and $Y = (y, y_{n+1})$, consider

$$E(Y, b) = \{X : |X - Y| + k(x_{n+1} - y_{n+1}) = b\}.$$

$E(Y, b)$ is an ellipsoid of revolution with upper focus (y, y_{n+1}), lower focus $\left(y, y_{n+1} - \dfrac{2\kappa b}{1 - \kappa^2}\right)$, major semi axis $b/(1 - \kappa^2)$, minor semi axis $b/\sqrt{1 - \kappa^2}$, and eccentricity $\kappa = n_1/n_2 < 1$. Therefore the projection of $E(Y, b)$ over $\mathbb{R}^n$ is the ball $B_{b/\sqrt{1-\kappa^2}}(y)$. We define the lower and upper parts of $E(Y, b)$ by

$$E^-(Y, b) = \left\{X \in E(Y, b) : x_{n+1} \leq y_{n+1} - \frac{\kappa b}{1 - \kappa^2}\right\},$$

$$E^+(Y, b) = \left\{X \in E(Y, b) : x_{n+1} \geq y_{n+1} - \frac{\kappa b}{1 - \kappa^2}\right\},$$

respectively. We can regard $E^-(Y, b)$ as the graph of the function

$$(2.2) \qquad \phi(x) = y_{n+1} - \frac{\kappa b}{1 - k^2} - \sqrt{\frac{b^2}{(1 - \kappa^2)^2} - \frac{|x - y|^2}{1 - k^2}},$$

for $x \in B_{b/\sqrt{1-\kappa^2}}(y)$. If $X = (x, \phi(x))$, $x_{n+1} = \phi(x)$, then $D_x\phi(x) = \dfrac{x - y}{y_{n+1} - x_{n+1} - k|X - Y|}$.

Suppose that the interior of $E(Y, b)$ has refractive index n_2 and the exterior refractive index n_1. Since $\kappa < 1$, we get from [GH09, Section 2.2] and reversibility of optical paths that, each ray with direction e_{n+1} that strikes from below the graph of ϕ at the point $X = (x, \phi(x))$ is refracted into a ray passing through the upper focus Y, and therefore lies along the ray $X + s\Lambda(v)$ with $v = D\phi(x)$, with $\Lambda(v)$ given by (2.1). Viceversa, if $X = (x, \phi(x))$ and the focus Y can be written as $Y = X + s\Lambda(v)$ for some $s > 0$ and $v \in \mathbb{R}^n$, then $v = D\phi(x)$; see Figure 2.
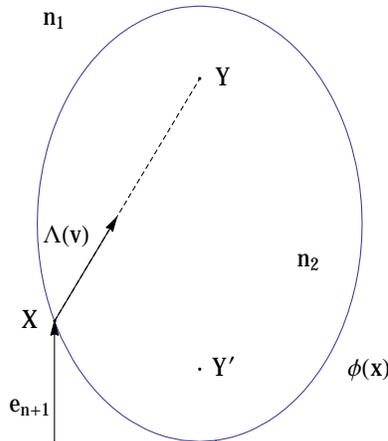

FIGURE 2. Refraction property of the ellipsoid with eccentricity κ; Y' denotes the lower focus.

Given $X, Y \in \mathbb{R}^{n+1}$, let us define

(2.3)
$$c(X, Y) = |X - Y| + k(x_{n+1} - y_{n+1}),$$

and we have

(2.4)
$$(1 - \kappa)|X - Y| \leq c(X, Y) \leq (1 + \kappa)|X - Y|.$$

Given $X_0, Y \in \mathbb{R}^{n+1}$ with $X_0 \neq Y$, then $E(Y, c(X_0, Y))$ is the unique ellipsoid passing through X_0 and having upper focus at Y.

We have that

(2.5)
$$X \in E^-(Y, b) \text{ iff } y_{n+1} - x_{n+1} - \kappa|X - Y| \geq 0.$$

and

$$X \in E^+(Y, b) \text{ iff } y_{n+1} - x_{n+1} - \kappa|X - Y| \leq 0.$$

2.3. **Set up.** We fix a bounded domain $\Omega \subset \mathbb{R}^n$ and a cylinder set $\mathscr{C}_\Omega = \Omega \times (0, M)$; points in $\mathscr{C}_\Omega$ are denoted by the letter X.

Given $0 < \delta < 1$ we define the region

(2.6)
$$\mathcal{T} = \left\{ Y \in \mathbb{R}^{n+1} : \text{for each } X_0 \in \mathscr{C}_\Omega \text{ we have } X_0 \in E^-(Y, c(X_0, Y)) \text{ and } \Omega \subset B_{\delta c(X_0, Y)/\sqrt{1-\kappa^2}}(y) \right\}.$$

The region $\mathcal{T}$ is open and unbounded. Notice that if $(y, y_{n+1}) \in \mathcal{T}$, then $(y, y'_{n+1}) \in \mathcal{T}$, for all $y'_{n+1} \geq y_{n+1}$. Also, if $Y \in \mathcal{T}$ and $X_0 \in \mathscr{C}_\Omega$, then $c(X_0, Y) \geq c_n |\Omega|^{1/n} \sqrt{1 - \kappa^2}/\delta$ and so from (2.4) we get

(2.7)
$$\text{dist}(\mathcal{T}, \mathscr{C}_\Omega) \geq C(\delta, \kappa, |\Omega|, n) > 0.$$

We have that the function

(2.8)
$$\phi(x, Y, X_0) = y_{n+1} - \frac{k \, c(X_0, Y)}{1 - k^2} - \sqrt{\frac{c(X_0, Y)^2}{(1 - k^2)^2} - \frac{|x - y|^2}{1 - k^2}},$$

$X_0 = (x_0, x_{0_{n+1}})$, defines the lower part of the ellipsoid $E(Y, c(X_0, Y))$ for $x \in B_{c(X_0, Y)/\sqrt{1-\kappa^2}}(y)$; recall that Y is the upper focus of E and X_0 is point on E. We then have for $X_0 \in \mathscr{C}_\Omega$, $Y \in \mathcal{T}$ and $X \in E^-(Y, c(X_0, Y))$, $X = (x, x_{n+1})$, with $x \in \Omega$ that

(2.9)
$$y_{n+1} - x_{n+1} - \kappa|X - Y| \geq \beta \, c(X, Y),$$

with $\beta = \sqrt{1 - \delta^2}$. From the definition of $\mathcal{T}$, the tangents to the lower part of the ellipsoids considered are never vertical. Since the target Σ will be placed inside $\mathcal{T}$, this implies uniform Lipschitz bounds for supporting ellipsoids to the refractor.

2.4. **Estimates for the derivatives of ϕ.** We show that the function $\phi(x, Y, X_0)$ is differentiable of any order in all variables for $x \in \Omega$, $Y \in \mathcal{T}$, and $X_0 \in \mathscr{C}_\Omega$. Let us calculate first $\dfrac{\partial \phi}{\partial x_i}$, $1 \leq i \leq n$. We have

$$\frac{\partial \phi}{\partial x_i}(x, Y, X_0) = \frac{1}{1 - \kappa^2} \frac{x_i - y_i}{\sqrt{\dfrac{c(X_0, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}}}.$$

We have $c(X, Y) = |X - Y| + \kappa (x_{n+1} - y_{n+1}) = c(X_0, Y)$, with $X = (x, \phi(x, Y, X_0))$, $x \in \Omega$, then

$$y_{n+1} - x_{n+1} - \kappa |X - Y| = (1 - \kappa^2)(y_{n+1} - x_{n+1}) - \kappa c(X_0, Y)$$

$$(2.10) \qquad = (1 - \kappa^2) \sqrt{\frac{c(X_0, Y)^2}{(1 - \kappa^2)^2} - \frac{|x - y|^2}{1 - k^2}} \geq \sqrt{1 - \delta^2} \, c(X_0, Y),$$

for all $x \in \Omega$ from (2.9). Therefore $\left| \dfrac{\partial \phi}{\partial x_i}(x, Y, X_0) \right| \leq \dfrac{1}{\sqrt{1 - \delta^2}} \dfrac{|x_i - y_i|}{c(X_0, Y)}$. Let us now calculate $\dfrac{\partial^2 \phi}{\partial x_i \partial x_j}(x, Y, X_0)$, $1 \leq i, j \leq n$. We have

$$(2.11) \quad \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x, Y, X_0) = \frac{\delta_{ij} \sqrt{\dfrac{c(X_0, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}} + \dfrac{x_i - y_i}{1 - \kappa^2} \dfrac{x_j - y_j}{\sqrt{\dfrac{c(X_0, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}}}}{(1 - \kappa^2)\left(\sqrt{\dfrac{c(X_0, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}} \right)^2}.$$

So

$$\left| \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x, Y, X_0) \right| \leq \frac{1}{\sqrt{1 - \delta^2} \, c(X_0, Y)} + \frac{(1 - \kappa^2)|x - y|^2}{\left(\sqrt{1 - \delta^2} \, c(X_0, Y) \right)^3} \leq \frac{C(\kappa, \delta)}{c(X_0, Y)} \leq C$$

for $x \in \Omega$, $Y \in \mathcal{T}$ and $X_0 \in \mathscr{C}_\Omega$ by (2.7). Next we estimate the derivatives $\dfrac{\partial^2 \phi}{\partial x_i \partial y_j}(x, Y, X_0)$, $1 \leq i \leq n$, $1 \leq j \leq n + 1$. We have

$$\frac{\partial^2 \phi}{\partial x_i \partial y_j}(x, Y, X_0) = \frac{1}{1 - \kappa^2} \frac{-\delta_{ij} \sqrt{\dfrac{c(X_0, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}} - (x_i - y_i)\left(\dfrac{c(X_0, Y) \dfrac{\partial c}{\partial y_j}(X_0, Y)}{(1 - \kappa^2)^2} - \dfrac{x_j - y_j}{1 - \kappa^2}}{\sqrt{\dfrac{c(X_0, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}}} \right)}{\left(\sqrt{\dfrac{c(X_0, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}} \right)^2}.$$

Now $\dfrac{\partial c(X, Y)}{\partial y_j} = -\dfrac{x_j - y_j}{|X - Y|} - \kappa\, \delta_{j(n+1)}$. Hence

$$\left| \frac{\partial^2 \phi}{\partial x_i \partial y_j}(x, Y, X_0) \right| \leq \frac{1}{\sqrt{1 - \delta^2}\, c(X_0, Y)} + \frac{|x - y| \left(\dfrac{(1 + \kappa)\, c(X_0, Y)}{1 - k^2} + |x - y| \right)}{\left(\sqrt{1 - \delta^2}\, c(X_0, Y) \right)^2}$$

$$\leq C(\kappa, \delta) \left(\frac{1}{c(X_0, Y)} + 1 \right) \leq C$$

for $x \in \Omega$, $Y \in \mathcal{T}$ and $X_0 \in \mathscr{C}_\Omega$ by (2.7).

Therefore, we obtain $\left| \dfrac{\partial^2 \phi}{\partial x_i \partial y_j}(x, Y, X_0) \right| \leq C(\delta, K, M, |\Omega|, n)$ for all $x \in \Omega$, $X_0 \in \mathscr{C}_\Omega$, and $Y \in K$. Moreover, these estimates also hold for any X_0 such that there exists $\bar{X} \in C_\Omega$ with $c(X_0, Y) = c(\bar{X}, Y)$.

Continuing in this manner, we get that the function $\phi(x, Y, X_0)$ is C^∞ in all its $3n + 2$ variables on $\Omega \times \mathcal{T} \times \mathscr{C}_\Omega$ and $|D^\alpha \phi(x, Y, X_0)| \leq C$, for any multi-index $\alpha = (\alpha_1, ..., \alpha_{3n+2})$ with a constant C depending only on $\delta, \kappa, M, |\alpha|$ and $|\Omega|$.

2.5. **Assumptions on the target Σ.** We will frequently use the following fact:

(2.12) if the upper focus Y of the ellipsoid defined by $\phi(x, Y, X_0)$ in (2.8) satisfies
$$Y = X_0 + s\Lambda(v) \text{ for some } s > 0 \text{ and } v \in \mathbb{R}^n, \text{ then } D_x \phi(x_0, Y, X_0) = v.$$

Here $\Lambda(v)$ is the unit vector given by (2.1).

We assume the convex hull of the target Σ is contained in $\mathcal{T}$, where $\mathcal{T}$ is given by (2.6). For each fixed $X \in \mathscr{C}_\Omega$, we assume each $Y \in \Sigma$ can be represented parametrically with respect to X by the equation $Y = X + s_X(\Lambda)\Lambda$, with $|\Lambda| = 1$, where the function s_X varies with the point X, and $s_X(\Lambda)$ is Lipschitz in Λ for each $X \in \mathscr{C}_\Omega$. In the sequel, C denotes a structural constant, that is, depending on the constants $\delta, \kappa, M, \Omega$; if we write $C(X_0)$ we mean the constant depends also on the point X_0.

Lemma 2.1. Let X_0 be an arbitrary point in $\mathscr{C}_\Omega$, such that there are $\bar{Y}, \hat{Y} \in \Sigma$, $\bar{v}, \hat{v} \in \mathbb{R}^n$, and $\bar{s}, \hat{s} > 0$ such that $\bar{Y} = X_0 + \bar{s}\,\Lambda(\bar{v})$, and $\hat{Y} = X_0 + \hat{s}\,\Lambda(\hat{v})$, with Λ defined by (2.1). Then there exists a constant $C = C(X_0) \geq 1$

(2.13) $$\frac{1}{C}|\bar{Y} - \hat{Y}| \leq |\bar{v} - \hat{v}| \leq C|\bar{Y} - \hat{Y}|.$$

This implies that if $\bar{Y} \neq \hat{Y}$, are both in Σ, then the points $\bar{Y}, \hat{Y}, X_0$ cannot be aligned, in other words, from each point X_0 one can see at most a point in Σ on any straight line from X_0.

Proof. Since the function $s_{X_0}(\Lambda)$ is Lipschitz in Λ, then the left inequality in (2.13) follows. Indeed,

$$
\begin{aligned}
|\bar{Y} - \hat{Y}| &= \left| s_{X_0}(\Lambda(\bar{v})) \, \Lambda(\bar{v}) - s_{X_0}(\Lambda(\hat{v})) \, \Lambda(\hat{v}) \right| \\
&\leq \left| s_{X_0}(\Lambda(\bar{v})) \, \Lambda(\bar{v}) - s_{X_0}(\Lambda(\bar{v})) \, \Lambda(\hat{v}) \right| + \left| s_{X_0}(\Lambda(\bar{v})) \, \Lambda(\hat{v}) - s_{X_0}(\Lambda(\hat{v})) \, \Lambda(\hat{v}) \right| \\
&\leq \left| s_{X_0}(\Lambda(\bar{v})) \right| \, |\Lambda(\bar{v}) - \Lambda(\hat{v})| + \left| s_{X_0}(\Lambda(\bar{v})) - s_{X_0}(\Lambda(\hat{v})) \right|.
\end{aligned}
$$

To show the right inequality in (2.13), from (2.12) we can write that ($x_0 \in \Omega$)

$$
\begin{aligned}
|\bar{v}_i - \hat{v}_i| &= \left| D_i\phi(x_0, \bar{Y}, X_0) - D_i\phi(x_0, \hat{Y}, X_0) \right| \\
&= \left| \frac{x_{0_i} - \bar{y}_i}{\bar{y}_{n+1} - x_{0_{n+1}} - k|X_0 - \bar{Y}|} - \frac{x_{0_i} - \hat{y}_i}{\hat{y}_{n+1} - x_{0_{n+1}} - k|X_0 - \hat{Y}|} \right|,
\end{aligned}
$$

$1 \leq i \leq n$.

We write

$$
D_i\phi(x_0, \bar{Y}, X_0) - D_i\phi(x_0, \hat{Y}, X_0) = D_Y(D_i\phi)(x_0, \tilde{Y}, X_0) \cdot (\bar{Y} - \hat{Y})
$$

with $\tilde{Y}$ on the segment $[\bar{Y}, \hat{Y}]$ joining $\bar{Y}$ and $\hat{Y}$. This segment is contained in the convex hull of Σ, and if this convex hull is bounded and contained in $\mathcal{T}$, then the desired estimate follows from the estimates for the derivatives of ϕ proved in Subsection (2.4). $\qquad \square$

The following two lemmas are a consequence of the inequalities for the derivatives of ϕ.

Lemma 2.2. *Let $\bar{X} \in \mathscr{C}_\Omega$, $Y \in \mathcal{T}$ and $x_0 \in \Omega$. Let $x_{0_{n+1}} = \phi(x_0, Y, \bar{X})$ and set $X_0 = (x_0, x_{0_{n+1}})$. Assume that $X_0^\star = (x_0, x_{0_{n+1}} - h) \in \mathscr{C}_\Omega$. Then there is a constant $C > 0$ such that*

$$
0 \leq \phi(x, Y, X_0) - \phi(x, Y, X_0^\star) \leq C h,
$$

for all $x \in \Omega$

Proof. We have $0 \leq \phi(x, Y, X_0) - \phi(x, Y, X_0^\star) = \dfrac{\partial \phi}{\partial x_{0_{n+1}}}(x, Y, \bar{X}_0) h$ for some $\bar{X}_0 \in [X_0^\star, X_0]$. Since $c(\bar{X}_0, Y) = c(\hat{X}, Y)$ for some $\hat{X} \in C_M$, it follows from the estimates in Subsection (2.4) that $\left| \dfrac{\partial \phi}{\partial x_{0_{n+1}}}(x, Y, \bar{X}_0) \right| \leq C$ for all $x \in \Omega$, where C depends on δ, κ, M and $|\Omega|$. $\qquad \square$

Lemma 2.3. *There exists a constant $C > 0$ such that for all $Y, \bar{Y} \in \mathcal{T}$ with the straight segment $[Y, \bar{Y}] \subset \mathcal{T}$, and $X_0 \in \mathscr{C}_\Omega$, we have*

$$
|\phi(x, Y, X_0) - \phi(x, \bar{Y}, X_0)| \leq C |x - x_0| |Y - \bar{Y}|
$$

for all $x \in \Omega$, which is assumed convex.

Proof. Since $\phi(x_0, Y, X_0) = x_{0_{n+1}}$ for all $Y \in \Sigma$, it follows that $\dfrac{\partial \phi(x_0, Y, X_0)}{\partial y_j} = 0$. We then write

$$\phi(x, Y, X_0) - \phi(x, \bar{Y}, X_0) = \sum_{j=1}^{n+1} \frac{\partial \phi(x, \xi, X_0)}{\partial y_j}(Y_j - \bar{Y}_j), \quad \text{for some } \xi \in [Y, \bar{Y}]$$

$$= \sum_{j=1}^{n+1} \left(\frac{\partial \phi(x, \xi, X_0)}{\partial y_j} - \frac{\partial \phi(x_0, \xi, X_0)}{\partial y_j} \right)(Y_j - \bar{Y}_j)$$

$$= \sum_{j=1}^{n+1} \sum_{i=1}^{n} \frac{\partial^2 \phi(\zeta_i, \xi, X_0)}{\partial x_i \partial y_j}(x_i - x_{0_i})(Y_j - \bar{Y}_j)$$

for some $\zeta_i \in [x_0, x]$. Then the lemma follows from the estimates of the derivatives of ϕ proved in Subsection (2.4). □

3. REGULARITY HYPOTHESIS ON THE TARGET SET Σ

Given $\bar{Y}, \hat{Y} \in \mathcal{T}$ and $X_0 \in \mathscr{C}_\Omega$, let $\bar{v} = D_x \phi(x_0, \bar{Y}, X_0)$, $\hat{v} = D_x \phi(x_0, \hat{Y}, X_0)$, and $v(\lambda) = (1 - \lambda)\bar{v} + \lambda\hat{v}$, with $\lambda \in [0, 1]$. We consider the set of points

$$C(X_0, \bar{Y}, \hat{Y}) = \{X_0 + s\,\Lambda(v(\lambda)) : s > 0, \lambda \in [0, 1]\},$$

where $\Lambda(v(\lambda))$ is defined by (2.1). This set is a two dimensional wedge-shaped surface, in general non-planar, containing all rays having directions $\Lambda(v(\lambda))$, $0 \leq \lambda \leq 1$, emanating from X_0. The curve describing the tip of the vector $\Lambda(v(\lambda))$, pictured in Figure 3, is not contained in the plane generated by the rays with directions $\Lambda(\bar{v})$ and $\Lambda(\hat{v})$. If $Y \in C(X_0, \bar{Y}, \hat{Y})$, then $X_0 \in E^-(Y, c(X_0, Y))$. Hence, if $Y(\lambda) = X_0 + s\,\Lambda(v(\lambda)) \in C(X_0, \bar{Y}, \hat{Y})$, then by (2.12) $D_x \phi(x_0, Y(\lambda), X_0) = v(\lambda)$.

For $\bar{Y}, \hat{Y} \in \mathcal{T}$, and $X_0 \in C_\Omega$, let us define

(3.1) $$[\bar{Y}, \hat{Y}]_{X_0} := \Sigma \cap C(X_0, \bar{Y}, \hat{Y}).$$

From the parametrization of the target explained in Subsection 2.5, and consequently from (2.13), each ray $X_0 + s\,\Lambda(v(\lambda)) \in C(X_0, \bar{Y}, \hat{Y})$ intersects Σ in at most one point for each λ. Points in $[\bar{Y}, \hat{Y}]_{X_0}$ have the form

$$Y(\lambda) = X_0 + s_{X_0}(\Lambda(v(\lambda)))\,\Lambda(v(\lambda)), \qquad 0 \leq \lambda \leq 1.$$

We will introduce the following local definition on the target Σ.

Definition 3.1. *If $X_0 \in \mathscr{C}_\Omega$ we say that the target Σ is regular from X_0 if there exists a neighborhood U_{X_0} and positive constants C_1, C_2, depending on U_{X_0}, such that for all $\bar{Y}, \hat{Y} \in \Sigma$ and $Z = (z, z_{n+1}) \in U_{X_0}$ we have*

(3.2) $$\phi(x, Y_Z(\lambda), Z) \geq \min\left\{\phi(x, \bar{Y}, Z), \phi(x, \hat{Y}, Z)\right\} + C_1 |\bar{Y} - \hat{Y}|^2 |x - z|^2$$

for all $x \in \Omega$ with $|x - z| \leq C_2$, $1/4 \leq \lambda \leq 3/4$, and $Y_Z(\lambda) = Z + s_Z(\Lambda(v(\lambda)))\,\Lambda(v(\lambda))$. Here $\bar{v} = D_x \phi(z, \bar{Y}, Z)$, $\hat{v} = D_x \phi(z, \hat{Y}, Z)$, and $v(\lambda) = (1 - \lambda)\bar{v} + \lambda\hat{v}$.
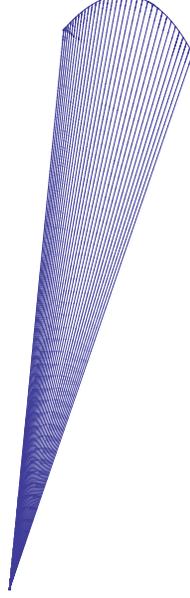
FIGURE 3. The edges of the wedge are $\Lambda(\bar{v})$ and $\Lambda(\hat{v})$, with $\bar{v} = (1, 0)$, $\hat{v} = (-2, 1)$, $n_1 = 1$, $n_2 = 3/2$

We introduce below a local differential condition (3.3), that resembles condition (A3) of Ma, Trudinger and Wang [MTW05] for cost functions, see Remark 3.3. Assuming that the function s_Z in the parametrization of the target is C^2,[†] and the set $[\bar{Y}, \hat{Y}]_Z$ given in (3.1) is a curve for each $Z \in U_{X_0}$, we prove in Theorem (3.3) that (3.2) and (3.3) are equivalent. The quantity on the left hand side of (3.3) depends on the target Σ. We emphasize that in this condition the constant depends on the point X_0 under consideration. The local Theorem 3.2 resembles a result of Loeper, [Loe09, Proposition 5.1].

Theorem 3.2. *The target Σ is regular from $X_0 \in \mathscr{C}_\Omega$ in the sense of Definition 3.1 if and only if there exists a neighborhood U_{X_0} and a positive constant $C(X_0)$ such that for all $Y_0 \in \Sigma$ and for all $Z \in U_{X_0}$ and for all vectors ξ and η such that $\xi \perp \eta$ we have*

(3.3)
$$\frac{d^2}{d\epsilon^2} \left\langle D_x^2 \phi(z, Y_\epsilon, Z)\eta, \eta \right\rangle \Big|_{\epsilon=0} \leq -C(X_0)|\xi|^2|\eta|^2,$$

where $Y_\epsilon = Z + s_Z(\Lambda(v + \epsilon\xi))\Lambda(v + \epsilon\xi)$ and $v = D\phi(z, Y_0, Z)$.

Proof. Let $\bar{Y}, \hat{Y} \in \Sigma$, $X_0 \in \mathscr{C}_\Omega$, and $Z \in U_{X_0}$. Let $v(\lambda) = (1 - \lambda)D\phi(z, \bar{Y}, Z) + \lambda D\phi(z, \hat{Y}, Z)$, $\xi = D\phi(z, \hat{Y}, Z) - D\phi(z, \bar{Y}, Z) = \hat{v} - \bar{v}$, and $Y(\lambda) = Z + s_Z(\Lambda(v(\lambda)))\Lambda(v(\lambda))$.

[†]The assumption $s_Z \in C^2$ is need to formulate (3.2) and used to prove that (3.2) and (3.2) are equivalent. However, to formulate (3.2) and for the set up in Section 2.5 it is enough to assume that s_Z is Lipschitz. Also Lemmas 5.1 and 5.2 hold true for s_Z Lipschitz because only (3.2) is used.

Notice that for η perpendicular to ξ we have

$$(3.4) \qquad \frac{d^2}{d\lambda^2} \left\langle D_x^2\phi(z, Y(\lambda), Z)\eta, \eta \right\rangle \leq -C(X_0)|\xi|^2|\eta|^2,$$

for all $\lambda \in [0,1]$. Indeed, fix $\lambda \in (0,1)$ and $\xi, \eta \in \mathbb{R}^n$, $\xi \perp \eta$. Applying (3.3) with $Y_0 \rightsquigarrow Y(\lambda)$, $v_0 \rightsquigarrow v(\lambda) := \bar{v} + \lambda\xi$ (notice that $\bar{v} + \lambda\xi = D_x\phi(z, Y(\lambda), Z)$ by (2.12)), and $Y_\epsilon \rightsquigarrow Z + s_Z(\Lambda(v(\lambda) + \epsilon\xi))\Lambda(v(\lambda) + \epsilon\xi)$, we obtain $\frac{d^2}{d\epsilon^2}\langle D^2\phi(z, Y_\epsilon, Z)\eta, \eta\rangle|_{\epsilon=0} \leq -C(X_0)|\xi|^2|\eta|^2$. Since $Y_\epsilon = Y(\epsilon + \lambda)$, (3.4) follows.

For $x \in \mathbb{R}^n$, let x' denote the orthogonal projection of x on the hyperplane through z with normal ξ, so $x' - z$ is perpendicular to ξ. We let $\eta = x' - z$.

We will first show that there exist positive constants $C_1(X_0)$ and $C_2(X_0)$ such that

$$(3.5) \qquad \begin{aligned} &\left\langle D_x^2\phi(z, Y(\lambda), Z)(x - z), x - z \right\rangle \\ &\geq \left\langle ((1 - \lambda)D_x^2\phi(z, \bar{Y}, Z) + \lambda D_x^2\phi(z, \hat{Y}, Z))(x - z), x - z \right\rangle \\ &\qquad + \lambda(1 - \lambda)|\xi|^2(C_1(X_0)|x - z|^2 - C_2(X_0)|x - x'|^2), \end{aligned}$$

for all $x \in \Omega$. In fact, fix $x \in \Omega$ and let $f(\lambda) = -\langle D_x^2\phi(z, Y(\lambda), Z)(x' - z), x' - z\rangle$. From (3.4), we have $f''(\lambda) \geq C(X_0)|x' - z|^2|\xi|^2$. We claim that $f(\lambda) \leq (1 - \lambda)f(0) + \lambda f(1) - C(X_0)|x' - z|^2|\xi|^2\lambda(1 - \lambda)$ for all $\lambda \in [0,1]$. To prove this claim, fix $\bar{\lambda} \in (0,1)$. By Taylor's theorem, we have that $f(\lambda) \geq f(\bar{\lambda}) + f'(\bar{\lambda})(\lambda - \bar{\lambda}) + C(X_0)/2\,|x' - z|^2|\xi|^2(\lambda - \bar{\lambda})^2$ for all $\lambda \in [0,1]$. Applying this inequality, first for $\lambda = 0$ and then for $\lambda = 1$, multiplying the first by $1 - \bar{\lambda}$ and the second by $\bar{\lambda}$, and then adding yields $(1 - \bar{\lambda})f(0) + \bar{\lambda}f(1) \geq f(\bar{\lambda}) + C(X_0)|x' - z|^2|\xi|^2\bar{\lambda}(1 - \bar{\lambda})$ which proves the claim.

Let $g(\lambda) = \langle D_x^2\phi(z, Y(\lambda), Z)(x' - z), x' - z\rangle - \langle D_x^2\phi(z, Y(\lambda), Z)(x - z), x - z\rangle$. Since s_Z is C^2, we have that $\left|\dfrac{d^2\, D_{ij}\phi(z, Y(\lambda), Z)}{d\lambda^2}\right| \leq C\,|\xi|^2$ and therefore $|g''(\lambda)| \leq C|\xi|^2|x - x'||x - z|$, since $|x' - z| \leq |x - z|$. Hence $g(\lambda) \leq (1 - \lambda)g(0) + \lambda g(1) + C\lambda(1 - \lambda)|\xi|^2|x - x'||x - z|$, for $0 \leq \lambda \leq 1$. Therefore we get

$$\begin{aligned} &-\left\langle D_x^2\phi(z, Y(\lambda), Z)(x - z), x - z \right\rangle = f(\lambda) + g(\lambda) \\ &\leq (1 - \lambda)(f(0) + g(0)) + \lambda(f(1) + g(1)) + \lambda(1 - \lambda)|\xi|^2\left(C\,|x - x'||x - z| - C(X_0)\,|x' - z|^2\right) \\ &\leq -\left\langle \left((1 - \lambda)D_x^2\phi(z, \bar{Y}, Z) + \lambda D_x^2\phi(z, \hat{Y}, Z)\right)(x - z), x - z \right\rangle \\ &\qquad + \lambda(1 - \lambda)|\xi|^2(C_1(X_0)|x - x'|^2 - C_2(X_0)|x - z|^2), \end{aligned}$$

since $|x' - z|^2 = |x - z|^2 - |x - x'|^2$. This finishes the proof of (3.5).

We now prove (3.2). Let $\lambda' \in [0, 1]$ to be chosen later, and let $\lambda \in [1/4, 3/4]$. We have

$$\min\{\phi(x, \bar{Y}, Z), \phi(x, \hat{Y}, Z)\} - \phi(x, Y(\lambda), Z)$$

$$\leq (1 - \lambda')\phi(x, \bar{Y}, Z) + \lambda'\phi(x, \hat{Y}, Z) - \phi(x, Y(\lambda), Z)$$

$$= \left\langle ((1 - \lambda')D_x\phi(z, \bar{Y}, Z) + \lambda'D_x\phi(z, \hat{Y}, Z)) - D_x\phi(z, Y(\lambda), Z), x - z \right\rangle$$

$$+ \frac{1}{2} \left\langle ((1 - \lambda')D_x^2\phi(z, \bar{Y}, Z) + \lambda'D_x^2\phi(z, \hat{Y}, Z) - D_x^2\phi(z, Y(\lambda), Z))(x - z), x - z \right\rangle$$

$$+ \frac{1}{6} \sum_{i,j,k=1}^{n} \left((1 - \lambda')D_{i,j,k}\phi(\tau, \bar{Y}, Z) + \lambda'D_{i,j,k}\phi(\tau, \hat{Y}, Z) - D_{i,j,k}\phi(\tau, Y(\lambda), Z) \right)(x_i - z_i)(x_j - z_j)(x_k - z_k)$$

for some $\tau \in [z, x]$. Using (3.5) in λ' we get that the last sum is less than or equal to

$$\left\langle (D\phi(z, Y(\lambda'), Z) - D\phi(z, Y(\lambda), Z)), x - z \right\rangle$$

$$+ \frac{1}{2} \left\langle (D_x^2\phi(z, Y(\lambda'), Z) - D_x^2\phi(z, Y(\lambda), Z))(x - z), x - z \right\rangle$$

$$- C_2(X_0)\lambda'(1 - \lambda')|\xi|^2|x - z|^2 + C_1(X_0)\lambda'(1 - \lambda')|\xi|^2|x - x'|^2$$

$$+ \frac{1}{6} \sum_{i,j,k=1}^{n} \left((1 - \lambda')D_{i,j,k}\phi(\tau, \bar{Y}, Z) + \lambda'D_{i,j,k}\phi(\tau, \hat{Y}, Z) - D_{i,j,k}\phi(\tau, Y(\lambda), Z) \right)(x_i - z_i)(x_j - z_j)(x_k - z_k).$$

Notice that $\langle (D\phi(z, Y(\lambda'), Z) - D\phi(z, Y(\lambda), Z)), x - z \rangle = (\lambda' - \lambda)\langle \xi, x - z \rangle = (\lambda' - \lambda)\langle \xi, x - x' \rangle$ by orthogonality. Hence we get that

$$\min\{\phi(x, \bar{Y}, Z), \phi(x, \hat{Y}, Z)\} - \phi(x, Y(\lambda), Z)$$

$$\leq -C_2(X_0)\lambda(1 - \lambda)|\xi|^2|x - z|^2 + (\lambda' - \lambda)\langle \xi, x - x' \rangle$$

$$+ C_1(X_0)\lambda(1 - \lambda)|\xi|^2|x - x'|^2 + C_2(X_0)|\xi|^2|x - z|^2(\lambda(1 - \lambda) - \lambda'(1 - \lambda'))$$

$$+ C_1(X_0)|\xi|^2|x - x'|^2(\lambda'(1 - \lambda') - \lambda(1 - \lambda))$$

$$+ \left\langle (D_x^2\phi(z, Y(\lambda'), Z) - D_x^2\phi(z, Y(\lambda), Z))(x - z), x - z \right\rangle$$

$$+ \sum_{i,j,k=1}^{n} ((1 - \lambda')D_{i,j,k}\phi(\tau, \bar{Y}, Z) + \lambda'D_{i,j,k}\phi(\tau, \hat{Y}, Z) - D_{i,j,k}\phi(\tau, Y(\lambda), Z))(x_i - z_i)(x_j - z_j)(x_k - z_k).$$

We will now choose λ' and estimate each of the terms. We have that

$$A := (\lambda' - \lambda)\langle \xi, x - x' \rangle + C_1(X_0)\lambda(1 - \lambda)|\xi|^2|x - x'|^2$$

$$= \left\langle x - x', (\lambda' - \lambda)\xi + C_1(X_0)\lambda(1 - \lambda)|\xi|^2(x - x') \right\rangle$$

$$= \left\langle x - x', |\xi| \left((\lambda' - \lambda)\frac{\xi}{|\xi|} + C_1(X_0)\lambda(1 - \lambda)|\xi|(x - x') \right) \right\rangle.$$

Notice that either $\dfrac{x - x'}{|x - x'|} = \dfrac{\xi}{|\xi|}$ or $\dfrac{x - x'}{|x - x'|} = -\dfrac{\xi}{|\xi|}$. If the $+$ sign holds, we choose $\lambda' = \lambda - C_1(X_0)\lambda(1 - \lambda)|\xi||x - x'|$, and if the $-$ sign holds, we choose $\lambda' = \lambda +$

$C_1(X_0)\lambda(1 - \lambda)|\xi||x - x'|$, so in either case $A = 0$. Notice that since $1/4 \leq \lambda \leq 3/4$, we have from Lemma 2.1 that $|\lambda' - \lambda| \leq \frac{C_1(X_0)}{16}|\xi||x - x'| \leq C(X_0)|\bar{Y} - \hat{Y}||x - z| \leq C(X_0)|x - z| \leq \frac{1}{4}$, if $|x - z| < 1/4C(X_0)$. Hence $\lambda' \in [0, 1]$.

We next estimate the remaining terms. We have $C_2(X_0)|\xi|^2|x - z|^2(\lambda(1 - \lambda) - \lambda'(1 - \lambda')) \leq 3C_2(X_0)|\xi|^2|x - z|^2|\lambda' - \lambda| \leq C(X_0)|\bar{Y} - \hat{Y}|^3|x - z|^3$, again by Lemma 2.1. Similarly, $C_1(X_0)|\xi|^2|x - x'|^2(\lambda'(1 - \lambda') - \lambda(1 - \lambda)) \leq C(X_0)|\bar{Y} - \hat{Y}|^3|x - z|^3$.

We also have from the estimates at the end of Subsection 2.4 that

$$\left\langle (D_x^2\phi(z, Y(\lambda'), Z) - D_x^2\phi(z, Y(\lambda), Z))(x - z), x - z \right\rangle$$
$$\leq C|Y(\lambda') - Y(\lambda)| |x - z|^2 \leq C|v(\lambda') - v(\lambda)||x - z|^2$$
$$= |\lambda' - \lambda| |\bar{v} - \hat{v}| |x - z|^2 \leq C|\lambda' - \lambda| |\bar{Y} - \hat{Y}| |x - z|^2 \leq C(X_0)|\bar{Y} - \hat{Y}|^2|x - z|^3.$$

To estimate the cubic form, let $h(\lambda) = D_{i,j,k}\phi(\tau, Y(\lambda), Z)$. As in the estimate of g'' above, we have $|h''(\lambda)| \leq C|\xi|^2$. We have

$$\left|((1 - \lambda')D_{i,j,k}\phi(\tau, \bar{Y}, Z) + \lambda'D_{i,j,k}\phi(\tau, \hat{Y}, Z) - D_{i,j,k}\phi(\tau, Y(\lambda), Z))(x_i - z_i)(x_j - z_j)(x_k - z_k)\right|$$
$$\leq C|x - z|^3|(1 - \lambda')h(0) + \lambda'h(1) - h(\lambda)|$$
$$\leq C|x - z|^3|(1 - \lambda')h(0) + \lambda'h(1) - h(\lambda')| + C|x - z|^3|h(\lambda') - h(\lambda)|$$
$$\leq C|x - z|^3|\xi|^2 + C|x - z|^3|\lambda' - \lambda| |\xi| \leq C|x - z|^3|\bar{Y} - \hat{Y}|^2 + C(X_0)|x - z|^4|\bar{Y} - \hat{Y}|^2.$$

Combining all these estimates we obtain

$$\min\{\phi(x, \bar{Y}, Z), \phi(x, \hat{Y}, Z)\} - \phi(x, Y(\lambda), Z)$$
$$\leq -C_2(X_0)|\bar{Y} - \hat{Y}|^2|x - z|^2 + C(X_0)(|\bar{Y} - \hat{Y}|^3|x - z|^3 + |\bar{Y} - \hat{Y}|^2|x - z|^3 + |\bar{Y} - \hat{Y}|^2|x - z|^4),$$

To obtain our desired estimate we write

$$C(X_0)\left(|\bar{Y} - \hat{Y}|^3|x - z|^3 + |\bar{Y} - \hat{Y}|^2|x - z|^3 + |\bar{Y} - \hat{Y}|^2|x - z|^4\right)$$
$$= C(X_0)|x - z|^2|\bar{Y} - \hat{Y}|^2\left(|\bar{Y} - \hat{Y}||x - z| + |x - z| + |x - z|^2\right)$$
$$\leq \frac{C_2(X_0)}{2}|x - z|^2|\bar{Y} - \hat{Y}|^2,$$

provided we choose $|x - z| \leq C(X_0)$ sufficiently small.

We finally show that Definition 3.1 implies (3.3). Fix $X_0 \in C_\Omega$, let U_{X_0} be the neighborhood in Definition 3.1, and $Z \in U_{X_0}$. Let $Y_0 \in \Sigma$ and let ξ and η be perpendicular vectors in $\mathbb{R}^n$. Set $v_0 = D\phi(z, Y_0, Z)$ and hence we can write $Y_0 = Z + s_Z(\Lambda(v_0))\Lambda(v_0)$. Let $Y_\epsilon \in \Sigma$ be given by $Y_\epsilon = Z + s(\Lambda(v_0 + \epsilon\xi))\Lambda(v_0 + \epsilon\xi)$ and notice that $Y_0 \in [Y_{-\epsilon}, Y_\epsilon]_Z$ for the value $\lambda = \frac{1}{2}$ and for all $\epsilon > 0$. Therefore from (3.2) we have $\phi(x, Y_0, Z) \geq \min\{\phi(x, Y_{-\epsilon}, Z), \phi(x, Y_\epsilon, Z)\} + C_1(X_0)\epsilon^2|\xi|^2|x - z|^2$ for $|x - z| \leq C_2(X_0)$.

Let $S_\epsilon = \{x \in \Omega : \phi(x, Y_{-\epsilon}, Z) = \phi(x, Y_\epsilon, Z)\}$. Notice that $D\phi(z, Y_\epsilon, Z) - D\phi(z, Y_{-\epsilon}, Z) = 2\epsilon\xi$ is a normal vector to S_ϵ at z.

Let γ be a curve contained in S_ϵ such that $\gamma(0) = z$ and $\gamma'(0) = \eta$. We then have that $\phi(\gamma(t), Y_0, Z) \geq \frac{1}{2}\phi(\gamma(t), Y_{-\epsilon}, Z) + \frac{1}{2}\phi(\gamma(t), Y_\epsilon, Z) + C_1(X_0)\epsilon^2|\xi|^2|\gamma(t) - z|^2$ for all $|t|$ small enough.

Let $g(t) = \phi(\gamma(t), Y_0, Z) - \frac{1}{2}\phi(\gamma(t), Y_{-\epsilon}, Z) - \frac{1}{2}\phi(\gamma(t), Y_\epsilon, Z) - C_1(X_0)\epsilon^2|\xi|^2|\gamma(t) - z|^2$.
We have $g'(0) = 0$ and $g''(0) \geq 0$, that is,

$$\left\langle (D_x^2\phi(z, Y_0, Z) - \frac{1}{2}D_x^2\phi(z, Y_{-\epsilon}, Z) - \frac{1}{2}D_x^2\phi(z, Y_\epsilon, Z) - 2C_1(X_0)\epsilon^2|\xi|^2\,Id)\eta, \eta \right\rangle \geq 0$$

and this inequality holds for all ϵ small enough. Letting $h(\epsilon) = \langle (D_x^2\phi(z, Y_\epsilon, Z))\eta, \eta\rangle$,
we get that $h(0) - \frac{1}{2}h(-\epsilon) - \frac{1}{2}h(\epsilon) \geq 2C_1(X_0)\epsilon^2|\xi|^2|\eta|^2$, implying that $h''(0) \leq -4C_1|\xi|^2|\eta|^2$.
Therefore we obtain that $\dfrac{d^2}{d\epsilon^2}\left\langle D_x^2\phi(z, Y_\epsilon, Z)\eta, \eta\right\rangle\Big|_{\epsilon=0} \leq -C(X_0)|\xi|^2|\eta|^2$. $\square$

Remark 3.3. We explain the difference between the condition (A3) of Ma, Trudinger
and Wang and condition (3.3). In a mass transport problem with a cost c, one de-
termines the mapping y from the equation $D_x c(x, y) = v$ and obtains $y = y(x, v)$.
Then the (A3) condition looks at the quantity $\dfrac{d^2}{d\epsilon^2}\,c_{x_ix_j}(x, y(x, v + \epsilon\xi))\eta_i\eta_j\Big|_{\epsilon=0}$ for
all perpendicular vectors ξ and η, and for each x and v. This quantity depends
on the cost c. Since the parallel refractor is not an optimal mass transport prob-
lem, in our condition (3.3), the mapping Y, now a point in the target Σ, is de-
termined from the equation $D_x\phi(x, Y, X) = v$ and the target Σ. We recall that
$\phi(x, Y, X)$ is defined by (2.8) and $D_x\phi(x, Y, X) = \dfrac{x - y}{y_{n+1} - x_{n+1} - \kappa|X - Y|}$. Then we
can write $Y = Y(X, v) = X + s_X(\Lambda(v))\Lambda(v)$, and we now look at the quantity
$\dfrac{d^2}{d\epsilon^2}\,\phi_{x_ix_j}(x, Y(X, v + \epsilon\xi), X)\eta_i\eta_j\Big|_{\epsilon=0}$ for all perpendicular ξ and η, and for each X
and v.

4. LOCAL AND GLOBAL REFRACTORS

4.1. **Refractors.** Let $u : \Omega \to [0, M]$, and assume that the convex hull of $\Sigma \subset \mathcal{T}$,
and Ω is connected. Given $x_0 \in \Omega$, set $X_0 = (x_0, u(x_0))$. We define

(4.1) $F_u(x_0) = \{Y \in \Sigma : u(x) \leq \phi(x, Y, X_0) \text{ for all } x \in \Omega\}$.

The function u is a *parallel refractor* if $F_u(x_0) \neq \emptyset$ for all $x_0 \in \Omega$.

We notice that, from the estimates of the derivatives $\partial_{x_i}\phi$ from Subsection 2.4,
any refractor is a Lipschitz function in Ω with a Lipschitz constant depending
only on δ in (2.6).

Suppose that u is a parallel refractor in Ω. In general, a local supporting ellipsoid
might not support the refractor in all of Ω. For example, rotating the Figure 4
around the z-axis, we obtain a refractor in 3d that has a local supporting ellipsoid
that is not global. In this case, the target is composed of the circle $x^2 + y^2 = (.03)^2$
with $z = 0$, and the point $(0, 0, 10)$.

The purpose of this section is to see that under condition (4.2) below, a local
supporting ellipsoid is also global. This will be used later in the proof of Theorem
5.3. We mention that in a different context, that is, for the near field reflector
problem with a source point, it was observed in [KW10, Section 5] that a local (A3)
type condition is enough to infer that local supporting ellipsoids are also global.
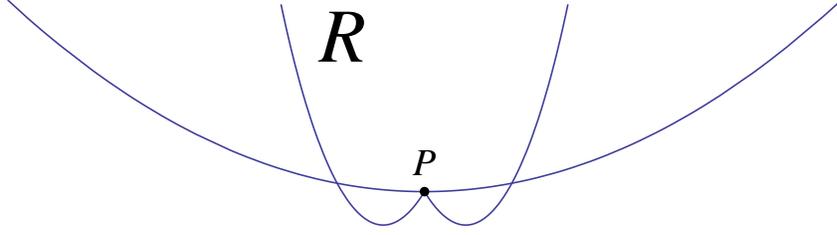
FIGURE 4. The refractor R is composed of the minimum of the ellipsoids given by $\phi(x, Y_1, P)$, and $\phi(x, Y_2, P)$, with $Y_1 = (.03, 5)$, $Y_2 = (-.03, 5)$ and $P = (0, 4.70456)$. The ellipsoid given by $\phi(x, Y_3, P)$ with $Y_3 = (0, 10)$, supports R at P locally but not globally; $\kappa = 2/3$.

Similarly, for the near field reflector with an extended source and collimated beams, see [Kar14, Lemma 8.1].

The target Σ satisfies the condition AW from $X_0 \in \mathscr{C}_\Omega$ if for all $Y_0 \in \Sigma$, and for all vectors ξ and η such that $\xi \perp \eta$ we have

$$(4.2) \qquad \frac{d^2}{d\epsilon^2} \left\langle D_x^2 \phi(x_0, Y_\epsilon, X_0)\eta, \eta \right\rangle \Big|_{\epsilon=0} \leq 0,$$

where $Y_\epsilon = X_0 + s(\Lambda(v + \epsilon\xi))\Lambda(v + \epsilon\xi)$ and $v = D\phi(x_0, Y_0, X_0)$. Clearly (3.3) implies (4.2). We will show in Proposition 4.1 that (4.2) implies that for all $\bar{Y}, \hat{Y} \in \Sigma$ we have

$$(4.3) \qquad \phi(x, Y_{X_0}(\lambda), X_0) \geq \min \left\{ \phi(x, \bar{Y}, X_0), \phi(x, \hat{Y}, X_0) \right\}$$

for all $x \in \Omega$, and $0 \leq \lambda \leq 1$, with $Y_{X_0}(\lambda) = X_0 + s_Z(\Lambda(v(\lambda))) \Lambda(v(\lambda))$. Here $\bar{v} = D_x\phi(x_0, \bar{Y}, X_0)$, $\hat{v} = D_x\phi(x_0, \hat{Y}, X_0)$, and $v(\lambda) = (1 - \lambda)\bar{v} + \lambda\hat{v}$.

Let

$$(4.4) \qquad H(v, X) := s_X(\Lambda(v))Q(v),$$

see (2.1) and Subsection 2.5. Condition (4.2) means that for all $\eta, \xi \in \mathbb{R}^n$, with $\eta \perp \xi$, we have

$$(4.5) \qquad D_{v_l, v_k}(D_{x_i, x_j}\phi(x, Y(v), X)\eta_i\eta_j\xi_k\xi_l \leq 0,$$

with $X = X_0 = (x_0, x_{n+1}^0)$. We are going to rewrite this condition in terms of the function H. We will show that this is equivalent to

$$(4.6) \qquad \langle D_v^2(1/H(v, X)) \, \xi, \xi \rangle \leq 0, \qquad \text{for all } \xi \in R^n.$$

Recall that from Subsection 2.4 we have that $D_{x_i}\phi(x, Y, X) = \dfrac{x_i - y_i}{y_{n+1} - x_{n+1} - k|X - Y|}$, where $x_{n+1} = \phi(x, Y, X)$. Let us set

$$J(Y, X, \eta) := \langle D_x^2\phi(x, Y, X)\eta, \eta \rangle = \sum_{i,j=1}^{n} D_{x_i, x_j}\phi(x, Y, X)\eta_i\eta_j.$$

From (2.11) and (2.10) we have

$$D_{x_i, x_j}\phi(x, Y, X) = \delta_{ij}\left(y_{n+1} - x_{n+1} - \kappa|X - Y|\right)^{-1} + (1 - \kappa^2)\left(y_{n+1} - x_{n+1} - \kappa|X - Y|\right)^{-3}(x_i - y_i)(x_j - y_j).$$

Therefore, for $|\eta| = 1$ we get

$$J(Y, X, \eta) = \left(y_{n+1} - x_{n+1} - \kappa|X - Y|\right)^{-1} + (1 - \kappa^2)\left(y_{n+1} - x_{n+1} - \kappa|X - Y|\right)^{-3}\langle x - y, \eta\rangle^2.$$

From (2.1), $\Lambda(v) = (-Q(v)v, Q(v) + \kappa)$. Then

$$Y_\epsilon - X = s_X(\Lambda(v))\left(-Q(v)v, Q(v) + \kappa\right).$$

Therefore, $J(Y_\epsilon, X, \eta) = \dfrac{1 + (1 - \kappa^2)\langle v, \eta\rangle^2}{s_X(\Lambda(v))\,Q(v)} := F(v, X, \eta)$. We will show that $\dfrac{d^2}{d\epsilon^2}F(v + \epsilon\xi, X, \eta)|_{\epsilon=0} \leq 0$. Calculating the second derivative with respect to ϵ, we have that the last inequality is equivalent to

$$(4.7) \qquad \frac{d^2}{d\epsilon^2}F(v + \epsilon\xi, X, \eta)|_{\epsilon=0} = \sum_{k,\ell=1}^{n} D_{v_k, v_\ell}F(v, X, \eta)\xi_k\xi_\ell \leq 0$$

for all vectors $\xi \perp \eta$.

If we set $G = 1/H$, then $F(v, X, \eta) = G(v, X)(1 + (1 - \kappa^2)\langle v, \eta\rangle^2)$. A calculation gives that

$$\sum_{k,\ell=1}^{n} D_{v_k, v_\ell}F(v, X, \eta)\xi_k\xi_\ell = (1 + (1 - \kappa^2)\langle v, \eta\rangle^2)\langle D^2 G(v, X)\xi, \xi\rangle$$

$$+ 4(1 - \kappa^2)\langle v, \eta\rangle\langle\xi, \eta\rangle\langle DG(v, X), \xi\rangle + 2(1 - \kappa^2)\langle\eta, \xi\rangle^2 G(v, X)$$

$$= (1 + (1 - \kappa^2)\langle v, \eta\rangle^2)\langle D^2 G(v, X)\xi, \xi\rangle$$

since $\xi \perp \eta$.

Therefore we have shown that condition (4.2) is equivalent to (4.6).

For simplicity in the notation we assume that $X_0 = 0$ and consider the solid ellipsoids

$$E(Y) = \{X \in \mathbb{R}^{n+1} : c(X, Y) \leq c(0, Y)\},$$

where $c(X, Y)$ is defined by (2.3). From Subsection 2.5, we recall that the target Σ is given parametrically from the origin by

$$Y = s(\Lambda(v))\Lambda(v),$$

where $\Lambda(v) = (-Q(v)v, Q(v) + \kappa)$ and $|\Lambda(v)| = 1$, with $Q(v)$ given in (2.1).

Proposition 4.1. *Suppose* (4.2) *holds from* X_0 *(assumed for simplicity 0). Let* $\bar{Y}, \hat{Y} \in \Sigma$ *be given by* $\bar{Y} = s(\Lambda(\bar{v}))\Lambda(\bar{v})$ *and* $\hat{Y} = s(\Lambda(\hat{v}))\Lambda(\hat{v})$, *and let* $v_\lambda = (1 - \lambda)\bar{v} + \lambda\hat{v}$ *for some* $\lambda \in (0, 1)$, *and let* $Y_\lambda = s(\Lambda(v_\lambda))\Lambda(v_\lambda)$. *Set* $H(v, 0) = H(v)$, *given in* (4.4). *Then*

$$(4.8) \qquad \frac{1}{H(v_\lambda)} \geq (1 - \lambda)\frac{1}{H(\bar{v})} + \lambda\frac{1}{H(\hat{v})},$$

and

$$E(Y_\lambda) \subseteq E(\bar{Y}) \cup E(\hat{Y}),$$

and in particular, $\phi(x, Y_\lambda, X_0) \geq \min\{\phi(x, \bar{Y}, X_0), \phi(x, \hat{Y}, X_0)\}$ for all x in their common domain (in particular for $x \in \Omega$).

Proof. Inequality (4.8) follows from (4.6) which is equivalent to (4.2).

We first notice that the set $bdryE(\hat{Y}) \cap bdryE(Y_\lambda)$ is contained on a hyperplane $\hat{T}$. Indeed, for $X \in bdryE(\hat{Y}) \cap bdryE(Y_\lambda)$ we have that $c(X, \hat{Y}) = c(0, \hat{Y})$, so $|X - \hat{Y}|^2 = (|\hat{Y}| - \kappa x_{n+1})^2$ which gives that $|X|^2 - 2\langle X, \hat{Y}\rangle + 2\kappa x_{n+1}|\hat{Y}| - \kappa^2 x_{n+1}^2 = 0$. Also $c(X, Y_\lambda) = c(0, Y_\lambda)$, and so $|X|^2 - 2\langle X, Y_\lambda\rangle + 2\kappa x_{n+1}|Y_\lambda| - \kappa^2 x_{n+1}^2 = 0$. Subtracting these identities yields $\langle X, \hat{\eta}\rangle = 0$ where $\hat{\eta} = \hat{Y} - \kappa|\hat{Y}|e_{n+1} - (Y_\lambda - \kappa|Y_\lambda|e_{n+1})$, and so $bdryE(\hat{Y}) \cap bdryE(Y_\lambda) \subseteq \hat{T} := \{X : \langle X, \hat{\eta}\rangle = 0\}$. In the same way, $bdryE(\bar{Y}) \cap bdryE(Y_\lambda) \subseteq \bar{T}$, where $\bar{T} = \{X : \langle X, \bar{\eta}\rangle = 0\}$ and $\bar{\eta} = \bar{Y} - \kappa|\bar{Y}|e_{n+1} - (Y_\lambda - \kappa|Y_\lambda|e_{n+1})$.

From (2.1) and the definition of H, we can write $\hat{\eta} = (H(v_\lambda)v_\lambda - H(\hat{v})\hat{v}, H(\hat{v}) - H(v_\lambda))$ and $\bar{\eta} = (H(v_\lambda)v_\lambda - H(\bar{v})\bar{v}, H(\bar{v}) - H(v_\lambda))$.

The proposition will follow from the following claims:

Claim 1: If $\langle X, \hat{\eta}\rangle \geq 0$ and $X \in E(Y_\lambda)$, then $X \in E(\hat{Y})$.
Claim 2: If $\langle X, \bar{\eta}\rangle \geq 0$ and $X \in E(Y_\lambda)$, then $X \in E(\bar{Y})$.
Claim 3: If $\langle X, \hat{\eta}\rangle < 0$ and $\langle X, \bar{\eta}\rangle < 0$, then $X \notin E(Y_\lambda)$.

Only the proof of the third claim uses condition AW, i.e., (4.8).

We prove Claim 1. Let $\langle X, \hat{\eta}\rangle \geq 0$ and $X \in E(Y_\lambda)$. Since $X \in E(Y_\lambda)$, we have $|X|^2 - 2\langle X, Y_\lambda\rangle + 2\kappa x_{n+1}|Y_\lambda| - \kappa^2 x_{n+1}^2 \leq 0$; and since $\langle X, \hat{\eta}\rangle \geq 0$ we have $\langle X, Y_\lambda - \hat{Y}\rangle - \kappa x_{n+1}(|Y_\lambda| - |\hat{Y}|) \leq 0$. Therefore,

$$|X|^2 - 2\langle X, \hat{Y}\rangle + 2\kappa x_{n+1}|\hat{Y}| - \kappa^2 x_{n+1}^2$$
$$= |X|^2 - 2\langle X, Y_\lambda\rangle + 2\kappa x_{n+1}|Y_\lambda| - \kappa^2 x_{n+1}^2 + 2\langle X, Y_\lambda - \hat{Y}\rangle - 2\kappa x_{n+1}(|Y_\lambda| - |\hat{Y}|) \leq 0.$$

It follows that $|X - \hat{Y}|^2 \leq (|\hat{Y}| - \kappa x_{n+1})^2$. In this inequality, writing $X = (x, x_{n+1})$, $\hat{Y} = (\hat{y}, \hat{y}_{n+1})$, and completing the squares we obtain, since $0 < \kappa < 1$, that $|\hat{Y}| - \kappa x_{n+1} \geq 0$ and hence $|X - \hat{Y}| \leq |\hat{Y}| - \kappa x_{n+1}$, which means $X \in E(\hat{Y})$.

The proof of Claim 2 is exactly the same.

We now prove Claim 3.

Assume that $\langle X, \hat{\eta}\rangle < 0$ and $\langle X, \bar{\eta}\rangle < 0$. Notice that $E(Y_\lambda) \setminus 0 \subseteq \{X : \langle X, N_\lambda\rangle < 0\}$, where $N_\lambda = (v_\lambda, -1)$ and hence, it is enough to show that $\langle X, N_\lambda\rangle \geq 0$.

We first assume that $H(v_\lambda) - H(\bar{v}) \neq 0$ and $H(v_\lambda) - H(\hat{v}) \neq 0$. We will show that we can write

$$N_\lambda = \frac{(1-t)}{H(v_\lambda) - H(\bar{v})}\bar{\eta} + \frac{t}{H(v_\lambda) - H(\hat{v})}\hat{\eta},$$

with $\dfrac{(1-t)}{H(v_\lambda) - H(\bar{v})} \leq 0$ and $\dfrac{t}{H(v_\lambda) - H(\hat{v})} \leq 0$, for some t. The above equality holds if and only if

$$\frac{(1-t)(H(v_\lambda)v_\lambda - H(\bar{v})\bar{v})}{H(v_\lambda) - H(\bar{v})} + \frac{t(H(v_\lambda)v_\lambda - H(\hat{v})\hat{v})}{H(v_\lambda) - H(\hat{v})} = v_\lambda,$$

which holds if and only if

$$(1-t)\left(v_\lambda + \frac{H(\bar{v})\,(v_\lambda - \bar{v})}{H(v_\lambda) - H(\bar{v})}\right) + t\left(v_\lambda + \frac{H(\hat{v})\,(v_\lambda - \hat{v})}{H(v_\lambda) - H(\hat{v})}\right) = v_\lambda,$$

which is true if and only if

$$(\hat{v} - \bar{v})\left\{\frac{(1-t)\lambda H(\bar{v})}{H(v_\lambda) - H(\bar{v})} - \frac{t(1-\lambda)H(\hat{v})}{H(v_\lambda) - H(\hat{v})}\right\} = 0.$$

Therefore we choose t such that $\dfrac{(1-t)\lambda H(\bar{v})}{H(v_\lambda) - H(\bar{v})} = \dfrac{t(1-\lambda)H(\hat{v})}{H(v_\lambda) - H(\hat{v})}$. Since $Q(v) > 0$,
we have that $\lambda H(\bar{v}) > 0$ and $(1-\lambda)H(\hat{v}) > 0$. Then it follows that $\dfrac{(1-t)}{H(v_\lambda) - H(\bar{v})}$
and $\dfrac{t}{H(v_\lambda) - H(\hat{v})}$ have both the same sign. From the last identity containing t we
obtain

$$\lambda H(\bar{v}) = \frac{t}{H(v_\lambda) - H(\hat{v})} \left\{H(v_\lambda)\left((1-\lambda)H(\hat{v}) + \lambda H(\bar{v})\right) - H(\bar{v})H(\hat{v})\right\}.$$

From (4.8) we get

$$H(v_\lambda)\left((1-\lambda)H(\hat{v}) + \lambda H(\bar{v})\right) - H(\bar{v})H(\hat{v}) \le 0,$$

hence $\dfrac{t}{H(v_\lambda) - H(\hat{v})} \le 0$, and therefore also $\dfrac{(1-t)}{H(v_\lambda) - H(\bar{v})} \le 0$.

Next we assume that $H(v_\lambda) - H(\hat{v}) = 0$ and $H(v_\lambda) - H(\bar{v}) \ne 0$. From (4.8),
this implies that $H(v_\lambda) < H(\bar{v})$. If we write $N_\lambda = \dfrac{1}{H(v_\lambda) - H(\bar{v})}\bar{\eta} + t\,\hat{\eta}$, then $t =$
$\dfrac{\lambda H(\bar{v})}{(H(v_\lambda) - H(\bar{v}))(1-\lambda)H(\hat{v})}$, and so $t < 0$.

Finally, if $H(v_\lambda) = H(\hat{v}) = H(\bar{v})$, then $\bar{\eta} = \lambda H(\bar{v})(\hat{v} - \bar{v}, 0)$, and $\hat{\eta} = (1-\lambda)H(\bar{v})(\bar{v} - \hat{v}, 0)$. So in this case, both inequalities $\langle X, \hat{\eta}\rangle < 0$ and $\langle X, \bar{\eta}\rangle < 0$ are impossible.

This proves Claim 3 and hence the proof of the proposition is complete.

$\square$

Proposition 4.2. *Suppose that u is a parallel refractor in Ω, $x_0 \in \Omega$, and assume that (4.2) holds from $X_0 = (x_0, u(x_0))$. If there exist $Y_0 \in \Sigma$ and $\epsilon > 0$ such $u(x) \le \phi(x, Y_0, X_0)$ for all $x \in B_\epsilon(x_0)$, then $u(x) \le \phi(x, Y_0, X_0)$ for all $x \in \Omega$.*

Proof. We define

$$\partial u(x_0) = \{v \in \mathbb{R}^n : u(x) \le u(x_0) + v \cdot (x - x_0) + o(|x - x_0|); \forall x \in B_\epsilon(x_0)\}.$$

We prove that $\{Y(X_0, v) : v \in \partial u(x_0)\} \subset F_u(x_0)$, where $Y(X_0, v) = X_0 + s_{X_0}(\Lambda(v))\Lambda(v)$. Since from (2.12), $v = D_x\phi(x, Y, X)$ with $Y \in \Sigma$ if and only if $Y = Y(X, v)$, the inclusion is equivalent to show that $\partial u(x_0) \subset \{D_x\phi(x_0, Y, X_0) : Y \in F_u(x_0)\} := B$. It is then enough to show that the extremal points of $\partial u(x_0)$ are contained in B and that B is convex. Let v_0 be a extremal point of $\partial u(x_0)$, then there exist $x_n \to x_0$ with u differentiable at x_n and $v_n = Du(x_n) \to v_0$, see [Cla90, Theorem 2.5.1]. Let $X_n = (x_n, u(x_n))$ and let $Y_n \in F_u(x_n)$. Since u is differentiable at x_n it follows that

$Y_n = Y(X_n, v_n)$ and then $Y_n \to Y(X_0, v_0) := Y_0$. We have that $u(x) \le \phi(x, Y_n, X_n)$ for all $x \in \Omega$. Letting $n \to \infty$ yields $u(x) \le \phi(x, Y_0, X_0)$ for all $x \in \Omega$, i.e., $Y_0 \in F_u(x_0)$ and $D_x\phi(x_0, Y_0, X_0) = v_0$.

To show that B is convex, let $Y_1, Y_2 \in F_u(x_0)$ and let $v_i = D_x\phi(x_0, Y_i, X_0)$, $i = 1, 2$. Consider $v_\lambda = (1 - \lambda)v_1 + \lambda v_2$ and $Y_\lambda = Y(X_0, v_\lambda)$. Since $v_\lambda = D_x\phi(x_0, Y_\lambda, X_0)$, we need to show that $Y_\lambda \in F_u(x_0)$. We have

$$u(x) \le \min\{\phi(x, Y_1, X_0), \phi(x, Y_2, X_0)\} \le \phi(x, Y_\lambda, X_0)$$

for all $x \in \Omega$ by Proposition 4.1. This completes the proof. $\square$

5. Main results

Lemma 5.1. *If u is a refractor, then for each $\bar{x}, \hat{x} \in \Omega$ and each $s \in [0, 1]$ we have $u((1 - s)\bar{x} + s\hat{x}) \ge (1 - s)u(\bar{x}) + su(\hat{x}) - C|\bar{x} - \hat{x}|^2 s(1 - s)$, with C is a structural constant.*

Proof. We use the fact from Subsection 2.4 that $|D_{x_i x_j}\phi(x, Y, \bar{X})| \le C$ for any $x \in \Omega$, $\bar{X} \in \mathscr{C}_\Omega$ and $Y \in \mathcal{T}$.

Given $\tilde{x} \in \Omega$, let $\tilde{Y} \in F_u(\tilde{x})$ and set $\tilde{X} = (\tilde{x}, u(\tilde{x}))$. We have $u(x) \le \phi(x, \tilde{Y}, \tilde{X}) \le \phi(\tilde{x}, \tilde{Y}, \tilde{X}) + \langle D\phi(\tilde{x}, \tilde{Y}, \tilde{X}), x - \tilde{x}\rangle + C|x - \tilde{x}|^2 = u(\tilde{x}) + \langle p, x - \tilde{x}\rangle + C|x - \tilde{x}|^2$ for all $x \in \Omega$. Next, given $s \in [0, 1]$, let $x_s = (1 - s)\bar{x} + s\hat{x}$, $Y \in F_u(x_s)$, and $X_s = (x_s, u(x_s))$. Applying the previous inequality $\tilde{x} \rightsquigarrow x_s$, $\tilde{Y} \rightsquigarrow Y$, and $p \rightsquigarrow D\phi(x_s, Y, X_s)$, we get $u(x) \le u(x_s) + \langle p, x - x_s\rangle + C|x - x_s|^2$ for all $x \in \Omega$.

Let $\psi(t) = u(x_t)$. We have $\psi(t) \le u(x_s) + \langle p, x_t - x_s\rangle + C|x_t - x_s|^2 = \psi(s) + (t - s)\langle p, \hat{x} - \bar{x}\rangle + C(t - s)^2|\bar{x} - \hat{x}|^2$. Therefore $\psi(0) \le \psi(s) - s\langle p, \hat{x} - \bar{x}\rangle + Cs^2|\bar{x} - \hat{x}|^2$ and $\psi(1) \le \psi(s) + (1 - s)\langle p, \hat{x} - \bar{x}\rangle + C(1 - s)^2|\bar{x} - \hat{x}|^2$. Proceeding as in Theorem 3.2, we obtain $(1 - s)\psi(0) + s\psi(1) \le \psi(s) + Cs(1 - s)|\bar{x} - \hat{x}|^2$ and the lemma is proved. $\square$

We will next prove our main lemma.

Lemma 5.2. *Suppose u is a parallel refractor and the target Σ is regular from $X^\star = (x^\star, u(x^\star))$ in the sense of Definition 3.1. There exist constants δ, C_1 and C_2 depending on $X^\star$, such that $B_\delta(x^\star) \subset \Omega$, and if $\bar{x}, \hat{x} \in B_\delta(x^\star)$, $\bar{Y} \in F_u(\bar{x})$, $\hat{Y} \in F_u(\hat{x})$ with $|\bar{Y} - \hat{Y}| \ge |\bar{x} - \hat{x}|$, then there exists $x_0 \in \overline{\bar{x}\hat{x}}$ (the segment from $\bar{x}$ to $\hat{x}$) such that if $X_0^\star = (x_0, u(x_0))$, then*

$$(5.1) \quad u(x) - \phi(x, Y, X_0^\star) \le C\,|\bar{Y} - \hat{Y}|\,|\bar{x} - \hat{x}| + C\,|Y(\lambda) - Y|\,|x - x_0| - C_1\,|\bar{Y} - \hat{Y}|^2|x - x_0|^2,$$

for all $Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0^\star}$, $1/4 \le \lambda \le 3/4$, $Y \in \Sigma$ and for all $x \in \Omega \cap B_{C_2}(x_0)$.

We remark that C is a structural constant, depending only on the bounds for the derivatives of ϕ.

Proof. Since Σ is regular from $X^\star$, there exists a neighborhood $U_{X^\star}$ of $X^\star$ such that (3.2) holds for all $\bar{Y}, \hat{Y} \in \Sigma$ and all $Z \in U_{X^\star}$. Since parallel refractors are uniformly Lipschitz in Ω, there exists $\delta > 0$ such that $(x, u(x)) \in U_{X^\star}$ for all $x \in B_\delta(x^\star)$.

If $\bar{Y} \in F_u(\bar{x})$ and $\hat{Y} \in F_u(\hat{x})$, it follows that $u(x) \le \min\{\phi(x, \bar{Y}, \bar{X}), \phi(x, \hat{Y}, \hat{X})\}$ for all $x \in \Omega$ with $\bar{X} = (\bar{x}, u(\bar{x}))$, $\hat{X} = (\hat{x}, u(\hat{x}))$. By continuity there exists $x_0 \in [\bar{x}, \hat{x}]$ such that $\phi(x_0, \bar{Y}, \bar{X}) = \phi(x_0, \hat{Y}, \hat{X})$. Indeed, setting $h(x) = \phi(x, \bar{Y}, \bar{X}) - \phi(x, \hat{Y}, \hat{X})$, we have that $h(\bar{x}) = u(\bar{x}) - \phi(\bar{x}, \hat{Y}, \hat{X}) \le 0$ and $h(\hat{x}) = \phi(\hat{x}, \bar{Y}, \bar{X}) - u(\hat{x}) \ge 0$.

Now set $x_{0_{n+1}} = \phi(x_0, \bar{Y}, \bar{X}) = \phi(x_0, \hat{Y}, \hat{X})$, $X_0 = (x_0, x_{0_{n+1}})$, and recall we have set $X_0^\star = (x_0, u(x_0)) \in U_{X^\star}$ and $u(x_0) \le x_{0_{n+1}}$. By definition $\phi(x, \hat{Y}, \hat{X}) = \phi(x, \hat{Y}, X_0)$ and $\phi(x, \bar{Y}, \bar{X}) = \phi(x, \bar{Y}, X_0)$ for all $x \in \Omega$. Hence we can write

$$u(x) \le \min\left\{\phi(x, \bar{Y}, X_0), \phi(x, \hat{Y}, X_0)\right\} = \min\left\{\phi(x, \bar{Y}, X_0^\star), \phi(x, \hat{Y}, X_0^\star)\right\} + E.$$

We now estimate E. First, notice that by Lemma (2.2), we have $0 \le E \le C(x_{0_{n+1}} - u(x_0))$. We claim that

$$(5.2) \qquad\qquad x_{0_{n+1}} - u(x_0) \le C\,|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}|.$$

We can write $x_0 = (1 - s)\bar{x} + s\hat{x}$ for some $0 < s < 1$. Then, by Lemma (5.1), we have $u(x_0) \ge (1 - s)u(\bar{x}) + su(\hat{x}) - C|\bar{x} - \hat{x}|^2 s(1 - s)$. Since $u(\bar{x}) = \phi(\bar{x}, \bar{Y}, X_0)$ and $u(\hat{x}) = \phi(\hat{x}, \hat{Y}, X_0)$, we get

$$(5.3) \quad x_{0_{n+1}} - u(x_0) \le x_{0_{n+1}} - \left((1 - s)\phi(\bar{x}, \bar{Y}, X_0) + s\phi(\hat{x}, \hat{Y}, X_0)\right) + C|\bar{x} - \hat{x}|^2 s(1 - s).$$

On the other hand, $\phi(\hat{x}, \hat{Y}, X_0)) \ge \phi(x_0, \hat{Y}, X_0)) + \langle D\phi(x_0, \hat{Y}, X_0), \hat{x} - x_0\rangle$, and $\phi(\bar{x}, \bar{Y}, X_0)) \ge \phi(x_0, \bar{Y}, X_0)) + \langle D\phi(x_0, \bar{Y}, X_0), \bar{x} - x_0\rangle$ by convexity. Using that $\bar{x} - x_0 = s(\bar{x} - \hat{x})$, and $\hat{x} - x_0 = -(1 - s)(\bar{x} - \hat{x})$ we obtain

$$\phi(\hat{x}, \hat{Y}, X_0) \ge x_{0_{n+1}} - (1 - s)\langle D\phi(x_0, \hat{Y}, X_0), \bar{x} - \hat{x}\rangle, \text{ and}$$
$$\phi(\bar{x}, \bar{Y}, X_0)) \ge x_{0_{n+1}} + s\langle D\phi(x_0, \bar{Y}, X_0), \bar{x} - \hat{x}\rangle.$$

It follows that

$$(1 - s)\phi(\bar{x}, \bar{Y}, X_0)) + s\phi(\hat{x}, \hat{Y}, X_0)$$
$$\ge x_{0_{n+1}} + (1 - s)s\left\langle\left(D\phi(x_0, \bar{Y}, X_0) - D\phi(x_0, \hat{Y}, X_0)\right), \bar{x} - \hat{x}\right\rangle$$
$$\ge x_{0_{n+1}} - s(1 - s)\left|\left\langle\left(D\phi(x_0, \bar{Y}, X_0) - D\phi(x_0, \hat{Y}, X_0)\right), \bar{x} - \hat{x}\right\rangle\right|$$
$$\ge x_{0_{n+1}} - s(1 - s)\left|D\phi(x_0, \bar{Y}, X_0) - D\phi(x_0, \hat{Y}, X_0)\right||\bar{x} - \hat{x}|$$
$$\ge x_{0_{n+1}} - s(1 - s)C\,|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}|,$$

from the estimates for the derivatives of ϕ in Y. Therefore, inserting the last estimate in (5.3) we obtain $x_{0_{n+1}} - u(x_0) \le Cs(1 - s)\left(|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| + |\bar{x} - \hat{x}|^2\right) \le C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}|$, where in the last inequality we have used that $s \in [0, 1]$ and $|\bar{x} - \hat{x}| \le |\bar{Y} - \hat{Y}|$ by assumption. We then obtain claim (5.2).

This yields

$$u(x) \le \min\{\phi(x, \bar{Y}, X_0^\star), \phi(x, \hat{Y}, X_0^\star)\} + C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}|.$$

If $\bar{x}, \hat{x} \in B_\delta(x^\star)$, then $X_0^\star \in U_{X^\star}$. So if $Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0^\star}$, then we can apply (3.2) to get $\min\{\phi(x, \bar{Y}, X_0^\star), \phi(x, \hat{Y}, X_0^\star)\} \le \phi(x, Y(\lambda), X_0^\star) - C_1\,|\bar{Y} - \hat{Y}|^2|x - x_0|^2$, for $x \in B_{C_2}(x_0) \cap \Omega$ where the constants C_1, C_2 depend on $X^\star$. And so

$$u(x) \le \phi(x, Y(\lambda), X_0^\star) + C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| - C_1\,|\bar{Y} - \hat{Y}|^2|x - x_0|^2.$$

Finally, from Lemma (2.3), we have that $|\phi(x, Y(\lambda), X_0^\star) - \phi(x, Y, X_0^\star)| \le C|Y(\lambda) - Y||x - x_0|$, and consequently $u(x) - \phi(x, Y, X_0^\star) \le C\,|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| + C\,|Y(\lambda) - Y||x - x_0| -$

$C_1 |\bar{Y} - \hat{Y}|^2 |x - x_0|^2$, where the constant C is structural. This completes the proof of the lemma.

□

We are now in a position to prove our main theorem.

Theorem 5.3. *Suppose u is a parallel refractor, and the target Σ is regular from $X^\star = (x^\star, u(x^\star))$ in the sense of Definition 3.1. Let C, C_1, C_2 and δ be the constants of Lemma 5.2. There exists a constant M depending on C and C_1 such that if $\hat{x}, \bar{x} \in B_{\frac{\delta}{2}}(x^\star)$, $\bar{Y} \in F_u(\bar{x}), \hat{Y} \in F_u(\hat{x})$ are such that*

(5.4)
$$\frac{|\bar{Y} - \hat{Y}|}{|\bar{x} - \hat{x}|} \geq \max\left\{1, \left(\frac{2M}{\delta}\right)^2, \left(\frac{2M}{C_2}\right)^2\right\},$$

then there exists x_0 on the straight segment $\overline{\bar{x}\hat{x}}$ such that we have

$$N_\mu\left(\left\{Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0^\star} : \lambda \in [1/4, 3/4]\right\}\right) \cap \Sigma \subseteq F_u(B_\eta(x_0)),$$

with $X_0^\star = (x_0, u(x_0))$, $\mu = |\bar{Y} - \hat{Y}|^{\frac{3}{2}}|\bar{x} - \hat{x}|^{\frac{1}{2}}$ and $\eta = M \dfrac{|\bar{x} - \hat{x}|^{\frac{1}{2}}}{|\bar{Y} - \hat{Y}|^{\frac{1}{2}}}$.

Proof. From the assumption, we have $|\bar{Y} - \hat{Y}| \geq |\bar{x} - \hat{x}|$ and so Lemma 5.2 is applicable. Let x_0 be the point in that lemma.

Fix $Y \in N_\mu(\{Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0^\star} : \lambda \in [\frac{1}{4}, \frac{3}{4}]\}) \cap \Sigma$. So, there exists $Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0^\star}$ with $\lambda \in [\frac{1}{4}, \frac{3}{4}]$ such that $|Y(\lambda) - Y| < \mu$.

We then have from Lemma 5.2 that

$$u(x) - \phi(x, Y, X_0^\star) \leq C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| + C|Y(\lambda) - Y||x - x_0| - C_1\lambda(1 - \lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2$$

$$\leq C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| + C\mu|x - x_0| - \frac{C_1}{16}|\bar{Y} - \hat{Y}|^2|x - x_0|^2,$$

for all $x \in \Omega$ such that $|x - x_0| < C_2$.

The right hand side in the last inequality is strictly negative for all $x \in \Omega$, with $|x - x_0| \geq \eta$, if we choose $\eta > \dfrac{8C\mu + 4\sqrt{4C^2\mu^2 + CC_1|\bar{Y} - \hat{Y}|^3|\bar{x} - \hat{x}|}}{C_1|\bar{Y} - \hat{Y}|^2}$. We pick $\mu = |\bar{Y} - \hat{Y}|^{\frac{3}{2}}|\bar{x} - \hat{x}|^{\frac{1}{2}}$, and $\eta := M\dfrac{|\bar{x} - \hat{x}|^{\frac{1}{2}}}{|\bar{Y} - \hat{Y}|^{\frac{1}{2}}}$ with $M := 2\left(\dfrac{8C + 4\sqrt{4C^2 + CC_1}}{C_1}\right)$. Since $x_0 \in B_{\delta/2}(x^\star)$, it follows from (5.4) that $B_\eta(x_0) \subset B_\delta(x^\star) \subset \Omega$ and $\eta \leq \dfrac{C_2}{2}$. Therefore $u(x) - \phi(x, Y, X_0^\star) < 0$ for all $x \in (\Omega \cap B_{C_2}(x_0)) \setminus B_\eta(x_0)$.

We now show $Y \in F_u(B_\eta(x_0))$. Notice that by definition of $X_0^\star$ we have $u(x_0) = \phi(x_0, Y, X_0^\star)$.

Let G_u denote the graph of u in $B_{C_2}(x_0)$, that is, $G_u = \{(x, u(x)) : x \in \Omega \cap B_{C_2}(x_0)\}$. Consider

(5.5)
$$\inf\{c(X, Y) - c(X_0^\star, Y) : X \in G_u\},$$

recalling that $c(X, Y) = |X - Y| + k(x_{n+1} - y_{n+1})$. We claim that $c(X, Y) - c(X_0^\star, Y) \geq 0$ for $X = (x, u(x))$ with $|x - x_0| \geq \eta$. Indeed, let $\bar{X} = (x, \phi(x, Y, X_0^\star))$ and notice that $c(X_0^\star, Y) = c(\bar{X}, Y)$. Since $\Sigma \subset \mathcal{T}$, with $\mathcal{T}$ given by (2.6), we have $X \in E^-(Y, c(X, Y))$. Since $u(x) - \phi(x, Y, X_0^\star) \leq 0$ for $|x - x_0| \geq \eta$, we have that X is below $\bar{X}$, and therefore we must have $c(X, Y) \geq c(\bar{X}, Y)$, and the claim follows. Therefore, the infimum in (5.5) is attained at some point $\tilde{X} = (\tilde{x}, u(\tilde{x}))$ with $\tilde{x} \in B_\eta(x_0)$.

We show that $Y \in F_u(\tilde{x})$. Indeed, we have $c(X, Y) \geq c(\tilde{X}, Y)$ for all $X \in G_u$. Writing $X = (x, u(x))$ we have that $(|x - y|^2 + (y_{n+1} - u(x))^2)^{1/2} - k(y_{n+1} - u(x)) \geq c(\tilde{X}, Y)$ and

noticing that $y_{n+1} \geq u(x)$, we get $u(x) \leq y_{n+1} - \dfrac{kc(\tilde{X}, Y)}{1 - k^2} - \sqrt{\dfrac{c(\tilde{X}, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}} =$

$\phi(x, Y, \tilde{X})$ for $x \in \Omega \cap B_{C_2}(x_0)$. Since $\eta < C_2/2$, we have that $B_{C_2/2}(\tilde{x}) \subset B_{C_2}(x_0)$. We therefore obtain the local estimate $u(x) \leq \phi(x, Y, \tilde{X})$ for all $x \in B_\epsilon(\tilde{x})$, with ϵ small.

Since $\tilde{x} \in B_\delta(x^\star)$, it follows from the choice of δ in Lemma 5.2 that $\tilde{X} = (\tilde{x}, u(\tilde{x})) \in U_{X^\star}$, the neighborhood in Theorem 3.2. Therefore we can apply (3.3) with $Z = \tilde{X}$, and in particular (4.2) holds at $\tilde{X}$. Hence applying Proposition 4.2 with $X_0 \rightsquigarrow \tilde{X}$, we obtain that $u(x) \leq \phi(x, Y, \tilde{X})$ holds for all $x \in \Omega$ obtaining that $Y \in F_u(\tilde{x})$.

$\square$

5.1. A property of refractors.

Let σ denote the Borel measure given on the target Σ and let μ be a Borel measure in Ω. We say $x \in \mathcal{T}_u(Y)$ if and only if $Y \in F_u(x)$, where F_u is defined by (4.1). Assuming $\mu = f\,dx$ with $f \in L^1(\Omega)$ and the energy conservation condition

$$\mu(\Omega) = \sigma(\Sigma),$$

it is proved in [GT13] the existence of a refractor u such that

(5.6) $\qquad \mu(\mathcal{T}_u(E^\star)) = \sigma(E^\star), \qquad$ for all Borel subsets $E^\star \subset \Sigma$.

The purpose of this subsection is to show the following proposition.

Proposition 5.4. *Suppose u is a refractor solving* (5.6), *and define*

$$S = \{x \in \Omega : \text{ there exists } \bar{x} \neq x, \bar{x} \in \Omega, \text{ such that } F_u(x) \cap F_u(\bar{x}) \neq \emptyset\}.$$

Let us assume the following conditions on Σ and σ:

   (a) *Σ is the graph of a C^1 function, say $\Sigma = \{(y, \psi(y)) : y \in \Omega^\star\}$ with Ω^* some domain in $\mathbb{R}^n$;*

   (b) *Given $Y \in \Sigma$, let T_Y denote the tangent plane to Σ at Y. Assume that for each $Y \in \Sigma$ and for each $X \in \mathscr{C}_\Omega$, the line $\{X + s(Y - X), s \in R\}$ is not contained in T_Y, that is, this line intersects T_Y only at the point Y.* [‡];

   (c) *if $E \subset \Sigma$ with $|\{y \in \Omega^* : (y, \psi(y)) \in E\}| = 0$, then $\sigma(E) = 0$.*

Then $\mu(S) = 0$ and we have the inequality

(5.7) $\qquad\qquad \sigma(F_u(B)) \leq \mu(B), \qquad$ *for all balls $B \subset \Omega$.*

[‡]This condition is implied by the visibility condition in Lemma 2.1 because $Y \in \Sigma$ and some $X_0 \in C_\Omega$ the line joining Y and X_0 is contained in T_Y, then there is ball B centered at X_0 with $B \subset C_\Omega$. By the visibility condition the convex hull C of Y and B intersects Σ only at Y. But then the line joining Y and X_0 is contained in C and T_Y. Therefore Σ is not differentiable at Y.

Proof. We first notice that $F_u(B)$ is a Borel set for each closed ball B. Because if $K \subseteq \Omega$ is closed, then $F_u(K)$ is closed in Σ. In fact, let $Y_k \in F_u(x_k)$ with $x_k \in K$ and assume $Y_k \to Y$ with $Y \in \Sigma$. There exists a subsequence $x_{k_j} \to \bar{x}$ for some $\bar{x} \in K$. Setting $X_{k_j} = (x_{k_j}, u(x_{k_j}))$, we have $X_{k_j} \to \bar{X}$ where $X = (\bar{x}, u(\bar{x}))$. Since $u(x) \leq \phi(x, Y_k, X_k)$ for all $x \in \Omega$ and for all k, we have $u(x) \leq \phi(x, Y, \bar{X})$. Therefore $Y \in F_u(K)$.

Let us assume for a moment that $\mu(S) = 0$. It is easy to see that that $\mathcal{T}_u(F_u(B)) \subseteq B \cup S$, and therefore, (5.7) follows.

To prove that $\mu(S) = 0$, let us consider the set

$$S^\star = \{Y \in \Sigma : \exists x, \bar{x} \in \Omega, x \neq \bar{x} \ Y \in F_u(x) \cap F_u(\bar{x})\}.$$

We have that $\mathcal{T}_u(S^\star) = S$ and therefore $\mu(S) = \sigma(S^\star)$. Under the assumptions (a), (b) and (c) above, we are going to show that $\sigma(S^\star) = 0$. To this end, we define $u^\star : \Omega^\star \to R$ by

$$u^\star(y) = \inf\{c(X, Y) : X \in G_u, Y = (y, \psi(y))\},$$

where G_u is the graph of u. We claim $u^\star$ is Lipschitz in $\Omega^\star$. Indeed, say $u^\star(y_0) = c(X_0, Y_0)$ and $u^\star(y_1) = c(X_1, Y_1)$, then $u^\star(y_1) - u^\star(y_0) \leq c(X_0, Y_1) - c(X_0, Y_0) \leq C|Y_1 - Y_0| \leq C|y_1 - y_0|$, since ψ is Lipschitz. Similarly, we get the other inequality.

Now fix $Y_0 \in S^\star$, so $Y_0 \in F_u(x) \cap F_u(\bar{x})$ with $x \neq \bar{x}$. Write $Y_0 = (y_0, \psi(y_0))$ with $y_0 \in \Omega^\star$, $X = (x, u(x))$ and $\bar{X} = (\bar{x}, u(\bar{x}))$. We claim that $u^\star$ is not differentiable at y_0. Suppose by contradiction that $u^\star$ is differentiable at y_0. It is easy to see that $u^\star(y_0) = c(X, Y_0)$ and also $u^\star(y_0) = c(\bar{X}, Y_0)$. From $u^\star(y_0) = c(X, Y_0)$, it follows that $u^\star(y) \leq c(X, Y)$ for all $y \in \Omega^\star$ with equality at y_0. Hence we have

$$Du^\star(y_0) = D_y(c(X, (y, \psi(y))))(y_0) = \frac{y_0 - x + (\psi(y_0) - x_{n+1})D\psi(y_0)}{|Y_0 - X|} - kD\psi(y_0).$$

Also from $u^\star(y_0) = c(\bar{X}, Y_0)$, we deduce that

$$Du(y_0) = D_y(c(\bar{X}, (y, \psi(y))))(y_0) = \frac{y_0 - \bar{x} + (\psi(y_0) - \bar{x}_{n+1})D\psi(y_0)}{|Y_0 - \bar{X}|} - kD\psi(y_0).$$

This implies that $\dfrac{y_0 - x + (\psi(y_0) - x_{n+1})D\psi(y_0)}{|Y_0 - X|} = \dfrac{y_0 - \bar{x} + (\psi(y_0) - \bar{x}_{n+1})D\psi(y_0)}{|Y_0 - \bar{X}|}$.

Let us set $\Gamma = \dfrac{X - Y_0}{|X - Y_0|} = (\xi, \xi_{n+1})$ and $\bar{\Gamma} = \dfrac{\bar{X} - Y_0}{|\bar{X} - Y_0|} = (\bar{\xi}, \bar{\xi}_{n+1})$. So $\bar{\xi} + \bar{\xi}_{n+1}D\psi(y_0) = \xi + \xi_{n+1}D\psi(y_0)$ and hence $\bar{\xi} - \xi = D\psi(y_0)(\xi_{n+1} - \bar{\xi}_{n+1})$.

If $\xi_{n+1} = \bar{\xi}_{n+1}$, then $\xi = \bar{\xi}$. Hence $\Gamma = \bar{\Gamma}$ but this implies that $X = \bar{X}$, since $X, \bar{X} \in E^-(Y_0, b)$ where $b = c(X, Y_0) = c(\bar{X}, Y_0)$. Since by assumption $X \neq \bar{X}$, we obtain a contradiction.

Therefore, $\xi_{n+1} \neq \bar{\xi}_{n+1}$ and so we can write $D\psi(y_0) = \dfrac{\bar{\xi} - \xi}{\xi_{n+1} - \bar{\xi}_{n+1}}$. We claim that the line $L := \left\{Y_0 + s\left(\frac{\Gamma}{2} + \frac{\bar{\Gamma}}{2}\right) : s \in R\right\}$ is contained in T_{Y_0}. Indeed, if $Y = Y_0 + s\left(\frac{\Gamma}{2} + \frac{\bar{\Gamma}}{2}\right)$, then a simple calculation shows that $\langle Y - Y_0, (-D\psi(y_0), 1)\rangle = 0$. On the other hand, the line L clearly intersects $\mathscr{C}_\Omega$. We then obtain a contradiction with the assumption (b) above, and therefore $u^\star$ is not differentiable at y_0.

If we set $P^\star = \{y \in \Omega^\star : (y, \psi(y)) \in S^\star\}$, then we proved that $y \in P^\star$ implies that $u^\star$ is not differentiable at y. Since $u^\star$ is Lipschitz in $\Omega^\star$, we get $|P^\star| = 0$. Therefore from (c) we obtain $\sigma(S^\star) = 0$ which completes the proof of the proposition.

□

6. Hölder continuity of the gradient of the refractor

We introduce the following local condition at $X_0 \in C_\Omega$ between the measure σ and target Σ: There exist a neighborhood U_{X_0} and a constant $\hat{C} > 0$ depending on X_0 such that

(6.1) $$\sigma\left(N_\mu\left(\{[\bar{Y}, \hat{Y}]_Z : \lambda \in [1/4, 3/4]\}\right) \cap \Sigma\right) \geq \hat{C}\, \mu^{n-1}\, |\bar{Y} - \hat{Y}|$$

for any $\bar{Y}, \hat{Y} \in \Sigma$, $Z \in U_{X_0}$ and for all $\mu > 0$ small (depending on X_0). Here $N_\mu(E)$ denotes the μ- neighborhood of the set E in R^{n+1}.

Theorem 6.1. *Suppose u is a parallel refractor, the target Σ is regular from $X^\star = (x^\star, u(x^\star))$ in the sense of Definition 3.1, and there exist constants $C_0 > 0$ and $1 \leq q < \dfrac{n}{n-1}$ such that*

(6.2) $$\sigma\left(F_u(B_\eta)\right) \leq C_0\, \eta^{n/q}$$

for all balls $B_\eta \subseteq \Omega$. Suppose in addition that the local condition (6.1) is satisfied at $X^\star$.

Then there exist constants δ, $M > 0$ and $C_2 > 0$ depending on $X^\star$, such that if $\hat{x}, \bar{x} \in B_{\frac{\delta}{2}}(x^\star)$, $\bar{Y} \in F_u(\bar{x})$, $\hat{Y} \in F_u(\hat{x})$ are such that

(6.3) $$\frac{|\bar{Y} - \hat{Y}|}{|\bar{x} - \hat{x}|} \geq \max\left\{1, \left(\frac{2M}{\delta}\right)^2, \left(\frac{2M}{C_2}\right)^2\right\},$$

then we have $|\bar{Y} - \hat{Y}| \leq C_1 |\bar{x} - \hat{x}|^\alpha$ with $\alpha = \dfrac{\dfrac{n}{2q} - \dfrac{n-1}{2}}{1 + \dfrac{3}{2}(n-1) + \dfrac{n}{2q}}$, where C_1 depends only on C_0 and $\hat{C}$ in (6.1), and therefore from $X^\star$.

Proof. From Theorem 5.3 we have $N_\mu(\{Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0^\star} : \lambda \in [\frac{1}{4}, \frac{3}{4}]\}) \cap \Sigma \subseteq F_u(B_\eta(x_0))$ where $\mu = |\bar{Y} - \hat{Y}|^{\frac{3}{2}}|\bar{x} - \hat{x}|^{\frac{1}{2}}$ and $\eta = M\dfrac{|\bar{x} - \hat{x}|^{\frac{1}{2}}}{|\bar{Y} - \hat{Y}|^{\frac{1}{2}}}$. Therefore from (6.1) and (6.2), we obtain $|\bar{Y} - \hat{Y}|^{1+\frac{3}{2}(n-1)+\frac{n}{2q}} \leq C|\bar{x} - \hat{x}|^{\frac{n}{2q} - \frac{n-1}{2}}$ and the theorem follows.

□

Under all previous hypotheses on the target, we now show interior $C^{1,\alpha}$ estimates.

Theorem 6.2. *With the assumptions of Theorem 6.1, there exist positive constants δ and C, depending on $X^\star$, such that $u \in C^{1,\alpha}(B_\delta(x^\star))$ with $|Du(\bar{x}) - Du(\hat{x})| \leq C|\bar{x} - \hat{x}|^\alpha$ for all $\hat{x}, \bar{x} \in B_\delta(x^\star)$.*

Proof. By Theorem 6.1 we get that $F_u(x)$ is a singleton for each $x \in B_\delta(x^\star)$. Take $\bar{x} \in B_{\delta/2}(x^\star)$.

We first show that if $\bar{Y} \in F_u(\bar{x})$, then u is differentiable at $\bar{x}$, and $D_i u(\bar{x}) = D_i \phi(\bar{x}, \bar{Y}, \bar{X})$ where $\bar{X} = (\bar{x}, u(\bar{x}))$.

For $0 < h < \delta/2$ we have

$$\frac{u(\bar{x} + he_i) - u(\bar{x})}{h} - D_i\phi(\bar{x}, \bar{Y}, \bar{X}) \le \frac{\phi(\bar{x} + he_i, \bar{Y}, \bar{X}) - \phi(\bar{x}, \bar{Y}, \bar{X})}{h} - D_i\phi(\bar{x}, \bar{Y}, \bar{X})$$

$$= D_i\phi(\bar{x} + \tilde{h}e_i, \bar{Y}, \bar{X}) - D_i\phi(\bar{x}, \bar{Y}, \bar{X}), \quad \text{for some } 0 \le \tilde{h} \le h$$

$$= D_{i,i}\phi(\bar{x} + \hat{h}e_i, \bar{Y}, \bar{X})\tilde{h},$$

for some $0 \le \hat{h} \le \tilde{h}$. Hence $\dfrac{u(\bar{x} + he_i) - u(\bar{x})}{h} - D_i\phi(\bar{x}, \bar{Y}, \bar{X}) \le Ch$.

To prove the inequality in the opposite direction, let $\hat{x} = \bar{x} + he_i$, $\hat{Y} \in F_u(\hat{x})$, and $\hat{X} = (\hat{x}, u(\hat{x}))$. We have that

$$D_i\phi(\bar{x}, \bar{Y}, \bar{X}) - \frac{(u(\hat{x}) - u(\bar{x}))}{h} \le D_i\phi(\bar{x}, \bar{Y}, \bar{X}) - \frac{(\phi(\hat{x}, \hat{Y}, \hat{X}) - \phi(\bar{x}, \hat{Y}, \hat{X}))}{h}$$

$$= D_i\phi(\bar{x}, \bar{Y}, \bar{X}) - D_i\phi(\tilde{x}, \hat{Y}, \hat{X})$$

for some $\tilde{x} \in [\bar{x}, \hat{x}]$. From the estimates for the derivatives of ϕ from Subsection (2.4), we can also write

$$D_i\phi(\bar{x}, \bar{Y}, \bar{X}) - D_i\phi(\tilde{x}, \hat{Y}, \hat{X})$$

$$= D_i\phi(\bar{x}, \bar{Y}, \bar{X}) - D_i\phi(\tilde{x}, \bar{Y}, \bar{X}) + D_i\phi(\tilde{x}, \bar{Y}, \bar{X}) - D_i\phi(\tilde{x}, \hat{Y}, \bar{X}) + D_i\phi(\tilde{x}, \hat{Y}, \bar{X}) - D_i\phi(\tilde{x}, \hat{Y}, \hat{X})$$

$$\le C(|\bar{x} - \hat{x}| + |\bar{Y} - \hat{Y}| + |\bar{X} - \hat{X}|),$$

with C a structural constant. On the other hand, since u is Lipschitz (with a constant depending only on structure), we have $|\bar{X} - \hat{X}| \le |\bar{x} - \hat{x}| + |u(\bar{x}) - u(\hat{x})| \le C|\bar{x} - \hat{x}|$. In addition, using Theorem 6.1, we get that $|\bar{Y} - \hat{Y}| \le C \max\{|\bar{x} - \hat{x}|, |\bar{x} - \hat{x}|^\alpha\} \le Ch^\alpha$ where C is a structural constant depending also on $X^\star$. This yields $\dfrac{u(\bar{x} + he_i) - u(\bar{x})}{h} - D_i\phi(\bar{x}, \bar{Y}, \bar{X}) \ge -Ch^\alpha$, completing the proof that $D_i u(\bar{x}) = D_i\phi(\bar{x}, \bar{Y}, \bar{X})$.

We finally prove that $u \in C^{1,\alpha}(B_{\delta/2}(x^\star))$. Let $\bar{x}, \hat{x} \in B_{\frac{\delta}{2}}(x^\star)$, $\bar{Y} \in F_u(\bar{x})$, $\hat{Y} \in F_u(\hat{x})$, and set $\bar{X} = (\bar{x}, u(\bar{x}))$, $\hat{X} = (\hat{x}, u(\hat{x}))$. We then have $|D_i u(\bar{x}) - D_i u(\hat{x})| = |D_i\phi(\bar{x}, \bar{Y}, \bar{X}) - D_i\phi(\hat{x}, \hat{Y}, \hat{X})| \le |D_i\phi(\bar{x}, \bar{Y}, \bar{X}) - D_i\phi(\hat{x}, \bar{Y}, \bar{X})| + |D_i\phi(\hat{x}, \bar{Y}, \bar{X}) - D_i\phi(\hat{x}, \hat{Y}, \bar{X})| + |D_i\phi(\hat{x}, \hat{Y}, \bar{X}) - D_i\phi(\hat{x}, \hat{Y}, \hat{X})| \le C\{|\bar{x} - \hat{x}| + |\bar{Y} - \hat{Y}| + |\bar{X} - \hat{X}|\} \le C|\bar{x} - \hat{x}|^\alpha$, with a structural constant C depending also on $X^\star$.

$\square$

Corollary 6.3. *Suppose u is a refractor solving (5.6), conditions (a), (b), and (c) from Proposition 5.4 hold, and $\mu = f\,dx$ with $f \in L^p(\Omega)$ for some $p > n$. Suppose that the target Σ is regular from $X^\star = (x^\star, u(x^\star))$ in the sense of Definition 3.1, and the local condition (6.1) is satisfied at $X^\star$. Then there exist δ and C depending on $X^\star$ such that $u \in C^{1,\alpha}(B_\delta(x^\star))$ with α given in Theorem 6.1, and $1/p + 1/q = 1$.*

Proof. By Hölder's inequality $\mu(B) \le \|f\|_p |B|^{1/q}$, for all balls $B \subset \Omega$. Therefore from (5.7) we obtain condition (6.2), and the corollary follows.   □

7. Example of a target for refraction

If we assume the target set Σ is given by the graph of a C^2 function $y_{n+1} = \psi(y)$, then we are going to find a condition on ψ so that (3.3) holds locally. We will see that for (3.3) to hold, the graph of the target needs to satisfy a quantitative condition, see (7.3). As in Subsection 4, we set $H(v, X) = s_X(\Lambda(v))Q(v)$. We have for $Y \in \Sigma$ that $Y = X + s_X(\Lambda(v))(-Q(v)v, Q(v) + \kappa)$. Then we can write $y_{n+1} = x_{n+1} + s_X(\Lambda(v))(Q(v) + k) = \psi(x - s_X(\Lambda(v)) Q(v)v)$. We therefore get that H satisfies the implicit equation

$$(7.1) \qquad x_{n+1} + H(v, X)\left(\frac{Q(v) + \kappa}{Q(v)}\right) = \psi(x - H(v, X)v).$$

Proceeding as in Subsection 4, setting $G = 1/H$ we need to check that

$$(7.2) \qquad \langle D^2 G(0, X)\xi, \xi \rangle < 0$$

for all $|\xi| = 1$. We will prove (7.2) for $X = 0$ and by continuity (7.2) will hold for X in a neighborhood of $X = 0$. Indeed, we next compute $D_{ij}G(0, 0)$ in terms of ψ. Since $G = 1/H$, we first proceed to calculate the derivatives of H. From (2.1), $Q(v) = \dfrac{\sqrt{1 + (1 - \kappa^2)|v|^2} - \kappa}{1 + |v|^2}$ and so $Q(0) = 1 - \kappa$, $D_i Q(0) = 0$ and $D_{ij}Q(0) = -\delta_{ij}(1 - \kappa)^2$. From (7.1), we get $H(0, 0) = (1 - \kappa)\psi(0)$, $D_i H(0, 0) = -(1 - \kappa)^2 D_i\psi(0) \psi(0)$, and

$$D_{ij}H(0, 0) = (1 - \kappa)^3 \psi(0)\left(\frac{-\kappa}{1 - \kappa}\delta_{ij} + 2 D_i\psi(0) D_j\psi(0) + D_{ij}\psi(0) \psi(0)\right).$$

Now, noticing that $D_{ij}G = H^{-3}(2D_i H D_j H - H D_{ij}H)$ and inserting the above expressions yields

$$D_{ij}G(0, 0) = \frac{1 - \kappa}{\psi(0)}\left(\frac{\kappa}{1 - \kappa}\delta_{ij} - \psi(0) D_{ij}\psi(0)\right).$$

We therefore obtain that

$$\langle D^2 G(0, 0)\xi, \xi \rangle = \frac{1 - \kappa}{\psi(0)}\left(\frac{\kappa}{1 - \kappa} - \psi(0)\langle D^2\psi(0)\xi, \xi \rangle\right).$$

Thus (7.2) is equivalent to

$$(7.3) \qquad \frac{\kappa}{1 - \kappa} < \psi(0) \langle D^2\psi(0)\xi, \xi \rangle,$$

for all unit vectors ξ.

We have then proved that for Σ the graph of ψ, and for $X_0 = (0, 0), Y_0 = (0, \psi(0))$, the condition $\dfrac{d^2}{d\epsilon^2}\langle D^2\phi(x_0, Y_\epsilon, X_0)\eta, \eta \rangle|_{\epsilon=0} < 0$ holds for all unit vectors $\xi \perp \eta$, provided (7.3) holds. By continuity we have that $\dfrac{d^2}{d\epsilon^2}\langle D^2\phi(x, Y(\epsilon), X)\eta, \eta \rangle|_{\epsilon=0} \le -C|\xi|^2|\eta|^2$ for all vectors $\xi \perp \eta$ and for all X close to X_0 and Y close to Y_0.

Remark 7.1. We show that if the target Σ is a horizontal plane, then (3.2) does not hold when $n \geq 2$. Indeed, we show it is not true that $\phi(x, Y(\lambda), X_0) \geq \min\{\phi(x, \bar{Y}, X_0), \phi(x, \hat{Y}, X_0)\}$. In $\mathbb{R}^3$, let $\bar{Y} = (0, -1, 0)$, $\hat{Y} = (0, 1, 0)$, $\bar{\phi}(x, y) = -\dfrac{kb}{1 - k^2} - \left(\dfrac{b^2}{(1 - k^2)^2} - \dfrac{x^2 + (y + 1)^2}{1 - k^2}\right)^{1/2}$ and $\hat{\phi}(x, y) = -\dfrac{kb}{1 - k^2} - \left(\dfrac{b^2}{(1 - k^2)^2} - \dfrac{x^2 + (y - 1)^2}{1 - k^2}\right)^{1/2}$, with b sufficiently large. Let $X_0 = (0, 0, \bar{\phi}(0, 0)) = (0, 0, \hat{\phi}(0, 0))$ and we have $\bar{\phi}(x, y) = \phi((x, y), \bar{Y}, X_0)$ $\hat{\phi}(x, y) = \phi((x, y), \hat{Y}, X_0)$. We also have that $\Lambda(v(\lambda))$ is on the two dimensional plane $x = 0$, and so $[\bar{Y}, \hat{Y}]_{X_0} = [\bar{Y}, \hat{Y}]$ is the straight segment from $\bar{Y}$ to $\hat{Y}$. We have $Y(1/2) = (0, 0, 0)$. Let $\phi_0(x, y) := \phi((x, y), Y(1/2), X_0) = -\dfrac{kb_0}{1 - k^2} - \left(\dfrac{b_0^2}{(1 - k^2)^2} - \dfrac{(x^2 + y^2)}{1 - k^2}\right)^{1/2}$, where b_0 is chosen so that $\phi_0(0, 0) = \bar{\phi}(0, 0)) = \hat{\phi}(0, 0)$. That is, $b_0 = \dfrac{kb + (b^2 - (1 - k^2))^{\frac{1}{2}}}{1 + k}$. We claim that $\phi_0(x, 0) < \bar{\phi}(x, 0) = \hat{\phi}(x, 0)$ for $x \neq 0$, x small. Let $g(x) = \phi_0(x, 0)$ and $h(x) = \bar{\phi}(x, 0)$. Then one can check that $(g - h)'(0) = 0$ and $(g - h)''(0) < 0$ which gives the claim.

8. On the definition of refractor

We can define refractor with ellipsoids touching u from below, that is, the ellipsoids enclose u. In fact, we can define analogously to (4.1)

$$(8.1) \qquad \tilde{F}_u(x_0) = \{Y \in \Sigma : u(x) \geq \phi(x, Y, X_0) \text{ for all } x \in \Omega\},$$

and we can say u is a *refractor* if $\tilde{F}_u(x_0) \neq \emptyset$ for all $x_0 \in \Omega$. With this new definition, we can obtain the same regularity results as with definition (4.1) by changing the inequalities accordingly. We indicate the changes. In Definition 3.1, condition (3.2) is replaced by:

$$(8.2) \qquad \phi(x, Y_Z(\lambda), Z) \leq \max\left\{\phi(x, \bar{Y}, Z), \phi(x, \hat{Y}, Z)\right\} - C_1 |\bar{Y} - \hat{Y}|^2 |x - z|^2.$$

Inequality (3.3) is replaced by:

$$(8.3) \qquad \frac{d^2}{d\epsilon^2}\langle D^2\phi(z, Y_\epsilon, Z)\eta, \eta\rangle|_{\epsilon=0} \geq C(X_0) |\xi|^2 |\eta|^2.$$

Lemma 5.1 is replaced by the convexity of u. The inequality (5.1) is replaced by:

$$(8.4) \quad u(x) - \phi(x, Y, X_0^\star) \geq -C_0 |\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| - C_1 |Y(\lambda) - Y||x - x_0| + C_2 |\bar{Y} - \hat{Y}|^2 |x - x_0|^2,$$

and in the proof of Lemma 5.2, min is replaced by max with the corresponding changes in the inequalities. For the example in Section 7, we now get that condition (7.3) is replaced by

$$(8.5) \qquad \frac{\kappa}{1 - \kappa} > \psi(0)\langle D^2\psi(0)\xi, \xi\rangle,$$

for all unit vectors ξ.

9. Regularity results for the parallel reflector problem in the near field case

In this section we shall prove results that are similar to the ones proved in the previous sections but for the reflector problem. Since the arguments are similar, we will omit most details. For results in both the near field and far field reflector with one point source see [CO08] and [KO97].

9.1. **Reflection.** We first review the process of reflection. Our setting is $\mathbb{R}^{n+1}$, and points will be denoted by $X = (x, x_{n+1})$. We consider parallel rays moving in the direction e_{n+1}. Let T be a hyperplane in $\mathbb{R}^{n+1}$ with upper unit normal N and let $X \in T$. By Snell law of reflection, a ray coming from below with direction e_{n+1} that hits T at X is reflected in the unit direction $\Lambda = e_{n+1} - 2(e_{n+1} \cdot N)N$. In particular, if $v \in \mathbb{R}^n$ and $N = \dfrac{(-v, 1)}{(1 + |v|^2)^{\frac{1}{2}}}$, then the reflected direction is the unit vector

$$(9.1) \qquad \Lambda(v) = \left(\frac{2v}{1 + |v|^2}, \frac{|v|^2 - 1}{1 + |v|^2} \right).$$

The reflected ray consists of the points $Y = X + s\Lambda$, for $s > 0$. We have in mind here that $v = Du(x)$ and $X = (x, u(x))$, where u is a reflector.

If $b > 0$, and $Y \in \mathbb{R}^{n+1}$, then the set of $X \in \mathbb{R}^{n+1}$ with $|X - Y| + x_{n+1} - y_{n+1} = b$ is a downwards paraboloid with focus at Y. It can be written as the graph of the function $p(x, Y) = y_{n+1} + \dfrac{b^2 - |x - y|^2}{2b}$. The ray with direction e_{n+1} that hits the graph of p at $X = (x, p(x, Y))$ is reflected in direction $Y - X$. If $Y, X_0 \in \mathbb{R}^{n+1}$ with X_0 not in the vertical ray with direction $-e_{n+1}$ emanating from Y, then there exists a unique paraboloid with focus at Y passing through X_0. Such a paraboloid is described by the function $p(x, Y, X_0) = y_{n+1} + \dfrac{b^2 - |x - y|^2}{2b}$ where $b = |X_0 - Y| + x_{0_{n+1}} - y_{n+1}$. We will frequently use the following fact:

(9.2) \qquad if the focus Y of the paraboloid defined by $p(x, Y, X_0)$ satisfies

$\qquad Y = X_0 + s\Lambda(v)$ for some $s > 0$ and $v \in \mathbb{R}^n$, then $D_x p(x_0, Y, X_0) = v$,

where $\Lambda(v)$ is given by (9.1).

9.2. **Set up.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\mathscr{C}_\Omega$ be the cylinder $\mathscr{C}_\Omega = \Omega \times [0, M]$. For a fixed number $\beta > 0$ we define the region

$$\mathcal{T} = \{Y \in \mathbb{R}^{n+1} : |X - Y| + x_{n+1} - y_{n+1} \geq \beta \text{ for all } X \in \mathscr{C}_\Omega\}.$$

The set $\mathcal{T}$ consists of the points Y such that the cylinder $\mathscr{C}_\Omega$ is contained outside the interior of the paraboloid $|X - Y| + x_{n+1} - y_{n+1} = \beta$. We will assume that the target Σ has convex hull bounded and contained in $\mathcal{T}$.

Proceeding as in Subsection 2.4, it is easy to see that $\left| \dfrac{\partial p}{\partial x_{0_{n+1}}}(x, Y, X_0) \right|, \left| \dfrac{\partial^2 p}{\partial x_i \partial y_j}(x, Y, X_0) \right|$ are bounded uniformly for all $x \in$, $Y \in K \subset \mathcal{T}$, and $X_0 \in \mathscr{C}_\Omega$ for $1 \leq i \leq n$ and

$1 \leq j \leq n + 1$, where K is compact. Hence as in Lemmas 2.2 and 2.3, we obtain the following.

Lemma 9.1. *If $X_0 \in \mathscr{C}_\Omega$ with $X_0 + he_{n+1} \in \mathscr{C}_\Omega$, and $Y, \bar{Y} \in \Sigma$, then we have*

$$|p(x, Y, X_0) - p(x, Y, X_0 + he_{n+1})| \leq C|h|,$$

and

$$|p(x, Y, X_0) - p(x, \bar{Y}, X_0)| \leq C|x - x_0||Y - \bar{Y}|,$$

for all $x \in \Omega$.

9.3. **Hypothesis on the target set.** Given $\bar{Y}, \hat{Y} \in \Sigma$, $X_0 \in \mathscr{C}_\Omega$, we let $\bar{v} = D_x p(x_0, \bar{Y}, X_0)$, $\hat{v} = D_x p(x_0, \hat{Y}, X_0)$, and $v(\lambda) = (1 - \lambda)\bar{v} + \lambda\hat{v}$, for $\lambda \in [0, 1]$. Consider the set of points

$$C(X_0, \bar{Y}, \hat{Y}) = \{Y = X_0 + s\Lambda(v(\lambda)) : s > 0, \lambda \in [0, 1]\},$$

where Λ is given by (9.1). Notice that from (9.2), if $Y = X_0 + s\Lambda(v(\lambda))$, then $D_x p(x_0, Y, X_0) = v(\lambda)$.

For $X_0 \in \mathscr{C}_\Omega$ and for $\bar{Y}, \hat{Y} \in \Sigma$ we assume $[\bar{Y}, \hat{Y}]_{X_0} := C(X_0, \bar{Y}, \hat{Y}) \cap \Sigma$ is a curve joining $\bar{Y}$ and $\hat{Y}$.

We introduce the following condition on the target Σ, similar to Definition 3.1 for refractors.

Definition 9.2. *If $X_0 \in \mathscr{C}_\Omega$ we say that the target Σ is regular from X_0 if there exists a neighborhood U_{X_0} and positive constants C_{X_0}, depending on U_{X_0}, such that for all $\bar{Y}, \hat{Y} \in \Sigma$ and $Z = (z, z_{n+1}) \in U_{X_0}$ we have*

$$(9.3) \qquad \max\left\{p(x, \bar{Y}, Z), p(x, \hat{Y}, Z)\right\} \geq p(x, Y_Z(\lambda), Z) + C_{X_0} |\bar{Y} - \hat{Y}|^2|x - z|^2$$

for all $x \in \Omega$, $1/4 \leq \lambda \leq 3/4$, and $Y_Z(\lambda) = Z + s_Z(\Lambda(v(\lambda))) \Lambda(v(\lambda))$. Here $\bar{v} = D_x\phi(z, \bar{Y}, Z)$, $\hat{v} = D_x\phi(z, \hat{Y}, Z)$, and $v(\lambda) = (1 - \lambda)\bar{v} + \lambda\hat{v}$.

As in the case of refractors, we also have a differential condition that is equivalent to (9.3). This is the contents of the following theorem.

Theorem 9.3. *The target Σ is regular from $X_0 \in \mathscr{C}_\Omega$ in the sense of Definition 9.2 if and only if there exists a neighborhood U_{X_0} and a positive constant C depending on X_0 such that for all $Y_0 \in \Sigma$ and for all $Z \in U_{X_0}$ and for all vectors ξ and η such that $\xi \perp \eta$ we have*

$$(9.4) \qquad \frac{d^2}{d\epsilon^2} \left\langle D_x^2 p(z, Y_\epsilon, Z)\eta, \eta\right\rangle\Big|_{\epsilon=0} \geq C |\xi|^2|\eta|^2.$$

where $Y_\epsilon = Z + s(\Lambda(v + \epsilon\xi))\Lambda(v + \epsilon\xi)$ and $v = Dp(z, Y_0, Z)$.

Proof. That (9.2) implies (9.4), follows in the same way as (3.2) implies (3.3) in Theorem 3.2.

Let $X_0 \in \mathscr{C}_\Omega$, and $Z \in U_{X_0}$. We first show that if (9.4) holds for $\xi \perp \eta$, then it holds for all vectors ξ, η. In fact, we have

$$(9.5) \qquad \left\langle D_x^2 p(z, Y_\epsilon, Z)\eta, \eta\right\rangle = -\frac{|\eta|^2}{2} \frac{1 + |v_0 + \epsilon \xi|^2}{s_Z\left(\Lambda(v_0 + \epsilon \xi)\right)}.$$

So if $\xi \cdot \eta \neq 0$, we pick η' with $|\eta'| = |\eta|$ and $\eta' \cdot \xi = 0$, and we have $\langle D_x^2 p(z, Y_\epsilon, Z) \eta, \eta \rangle = \langle D_x^2 p(z, Y_\epsilon, Z) \eta', \eta' \rangle$. Let $\bar{Y}, \hat{Y} \in \Sigma$, $\bar{v} = Dp(z, \bar{Y}, Z)$, $\hat{v} = Dp(z, \hat{Y}, Z)$, $\xi = \hat{v} - \bar{v}$, and $Y_Z(\lambda) = Z + s_Z \left(\Lambda(\bar{v} + \lambda\xi)\right) \Lambda(\bar{v} + \lambda\xi)$. We then have

$$\frac{d}{d\lambda^2} \left\langle D_x^2 p\left(z, Y_Z(\lambda), Z\right) \eta, \eta \right\rangle \geq C |\xi|^2 |\eta|^2,$$

for $0 \leq \lambda \leq 1$. Fix $x \in \Omega$, and let $f(\lambda) = \langle D_x^2 p(z, Y_Z(\lambda), Z) (x - z), (x - z) \rangle$. Since $f''(\lambda) \geq C|\xi|^2 |x - z|^2 \geq C|\bar{Y} - \hat{Y}|^2 |x - z|^2$, where the second inequality follows from the analogue of (2.13) for reflectors, it follows that $(1 - \lambda)f(0) + \lambda f(1) \geq f(\lambda) + C\lambda(1 - \lambda)|x - z|^2 |\bar{Y} - \hat{Y}|^2$. Therefore,

$$\max\left\{ p(x, \bar{Y}, Z), p(x, \hat{Y}, Z) \right\} - p(x, Y_Z(\lambda), Z)$$

$$\geq (1 - \lambda)p(x, \bar{Y}, Z) + \lambda p(x, \hat{Y}, Z) - p(x, Y_Z(\lambda), Z)$$

$$= \frac{1}{2} \left\langle \left((1 - \lambda) D_x^2 p(z, \bar{Y}, Z) + \lambda D_x^2 p(z, \hat{Y}, Z) - D_x^2 p(z, Y_Z(\lambda), Z)\right)(x - z), x - z \right\rangle$$

$$\geq C\lambda(1 - \lambda)|x - z|^2 |\bar{Y} - \hat{Y}|^2,$$

for all $x \in \Omega$, and $0 \leq \lambda \leq 1$. This completes the proof of the theorem.

$\square$

9.4. Definition of parallel reflector and hypothesis on the measures.

We say $u : \Omega \to [0, M]$ is a parallel reflector from Ω to Σ if for each $x_0 \in \Omega$, there exists $Y \in \Sigma$ such that $u(x) \geq p(x, Y, X_0)$ for all $x \in \Omega$, where $X_0 = (x_0, u(x_0))$. In this case, we say $Y \in F_u(x_0)$. Any reflector is Lipschitz in Ω with a uniform Lipschitz constant depending on the bounds for the derivatives of p, which are uniform for $Y \in K \Subset \mathcal{T}$, $x \in \Omega$ and $X_0 \in \mathscr{C}_\Omega$.

Existence of solutions with this definition of parallel reflector can be proved in a way similar to the existence of parallel refractors as done in [GT13]. We omit the corresponding details.

We make the following hypothesis on the measures.

Also similarly to (6.1), we introduce the following local condition at $X_0 \in \mathscr{C}_\Omega$ between the measure σ and target Σ: There exist a neighborhood U_{X_0} and a constant $\hat{C} > 0$ depending on X_0 such that

$$(9.6) \qquad \sigma\left(N_\mu\left(\left\{[\bar{Y}, \hat{Y}]_Z : \lambda \in [1/4, 3/4]\right\}\right) \cap \Sigma\right) \geq \hat{C} \mu^{n-1} |\bar{Y} - \hat{Y}|$$

for any $\bar{Y}, \hat{Y} \in \Sigma$, $Z \in U_{X_0}$ and for all $\mu > 0$ small (depending on X_0). Here $N_\mu(E)$ denotes the μ- neighborhood of the set E in R^{n+1}.

The following lemma is the analogue of Lemma 5.1.

Lemma 9.4. *If u is a parallel reflector, then for any $\bar{x}, \hat{x} \in \Omega$ and for any $s \in [0, 1]$ we have $u((1 - s)\bar{x} + s\hat{x}) \leq (1 - s)u(\bar{x}) + su(\hat{x}) + C|\bar{x} - \hat{x}|^2 s(1 - s)$, where C is a structural constant.*

The following lemma is the analogue of Lemma 5.2.

Lemma 9.5. *Let u be a parallel reflector such that the target Σ is regular from $X^\star = (x^\star, u(x^\star))$ in the sense of Definition 9.2. There exist constants δ and C_1, depending on $X^\star$ such that if $\bar{x}, \hat{x} \in B_\delta(x^\star) \subset \Omega$, $\bar{Y} \in F_u(\bar{x})$, $\hat{Y} \in F_u(\hat{x})$ and $|\bar{Y} - \hat{Y}| \geq |\bar{x} - \hat{x}|$, then there exists $x_0 \in \overline{\bar{x}\hat{x}}$ such that if $X_0^\star = (x_0, u(x_0))$, then we have for all $Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0^\star}$ and for all $Y \in \Sigma$ and for all $x \in \Omega$ that the following inequality holds*

$$u(x) - p(x, Y, X_0^\star) \geq -C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| - C|Y(\lambda) - Y||x - x_0| + C_1\lambda(1-\lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2.$$

We remark that C is a structural constant.

Proof. The proof is very much the same as in the refractor problem. We indicate the main points.

Since Σ is regular from $X^\star$, there exists a neighborhood $U_{X^\star}$ of $X^\star$ such that (9.3) holds for all $\bar{Y}, \hat{Y} \in \Sigma$ and all $Z \in U_{X^\star}$. Since parallel reflectors are uniformly Lipschitz in Ω, there exists $\delta > 0$ such that $(x, u(x)) \in U_{X^\star}$ for all $x \in B_\delta(x^\star)$. For $x \in \Omega$ we have $u(x) \geq \max\{p(x, \bar{Y}, \bar{X}), p(x, \hat{Y}, \hat{X})\}$, and there exists $x_0 \in [\bar{x}, \hat{x}]$ such that $p(x_0, \bar{Y}, \bar{X}) = p(x_0, \hat{Y}, \hat{X}) := x_{0_{n+1}}$. Set $X_0 = (x_0, x_{0_{n+1}})$, and $X_0^\star = (x_0, u(x_0)) \in U_{X^\star}$ and notice that $u(x_0) \geq x_{0_{n+1}}$. Similarly as in Lemma 5.2 we get

$$u(x) \geq \max\{p(x, \bar{Y}, \bar{X}), p(x, \hat{Y}, \hat{X})\} = \max\{p(x, \bar{Y}, X_0), p(x, \hat{Y}, X_0)\}$$
$$= \max\{p(x, \bar{Y}, X_0^\star), p(x, \hat{Y}, X_0^\star)\} - E.$$

Using Lemmas 9.1 and 9.4, and proceeding as in the proof of claim (5.2), we get $0 \leq E \leq C\left(u(x_0) - x_{0_{n+1}}\right) \leq C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}|$. Hence $\max\{p(x, \bar{Y}, X_0^\star), p(x, \hat{Y}, X_0^\star)\} - E \geq \max\{p(x, \bar{Y}, X_0^\star), p(x, \hat{Y}, X_0^\star)\} - C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| \geq p(x, Y_{X_0^\star}(\lambda), X_0^\star) + C_1\lambda(1-\lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2 - C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}|$, where we have used (9.3). Now using the second inequality in Lemma 9.1, we get $p(x, Y_{X_0^\star}(\lambda), X_0^\star) + C_1\lambda(1-\lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2 - C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| \geq p(x, Y, X_0^\star) - C|x - x_0||Y - Y(\lambda)| + C_1\lambda(1-\lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2 - C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}|$, which proves the lemma. $\square$

We then obtain results similar to Theorems 5.3 and 6.1 for parallel reflectors. We remark that from Theorem 9.3, inequality (9.4) implies (9.3) for all $x \in \Omega$, and therefore for parallel reflectors the arguments from Section 4 are not needed.

Theorem 9.6. *Suppose u is a parallel reflector, the target Σ is regular from $X^\star = (x^\star, u(x^\star))$ in the sense of Definition 9.2. There exist a ball $B_\delta(x^\star) \subset \Omega$, and a constant $M > 0$ depending on $X^\star$ such that if $\hat{x}, \bar{x} \in B_{\delta/2}(x^\star)$, $\bar{Y} \in F_u(\bar{x})$, $\hat{Y} \in F_u(\hat{x})$ are such that*

(9.7)
$$\frac{|\bar{Y} - \hat{Y}|}{|\bar{x} - \hat{x}|} \geq \max\left\{1, \left(\frac{2M}{\delta}\right)^2\right\},$$

then there exists $x_0 \in \overline{\bar{x}, \hat{x}}$ such that we have

$$N_\mu\left(\left\{Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0^\star} : \lambda \in [1/4, 3/4]\right\}\right) \cap \Sigma \subseteq F_u(B_\eta(x_0)),$$

with $X_0^\star = (x_0, u(x_0))$, $\mu = |\bar{Y} - \hat{Y}|^{\frac{3}{2}}|\bar{x} - \hat{x}|^{\frac{1}{2}}$ and $\eta = M\dfrac{|\bar{x} - \hat{x}|^{\frac{1}{2}}}{|\bar{Y} - \hat{Y}|^{\frac{1}{2}}}$.

Theorem 9.7. *Suppose u is a parallel reflector, the target Σ is regular from $X^\star = (x^\star, u(x^\star))$ in the sense of Definition 9.2, and there exist constants $C_0 > 0$ and $1 \leq q < \dfrac{n}{n-1}$ such that*

(9.8) $$\sigma\left(F_u(B_\eta)\right) \leq C_0 \, \eta^{n/q}$$

for all balls $B_\eta \subseteq \Omega$. Suppose in addition that the local condition (9.6) is satisfied at $X^\star$.

Then there exist a ball $B_\delta(x^\star) \subset \Omega$, and a constant $M > 0$ depending on $X^\star$, such that if $\hat{x}, \bar{x} \in B_{\delta/2}(x^\star)$, $\bar{Y} \in F_u(\bar{x})$, $\hat{Y} \in F_u(\hat{x})$ are such that

(9.9) $$\frac{|\bar{Y} - \hat{Y}|}{|\bar{x} - \hat{x}|} \geq \max\left\{1, \left(\frac{2M}{\delta}\right)^2\right\},$$

then we have $|\bar{Y} - \hat{Y}| \leq C_1 \, |\bar{x} - \hat{x}|^\alpha$ with $\alpha = \dfrac{\dfrac{n}{2q} - \dfrac{n-1}{2}}{1 + \dfrac{3}{2}(n-1) + \dfrac{n}{2q}}$, where C_1 depends only on C_0 and $\hat{C}$ in (9.6), and therefore from $X^\star$.

We can then proceed in exactly the same way as in the refractor problem to get the analogues of Theorem 6.2, and Corollary 6.3 for parallel reflectors.

Remark 9.8. We show here that when the target Σ is contained in a non-vertical hyperplane, then condition (9.4) holds. Notice that from (9.5), it is sufficient to show that

(9.10) $$\left\langle D_v^2\left(\frac{1 + |v|^2}{s_X(\Lambda(v))}\right)\xi, \xi\right\rangle \leq -C$$

for all $v \in \mathbb{R}^n$ and $|\xi| = 1$ ($|\eta| = 1$). Let $w \in \mathbb{R}^n$, and $\Sigma = \{(y, y_{n+1}), y_{n+1} = y \cdot w\}$. Suppose $0 < C_1 \leq x_{n+1} - x \cdot w \leq C_2$ for all $X \in C_\Omega$. We calculate $\dfrac{1 + |v|^2}{s_X(\Lambda(v))}$. From (9.1) we have

$$y - x = s_X(\Lambda(v)) \frac{2v}{1 + |v|^2}$$

$$y_{n+1} - x_{n+1} = s_X(\Lambda(v)) \frac{|v|^2 - 1}{1 + |v|^2}.$$

Hence $\dfrac{1 + |v|^2}{s_X(\Lambda(v))} = \dfrac{2v \cdot w + 1 - |v|^2}{x_{n+1} - x \cdot w}$, and then $\left\langle D_v^2\left(\dfrac{1 + |v|^2}{s_X(\Lambda(v))}\right)\xi, \xi\right\rangle = -\dfrac{2}{x_{n+1} - x \cdot w} \leq -C$.

If $(y, y_{n+1}) \in \Sigma$ and Σ is a vertical plane, then $y \cdot w = 0$ and we get $\dfrac{1 + |v|^2}{s_X(\Lambda(v))} = -\dfrac{2v \cdot w}{x \cdot w}$, so $D_v^2\left(\dfrac{1 + |v|^2}{s_X(\Lambda(v))}\right) = 0$.

Remark 9.9. Similarly to Section 8, we can define parallel reflector by taking paraboloids enclosing the solution. In other words, we can say $u : \Omega \to [0, M]$ is a parallel reflector from Ω to Σ if for each $x_0 \in \Omega$, there exists $Y \in \Sigma$ such that $u(x) \leq p(x, Y, X_0)$ for all $x \in \Omega$, where $X_0 = (x_0, u(x_0))$. The same regularity results hold with this definition of solution by changing accordingly the inequalities in the conditions on the target. Indeed, with the notation in Subsection 9.3, condition (9.3) is replaced by

$$(9.11) \qquad \min \left\{ p(x, \bar{Y}, Z), p(x, \hat{Y}, Z) \right\} \leq p(x, Y_Z(\lambda), Z) - C_{X_0} |\bar{Y} - \hat{Y}|^2 |x - z|^2.$$

The analogue of condition (9.4) is now

$$\frac{d^2}{d\epsilon^2} \left\langle D_x^2 p(z, Y_\epsilon, Z) \eta, \eta \right\rangle \Big|_{\epsilon=0} \leq -C|\xi|^2 |\eta|^2,$$

and as before this is equivalent to (9.11). The concavity of u now replaces Lemma 9.4, the inequality in Lemma 9.5 is replaced by

$$u(x) - p(x, Y, X_0^\star) \leq C |\bar{Y} - \hat{Y}||\bar{x} - \hat{x}| + C |Y(\lambda) - Y||x - x_0| - C_1 \lambda (1 - \lambda)|\bar{Y} - \hat{Y}|^2 |x - x_0|^2,$$

and in the proof the max is replaced by the min. In contrast with Remark 9.8, in this case a target contained in a hyperplane cannot satisfy (9.11).

10. Summary of Notation

- $\Omega \subset \mathbb{R}^n$, extended source domain.
- $\Sigma \subset \mathbb{R}^{n+1}$, target or receiver.
- $\mathscr{C}_\Omega = \Omega \times (0, M)$.
- $\kappa = n_1/n_2 < 1$, n_i refractive indices of the media.
- $\Lambda(v) = (-Q(v)v, Q(v) + \kappa)$; $Q(v) = \dfrac{-\kappa + \sqrt{1 + (1 - \kappa^2)|v|^2}}{1 + |v|^2}$. $\Lambda(v)$ represents the unit vector direction of the refracted ray when the normal is $N = \dfrac{(-v, 1)}{\sqrt{1 + |v|^2}}$ and the incident direction is e_{n+1}, the $n+1$-th unit coordinate vector in $\mathbb{R}^{n+1}$.
- $c(X, Y) = |X - Y| + \kappa (x_{n+1} - y_{n+1})$; $X = (x, x_{n+1})$, $Y = (y, y_{n+1})$.
- $E(Y, b) = \{X : c(X, Y) = b\}$.
- $E^-(Y, b) = \left\{ X \in E(Y, b) : x_{n+1} \leq y_{n+1} - \dfrac{\kappa b}{1 - \kappa^2} \right\}$.
- Given $0 < \delta < 1$, the target or receiver is contained in the following admissible region:

$$\mathcal{T} = \left\{ Y \in \mathbb{R}^{n+1} : \forall X_0 \in \mathscr{C}_\Omega \text{ we have } X_0 \in E^-(Y, c(X_0, Y)) \text{ and } \Omega \subset B_{\delta c(X_0, Y)/\sqrt{1-\kappa^2}}(y) \right\}.$$

- $\phi(x, Y, X_0) = y_{n+1} - \dfrac{k c(X_0, Y)}{1 - k^2} - \sqrt{\dfrac{c(X_0, Y)^2}{(1 - k^2)^2} - \dfrac{|x - y|^2}{1 - k^2}}$.
- $C(X_0, \bar{Y}, \hat{Y}) = \{X_0 + s \Lambda(v(\lambda)) : s > 0, \lambda \in [0, 1]\}$ where $\bar{Y}, \hat{Y} \in \mathcal{T}$, $X_0 \in \mathscr{C}_\Omega$, $\bar{v} = D_x \phi(x_0, \bar{Y}, X_0)$, $\hat{v} = D_x \phi(x_0, \hat{Y}, X_0)$, and $v(\lambda) = (1 - \lambda)\bar{v} + \lambda \hat{v}$, with $0 \leq \lambda \leq 1$.
- $[\bar{Y}, \hat{Y}]_{X_0} = \Sigma \cap C(X_0, \bar{Y}, \hat{Y})$.
- $F_u(x_0) = \{Y \in \Sigma : u(x) \leq \phi(x, Y, X_0) \text{ for all } x \in \Omega\}$.

- Given $X \in \mathcal{C}_\Omega$, each $Y \in \Sigma$ has the parametrization $Y = X + s_X(\Lambda)\Lambda$, with $|\Lambda| = 1$; s_X varies with the point X, and is Lipschitz in Λ.

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