ON THE HARNACK INEQUALITY FOR VISCOSITY SOLUTIONS OF NON-DIVERGENCE EQUATIONS

Cristian E. Gutiérrez

These notes are devoted to understand and explore the method of Caffarelli-Krylov-Safonov, [**Ca1**], [**K-S**], to study the regularity properties of solutions of non-divergence linear elliptic equations. We give complete proofs of the results and we compare the different notions of solutions: strong and viscosity solutions. Also, clarifications and modifications of some arguments in the method are described. The method has been extended by Caffarelli to obtain results for non-linear partial differential equations, in particular for the Monge-Ampére equation, see [**Ca2**], [**Ca3**].

The differential operators considered here are of the form

$$Lu = a_{ij}(x)D_{ij}u,$$

where $A(x) = (a_{ij}(x))$ is an elliptic matrix whose entries are real-valued measurable functions of the variable $x \in R^n$, i.e., there exist positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2,$$

for all $\xi \in R^n$.

Let $\Omega \subset R^n$ an open set. Assuming that the positive semi-definite matrix $A(x)$ and the function $f(x)$ are well defined for *all* $x \in \Omega$, we may define the notion of viscosity solution of $Lu = f$. The function $u \in C(\Omega)$ is a viscosity sub-solution (super-solution) of the equation $Lu = f$ in Ω if given $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ attains a local maximum (minimum) at x_0 then $L\phi(x_0) \geq (\leq) f(x_0)$.

The function $u \in C(\Omega)$ is a viscosity solution of $Lu = f$ if it is both a viscosity sub-solution and a viscosity super-solution.

Remarks on the notion of viscosity solution. Viscosity solutions versus strong solutions

1. If $u \in C^2(\Omega)$ is a classical sub-solution (super-solution) then u is a viscosity sub-solution (super-solution). In fact, let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum at x_0. Since $u - \phi \in C^2(\Omega)$, we have $\nabla(u - \phi)(x_0) = 0$ and the Hessian matrix $D^2(u - \phi)(x_0) \leq 0$, (i.e, it is non-positive definite). Consequently,

$$L(u - \phi)(x_0) = \text{trace}(A(x_0)D^2(u - \phi)(x_0)) \leq 0,$$

since $A(x) \geq 0$. Therefore

$$Lu(x_0) \leq L\phi(x_0),$$

and since $Lu(x_0) \geq f(x_0)$, the remark follows.

2. If $u \in C^2(\Omega)$ and u is a viscosity sub-solution (super-solution) then u is a classical sub-solution (super-solution). In fact, given $x_0 \in \Omega$ this immediatly follows by taking $\phi = u$.

3. We may also define viscosity sub-solution (super-solution) in the following way. A function $u \in C(\Omega)$ is a viscosity sub-solution (super-solution) of the equation $Lu = f$ in Ω if given $x_0 \in \Omega$ and given a polynomial p of degree less than or equal to 2 such that $u - p$ attains a local maximum (minimum) at x_0 then $Lp(x_0) \geq (\leq) f(x_0)$. This definition is equivalent to the one given before. In fact, let $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum at x_0. By taking the second order Taylor approximation of ϕ we have that

$$\phi(x) = p(x) + \circ(|x - x_0|^2), \qquad \text{as } x \to x_0,$$

where

$$p(x) = \phi(x_0) + \nabla\phi(x_0) \cdot (x - x_0) + \frac{1}{2}\langle D^2\phi(x_0)(x - x_0), x - x_0\rangle.$$

Let $\epsilon > 0$ and set $p_\epsilon(x) = p(x) + \epsilon|x - x_0|^2$. The function $u - p_\epsilon$ has a local maximum at x_0 for all $\epsilon > 0$. Therefore

$$Lp_\epsilon(x_0) \geq f(x_0).$$

But $Lp_\epsilon(x_0) = Lp(x_0) + 2\epsilon \sum_{i=1}^n a_{ii}(x_0)$. Since $Lp(x_0) = L\phi(x_0)$, by letting $\epsilon \to 0$ the claim follows.

4. Gilbarg and Trudinger, [Gi-Tr], chap. 8, introduced the notion of "strong" solution. A function u is a strong sub-solution (super-solution) of the equation $Lu = f$ in Ω if $u \in W^{2,n}(\Omega)$ and the weak second order derivatives $D_{ij}u$ of u satisfy

$$Lu(x_0) = \sum_{i,j=1}^n a_{ij}(x_0) D_{ij}u(x_0) \geq (\leq) f(x_0)$$

for almost every $x_0 \in \Omega$. We show that if u is a viscosity sub-solution (super-solution) and $u \in W^{2,n}(\Omega)$ then u is a strong sub-solution (super-solution). Since $u \in W^{2,n}(\Omega)$, by Theorem 12, page 204, of [Cal-Z], the function u has derivatives of order 2 at almost every $x_0 \in \Omega$, i.e., there exists a polynomial $P(x, x_0)$ of degree ≤ 2 such that

$$u(x) = P(x, x_0) + \circ(|x - x_0|^2),$$

as $x \to x_0$. We also have that

$$P(x, x_0) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + \frac{1}{2}\langle D^2u(x_0)(x - x_0), x - x_0\rangle,$$

where $\nabla u(x_0)$ is the regular gradient at x_0 and $D^2u(x_0)$ is the Hessian in the sense of distributions.

Let $\phi(x) = P(x, x_0) + \epsilon|x - x_0|^2$, with $\epsilon > 0$. The function $u - \phi$ has a local maximum at x_0, and since u is a viscosity sub-solution we have

$$L\phi(x_0) \geq f(x_0).$$

But $L\phi(x_0) = \text{trace}(A(x_0)D^2u(x_0)) + 2\epsilon \sum_{i=1}^n a_{ii}(x_0)$ and the claim follows by letting $\epsilon \to 0$. Note that this argument only requieres that $u \in t_2^\infty(x_0)$ for almost all $x_0 \in \Omega$. By theorem 12 of [**Cal-Z**], if $u \in L_2^p(\Omega)$ with $p > n/2$ then $u \in t_2^q(x_0)$ for almost every $x_0 \in \Omega$ and $p \leq q \leq \infty$. The definition of the clases t_p^q is in [**Cal-Z**].

5. In case that the equation has continuous coefficients, we show the converse of 4. Let $\Omega \subset \mathbf{R}^n$ be an open set, we have:

Theorem. *Let $u \in W^{2,n}(\Omega) \cap C(\bar{\Omega})$ be a strong sub-solution(super-solution) of $Lu = f$ in Ω and $f \in C(\Omega)$. Then u is a viscosity sub-solution(super-solution) of $Lu = f$ in Ω.*

In order to prove this it is enough to show the following. Let B denote the open unit ball in $\mathbf{R}^n$.

Proposition. *Let $u \in W^{2,n}(B)$ be a strong sub- solution of $Lu = f$ in B. Then u is a viscosity sub-solution of $Lu = f$ at the origin.*

This will follow from three lemmas.

Lemma A. *Let u, u_k in $W^{2,n}(B)$ with $u_k \to u$ in $W^{2,n}(B)$. Then*

$$\int_B |\det(D^2u_k)|dx \to \int_B |\det(D^2u)|dx.$$

Lemma B. *$W^{2,n}(B) \subset C(B)$ and $u_k \to u$ in $W^{2,n}$ implies $u_k \to u$ locally uniformly on B.*

For $p \in \mathbf{R}^n$ set $u_p(x) = u(x) - p \cdot x$. For $u \in C(\bar{B})$ let Γ_ϵ denote the set of $x \in B$ such that for some $|p| \leq \epsilon$,

$$u_p(x) = \max\{u_p(y) : y \in \bar{B}\}.$$

Lemma C. *Let $u \in W^{2,n}(B) \cap C(\bar{B})$ and assume $\rho \equiv \max_{x \in \partial B} u(x) < u(0)$. If $\epsilon < u(0) - \rho$, then $|\Gamma_\epsilon| > 0$.*

Proof. Assume first $u \in C^2(B) \cap C(\bar{B})$. The idea is to show that the normal mapping of Γ_ϵ has positive measure. If $x \in \Gamma_\epsilon$ then $|\nabla u(x)| \leq \epsilon$ and $D^2u(x) \leq 0$. Thus $\nabla u(\Gamma_\epsilon)$ is a subset of the closed ball $\bar{B}_\epsilon(0)$. In fact it is onto $\bar{B}_\epsilon(0)$: if not there exists $p \in B_\epsilon(0)$ such that u_p is maximized at ∂B hence there exists an $x \in \partial B$ such that

$$\rho \geq u(x) \geq u(0) + p \cdot (x - 0) \geq u(0) - |p| \geq u(0) - \epsilon,$$

which contradicts the choice of ϵ. Therefore as in [**Gi-Tr**], page 221,

$$(*) \qquad \omega_n \epsilon^n = |B_\epsilon(0)| \leq \int_{\Gamma_\epsilon} |\det(D^2u)|\, dx.$$

Now if u is not C^2 choose $u_k \in C^2(B)$ approximating u in $W^{2,n}$. Since $u_k \to u$ pointwise it follows easily that

$$\Gamma_\epsilon \supset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \Gamma_\epsilon^k = \limsup_{k\to\infty} \Gamma_\epsilon^k.$$

By (*) applied to u_k we have

$$\omega_n \epsilon^n = |B_\epsilon(0)| \le \int_{\Gamma_\epsilon^k} |\det(D^2 u_k)|\, dx.$$

Since $u \in W^{2,n}$, by letting $k \to \infty$ in the last inequality and Fatou's lemma (version for $\limsup$) we have

$$\begin{aligned}
\omega_n \epsilon^n &\le \limsup_{k\to\infty} \int_{\Gamma_\epsilon^k} |\det(D^2 u_k)|\, dx \\
&\le \int_B \limsup_{k\to\infty} \chi_{\Gamma_\epsilon^k}(x) |\det(D^2 u_k)(x)|\, dx \\
&\le \int_B \limsup_{k\to\infty} \chi_{\Gamma_\epsilon^k}(x) |\det(D^2 u)(x)|\, dx \\
&+ \int_B \limsup_{k\to\infty} \chi_{\Gamma_\epsilon^k}(x) \left(|\det(D^2 u_k)(x)| - |\det(D^2 u)(x)| \right)\, dx \\
&\le \int_{\limsup_{k\to\infty} \Gamma_\epsilon^k} |\det(D^2 u)(x)|\, dx + \int_B \lim_{k\to\infty} \left(|\det(D^2 u_k)(x)| - |\det(D^2 u)(x)| \right)\, dx \\
&\le \int_{\Gamma_\epsilon} |\det(D^2 u)(x)|\, dx,
\end{aligned}$$

which shows that (*) holds for u and lemma C follows.

Proof of Proposition. Let $\phi \in C^2(\bar{B})$ and suppose $u - \phi$ has a strict absolute maximum at the origin over $\bar{B}$ (if the maximum is not strict we add a quadratic function to make it strict, i.e., $u(x) - (\phi(x) + \delta|x|^2)$ has a strict max at 0. We let $\delta \to 0$ at the end). Then $|\Gamma_\epsilon| > 0$, where Γ_ϵ corresponds to $u - \phi$, for ϵ small. If D is the set of points x where $Lu(x) \ge f(x)$ and $\nabla u(x)$ and $D^2 u(x)$ exist, then there are $x_\epsilon \in \Gamma_\epsilon \cap D$ which yields

$$(u - \phi)(x_\epsilon) \ge (u - \phi)(y) + p_\epsilon \cdot (x_\epsilon - y), y \in \bar{B}.$$

Here $|p_\epsilon| \le \epsilon$. Since the origin is a strict max, $x_\epsilon \to 0$ as $\epsilon \to 0$, $\nabla u(x_\epsilon) = \nabla \phi(x_\epsilon)$, $D^2 u(x_\epsilon) \le D^2 \phi(x_\epsilon)$. Thus $L\phi(x_\epsilon) \ge Lu(x_\epsilon) \ge f(x_\epsilon)$. Let $\epsilon \to 0$ to get result.

Question: remarks 4 and 5 raise the following question. If u is a viscosity solution, is it true that $u \in W^{2,n}$?

The convex envelope of a viscosity super-solution

The following lemma is the key to prove the Aleksandrov-Bakelman-Pucci maximum principle for viscosity super-solutions. This lemma implies pointwise estimates of the second order derivatives of the convex envelope of a viscosity super-solution in terms of the right-hand side of the equation.

LEMMA 1. *Let u be a viscosity super-solution of $Lu = f$ in the unit ball $B_1 = B_1(0)$, with f continuous in $\bar{B}_1$. Let $x_0 \in B_1$ and assume that $f(x_0) \leq \mu$ and there exists a convex function w in B_1 such that*

$$w(x) \leq u(x), \qquad \forall x \in B_1, \tag{1}$$

$$w(x_0) = u(x_0).$$

We then have

(a) $\mu \geq 0$,

and

(b) there exists a constant $C = C(\lambda, \Lambda)$ and an affine function $l(x)$ such that $l(x_0) = w(x_0)$ and

$$l(x) \leq w(x) \leq l(x) + C\mu|x - x_0|^2,$$

for $|x - x_0|$ sufficiently small. In case $\mu = 0$ then the last inequality holds with $C\mu$ replaced by $C\epsilon$ for any $\epsilon > 0$ and for $|x - x_0|$ sufficiently small (smallness possibly depending on ϵ).

Proof. Since w is convex, there is a supporting hyperplane at x_0, i.e., there exists an affine function $l(x)$ such that $l(x_0) = w(x_0)$ and

$$w(x) \geq l(x), \qquad \forall x \in B_1.$$

If we set $\tilde{w}(x) = w(x) - l(x)$, $\tilde{u}(x) = u(x) - l(x)$, then $\tilde{w}(x) \leq \tilde{u}(x)$, $\tilde{w}(x_0) = \tilde{u}(x_0) = 0$, and $\tilde{u}$ is a viscosity super-solution. In view of this we may substract from u and w the supporting hyperplane and assume that $u(x) \geq w(x) \geq 0$, and $w(x_0) = u(x_0) = 0$. For simplicity, we shall also assume that $x_0 = 0$, the changes in the argument below for $x_0 \neq 0$ are minor.

Let us first prove (a). Consider $\phi(x) = -\epsilon|x|^2$, where $\epsilon > 0$. We have $\phi \in C^2(B_1)$ and by (1)

$$u(x) - \phi(x) = u(x) + \epsilon|x|^2 \geq w(x) \geq 0 = u(0) - \phi(0),$$

i.e., $u - \phi$ has a local minimum at 0, and since u is a viscosity super-solution, $L\phi(0) \leq f(0)$. Also

$$L\phi(x) = -2\epsilon \sum_{i=1}^{n} a_{ii}(x) \geq -2\epsilon n\Lambda.$$

Then $-2\epsilon n\Lambda \leq \mu$ and (a) follows by letting $\epsilon \to 0$.

We prove (b) in the following way. Let us first assume that $\mu > 0$. By the uniform continuity of f there exists δ depending on μ such that

$$|f(x) - f(x_0)| < \frac{\mu}{2},$$

for $|x - x_0| \leq \delta$. Consider for $0 \leq \rho \leq d(x_0, \partial B_1)$

$$\alpha(\rho) = \sup_{|x-x_0|\leq\rho} w(x).$$

We claim that

$$\alpha(\rho) \leq \frac{4(1+\Lambda)}{\lambda}\mu\rho^2,$$

for every ρ such that

$$0 < \rho \leq \rho_0 = \min\{\frac{\delta}{\sqrt{M+1}}, d(x_0, \partial B_1)\},$$

where M depends only on n and the ellipticity constants (see definition below). The claim immediatly implies (b) with $C = \dfrac{4(1+\Lambda)}{\lambda}$, since $w(x) \leq \alpha(|x - x_0|)$.

We shall prove the claim assuming $x_0 = 0$, the proof is similar in other case.

Suppose that the claim is false, then there would exist a ρ such that $0 < \rho \leq \rho_0$ and we would have

$$\alpha(\rho) > \frac{4(1+\Lambda)}{\lambda}\mu\rho^2.$$

The function w is continuous and convex, therefore the maximum $\alpha(\rho)$ is attained at some point z, $|z| = \rho$. Let us assume that this point z has the form $z = \rho e_n$, where e_n is the n-th coordinate unit vector in R^n. We claim that the convexity of w forces that

$$w(x_1, ..., x_{n-1}, \rho) \geq \alpha(\rho).$$

In fact, take the supporting hyperplane to w at ρe_n, i.e., there exists $p \in \mathbf{R}^n$ such that

$$w(x) \geq p \cdot (x - \rho e_n) + w(\rho e_n)$$

for all $x \in B_1$. In particular, for $|x| = \rho$ we have

$$\alpha(\rho) = w(\rho e_n) \geq w(x) \geq p \cdot (x - \rho e_n) + w(\rho e_n),$$

i.e., $p \cdot (x - \rho e_n) \leq 0$ $\forall |x| = \rho$. This implies that $p = (0, \lambda\rho)$ for some $\lambda < 0$. Therefore, the supporting hyperplane has the form $\lambda\rho(x_n - \rho) + w(\rho e_n)$, and our claim follows.

Let $M = \left(4n\dfrac{1+\Lambda}{\lambda}\right)^{1/2}$, and consider the cylinder

$$R_\rho = \{x : |(x_1, ..., x_{n-1})| \leq M\rho, \qquad |x_n| \leq \rho\}$$

Note that by the choice of ρ_0 we have

$$R_\rho \subset B_\delta(0).$$

Also consider the polynomial

$$P(x) = (1+\Lambda)(x_n+\rho)^2 - \frac{\lambda}{n}\sum_{i=1}^{n-1} x_i^2.$$

The polynomial P satisfies the following properties:

(i) $LP(x) > 2\lambda$, for every x.

Property (i) follows from

$$LP(x) = a_{nn}(x)2(1+\Lambda) - 2\frac{\lambda}{n}\sum_{i=1}^{n-1} a_{ii}(x) \geq 2\lambda(1+\Lambda) - 2\frac{\lambda}{n}(n-1)\Lambda = 2\lambda(1+\frac{\Lambda}{n}).$$

(ii) $P(x) \leq 0$, for $x \in \partial R_\rho \setminus \{x : x_n = \rho\}$.

To show (ii), we first assume that $|(x_1, ..., x_{n-1})| = M\rho$ and $-\rho \leq x_n < \rho$. In this case we have

$$P(x) = (1+\Lambda)(x_n+\rho)^2 - \frac{\lambda}{n}(M\rho)^2 < 4(1+\Lambda)\rho^2 - \frac{\lambda}{n}M^2\rho^2 = 0,$$

by the choice of M. If $|(x_1, ..., x_{n-1})| \leq M\rho$ and $x_n = -\rho$, then $P(x) = -\frac{\lambda}{n}\sum_{i=1}^{n-1} x_i^2 \leq 0$.

(iii) $P(x) \leq 4(1+\Lambda)\rho^2$ on the set $\{x : x_n = \rho\}$.

Therefore, we have that

$$P(x) \leq 0 \leq w(x) \leq u(x), \qquad \forall x \in \partial R_\rho \setminus \{x : x_n = \rho\},$$

and for $x \in \{x : x_n = \rho\}$, since $\mu > 0$, we have

$$P(x) \leq 4(1+\Lambda)\rho^2 < \frac{\lambda}{\mu}\alpha(\rho) \leq \frac{\lambda}{\mu}w(x_1, ..., x_{n-1}, \rho) \leq \frac{\lambda}{\mu}u(x_1, ..., x_{n-1}, \rho).$$

Consequently

$$\frac{\mu}{\lambda}P(x) \leq w(x) \leq u(x), \qquad \forall x \in \partial R_\rho.$$

Therefore $u(x) - \frac{\mu}{\lambda}P(x) \geq 0$ on ∂R_ρ, and $u(0) - \frac{\mu}{\lambda}P(0) = -\frac{\mu}{\lambda}P(0) < 0$. Since u is continuous, this implies that the function $u(x) - \frac{\mu}{\lambda}P(x)$ must have an absolute negative minimum inside R_ρ, say at a point y. But, since u is a viscosity super-solution we must have

$$L(\frac{\mu}{\lambda}P)(y) \leq f(y),$$

and by property (i) this implies

$$2\mu < L(\frac{\mu}{\lambda}P)(y) \leq f(y).$$

But $R_\rho \subset B_\delta(0)$, therefore $|f(y) - f(0)| < \mu/2$, and combining the inequalities we obtain

$$2\mu < f(y) < \frac{3\mu}{2}$$

which is a contradiction and the claim is proved.

In case $\mu = 0$ we apply the argument above with $\mu = \epsilon > 0$.

REMARKS TO LEMMA 1.

Lemma 1 has a pointwise character, as it is stated the function w may vary with the point x_0. If we assume that w is the convex envelope of u then the points x_0 in lemma 1 are the contact points. In the following remarks we assume that w is the same for all points satisfying the hypotheses of lemma 1, in particular, we may take w to be the convex envelope of u. We set $H = \{$ the set of points x_0 satisfying the hypotheses of lemma 1$\}$, in case w is the convex envelope $H = \{$the contact points$\}$.

REMARK 1.

Given a point x_0 satisfying the hypotheses of Lemma 1 the convex function w has a unique supporting hyperplane at x_0. In fact, suppose that at a point x_0 there are two supporting hyperplanes H_1 and H_2 given by affine functions $l_1(x)$ and $l_2(x)$ respectively. By Lemma 1 we have that

$$0 \leq w(x) - l_i(x) \leq C_i|x - x_0|^2,$$

for $i = 1, 2$ and x in a neighborhood of x_0. Therefore,

$$|l_1(x) - l_2(x)| \leq |l_1(x) - w(x)| + |l_2(x) - w(x)| \leq C_1|x - x_0|^2 + C_2|x - x_0|^2.$$

Then $\nabla(l_1 - l_2)(x_0) = 0$, and since $l_i(x) = b_i(x_0) \cdot (x - x_0) + w(x_0)$, we obtain $b_1(x_0) = b_2(x_0)$.

REMARK 2.

Lemma 1 implies that w has gradient at each point x_0 satisfying the hypotheses and the gradient is locally a Lipschitz function, i.e, $w \in C^{1,1}$.

Let us first show that the gradient is well defined. At x_0 the function w has a supporting hyperplane given by the affine function $l(x) = b(x_0) \cdot (x - x_0) + w(x_0)$, $b(x_0) = (b_1(x_0), ..., b_n(x_0))$, and

$$0 \leq w(x) - l(x) \leq C|x - x_0|^2,$$

and

$$w(x) - w(x_0) = w(x) - l(x) + b(x_0) \cdot (x - x_0).$$

In particular,

$$\frac{w(x_0 + he_j) - w(x_0)}{h} = I + b_j(x_0),$$

with $|I| \leq C|h|$. So, by letting $h \to 0$ we get

$$D_j w(x_0) = b_j(x_0).$$

The next step is to show that $\nabla w(x)$ is locally a Lipschitz function. By this we exactly mean the following. Let $x_0 \in H$ and $f(x_0) \leq \mu$, then there exists a ball $B = B_\delta(x_0)$ such that for $x_1, x_2 \in B \cap H$ we have

$$|\nabla w(x_1) - \nabla w(x_2)| \leq C\mu|x_1 - x_2|.$$

In fact, if $f(x_0) \leq \mu$ then there exists a ball $B_{\delta_1}(x_0)$ such that $f(x) \leq 2\mu$ for all $x \in B_{\delta_1}(x_0)$, and we may assume that $d(B_{\delta_1}(x_0), \partial B_1) = d > 0$. By lemma 1 there exists $\delta > 0$ depending only on 2μ such that for each $z \in H \cap B_{\delta_1}(x_0)$

$$\nabla w(z) \cdot (x - z) + w(z) \leq w(x) \leq \nabla w(z) \cdot (x - z) + w(z) + 2C\mu|x - z|^2,$$

for any

$$|x - z| < \min\{\frac{\delta}{\sqrt{M+1}}, d(z, \partial B_1)\}.$$

Let

$$\delta_2 = \min\{\frac{\delta}{\sqrt{M+1}}, d, \delta_1\}$$

and we claim that for $x_1, x_2 \in B_{\delta_2/2}(x_0) \cap H$ we have

$$|\nabla w(x_1) - \nabla w(x_2)| \leq C\mu|x_1 - x_2|. \tag{3}$$

In fact, consider the supporting affine functions at each point:

$$l(x_i, x) = b(x_i) \cdot (x - x_i) + w(x_i), \quad i = 1, 2,$$

we shall first give an easy proof of (3) in one dimension. In such case we have $w'(x_i) = b(x_i)$, and by Lemma 1 we may write

$$w(x) = w(x_i) + w'(x_i)(x - x_i) + g(x_i, x)|x - x_i|^2,$$

where $g(x_i, x)$ is a function satisfying

$$0 \leq g(x_i, x) \leq 2C\mu,$$

whenever

$$|x - x_i| < \min\{\frac{\delta}{\sqrt{M+1}}, d(x_i, \partial B_1)\}.$$

If $|x_1 - x_2| < \delta_2/2$ we then may write

$$w(x_2) = w(x_1) + w'(x_1)(x_2 - x_1) + g(x_1, x_2)|x_2 - x_1|^2,$$

and

$$w(x_1) = w(x_2) + w'(x_2)(x_1 - x_2) + g(x_2, x_1)|x_1 - x_2|^2.$$

By comparison of these identities we get

$$(w'(x_1) - w'(x_2))(x_2 - x_1) = -g(x_1, x_2)|x_2 - x_1|^2 - g(x_2, x_1)|x_1 - x_2|^2,$$

which by the boundedness of g implies (3).

This proof does not seem to work when $n > 1$, in this case we shall use the following lemma due to Calderón and Zygmund, [**Cal-Z**], p. 182.

Lemma. *Let k be a non-negative integer. There exists a function $\phi \in C_0^\infty(R^n)$ with support contained in the unit ball centered at 0, such that for every polynomial P in R^n of degree less than or equal to k, and for every $\epsilon > 0$ we have*

$$\phi_\epsilon * P(x) = P(x), \qquad \forall x \in R^n.$$

As in one dimension we have

$$w(x) = w(x_i) + \nabla w(x_i) \cdot (x - x_i) + g(x_i, x)|x - x_i|^2,$$

$i = 1, 2$, and convolution of these identities with ϕ_ϵ gives

$$w * \phi_\epsilon(x) = w(x_i) + \nabla w(x_i) \cdot (x - x_i) + (g(x_i, x)|x - x_i|^2 * \phi_\epsilon)(x), \tag{4}$$

$i = 1, 2$. Let us consider a variable, say x_j, and take derivatives with respect to x_j in (4). This gives

$$D_j(w * \phi_\epsilon)(x) = D_j w(x_i) + D_j(g(x_i, x)|x - x_i|^2 * \phi_\epsilon)(x),$$

$i = 1, 2$. By setting $x = x - 1, x_2$ and by comparison of the identities we obtain

$$\begin{aligned}
&D_j w(x_1) - D_j w(x_2) \\
&= D_j(g(x_2, x)|x - x_2|^2 * \phi_\epsilon)(x_1) - D_j(g(x_1, x)|x - x_1|^2 * \phi_\epsilon)(x_1) \\
&= (g(x_2, x)|x - x_2|^2 * D_j\phi_\epsilon)(x_1) - (g(x_1, x)|x - x_1|^2 * D_j\phi_\epsilon)(x_1) \\
&= I - II.
\end{aligned}$$

We then have

$$\begin{aligned}
|I| &\le C \int_{|x_1 - z| \le \epsilon} |z - x_2|^2 \epsilon^{-n-1} |D_j\phi((x_1 - z)/\epsilon)| \, dz \\
&\le C' \int_{|x_1 - z| \le \epsilon} |z - x_2|^2 \epsilon^{-n-1} \, dz \\
&\le C' \int_{|x_2 - z| \le \epsilon + |x_1 - x_2|} |z - x_2|^2 \epsilon^{-n-1} \, dz.
\end{aligned}$$

By setting $\epsilon = |x_1 - x_2|$ we obtain

$$|I| \le C'|x_1 - x_2|.$$

Analogously for II we obtain

$$|II| \le C'|x_1 - x_2|,$$

and so

$$|D_j w(x_1) - D_j w(x_2)| \le C|x_1 - x_2|.$$

REMARK 3.

By a result of Aleksandrov, [**Ev-Ga**], p. 242, every convex function has second order derivatives almost everywhere. Lemma 1 says that in case that the convex function is the convex envelope of a super-solution its second order derivatives can be controlled a.e. by the RHS of the equation. In other words, Lemma 1 is used to prove that if x_0 is a point satisfying its hypotheses, then the second order derivatives of w exist at x_0 and are bounded by the coefficient of $|x - x_0|^2$. In particular, if $f(x_0) = 0$ then the second order derivatives of w at x_0 are bounded by $C\epsilon$ for any $\epsilon > 0$, and therefore are 0 at that point. This follows because by Remark 3 the function w is locally $C^{1,1}$ at the points x_0 for which the hypotheses of Lemma 1 hold. Note that at those points the gradient of w is locally a Lipschitz function with constant bounded by $C\mu$, therefore by Rademacher-Stepanov's theorem the second derivatives of w exist at almost all points x_0 and are bounded by $C\mu$. In case $f(x_0) > 0$ then we apply the Lemma to obtain that the second order derivatives of w at x_0 are bounded by $Cf(x_0)$. So in any case we obtain the following desired estimate:

$$|D_{ij}w(x_0)| \leq C \max\{f(x_0), 0\},$$

at almost any x_0 satisfying the hypotheses of Lemma 1.

The Aleksandrov-Bakelman-Pucci maximum principle for viscosity super-solutions

Let B be an open ball in R^n. Given $x_0 \in R^n$ and $t > 0$ we let

$$\Omega(x_0, t) = \{y \in R^n : y \cdot (\xi - x_0) + t > 0, \quad \forall \xi \in \bar{B}\}.$$

The set $\Omega(x_0, t)$ consists of the gradients of linear functions which have value t at x_0 and are positive in $\bar{B}$.

We define the normal mapping of a function u (or the subdifferential of u) in the usual way. That is, given a point $x_0 \in \Omega$ we consider the set

$$\chi_u(x_0) = \{p : u(x) \leq u(x_0) + p \cdot (x - x_0), \qquad \forall x \in \Omega\},$$

(note that this set may be empty) and the normal mapping of u or subdifferential of u at the point x_0 is defined by $\chi_u(x_0)$. Therefore $\chi_u : \Omega \to \mathcal{P}(R^n)$. In case the function u is differentiable at x_0 then $\chi_u(x_0)$ consists of only one point, i.e., $\chi_u(x_0) = \nabla u(x_0)$.

Let $\Gamma(u)(x)$ be the upper convex envelope of u in $\bar{B}$, i.e.,

$$\Gamma(u)(x) \geq u(x), \qquad \forall x \in \bar{B},$$

$-\Gamma(u)$ is convex in $\bar{B}$, and $\Gamma(u)$ is the minimal function with this property.

Let $\mathcal{C}(u)$ be the "contact set" of u, i.e.,

$$\mathcal{C}(u) = \{x \in B : \Gamma(u)(x) = u(x)\}.$$

LEMMA 2. *Let $u \in C(\bar{B})$ such that $u(x) \leq 0$ on ∂B, and $x_0 \in B$ with $u(x_0) > 0$. Then*

$$\Omega(x_0, u(x_0)) \subset \chi_{\Gamma(u)}(\mathcal{C}(u)).$$

Proof. Let $y \in \Omega(x_0, u(x_0))$, then

$$y \cdot (\xi - x_0) + u(x_0) > 0, \qquad \forall \xi \in \bar{B}.$$

Let

$$\lambda_0 = \inf\{\lambda : \lambda + y \cdot (\xi - x_0) \geq u(\xi), \forall \xi \in \bar{B}\}.$$

By continuity we have

$$\text{(a)} \qquad \lambda_0 + y \cdot (\xi - x_0) \geq u(\xi), \qquad \forall \xi \in \bar{B}.$$

Consider the minimum

$$\min_{\xi \in \bar{B}} \{\lambda_0 + y \cdot (\xi - x_0) - u(\xi)\},$$

this minimum is attained at some point $\xi \in \bar{B}$, and we have

$$\lambda_0 + y \cdot (\bar{\xi} - x_0) - u(\bar{\xi}) = 0.$$

Because on the contrary

$$\lambda_0 + y \cdot (\xi - x_0) - u(\xi) \geq \epsilon > 0, \qquad \forall \xi \in \bar{B},$$

and λ_0 would not be the infimum.

We claim that $\xi \in B$. Since $u|_{\partial B} \leq 0$, the claim will be proved if we show $u(\bar{\xi}) > 0$. By taking $\xi = x_0$ in (a) we get $u(x_0) \leq \lambda_0$, and consequently

$$y \cdot (\xi - x_0) + \lambda_0 > 0, \qquad \forall \xi \in \bar{B},$$

in particular, for $\xi = \bar{\xi}$

$$u(\bar{\xi}) = y \cdot (\bar{\xi} - x_0) + \lambda_0 > 0.$$

Therefore we proved that if $y \in \Omega(x_0, u(x_0))$ then there exists a point $\bar{\xi} \in B$ such that

$$u(\bar{\xi}) = y \cdot (\bar{\xi} - x_0) + \lambda_0,$$

and

$$u(\xi) \leq y \cdot (\xi - x_0) + \lambda_0, \qquad \forall \xi \in \bar{B}.$$

This means that $\lambda_0 + y \cdot (\xi - x_0)$ is a supporting hyperplane of u at $\bar{\xi}$. Therefore $\bar{\xi}$ is a point of concavity of u and we must have $\bar{\xi} \in \mathcal{C}(u)$. The fact that $\Gamma(u)$ is minimal implies that $\Gamma(u)(\xi) \leq y \cdot (\xi - x_0) + \lambda_0$, and consequently $y \in \chi_{\Gamma(u)}(\mathcal{C}(u))$ as we wished.

LEMMA 3. *Let v be a viscosity super-solution of the equation $Lu = f$ in B_1, the ball of radius 1 centered at the origin. Assume that $v(x) \geq 0, \forall x \in \partial B_1$, and let*

$$v^-(x) = -\min\{v(x), 0\},$$

i.e., v^- is the negative part of v. Let $\Gamma(v)(x)$ be the lower convex envelope of $-v^-(x)$ in B_1. We then have that

$$\max_{B_1} v^-(x) \leq C \left(\int_{\mathcal{C}(v)} f^+(x)^n \, dx \right)^{1/n},$$

where $\mathcal{C}(v) = \{x \in B_1 : v(x) = \Gamma(v)(x)\}$, and C is a constant depending only on the ellipticity and dimension.

Proof. We assume that there exists a point $x_0 \in B$ such that $v(x_0) < 0$, on the contrary there is nothing to prove. Consequently v^- is not identically 0.

We consider $\Gamma(v^-)(x)$ the upper convex envelope of v^-. We have $v^-(x) = 0$ on ∂B, and v^- is clearly continuous on $\bar{B}$. Let $M = \max_B v^-(x)$ which is attained at some point $x_0 \in B$. We may apply the previous lemma to obtain

$$\Omega(x_0, v^-(x_0)) \subset \chi_{\Gamma(v^-)}(\mathcal{C}(v^-)). \tag{6}$$

Let $w(x)$ be the lower convex envelope of $-v^-(x)$ in B_1. Note that $w(x) = -\Gamma(v^-)(x)$ and

$$w(x) \leq -v^-(x) \leq v(x),$$

and consequently $w(x) \leq 0$. The assumption that there is a point $x_0 \in B_1$ for which $v(x_0) < 0$, and the fact that w is convex imply that $w(x) < 0$ except possibly on the boundary of B_1. (Assume is not true, then there exists $z \in B_1$ such that $w(z) = 0$, and the line joining z and x_0 intersects the boundary at z_1. We have $w(z_1) \leq 0$, which gives contradiction with $w(z) = 0, w(x_0) < 0$ and the convexity.)

If $z \in \mathcal{C}(v^-)$ then $v(z) = w(z)$. This is because if $v(z) < 0$ then $v^-(z) = -v(z)$. If $v(z) \geq 0$ then $v^-(z) = 0 = -w(z)$ which by our assumption is impossible for points in the interior of B_1. Therefore

$$\mathcal{C}(v^-) \subset \{x \in B : v(x) = w(x)\} = H,$$

and since v is a super-solution, by Lemma 3 the function $w = -\Gamma(v^-)$ has gradient for $x \in H$, the gradient is locally a Lipschitz function and the second order derivatives exist a.e. in H and are bounded by $f^+(x)$. Note that the function $-w = \Gamma(v^-)$ is concave, and therefore $D^2(-w)(x) \leq 0$ for almost every $x \in \mathcal{C}(v^-)$.

On the other hand by (6) we have

$$|\Omega(x_0, v^-(x_0))| \leq |\chi_{\Gamma(v^-)}(\mathcal{C}(v^-))|,$$

and to control the RHS we use the formula of change of variables. In order to do this and due to the fact that the gradient of $\Gamma(v^-)$ is only locally Lipschitz in $\mathcal{C}(v^-)$ we use the following covering argument.

First of all the formula of change of variables we use is the following (see [Fe], p. 243). Let $f : R^n \to R^n$ be a Lipschitz function and $A \subset R^n$ a measurable set. If f is univalent in A then

$$|f(A)| = \int_A |J_f(x)|\, dx.$$

To simply the notation we set $h(x) = \Gamma(v^-)$ and $C = \mathcal{C}(v^-)$. Let $z \in C$ and let $B_\delta(z)$ be a ball such that ∇h is Lipschitz in $B_\delta(z) \cap C$. Then there exists a Lipschitz extension $\bar{h}$ of ∇h to $B_\delta(z)$, $\bar{h}(x) = \nabla h(x)$ for $x \in C \cap B_\delta(z)$, and the Lipschitz constant of $\bar{h}$ is bounded by a multiple of the constant of ∇h. Remember that ∇h is differentiable in C and $D^2h(x) \leq 0$ for a.e. $x \in C$. Let $\epsilon > 0$ and set

$$\Phi_\epsilon(x) = \bar{h}(x) - \epsilon x.$$

If $z \in C \cap B_\delta(z)$ is a density point then we have

$$D^2h(z) = J_{\bar{h}}(z).$$

Almost all points of C are density points. Now note that the mapping Φ_ϵ is univalent in $C \cap B_\delta(z)$ (or in the subset of density points of $C \cap B_\delta(z)$), in fact $D^2\Phi_\epsilon \leq \epsilon$. Therefore we may apply the formula of change of variables to obtain

$$|\Phi_\epsilon(C \cap B_\delta(z))| = \int_{C \cap B_\delta(z)} |J_{\Phi_\epsilon}(x)|\, dx.$$

Since $\Phi_\epsilon(x) \to \bar{h}(x)$, we have that

$$\nabla h(C \cap B_\delta(z)) = \liminf_{\epsilon \to 0} \Phi_\epsilon(C \cap B_\delta(z)),$$

and by Fatou's lemma

$$|\nabla h(C \cap B_\delta(z))| \leq \liminf_{\epsilon \to 0} |\Phi_\epsilon(C \cap B_\delta(z))|.$$

Consequently,

$$|\nabla h(C \cap B_\delta(z))| \leq \int_{C \cap B_\delta(z)} |\det(D^2h)(x)|\, dx.$$

Now for every $z \in C$ we have a ball $B_\delta(z)$ such that ∇h is Lipschitz in $C \cap B_\delta(z)$ and $D^2h(z) \leq 0$. By the Besicovith covering lemma there exists a countably family of balls

$B_k = B_{\delta_k}(z_k)$ covering C and with bounded overlaps. Then

$$\begin{aligned}
|\nabla h(C)| &\leq \sum_{k=1}^{\infty} |\nabla h(C \cap B_k)| \\
&\leq \sum_{k=1}^{\infty} \int_{C \cap B_k} |\det(D^2 h)(x)|\, dx \\
&= \sum_{k=1}^{\infty} \int_{C} \chi_{B_k}(x) |\det(D^2 h)(x)|\, dx \\
&\leq \int_{C} \left(\sum_{k=1}^{\infty} \chi_{B_k}(x) \right) |\det(D^2 h)(x)|\, dx \\
&\leq C \int_{C} |\det(D^2 h)(x)|\, dx.
\end{aligned}$$

Since $|\Omega(x_0, v^-(x_0))| = M^n |\Omega(x_0, 1)|$, the proof of Lemma 3 is complete.

The test function

LEMMA 4. *Let B_1, B_2 and B_3 balls in R^n with radius r_i, such that $B_1 \subset B_2 \subset B_3$, $d(B_1, \partial B_3) = \alpha > 3r_1$, and $r_2 < 2r_1$. There exists a function $p \in C^2(B_3)$ such that*

(i) $p(x) \geq 0, \forall x \in \partial B_3$.

(ii) $Lp(x) \leq 0, \forall x \notin B_1$, and for every elliptic operator L with ellipticity constants λ, Λ.

(iii) $p(x) \leq -1, \forall x \in B_2$.

(iv) $\|p\|_{C^2} \leq C(\lambda, \Lambda)$.

Note: an analogous version of this lemma holds with cubes instead of balls.

Proof. Let x_0 be the center of B_1, and $0 < \epsilon < r_1$. The function is defined in $B_3 \setminus B_1$ by

$$p(x) = K \left(\frac{1}{4} - \frac{1}{d(x, B_{r_1 - \epsilon}(x_0))} \right),$$

where K, ϵ, M are positive constants that will be chosen appropriatly to produce this marvelous function.

We set

$$d(x) = d(x, B_{r_1 - \epsilon}(x_0)),$$

and note that if $x \notin B_{r_1 - \epsilon}$ then

$$d(x) = |x - x_0| - (r_1 - \epsilon).$$

We have

$$\begin{aligned}
D_i d(x) =& \frac{x_i - x_i^0}{|x - x_0|} \\
D_{ij} d(x) =& \delta_{ij} |x - x_0|^{-1} - |x - x_0|^{-3} (x_i - x_i^0)(x_j - x_j^0) \\
D_i(d^{-M}) =& -M d^{-M-1} D_i d \\
D_{ij}(d^{-M}) =& M(M+1) d^{-M-2} D_i d D_j d - M d^{-M-1} D_{ij} d.
\end{aligned}$$

Therefore

$$Lp = -KM(M+1)d^{-M-2}\sum a_{ij}D_i d D_j d + KMd^{-M-1}\sum a_{ij}D_{ij}d = J,$$

and we want $J \leq 0$ for $x \notin B_1$. By the ellipticity we get

$$\begin{aligned} J &\leq -KM(M+1)d^{-M-2}\lambda + KMd^{-M-1}|x-x_0|^{-1}(-\lambda + n\Lambda) \\ &= KMd^{-M-2}\lambda\{-(M+1) + d|x-x_0|^{-1}(-\lambda + n\Lambda)\}, \end{aligned}$$

and if we pick M such that

$$\frac{M+1}{\gamma - 1} \geq 1,$$

where $\gamma = \Lambda/\lambda$, then the quantity between curly brackets is non-positive and so property (ii) follows.

To show (i) note that if $x \in \partial B_3$, since $d(B_1, \partial B_3) = \alpha > 3r_1$, we then have

$$d(x) \geq \alpha + \epsilon,$$

and therefore

$$d^{-M} \leq (\alpha + \epsilon)^{-M},$$

and since $K > 0$, (i) follows.

We now show that $p(x) \leq -1$ for $x \in B_2 \setminus B_1$ and for an appropriate choice of K. Let $x \in B_2$, since $B_1 \subset B_2$, we have $x_0 \in B_2$ and

$$\begin{aligned} d(x) &= |x - x_0| - (r_1 - \epsilon) \\ &\leq 2r_2 - r_1 - \epsilon < 2r_2 - \frac{r_2}{2} + \epsilon \\ &= \frac{3}{2}r_2 + \epsilon < 3r_1 + \epsilon < \alpha + \epsilon, \end{aligned}$$

and so

$$\max_{B_2} d(x) < \delta < \alpha + \epsilon.$$

By taking K such that

$$-\frac{1}{K} = \frac{1}{(\alpha+\epsilon)} - \frac{1}{\delta} \geq \frac{1}{(\alpha+\epsilon)} - \frac{1}{d(x)},$$

and we then obtain $p(x) \leq -1$ for $x \in B_2 \setminus B_{r_1-\epsilon}$.

Note that the function p defined above blows up on the boundary of $B_{r_1-\epsilon}$. The function p of the lemma is obtained by extending p from $B_3 \setminus B_{r_1-\frac{\epsilon}{2}}$ to all B_3 in such a way that the C^2-norm in B_3 is bounded by a multiple of the C^2-norm on $B_3 \setminus B_{r_1-\frac{\epsilon}{2}}$. This last norm depends only on the ellipticity constants.

First step in the proof of Harnack's inequality

LEMMA 5. *Let B_1, B_2 and B_3 balls in R^n such that $B_1 \subset B_2 \subset B_3$ and satisfying the hypotheses of Lemma 4. Let u be a non-negative viscosity super-solution of $Lu = f$ in the ball B_3 such that*

$$\inf_{B_2} u \leq 1.$$

Then there exist positive constants ϵ_0, M and μ depending only on the ellipticity constants Λ, λ of L such that if

$$\|f\|_{L^n(B_3)} \leq \epsilon_0$$

then

$$|\{x \in B_1 : u(x) < M\}| > \mu.$$

NOTE: this lemma holds with cubes instead of balls.

Proof. Let p be the test function of Lemma 4. Since p is C^2, the function $u + 2p$ is a viscosity super-solution of $Lu = f + 2Lp$ in B_3. In fact, let $x_0 \in B_3$ and let $\phi \in C^2(B_3)$ such that $u + 2p - \phi$ has a local minimum at x_0, then obviously $u - (\phi - 2p)$ has a local min at x_0, and since u is a viscosity super-solution

$$L(\phi - 2p)(x_0) \leq f(x_0),$$

and consequently the claim follows.

Also, $u(x) + 2p(x) \geq 0$ for $x \in \partial B_3$, and then we may apply Lemma 3 to $v = u + 2p$ in B_3. Set

$$v^-(x) = -\min(u(x) + 2p(x), 0),$$

we claim that

$$\max_{B_2} v^-(x) \geq 1.$$

In fact, since $\inf_{B_2} u \leq 1$, there exists $z \in B_2$ such that $u(z) \leq 1$ and consequently $u(z) + 2p(z) \leq -1$. Therefore $v^-(z) \geq 1$. Then application of Lemma 3 gives

$$1 \leq C \left(\int_{\mathcal{C}(v)} (g(x)^+)^n \, dx \right)^{1/n},$$

where $g(x) = f(x) + 2Lp(x)$. We recall that $\mathcal{C}(v) = \{x \in B_3 : v(x) = \Gamma(v)(x)\}$ where $\Gamma(v)(x)$ is the convex envelope of $-v^-$.

Set

$$\int_{\mathcal{C}(v)} (g(x)^+)^n \, dx = \int_{\mathcal{C}(v) \cap B_1} (g(x)^+)^n \, dx + \int_{\mathcal{C}(v) \cap B_1^c} (g(x)^+)^n \, dx = I + II.$$

Now, if $x \notin B_1$ then $Lp(x) \leq 0$, and so $g(x) \leq f(x)$ which gives

$$II \leq \int_{\mathcal{C}(v) \cap B_1^c} (f(x)^+)^n \, dx \leq \|f\|^n_{L^n(B_3)} \leq \epsilon_0^n.$$

On the other hand, for all $x \in B_3$ we have $Lp(x) \leq c$, and consequently $g(x) \leq f(x) + 2c$. So $g(x)^+ \leq (f(x) + 2c)^+ \leq f^+(x) + 2c$, and we obtain

$$\begin{aligned} I &\leq \int_{\mathcal{C}(v)\cap B_1} (f(x)^+ + 2c)^n \, dx \\ &\leq C_n \int_{\mathcal{C}(v)\cap B_1} (f(x)^+)^n \, dx + \int_{\mathcal{C}(v)\cap B_1} (2c)^n \, dx \\ &\leq C_n \|f\|_n^n + C(n, \lambda, \Lambda)|\mathcal{C}(v) \cap B_1| \\ &\leq C_n \epsilon^n + C(n, \lambda, \Lambda)|\mathcal{C}(v) \cap B_1|. \end{aligned}$$

By combining estimates we then get

$$1 \leq C_n \epsilon^n + C(n, \lambda, \Lambda)|\mathcal{C}(v) \cap B_1|,$$

and by picking $\epsilon_0 > 0$ such that $C_n \epsilon_0^n < 1/2$, we obtain that

$$|\mathcal{C}(v) \cap B_1| > \mu.$$

To end the proof of Lemma 5 we shall show that

$$\mathcal{C}(v) \cap B_1 \subset \{x \in B_1 : u(x) < M\},$$

for some $M > 0$ depending only on the ellipticity.

Since $-v^-(x) = \min(v(x), 0) \leq 0$, we always have $\Gamma(v)(x) \leq 0$. Let $x_0 \in \mathcal{C}(v)$, we claim that $v(x_0) < 0$. Since $v(x_0) = \Gamma(v)(x_0)$, we always have $v(x_0) \leq 0$. Suppose that $v(x_0) = 0$. Then $\Gamma(v)(x_0) = 0$, and we shall show that this implies that $\Gamma(v) \equiv 0$. If Γ were not identically zero there would exist a point x_1 such that $\Gamma(v)(x_1) < 0$. Let x_2 be a point on the line joining x_0 and x_1,but x_2 is outside the segment between x_0 and x_1. We have $\Gamma(v)(x_2) \leq 0$ and for some $0 < t < 1$ we have $x_0 = tx_1 + (1-t)x_2$ and by convexity

$$\begin{aligned} 0 = \Gamma(v)(x_0) &= \Gamma(v)(tx_1 + (1-t)x_2) \\ &\leq t\Gamma(v)(x_1) + (1-t)\Gamma(v)(x_2) < 0, \end{aligned}$$

a contradiction. Therefore Γ is identically zero and since $\Gamma \leq v$, we have $v \geq 0$. This last inequality is impossible since $\min_{B_2} v \leq -1$.

Consequently,

$$\begin{aligned} \mathcal{C}(v) \cap B_1 &\subset \{x \in B_1 : v(x) < 0\} \\ &= \{x \in B_1 : u(x) + 2p(x) < 0\} \\ &= \{x \in B_1 : u(x) < -2p(x)\}, \end{aligned}$$

but $p(x) \leq -1$ in B_1, so $m = \min_{B_1} p(x) \leq -1$, and

$$\{x \in B_1 : u(x) < -2p(x)\} \subset \{x \in B_1 : u(x) < -2m\},$$

and the claim is proved. Consequently the lemma holds with $M = -2m$.

The estimate of the distribution function of the super-solution u

Let Q be a cube in R^n. We divide Q into 2^n congruent subcubes Q^k with disjoint interiors such that $|Q| = 2^{kn}|Q^k|$. Given a cube Q^k its predecessor is denoted by $\tilde{Q}^k$, and Q^k is obtained from $\tilde{Q}^k$ by bisecting its sides, i.e., $|\tilde{Q}^k| = 2^n|Q^k|$.

Let A be a measurable set, and let Q be a cube in R^n. Suppose that $A \subset Q$ and

$$|A| \leq \delta|Q|,$$

for some $\delta > 0$. By the Calderón-Zygmund decomposition of A there exists a family $\{Q^k\}$ of dyadic subcubes of Q such that

$$A \subset \bigcup_{k=1}^{\infty} Q^k \qquad a.e.,$$

$$|A \cap Q^k| > \delta|Q^k|$$

and

$$|A \cap \tilde{Q}^k| \leq \delta|\tilde{Q}^k|$$

where $\tilde{Q}^k$ is the predecessor of Q^k.

LEMMA 6. *Let $A \subset B \subset Q^1$, measurable sets, Q^1 is a cube. Assume that there exists $\delta > 0$ such that*

(i) $|A| \leq \delta|Q^1|$

(ii) if Q^k is a dyadic cube of Q^1 from the Calderón-Zygmund decomposition of A which satisfies

$$|A \cap Q^i| > \delta|Q^i|$$

we then have $\tilde{Q}^i \subset B$.

Then we have $|A| \leq \delta|B|$.

LEMMA 7. *Let u be a super-solution satisfying the hypotheses of Lemma 5 over cubes. Then there exist positive constants $M > 1$ and $0 < \mu < 1$ depending only on the ellipticity constants such that*

$$|\{x \in Q_1 : u(x) \geq M^k\}| \leq (1-\mu)^k, \quad k = 1, ... \tag{5}$$

Proof. The proof is by induction. Suppose Q_1 is a unit cube, and let M and μ be the constants of Lemma 5. If $k = 1$ we have

$$Q_1 = \{x \in Q_1 : u(x) < M\} \cup \{x \in Q_1 : u(x) \geq M\},$$

and

$$\begin{aligned} 1 = |Q_1| =& |\{x \in Q_1 : u(x) < M\}| + |\{x \in Q_1 : u(x) \geq M\}| \\ \geq& \mu + |\{x \in Q_1 : u(x) \geq M\}|. \end{aligned}$$

Assume that (5) holds for $k-1$, and let

$$A = \{x \in Q_1 : u(x) \geq M^k\}, \qquad B = \{x \in Q_1 : u(x) \geq M^{k-1}\}.$$

Since $M > 1$, we have $A \subset B$ and by inductive hypothesis $|B| \leq (1-\mu)^{k-1}$. We claim that

$$|A| \leq (1-\mu)|B|.$$

To prove the claim we use Lemma 6. We need to show that (i) and (ii) of that lemma hold. Since $A \subset \{x \in Q_1 : u \geq M\}$, we have (i). To show (ii) let $\{Q^j\}$ be the Calderón-Zygmund decomposition of A with

$$|A \cap Q^j| > (1-\mu)|Q^j|, \tag{6}$$

and we want to show that the predecessor $\tilde{Q}^j$ of Q^j satisfies $\tilde{Q}^j \subset B$. Suppose by contradiction that this is not true, i.e., there exists a point $\bar{x} \in \tilde{Q}^j$ such that $\bar{x} \notin B$, that is $u(\bar{x}) < M^{k-1}$. Set

$$\bar{u}(x) = \frac{u(x)}{M^{k-1}}.$$

We shall apply Lemma 5 to $\bar{u}$, in order to do this we have to rescale. Let z_0 be the center of $\tilde{Q}^j$ and it edgelength is δ. Let Q_2 be the cube with side 2 center at the origin. We define the transformation

$$T(z) = \frac{\delta}{2} z + z_0,$$

and note that $T : Q_2 \to \tilde{Q}^j$. Consider

$$\tilde{u}(x) = \bar{u}(Tx).$$

If

$$\tilde{L} = \sum a_{ij}(Tz) D_{ij},$$

and

$$g(z) = \left(\frac{\delta}{2}\right) \frac{1}{M^{k-1}} f(Tz),$$

then we claim that $\tilde{u}$ is a viscosity super-solution of $\tilde{L}u = g$ in Q_2. In fact, let $x_0 \in Q_2$ and $\phi \in C^2(Q_2)$ such that $\tilde{u} - \phi$ has a local minimum at x_0. We write

$$\tilde{u}(x) - \phi(x) = \frac{u(Tx)}{M^{k-1}} - \phi(x) = \frac{u(Tx)}{M^{k-1}} - \phi(T^{-1}Tx),$$

and so $\dfrac{u}{M^{k-1}} - \phi \circ T^{-1}$ has a local minimum at Tx_0. Since u/M^{k-1} is a viscosity super-solution of $Lu = \dfrac{1}{M^{k-1}} f$, we have

$$L(\phi \circ T^{-1})(Tx_0) \leq \frac{1}{M^{k-1}} f(Tx_0),$$

but

$$L(\phi \circ T^{-1})(z) = (2/\delta)^2 a_{ij}(z) D_{ij}\phi(T^{-1}z),$$

and therefore

$$L(\phi \circ T^{-1})(Tx_0) = (2/\delta)^2 a_{ij}(Tx_0) D_{ij}\phi(x_0) = (2/\delta)^2 \tilde{L}\phi(x_0),$$

and the claim follows.

Since $\inf_{Q_2} \tilde{u} \leq 1$ and $\tilde{u} \geq 0$, we may apply Lemma 5 to $\tilde{u}$ provided $\|g\|_n \leq \epsilon$. We calculate

$$\begin{aligned}
\|g\|_n &= \left(\frac{\delta}{2}\right)^2 \frac{1}{M^{k-1}} \left(\int |f(Tz)|^n \, dz\right)^{1/n} \\
&= \frac{\delta}{2} \frac{1}{M^{k-1}} \left(\int |f(x)|^n \, dx\right)^{1/n} \\
&\leq \|f\|_n < \epsilon,
\end{aligned}$$

since $\delta < 1$ and $M > 1$.

We apply Lemma 5 to $\tilde{u}$ in the cube Q_1 such that $Q^j = T(Q_1)$ and we obtain

$$|\{x \in Q_1 : \tilde{u}(x) < M\}| > \mu.$$

Now note that if we set $E = \{x \in Q_1 : \tilde{u}(x) < M\}$ and $H = \{z \in Q^j : \dfrac{u(z)}{M^{k-1}} < M\}$ then $T^{-1}(H) = E$ and then by changing variables we get

$$|E| = \left(\frac{2}{\delta}\right)^n |H|.$$

Since $|Q^j| = (\delta/2)^n$, we obtain

$$|H| = |Q^j||E| > \mu|Q^j|.$$

This implies that

$$|\{x \in Q^j : u(x) \geq M^k\}| \leq (1-\mu)|Q^j|,$$

i.e.,

$$|A \cap Q^j| \leq (1-\mu)|Q^j|,$$

a contradiction with (6). Thus, the proof of the lemma is complete.

If a sub-solution is large at a point then it is large in a small neighborhood of that point

Lemma 8. *Let u be a sub-solution of $Lu = f$ in a cube Q_l. Assume that Q_1(unit cube) and Q_2 are sub-cubes of Q_l as in Lemma 5, and $u \geq 0$ in Q_l, $\inf_{Q_2} u \leq 1$ and $\|f\| \leq \epsilon$. Let M and μ be the constants of Lemma 5.*

There exist constants k_0 and c depending only on the ellipticity constants with the following property:

If $k > k_0$ and x_0 are such that
(1) $u(x_0) \geq M^k$
(2) $d(x_0, Q_1^c) \geq c(1-\mu)^{k/n}$,
then for $2\rho = c(1-\mu)^{k/n}$ we have

$$\sup_{B_\rho(x_0)} u \geq u(x_0)(1 + \frac{1}{M}). \tag{7}$$

Proof. The proof is by contradiction. Suppose (7) is not true and consider

$$w(x) = \frac{u(x_0)(1+\frac{1}{M}) - u(x)}{\frac{u(x_0)}{M}},$$

and the cube Q^* with center x_0 and edgelength $\dfrac{\rho}{8\sqrt{n}}$. By Lemma 7 we have

$$|A_1| = |\{x \in Q_1 : u(x) \geq M^{k-1}\}| \leq (1-\mu)^{k-1}.$$

Since we assume that (7) is not true, $\sup_{B_\rho(x_0)} u < u(x_0)(1+\frac{1}{M})$, and consequently $w(x) > 0$ in $B_\rho(x_0)$ and also $w(x_0) = 1$. Note also that $Q^* \subset B_\rho(x_0)$. We apply Lemma 5 to w in Q^*, note that since u is a sub-solution, the minus sign makes w a super-solution of $Lu = \frac{M}{u(x_0)} f$. To apply Lemma 5 we rescale Q^* to $Q_1(0)$ as in the proof of the previous Lemma, i.e., we find an affine transformation $A : Q_1(0) \to Q^*$ and we apply Lemma 5 to $w(Az)$ in $Q_1(0)$. We obtain

$$|A_2| = |\{x \in Q^* : w(x) \geq M\}| \leq (1-\mu)|Q^*|.$$

We claim that $Q^* \subset A_1 \cup A_2$. In fact, if $x_1 \notin A_1$ then $u(x_1) < M^{k-1}$ and since $w(x) = M + 1 - \dfrac{Mu(x)}{u(x_0)}$, we have $w(x_1) > M + 1 - \dfrac{M^k}{u(x_0)}$. Then by (1) $w(x_1) \geq M$, i.e., $x_1 \in A_2$.

Consequently, $|Q^*| \leq |A_1| + |A_2| \leq (1-\mu)^{k-1} + (1-\mu)|Q^*|$ which implies

$$|Q^*| \leq \frac{(1-\mu)^{k-1}}{\mu}.$$

By the definition of ρ we then have

$$\left(\frac{c}{16\sqrt{n}}(1-\mu)^{k/n}\right)^n \leq \frac{(1-\mu)^{k-1}}{\mu}.$$

This implies that

$$c \leq \frac{16\sqrt{n}}{((1-\mu)\mu)^{1/n}}.$$

To obtain a contradiction we pick

$$c > \frac{16\sqrt{n}}{((1-\mu)\mu)^{1/n}},$$

and k sufficiently large such that $c(1-\mu)^{k/n}$ is sufficiently small.

Harnack's inequality

THEOREM. *Assume that $u \geq 0$ is a viscosity solution of $Lu = f$ in the cube Q_l. Let Q_1 be a cube with edgelength 1, $Q_1 \subset Q_l$ and such that the cube Q_2 concentric with Q_1 and with edge length 2 is contained in Q_l. Then there exists a constant C depending only on the ellipticity constants such that*

$$\sup_{Q_1} u \leq C\left(\inf_{Q_1} u + \|f\|_{L^n(Q_l)}\right),$$

for any non-negative viscosity solution u.

PROOF.

We first make the following simplification.

We may assume that $\inf_{Q_1} u \leq 1$ and $\|f\|_{L^n(Q_l)} \leq \epsilon \leq 1$. Because if this were not true we set

$$v(x) = \epsilon\frac{u(x)}{\inf_{Q_1} u + \|f\|_{L^n(Q_l)}},$$

and

$$g(x) = \epsilon\frac{f(x)}{\inf_{Q_1} u + \|f\|_{L^n(Q_l)}},$$

and we have

$$Lv = g, \qquad \text{and} \qquad \|g\|_{L^n(Q_l)} \leq \epsilon.$$

Therefore, if we may show that under the assumptions there exists a constant C depending only on the ellipticity and such that

$$\sup_{Q_1} v \leq C,$$

then this would imply that

$$\epsilon \frac{\sup_{Q_1} u}{\inf_{Q_1} u + \|f\|_{L^n(Q_l)}} \leq C,$$

and the theorem would follow with constant C/ϵ.

We shall show that there exist constants $D_0 = D(\lambda, \Lambda)$ and $\delta = \delta(\lambda, \Lambda)$ such that if u is a non-negative solution of $Lu = f$ in Q_l, such that $\inf_{Q_2} u \leq 1$ then we have

$$u(x) \leq D_0 d(x, \partial Q_1)^{-\delta}, \qquad \forall x \in Q_1, \tag{8}$$

where Q_1 is any unit cube such that $Q_1 \subset Q_2(0)$. The inequality (8) blows up on the boundary of Q_1. Harnack's inequality in the unit cube $Q_1(0)$ follows from (8). In fact, we may cover the cube $Q_1(0)$ with a finite number of unit cubes $Q_1 \subset Q_2(0)$ and apply (8) in each Q_1.

Let us prove (8). Let M and μ be the constants of Lemma 5, and define $\delta > 0$ such that

$$\frac{1}{M} = (1-\mu)^{\delta/n}.$$

Let

$$D = \sup_{x \in Q_1} u(x) d(x, \partial Q_1)^{\delta}.$$

Since u is continuous there exists $x_0 \in Q_1$ such that

$$D = u(x_0) d(x_0, \partial Q_1)^{\delta}.$$

Let k be such that

$$M^k < u(x_0) \leq M^{k+1}.$$

We have

$$d(x_0, \partial Q_1) \geq \left(\frac{D}{M^{k+1}}\right)^{1/\delta} = \left(\frac{D}{M}\right)^{1/\delta} (1-\mu)^{k/n}. \tag{9}$$

Let c and k_0 be the constants of Lemma 8. If $\dfrac{D}{M} < c^{\delta}$ then

$$u(x) \leq c^{\delta} M d(x, \partial Q_1)^{-\delta},$$

and (8) follows.

If $\dfrac{D}{M} \geq c^{\delta}$ then either $k > k_0$ or $k \leq k_0$. If $k < k_0$ then $u(x_0) \leq M^{k_0+1}$, consequently $D \leq M^{k_0+1} d(x_0, \partial Q_1)^{\delta}$ and so

$$u(x) \leq M^{k_0+1} d(x_0, \partial Q_1)^{\delta} d(x, \partial Q_1)^{-\delta},$$

and (8) follows.

The worse case is then when

$$D \geq c^\delta M, \qquad k \geq k_0.$$

In this case we may apply Lemma 8 because (1) and (2) of that lemma hold. We then have

$$\sup_{B_\rho(x_0)} u \geq u(x_0)(1+\frac{1}{M}) = \frac{D}{d(x_0,\partial Q_1)^\delta}(1+\frac{1}{M}). \tag{10}$$

On the other hand, $B_\rho(x_0) \subset Q_1$ and consequently

$$\begin{aligned}\sup_{B_\rho(x_0)} u &= \sup_{B_\rho(x_0)} \left(\frac{u(x)d(x,\partial Q_1)^\delta}{d(x,\partial Q_1)^\delta}\right)\\ &\leq \frac{1}{d(B_\rho(x_0),\partial Q_1)} \sup_{B_\rho(x_0)} \left(u(x)d(x,\partial Q_1)^\delta\right)\\ &\leq \frac{D}{(d(x_0,\partial Q_1)-\rho)^\delta}.\end{aligned}$$

Now from the definition of ρ and (9) we get

$$d(x_0,\partial Q_1) \geq \left(\frac{D}{M}\right)^{1/\delta} \frac{2}{c}\rho,$$

which applied to the previous estimate gives

$$\sup_{B_\rho(x_0)} u \leq \frac{D}{d(x_0,\partial Q_1)^\delta}\left(1-\frac{c}{2}(\frac{M}{D})^{1/\delta}\right)^{-\delta}. \tag{11}$$

A comparison of (10) and (11) gives

$$1+\frac{1}{M} \leq \left(1-\frac{c}{2}(\frac{M}{D})^{1/\delta}\right)^{-\delta}.$$

This implies that

$$D \leq M\left(\frac{2}{c}\left(1-\left(1+\frac{1}{M}\right)^{-1/\delta}\right)\right)^{-\delta},$$

and the proof of (8) is complete.

References

[**Al-1**] A. D. Aleksandrov, *Uniqueness conditions and estimates for the solution of the Dirichlet problem*, A. M. S. Translations **68** (1968), 89-119.

[**Al-2**] _____, *Majorization of solutions of second-order linear equations*, A. M. S. Translations **68** (1968), 120-143.

[**Caf1**] L. A. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. of Math. **130** (1989), 189-213.

[**Caf2**] _____, *A localization property of solutions to the Monge-Ampére equation and their strict convexity*, Ann. of Math. **131** (1990), 129-134.

[**Caf2**] _____, *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampére equation*, Ann. of Math. **131** (1990), 135-150.

[**Cal-Z**] A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **XX** (1961), 171-225.

[**Cr-I-L**] M. G. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, preprint.

[**Ev-Ga**] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine properties of Functions*, CRC Press, Boca Raton, FL, 1992.

[**Fe**] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[**Gi-Tr**] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin Heildelberg, 1983.

[**K-S**] N. Krylov and M. Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Math. USSR Izv. **16** (1981), 151-164.

[**Pu**] C. Pucci, *Limitazioni per soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl. **74** (1966), 15-30.

[**R-V**] A. W. Roberts and D. L. Varberg, *Convex Functions*, Academic Press, New York, 1973.

Department of Mathematics, Temple University, Philadelphia, PA 19122
E-mail address: `gutier@euclid.math.temple.edu`