Reflection, refraction, and the Legendre Transform

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We construct in dimension two a mirror that reflects collimated rays into a set of directions that amplify the image, and an optical lens so that collimated rays are refracted into a set of directions with a prescribed magnification factor. The profiles of these optical surfaces are given by explicit formulas involving the Legendre transformation.

1. Introduction

Recently, R. A. Hicks and collaborators considered the problem of constructing reflective surfaces with certain properties and for certain important practical applications such us the design of a driver mirror without a blind spot, see [HB01] and [HP05]. In this note, we consider the problem of constructing, in dimension two, a mirror that reflects collimated rays into a set of directions that amplify the image. It turns out that, using elementary methods, the solution can be described by an explicit formula depending on the parameters in the initial configuration; in particular, the magnification factor required for the image. The problem leads to a nonlinear differential equation of first order that

after applying the Legendre transformation can be explicitly solved. The construction below can be used to design the passenger mirror in a car; by symmetry and a change in the parameters, this also gives the driver mirror.

With this very same method, we also construct surfaces that separate two media with different refractive indices such that collimated rays are refracted into a set of directions with a prescribed magnification factor. This yields an optical lens as the region cut above by such a surface and below by a horizontal plane. In Section 2, we solve explicitly the mirror problem, and in Section 3 the lens problem, both in the far field case.

2. The mirror problem

Let us fix a point $(a, 0)$ on the x-axis with $a < 0$, say this is the location of the observer or driver. Suppose from each point $(z, 0)$, with $a \leq z$, a ray with unit direction $\mathbf{i} = (i_1, i_2)$ emanates, i.e., all these rays emanate in a parallel fashion; assume $\mathbf{i}$ is in the first quadrant. We seek for a one dimensional curve in the x, y-plane, given by a differentiable function $y = f(x)$, $x > 0$, such that the ray emanating from the point $(z, 0)$, $a < z < 0$, and with direction (i_1, i_2) strikes the curve at the point $(x, f(x))$ and is then reflected into the point $(t(z - a), -k)$, with $k > 0$, and t a function given in advance, see Figure 1. Notice that since the ray emanating from $(a, 0)$ hits the curve at $(0, f(0))$, then $\frac{i_2}{i_1} = \frac{f(0)}{-a}$, that is, the direction of (i_1, i_2) is determined. Also by similarity $\frac{f(x)}{x - z} = \frac{i_2}{i_1}$, so the x-coordinate of the curve at the striking point satisfies

$$z = x - \frac{i_1}{i_2} f(x). \tag{2.1}$$

The unit normal to the curve at the striking point $(x, f(x))$ is $\mathbf{N} = \frac{(f'(x), -1)}{\sqrt{1 + (f'(x))^2}}$*. From the Snell law, the reflected ray has unit direction

$$\mathbf{v} = \mathbf{i} - 2(\mathbf{i} \cdot \mathbf{N})\mathbf{N}.$$

*This is the inner normal to the curve. Since in the Snell law the normal appears twice, it is irrelevant to choose the direction of the normal.

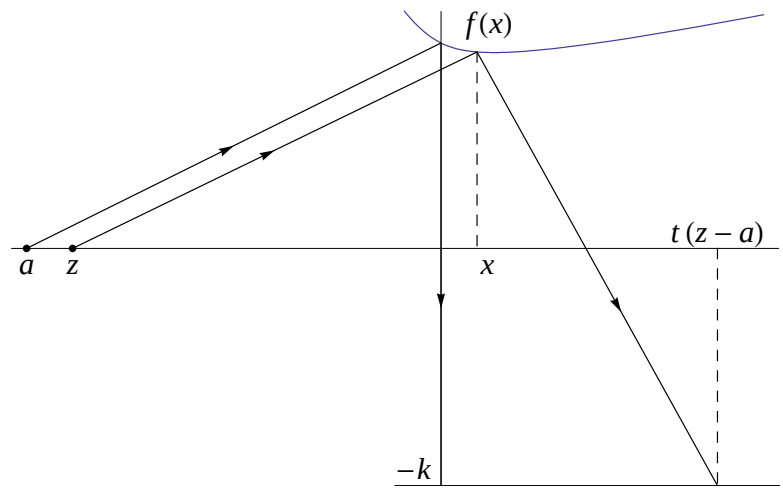

Fig. 1. Mirror problem

So the components v_1, v_2 of $\mathbf{v}$ are

$$v_1 = i_1 - \frac{2f'(x)}{1+(f'(x))^2}\,(i_1 f'(x) - i_2)$$
$$v_2 = i_2 + 2\frac{(i_1 f'(x) - i_2)}{1+(f'(x))^2}.$$

Let ℓ be the line with direction $\mathbf{v}$ that passes through $(x, f(x))$. We need to find when this line hits the point $(t(z-a), -k)$. That is, we need λ such that $x + \lambda v_1 = t(z-a)$ and $f(x) + \lambda v_2 = -k$. Solving the second equation yields $\lambda = -\dfrac{k+f(x)}{v_2}$ and substituting in the first one yields

$$x - (k + f(x))\frac{v_1}{v_2} = t(z-a).$$

Now

$$\frac{v_1}{v_2} = \frac{2i_2 f'(x) + i_1(1-(f'(x))^2)}{2i_1 f'(x) - i_2(1-(f'(x))^2)},$$

so we get the ode

$$x - (k + f(x))\left(\frac{2i_2 f'(x) + i_1(1-(f'(x))^2)}{2i_1 f'(x) - i_2(1-(f'(x))^2)}\right) = t(z-a), \tag{2.2}$$

with z given in (2.1). We will solve it explicitly using the Legendre transform (see [CH62, pp. 32-39, vol. II])

$$w(\xi) + f(x) = x\xi; \quad \xi = f', \quad x = w'. \tag{2.3}$$

As usual, this transformation interchanges the roles of the derivative, that in our equation appears in a complicated way, and the independent variable, which in our case appears in a simple way.

2.A. *Far field right mirror: $k \to +\infty$*

† Let us fix $\beta > 0$, and suppose $t(x) = \beta k x$. So we get the equation

$$-(k + f(x))\left(\frac{2i_2 f'(x) + i_1(1 - (f'(x))^2)}{2i_1 f'(x) - i_2(1 - (f'(x))^2)}\right) = \beta k(z - a) - x.$$

Dividing this equation by k and letting $k \to \infty$ yields the nonlinear ode

$$-\frac{2i_2 f'(x) + i_1(1 - (f'(x))^2)}{2i_1 f'(x) - i_2(1 - (f'(x))^2)} = \beta(z - a) = \beta\left(x - \frac{i_1}{i_2} f(x) - a\right). \tag{2.4}$$

Applying the Legendre transformation we obtain the linear ode

$$F(\xi) := -\frac{2i_2\xi + i_1(1 - \xi^2)}{2i_1\xi - i_2(1 - \xi^2)} = \beta\left(\left(1 - \frac{i_1}{i_2}\xi\right)w' + \frac{i_1}{i_2}w - a\right), \tag{2.5}$$

that is,

$$(i_2 - i_1\xi)\left(\frac{1}{i_2 - i_1\xi}w\right)' = w' + \frac{i_1}{i_2 - i_1\xi}w = \frac{i_2}{i_2 - i_1\xi}\left(\frac{1}{\beta}F(\xi) + a\right).$$

Therefore integrating in ξ

$$w(\xi) = (i_2 - i_1\xi)\left(C + \int \frac{i_2}{(i_2 - i_1\xi)^2}\left(\frac{1}{\beta}F(\xi) + a\right) d\xi\right).$$

This can be completely integrated. In the application to the passenger mirror, z is close to a, and so x is close to zero.‡ Therefore $f(x)$ is close to $f(0) = -ai_2/i_1$, and so $x - \frac{i_1}{i_2}f(x) - a \approx x$. This means that, in this case, the right hand side of (2.4) has magnitude βx for $x > 0$, small, and will solve the corresponding equation explicitly. That is, we solve the equation

$$-\frac{2i_2 f'(x) + i_1(1 - (f'(x))^2)}{2i_1 f'(x) - i_2(1 - (f'(x))^2)} = \beta x.$$

Applying the Legendre transformation we obtain the linear ode

$$F(\xi) = -\frac{2i_2\xi + i_1(1 - \xi^2)}{2i_1\xi - i_2(1 - \xi^2)} = \beta w'. \tag{2.6}$$

†The near field case also leads to a first order linear ode that can be solved explicitly, depending on t, again with the Legendre transform.

‡For the passenger mirror, this means the visual angle of the driver is small compared with the distance of the driver to the mirror.

The function $F(\xi)$ blows up at $\xi_1 = \dfrac{-i_1 - 1}{i_2}$ and $\xi_2 = \dfrac{-i_1 + 1}{i_2}$. It is easy to check that $F(\xi)$ is strictly increasing in the intervals $(-\infty, \xi_1)$, (ξ_1, ξ_2) and $(\xi_2, +\infty)$. In addition, F vanishes at the points $\dfrac{i_2 \pm 1}{i_1}$.

To figure out the initial condition for w, we require for the ray emanating from $(a, 0)$ to be reflected at the point $(0, f(0))$ in the direction $\mathbf{v} = (0, -1)$, that is, in the direction of the negative y-axis[§]. This means, by the Snell law, that the normal to the curve at $(0, f(0))$ must form the same angle with (i_1, i_2) and $(0, -1)$. That is, $\dfrac{(f'(0), -1)}{\sqrt{1 + (f'(0))^2}} \cdot (i_1, i_2) = \dfrac{(f'(0), -1)}{\sqrt{1 + (f'(0))^2}} \cdot (0, 1)$, and so $f'(0) = \dfrac{-1 + i_2}{i_1} := \xi_0$. We assume that $0 < i_1, i_2 < 1$, i.e., the unit vector $\mathbf{i}$ is in the first quadrant. We have $\xi_0 < \xi_2$ iff $i_1 + i_2 > 1$ which holds since $\mathbf{i}$ is in the first quadrant. Similarly $\xi_1 < \xi_0$. If $\xi_0' = \dfrac{i_2 + 1}{i_1}$, $\xi_0' > \xi_2$, and so F vanishes on (ξ_1, ξ_2) only at the point ξ_0. From the definition of Legendre transform $w(\xi_0) = -f(0)$. Therefore we need to solve (2.6) for $\xi \in (\xi_1, \xi_2)$ with the initial condition

$$w\left(\frac{-1 + i_2}{i_1}\right) = -f(0).$$

We have

$$\frac{2i_2\xi + i_1(1 - \xi^2)}{2i_1\xi - i_2(1 - \xi^2)} = -\frac{i_1}{i_2} + \frac{2\xi/i_2}{i_2\xi^2 + 2i_1\xi - i_2}$$

since $i_1^2 + i_2^2 = 1$. Then[¶]

$$\begin{aligned}
w(\xi) &= \frac{1}{\beta}\int\left(\frac{i_1}{i_2} - \frac{2\xi/i_2}{i_2\xi^2 + 2i_1\xi - i_2}\right)d\xi \\
&= \frac{1}{\beta}\frac{i_1}{i_2}\xi - \frac{2}{\beta i_2}\int\frac{\xi}{i_2\xi^2 + 2i_1\xi - i_2}\,d\xi \\
&= \frac{1}{\beta}\frac{i_1}{i_2}\xi - \frac{2}{\beta i_2}\left(\frac{1}{2i_2}\ln\left|i_2\xi^2 + 2i_1\xi - i_2\right| - \frac{2i_1}{2i_2}\int\frac{1}{i_2\xi^2 + 2i_1\xi - i_2}\,d\xi\right) + C \\
&= \frac{1}{\beta}\frac{i_1}{i_2}\xi - \frac{1}{\beta i_2^2}\ln\left|i_2\xi^2 + 2i_1\xi - i_2\right| + \frac{i_1}{\beta i_2^2}\ln\left|\frac{2i_2\xi + 2i_1 - 2}{2i_2\xi + 2i_1 + 2}\right| + C \\
&= \frac{1}{\beta}\frac{i_1}{i_2}\xi - \frac{1}{\beta i_2^2}\ln\left|i_2\xi^2 + 2i_1\xi - i_2\right| + \frac{i_1}{\beta i_2^2}\ln\left|\frac{i_2\xi + i_1 - 1}{i_2\xi + i_1 + 1}\right| + C.
\end{aligned}$$

[§]Of course, we can prescribe other directions of reflection at $(0, f(0))$ which changes the initial condition in the ode.

[¶] $\displaystyle\int\frac{x}{ax^2 + bx + c}\,dx = \frac{1}{2a}\ln\left|ax^2 + bx + c\right| - \frac{b}{2a}\int\frac{1}{ax^2 + bx + c}\,dx.$

And $\displaystyle\int\frac{1}{ax^2 + bx + c}\,dx = \frac{1}{\sqrt{b^2 - 4ac}}\ln\left|\frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right|$, if $4ac - b^2 < 0$.

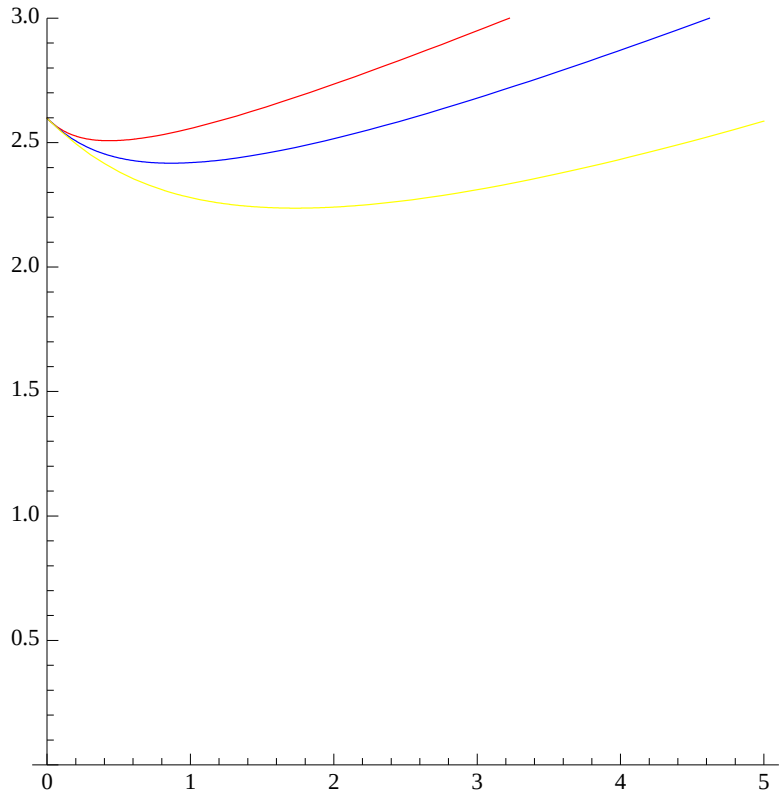

Fig. 2. Graphs of f when $\beta = 1, 2, 4$ from top to bottom

Notice that the function w is defined for all ξ in the interval (ξ_1, ξ_2). Now

$$\begin{aligned} w(\xi_0) &= w\left(\frac{-1+i_2}{i_1}\right) \\ &= \frac{(-1+i_2)}{\beta i_2} - \frac{1}{\beta i_2^2}\ln\left|\frac{2(i_2-1)}{i_1^2}\right| + \frac{i_1}{\beta i_2^2}\ln\left|\frac{1-(i_2+i_1)}{1-(i_2-i_1)}\right| + C \\ &:= C(i_1, i_2) + C = -f(0), \end{aligned}$$

so $C = -f(0) - C(i_1, i_2)$. Notice that from the configuration, $i_2 > 0$ and $i_1 \neq 0, -1$ and so $1 - (i_2 - i_1) \neq 0$, since $i_1^2 + i_2^2 = 1$, i.e., the constant $C(i_1, i_2)$ is finite. So

$$w(\xi) = -f(0) - C(i_1, i_2) + \frac{1}{\beta}\frac{i_1}{i_2}\xi - \frac{1}{\beta i_2^2}\ln\left|i_2\xi^2 + 2i_1\xi - i_2\right| + \frac{i_1}{\beta i_2^2}\ln\left|\frac{i_2\xi + i_1 - 1}{i_2\xi + i_1 + 1}\right|. \tag{2.7}$$

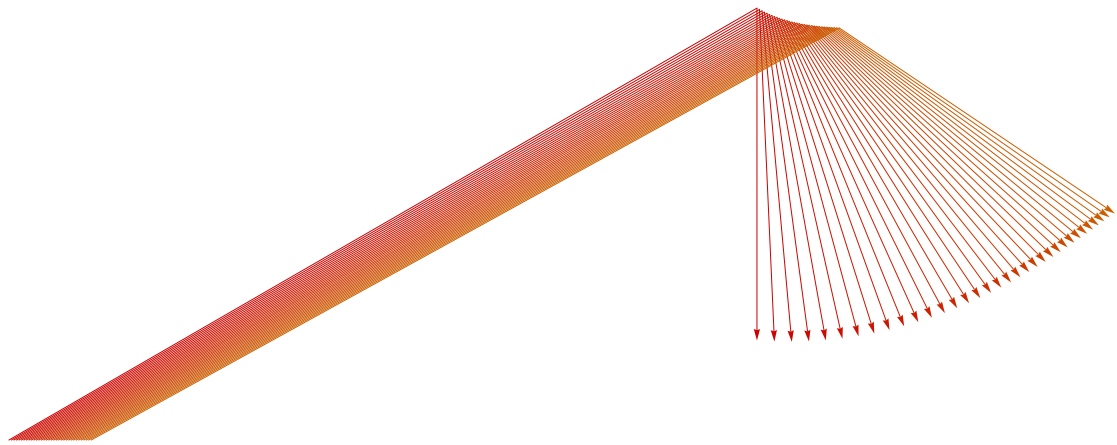

Fig. 3. Path of the reflected rays on the mirror when $\beta = 3$

Next we want to solve $w'(\xi) = x$ in ξ, that is, from the equation (2.6)

$$-F(\xi) = \frac{2i_2\xi + i_1(1-\xi^2)}{2i_1\xi - i_2(1-\xi^2)} = -\beta x. \tag{2.8}$$

The function F is strictly increasing and bijective from $[\xi_0, \xi_2)$ to $[0, +\infty)$ and so is invertible. Equation (2.8) is a quadratic equation in ξ whose solutions are

$$\frac{-(i_2 + i_1\beta x) \pm \sqrt{1 + (\beta x)^2}}{i_2\beta x - i_1}.$$

Since $F(\xi_0) = 0$, we must take the positive sign in the square root and we get

$$\xi = g(x) := \frac{-(i_2 + i_1\beta x) + \sqrt{1 + (\beta x)^2}}{i_2\beta x - i_1},$$

and $g : [0, +\infty) \to [\xi_0, \xi_2)$. We therefore obtain from (2.3) the explicit solution

$$f(x) = x\, g(x) - w(g(x)),$$

defined for all $x \in [0, +\infty)$ with w given by (2.7); and satisfying $f(0) = (-a)i_2/i_1$.

This function describes the profile of a passenger car mirror and by symmetry we also obtain driver's mirror.

As an example, let us assume $-a = 4.5$ft, and the angle between the line joining the location of the driver with the right mirror, and the back of the driver seat is $\pi/6$. So

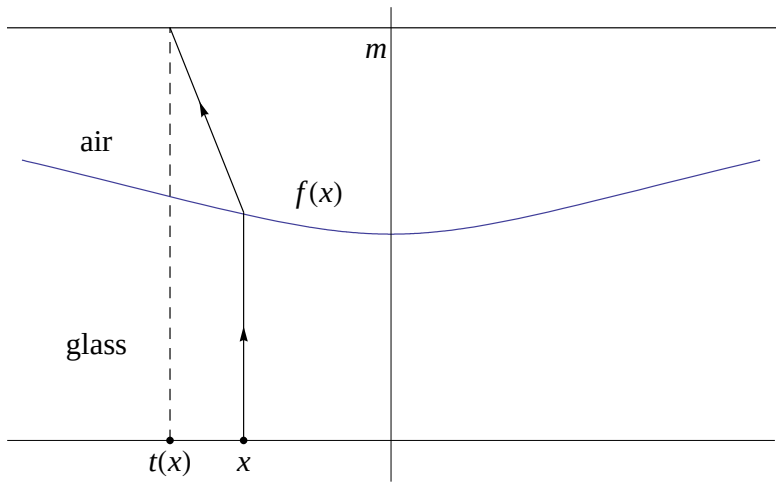

Fig. 4. Lens problem

$i_1 = \cos \pi/6 = \sqrt{3}/2$ and $i_2 = \sin \pi/6 = 1/2$. Therefore $f(0) = 4.5/\sqrt{3}$, which represents the distance of the right mirror to the back of the passenger seat. Figures (2) and (3) illustrate this configuration.

3. The lens problem

We look for a one dimensional curve in the x, y-plane with equation $y = f(x)$ that separates glass and air (or in general two materials with different refractive indices) and such that a ray emanating from the point $(x, 0)$ is refracted by the curve into the point $(t(x), m)$. We assume the incident ray emanating from $(x, 0)$ has direction $\mathbf{i} = (0, 1)$, that is, all emanating rays are parallel. The unit outer normal to the curve is $\mathbf{N} = \dfrac{(-f'(x), 1)}{\sqrt{1 + (f'(x))^2}}$. Suppose all points $(x, 0)$ are surrounded by glass and all points $(t(x), m)$ surrounded by air, see Figure 4. Glass has index of refraction n_1 and air has index of refraction $n_2 (\approx 1)$. Let $\kappa = n_2/n_1$, so in this case we have $\kappa < 1$. The geometry of the refracting surfaces depend on weather $\kappa < 1$ or $\kappa > 1$, see [GH09]. From the Snell law of refraction in vector form, see also [GH09, Section 2.1] for a discussion, if the incident ray has unit direction $\mathbf{i}$ then the unit direction of the refracted ray is $\mathbf{v}$ with

$$\mathbf{i} - \kappa\, \mathbf{v} = \lambda\, \mathbf{N}. \tag{3.9}$$

To figure out the equation of the curve we need to calculate λ. Making the dot product with $\mathbf{N}$ in (3.9) yields

$$\lambda = \mathbf{i} \cdot \mathbf{N} - \kappa\, \mathbf{v} \cdot \mathbf{N}$$

If θ_1 is the angle of incidence and θ_2 is the angle of refraction we have $\sin\theta_2 = \dfrac{1}{\kappa}\sin\theta_1$. So $\mathbf{v} \cdot \mathbf{N} = \cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - \kappa^{-2}\sin^2\theta_1}$, and $\sin^2\theta_1 = 1 - \cos^2\theta_1 = 1 - (\mathbf{i} \cdot \mathbf{N})^2$. So

$$\mathbf{v} \cdot \mathbf{N} = \sqrt{1 - \kappa^{-2}\left(1 - (\mathbf{i} \cdot \mathbf{N})^2\right)}$$

and therefore

$$\lambda = \mathbf{i} \cdot \mathbf{N} - \kappa\, \sqrt{1 - \kappa^{-2}\left(1 - (\mathbf{i} \cdot \mathbf{N})^2\right)}.$$

Notice that since $\kappa < 1$, then for refraction to occur the incidence angle θ_1 we must satisfy $\theta_1 \leq \theta_c$, with θ_c the critical angle, that is, $\sin\theta_c = \kappa$. Therefore $\sin\theta_1 \leq \kappa$ and so $1 - (\mathbf{i} \cdot \mathbf{N})^2 \leq \kappa^2$ and consequently $(\mathbf{i} \cdot \mathbf{N})^2 \geq 1 - \kappa^2$. Let us set $D = \sqrt{1 + (f'(x))^2}$, then

$$\mathbf{i} \cdot \mathbf{N} = \frac{1}{D},$$

and we obtain that

$$|f'(x)| \leq \frac{\kappa}{\sqrt{1 - \kappa^2}}.$$

Then

$$\begin{aligned}
\lambda &= \frac{1}{D} - \kappa\, \sqrt{1 - \kappa^{-2}\left(1 - \left(\frac{1}{D}\right)^2\right)} \\
&= \frac{1}{D}\left(1 - \kappa\sqrt{D^2 - \kappa^{-2}\left(D^2 - 1\right)}\right) \\
&= \frac{1}{\sqrt{1 + (f'(x))^2}}\left(1 - \kappa\sqrt{1 + (1 - \kappa^{-2})(f'(x))^2}\right).
\end{aligned}$$

If we let

$$\Phi(\xi) = \frac{1}{\sqrt{1 + \xi^2}}\left(1 - \kappa\sqrt{1 + (1 - \kappa^{-2})\xi^2}\right),$$

then

$$\lambda = \Phi(f'(x))$$

From (3.9)

$$\mathbf{v} = \frac{1}{\kappa}\left(\mathbf{i} - \lambda\mathbf{N}\right)$$

and if v_1, v_2 are the components of $\mathbf{v}$ we get

$$v_1 = \frac{1}{\kappa}\lambda\frac{f'(x)}{D} = \frac{1}{\kappa}\Phi(f'(x))\frac{f'(x)}{\sqrt{1+(f'(x))^2}}$$
$$v_2 = \frac{1}{\kappa}\left(1-\frac{\lambda}{D}\right) = \frac{1}{\kappa}\left(1-\frac{\Phi(f'(x))}{\sqrt{1+(f'(x))^2}}\right).$$

Let ℓ be the line with direction $\mathbf{v}$ that passes through $(x, f(x))$. We need to find when this line hits the point $(t(x), m)$. That is, we need μ such that $x + \mu v_1 = t(x)$ and $f(x) + \mu v_2 = m$. Solving the second equation yields $\mu = \dfrac{m - f(x)}{v_2}$ and substituting this value in the first one yields

$$(m - f(x))\frac{v_1}{v_2} = t(x) - x,$$

that is we get the ode

$$(m - f(x))\frac{f'(x)\Phi(f'(x))}{\sqrt{1+(f'(x))^2} - \Phi(f'(x))} = t(x) - x. \tag{3.10}$$

We solve it explicitly using once again the Legendre transform

$$w(\xi) + f(x) = x\xi; \quad \xi = f', \quad x = w'.$$

Suppose $t(x) = (\alpha + 1)x$, with $\alpha > 0$, i.e., the lens magnifies the image.[‖] Applying the Legendre change of variables to (3.10) yields

$$(m - (\xi w' - w))\,\frac{\xi\Phi(\xi)}{\sqrt{1+\xi^2} - \Phi(\xi)} = \alpha w'.$$

If we set $h(\xi) = \dfrac{\xi\Phi(\xi)}{\sqrt{1+\xi^2} - \Phi(\xi)}$, then we get the linear ode

$$w' - \frac{h(\xi)}{\xi h(\xi) + \alpha}w = m\frac{h(\xi)}{\xi h(\xi) + \alpha}.$$

We have

$$h(\xi) = \frac{\xi\left(1 - \kappa\sqrt{1+(1-\kappa^{-2})\xi^2}\right)}{\xi^2 + \kappa\sqrt{1+(1-\kappa^{-2})\xi^2}} = \frac{\xi\,(1-\kappa^2)}{\kappa\left(\kappa + \sqrt{1+(1-\kappa^{-2})\xi^2}\right)}. \tag{3.11}$$

[‖]Notice that if $t(x) = (\alpha + 1)x$, $\alpha > 0$, then for $x < 0$ the right hand side of (3.10) is negative and so on the left hand side $h(f'(x))$ must be negative. Since h is strictly increasing in the interval $\left(-\frac{\kappa}{\sqrt{1-\kappa^2}}, \frac{\kappa}{\sqrt{1-\kappa^2}}\right)$ (see (3.15)) and $h(0) = 0$, we must have $f'(x) < 0$. Similarly $f'(x) > 0$ for $x > 0$. So f decreases to the left of 0 and increases to the right of 0. The opposite thing happens if t reduces the size of the image.

If we set $\Delta = 1 - \kappa\sqrt{1 + (1-\kappa^{-2})\xi^2}$, then

$$\xi h(\xi) + \alpha = \frac{\xi^2\Delta}{\xi^2 - \Delta + 1} + \alpha = \frac{\xi^2(\alpha+\Delta) + \alpha(1-\Delta)}{\xi^2 - \Delta + 1},$$

and

$$\frac{h(\xi)}{\xi h(\xi) + \alpha} = \frac{\xi\Delta}{\xi^2(\alpha+\Delta) + \alpha(1-\Delta)}.$$

So the solution to the ode is

$$w(\xi) = -m + K\exp\left(\int \frac{h(\xi)}{\xi h(\xi) + \alpha}\, d\xi\right). \tag{3.12}$$

We let $1 - \kappa^{-2} = -d$, with $d > 0$ since $\kappa < 1$. So

$$\frac{h(\xi)}{\xi h(\xi) + \alpha} = \frac{\xi\left(1 - \kappa\sqrt{1 - d\xi^2}\right)}{\xi^2(\alpha + 1 - \kappa\sqrt{1-d\xi^2}) + \alpha\left(\kappa\sqrt{1-d\xi^2}\right)}.$$

To calculate the integral (3.12) we first make the change of variables $\xi = d^{-1}\sin\theta$ and the integrand is transformed into rational function of $\sin\theta$ and $\cos\theta$. To evaluate the resulting integral we make the substitution $\tan(\theta/2) = y$, so $\theta = 2\arctan y$, and so $\sin\theta = \dfrac{2y}{1+y^2}$, $\cos\theta = \dfrac{1-y^2}{1+y^2}$, and $d\theta = \dfrac{2}{1+y^2}\,dy$. Therefore the resulting integrand is a rational function of y which can be integrated by partial fractions.

3.A. *Far field case, $m \to \infty$*

Let $\beta > 0$ fixed. We assume $t(x) = \beta m x$. Inserting this value of $t(x)$ in the equation (3.10) yields the equation:

$$(m - f(x))\frac{f'(x)\Phi(f'(x))}{\sqrt{1 + (f'(x))^2} - \Phi(f'(x))} = (\beta m - 1)x$$

and dividing this equation by m and letting $m \to \infty$ yields the equation

$$\frac{f'(x)\Phi(f'(x))}{\sqrt{1 + (f'(x))^2} - \Phi(f'(x))} = \beta x. \tag{3.13}$$

Applying the Legendre transform change of variables to this equation we obtain the ode

$$\frac{\xi\Phi(\xi)}{\sqrt{1+\xi^2} - \Phi(\xi)} = \beta w',$$

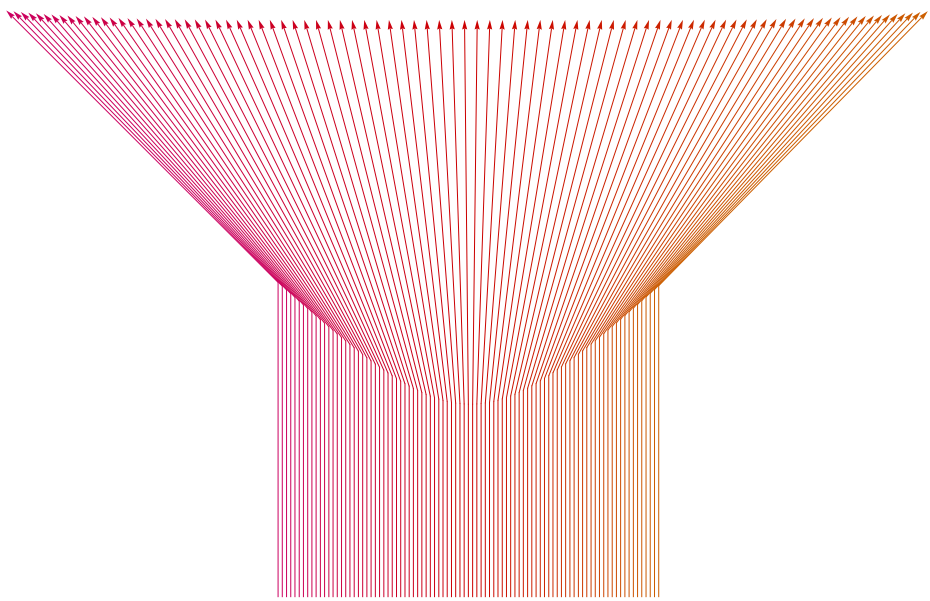

Fig. 5. Trajectories of refracted rays when $\beta = 2$, glass to air, $\kappa = 2/3$.

or with the notation before

$$w'(\xi) = \frac{1}{\beta} h(\xi). \tag{3.14}$$

Since the graph of f is symmetric around the origin, we get that $f'(0) = 0$ and so $w(0) = 0$. Therefore

$$w(\xi) = \frac{1}{\beta} \int_0^{\xi} h(s)\, ds.$$

Using the second identity in (3.11) we get that the integral of h equals

$$-\kappa \sqrt{1 + (1 - \kappa^{-2})\xi^2} + \kappa^2 \ln\left[\kappa + \sqrt{1 + (1 - \kappa^{-2})\xi^2}\right] + \kappa - \kappa^2 \ln(1 + \kappa).$$

As is the case of the mirror, to obtain the solution f, we need invert the equation $w'(\xi) = x$, which in view of (3.14) amounts to invert h. We have

$$h'(\xi) = \frac{\kappa^2(1 + \xi^2)\Delta}{(1 - \Delta)(\xi^2 + 1 - \Delta)^2}, \tag{3.15}$$

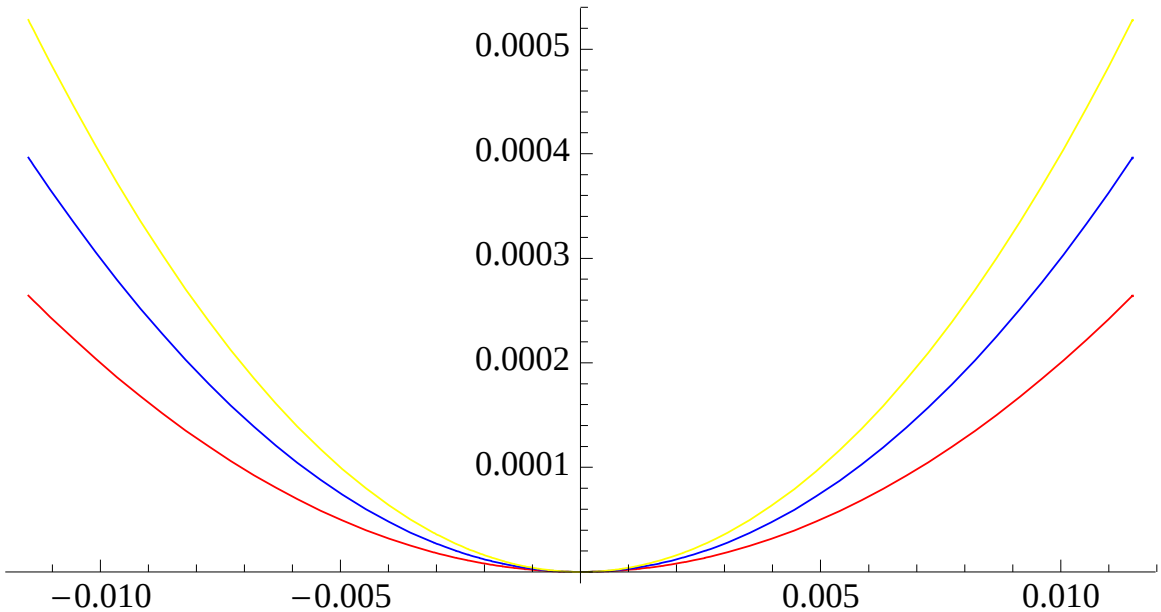

Fig. 6. Graphs of f when $\beta = 2, 3, 4$ from bottom to top; glass to air, $\kappa = 2/3$.

and since $0 \leq \Delta \leq 1$ for $|\xi| \leq \dfrac{\kappa}{\sqrt{1-\kappa^2}}$, we have that the function $h(\xi)$ is strictly increasing in the interval $-\dfrac{\kappa}{\sqrt{1-\kappa^2}} < \xi < \dfrac{\kappa}{\sqrt{1-\kappa^2}}$; and

$$h\left(\pm\frac{\kappa}{\sqrt{1-\kappa^2}}\right) = \pm\frac{\sqrt{1-\kappa^2}}{\kappa}. \tag{3.16}$$

Using the second identity in (3.11), is easy to see that the inverse of h can be calculated solving a quadratic equation and therefore h has an inverse function

$$g : \left[-\frac{\sqrt{1-\kappa^2}}{\kappa}, \frac{\sqrt{1-\kappa^2}}{\kappa}\right] \to \left[-\frac{\kappa}{\sqrt{1-\kappa^2}}, \frac{\kappa}{\sqrt{1-\kappa^2}}\right],$$

with

$$g(x) = \begin{cases} \dfrac{\kappa^2 x + \sqrt{\kappa^2(x^2+x^4)}}{1-\kappa^2+x^2}, & \text{for } 0 \leq x \leq \dfrac{\sqrt{1-\kappa^2}}{\kappa} \\[2ex] \dfrac{\kappa^2 x - \sqrt{\kappa^2(x^2+x^4)}}{1-\kappa^2+x^2}, & \text{for } -\dfrac{\sqrt{1-\kappa^2}}{\kappa} \leq x \leq 0 \end{cases}$$

where the signs in front of the square roots are chosen in accordance with (3.16). From (3.14) we then obtain that the inverse function of w' is $(w')^{-1}(x) = g(\beta x)$.

Consequently the solution to the lens problem is

$$f(x) = xg(\beta x) - w(g(\beta x)).$$

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