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Covering theorems, inequalities on metric spaces and applications to PDE's

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Abstract We establish a covering lemma of Besicovitch type for metric balls in the setting of Hölder quasimetric spaces of homogenous type and use it to prove a covering theorem for measurable sets. For families of measurable functions, we introduce the notions of power decay, critical density and double ball property and with the aid of the covering theorem we show how these notions are related. Next we present an axiomatic procedure to establish Harnack inequality that permits to handle both divergence and non divergence linear equations.

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1 Introduction

Harnack inequality for solutions to degenerate elliptic equations has received much attention in recent years, and in several cases the underlying geometry for their study relies on quite general metric structures of homogeneous type. Moser iteration technique has been extended to non Euclidean settings, yielding Harnack inequalities for solutions to second order degenerate elliptic equations in *divergence form* with underlying Carnot-Carathéodory metric structures, see [3,11,12] and references therein.

Caffarelli's technique [5,6], to prove Harnack's inequality for uniformly elliptic fully nonlinear equations has been extended in [8] to the linearized Monge-Ampère equation, see also [15, Chap. 2]. The role of the Euclidean balls in Caffarelli's original method is played, in [8], by the *sections* of a convex function. This makes clear the quasi metric character of some crucial parts of the procedure and it is one of the purposes of this paper to put these techniques in the frame of quasi metric spaces of homogeneous type. Our principal aim is to present an axiomatic procedure to the Harnack inequality that permits to handle both non divergence and divergence linear equations, with underlying Euclidean or non Euclidean metric structures. An important step in this approach is first to extend the covering lemma of Besicovitch type [7, Lemma 1] to the setting of Hölder quasi metric spaces defined in Sect. 2.2, an extension having independent interest, Lemma 3.1. Then assuming that the doubling Hölder metric space satisfies the *ring condition*, we prove a covering theorem for *measurable sets* analogue to the main Theorem in [7], see Theorem 3.3. With these tools we can handle families of just *measurable functions* and prove *power decay* (Definition 4.6) assuming *critical density* (Definition 4.1) and *double ball* (Definition 4.2) properties. Finally, we show two different proofs that power decay implies Harnack inequality in every quasi metric doubling space. We would like to stress that the present method to prove Harnack inequality avoids the use of any BMO inequality of John-Nirenberg type, which plays a crucial role in Moser iteration technique for divergence form operators. It is well known that this inequality is hard to prove in the settings of quasi metric doubling spaces, see [4].

We mention the recent paper [1] by Aimar, Forzani and Toledano extending the method from [8] to quasi metric spaces. These authors assume conditions involving metric rings and the boundary of metric balls and use a Vitali-type covering lemma by Aimar [2]. In addition, since their Calderón-Zygmund type decomposition, [1, Theorem 4.1], holds only for open sets, they can only handle semicontinuous functions and obtain Harnack inequality for continuous functions. In the applications, the semicontinuity of solutions is in general not known and it is more natural to make assumptions on the geometry of the underlying space rather than on the class of functions considered. In particular, our approach improves some conclusions in their paper.

The plan of the present work is as follows. In Sect. 2 we fix and recall some definitions and results from doubling quasi metric spaces theory. We also show some

explicit examples of spaces appearing in connection with PDE's, to which our results apply. Section 3 is devoted to the proof of our ϵ-Besicovitch Lemma and to the extension to metric settings of the main covering Theorem in [7]. We give two different versions of this theorem: the first one, Theorem 3.3, contains a stronger conclusion, but it requires the *ring condition*. The second one, Theorem 3.4, is weaker both in the assumption and in the conclusion: it requires that the measure of the ball is continuous with respect to the radius. In Sect. 4, we show that *critical density* and *double ball property* imply *power decay*, Theorem 4.7, assuming two different set of hypotheses on the spaces and the family of functions we are dealing with. Sections 5 and 6 contain two different proofs that in *every doubling quasi metric space* the Harnack inequality follows from the *power decay* property. Finally, in Sect. 7, we apply our abstract results to the X-elliptic operators, by giving a new proof of the Harnack inequality and Holder continuity of the weak solutions first appeared in [12]. We hope the axiomatic approach in the present paper might help to solve the problem of the validity of Harnack's inequality for X-elliptic operators in nondivergence form. We plan to return to this question in another occasion.

2 Metric definitions and preliminaries

2.1 Doubling quasi metric spaces

Given a nonempty set Y, a function $d : Y \times Y \to [0, \infty)$ is called a *quasi metric* if it is symmetric, strictly positive away from $\{x = y\}$ and such that for some constant $K \geq 1$,

$$d(x, y) \leq K \ (d(x, z) + d(z, y)),$$

for all $x, y, z \in Y$. The pair (Y, d) is called a *quasi metric space*. If $K = 1$, then d is a metric and (Y, d) is a *metric space*. The d-ball with center $x \in Y$ and radius $r > 0$ is given by

$$B(x, r) = B_r(x) = \{y \in Y : d(x, y) < r\}.$$

Definition 2.1 [9, p. 66] The quasi metric space (Y, d) is of *homogeneous type* if there exists a positive integer N such that for each x and for each $r > 0$ the ball $B(x, r)$ contains at most N points x_i satisfying $d(x_i, x_j) \geq r/2$ with $i \neq j$.

Definition 2.2 If (Y, d) is a quasi metric space and μ is a positive measure on a σ-algebra of subsets of Y containing the d-balls, then we say that μ satisfies the *doubling property* if there exists a positive constant C_D such that

$$0 < \mu(B(x, 2r)) \leq C_D \mu(B(x, r)), \quad \text{for all } x \in Y \text{ and } r > 0.$$

In this case, we say that (Y, d, μ) is a *a doubling quasi metric space*.

The following version of the doubling property is also useful:

$$\mu(B(x, R_2)) \leq C_D \left(\frac{R_2}{R_1}\right)^Q \mu(B(x, R_1)), \tag{2.1}$$

for $R_1 < R_2$ and $Q = \log_2 C_D$. As a consequence,

$$\mu(B(y, R)) \leq C_D \left(\frac{2KR}{r}\right)^Q \mu(B(x, r) \quad \text{whenever } B(x, r) \subset B(y, R). \tag{2.2}$$

Indeed, the inclusion $B(x, r) \subset B(y, R)$ implies $B(y, R) \subset B(x, 2KR)$, so that 2.2 follows from 2.1.

It is well known that every doubling quasi metric space is of homogeneous type, see [9, Remark on p. 67].

2.2 Hölder quasi metric spaces

Throughout the paper we will work with quasimetrics that are Hölder continuous in the sense of the following definition.

Definition 2.3 The quasi distance d is *Hölder continuous* if there exist positive constants β and $0 < \alpha \leq 1$ such that

$$|d(x, y) - d(x, z)| \leq \beta\, d(y, z)^\alpha \left(d(x, y) + d(x, z)\right)^{1-\alpha}, \tag{2.3}$$

for all $x, y, z \in Y$. In this case we say that (Y, d) is a *Hölder quasi metric space*.

Obviously, if d is a metric, then d is Hölder continuous with $\beta = \alpha = 1$.

By using the quasi-triangle inequality, is easy to see that (2.3) is equivalent to

$$|d(x, y) - d(x, z)| \leq \beta'\, d(y, z)^\alpha \left(d(x, z) + d(y, z)\right)^{1-\alpha}. \tag{2.4}$$

We would like to stress that the assumption that d is Hölder does not really affects the generality. Indeed, a theorem of Macías and Segovia [23, Theorem 2] states that given any quasi metric space (Y, d) there exists a quasi distance d' on Y that is Hölder continuous in the sense of the previous definition and it is equivalent to d. We would also like to note that if d is a Hölder quasi distance, then the d balls are *open* with respect to the topology generated by d. Then, if (X, d, μ) is a doubling quasi metric Hölder space, μ has to be a Borel measure.[1]

Remark 2.4 In $\mathbb{R}^N \times \mathbb{R}^N$ let us define

$$d_\delta(x, y) := |x - y|^\delta$$

[1] A set Ω is d-open iff any $x \in \Omega$ is the center of some ball contained in Ω. Since every doubling quasi metric space is separable, see [23], every open set is the union of a countable family of balls. Thus, if the metric balls are measurable, every open set is measurable.

where δ is a fixed strictly positive real number. If $\delta \leq 1$, then d_δ is a metric in $\mathbb{R}^N$ while, if $\delta > 1$, it is a Holder quasi metric with exponent $\alpha = \frac{1}{\delta}$ and constant $\beta = \delta$. This quasi distance appears in the fractal structures considered by Mosco in [21,22].

Remark 2.5 A remarkable example of a Hölder continuous quasi distance with exponent $\alpha = 1$ is the one induced by a homogeneous norm on a homogeneous Lie group $\mathbb{G}$. Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous group whose dilations $\{\delta_\lambda\}_{\lambda>0}$ are defined by

$$\delta_\lambda(x_1, \ldots, x_N) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_N} x_N),$$

$1 \leq \alpha_1 \leq \cdots \leq \alpha_N$, and let $d : \mathbb{R}^N \to \mathbb{R}$ be a continuous function, δ_λ-homogeneous of degree one, smooth and strictly positive outside the origin. Assume $d(x^{-1}) = d(x)$ for all $x \in \mathbb{G}$.[2] It is a standard fact that

$$d(x, y) := d(x^{-1} \circ y), \quad x, y \in \mathbb{R}^N,$$

is a quasi distance in $\mathbb{R}^N$. Moreover, if $|\cdot|$ denotes Lebesgue measure and we let $Q = \alpha_1 + \cdots + \alpha_N$, then

$$|B(x, r)| = |B(0, 1)|\, r^Q := w_Q\, r^Q, \tag{2.5}$$

for all $x \in \mathbb{R}^N$ and $r > 0$. Thus, $(\mathbb{R}^N, d, |\cdot|)$ is a doubling quasi metric space with doubling constant $C_D = 2^Q$. We now show that d satisfies (2.3) with $\alpha = 1$, that is,

$$d(x, y) \leq d(y, z) + \beta\, d(y, z), \quad \forall x, y, z \in \mathbb{R}^N.$$

A simple change of variables shows that this inequality is equivalent to

$$d(y \circ x) \leq d(x) + \beta\, d(y), \tag{2.6}$$

for all $x, y \in \mathbb{R}^N$. This inequality is trivial if $y = 0$, so assume $y \neq 0$. Using the fact that the function d is δ_λ homogeneous of degree one, (2.6) can be written as

$$d\left((\delta_{1/d(y)}(y))^{-1} \circ \delta_{1/d(y)}(x)\right) \leq d\left(\delta_{1/d(y)}(x)\right) + \beta\,,$$

or also keeping in mind that $d(z^{-1}) = d(z)$ it can be written as

$$d(\eta \circ \xi) - d(\eta) \leq \beta \quad \forall \xi, \eta \in \mathbb{R}^N, \quad d(\xi) = 1. \tag{2.7}$$

Since d is a quasi distance, there exists $K \geq 1$ such that

$$d(\eta \circ \xi) - d(\eta) \leq K\,(d(\eta) + d(\xi)) - d(\eta).$$

[2] We recall that in any Carnot group such an homogeneous norm exists.

So, if $d(\eta) < M$, then (2.7) holds with $C = (K-1)M + K$. Assume $d(\eta) \geq M$ and put $Y := \log(\xi)$, where log denotes the logarithm map from $\mathbb{G}$ to $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$. Consider the integral curve of Y starting from η:

$$\gamma(t) := \exp(tY)(\eta) = \eta \circ \exp(tY)(0).$$

Since $\gamma(0) = \eta$ and $\gamma(1) = \eta \circ \exp(Y)(0) = \eta \circ \exp(Y) = \eta \circ \xi$, it follows that

$$\begin{aligned} d(\eta \circ \xi) - d(\eta) = d(\gamma(1)) - d(\gamma(0)) &= \int_0^1 \langle \nabla d(\gamma(t)), \dot{\gamma}(t) \rangle \, \mathrm{d}t \\ &= \int_0^1 (Yd)(\gamma(t)) \, \mathrm{d}t. \end{aligned} \tag{2.8}$$

Let $Z_1, \ldots, Z_N$ be a basis of $\mathbb{G}$ and write

$$Y = \sum_{j=1}^{N} u_j(\xi) Z_j. \tag{2.9}$$

Since log is diffeomorphism, the real functions $\xi \to u_j(\xi)$ are smooth, and so

$$\sup_{d(\xi)=1} |u_j(\xi)| \leq C_1 < \infty, \quad j = 1, \ldots, N. \tag{2.10}$$

Moreover, we can relabel the Z_j's so that Z_j is δ_λ-homogeneous of degree α_j. Keeping in mind that d is δ_λ-homogeneous of degree one, then we get that the function $Z_j(d)$ is δ_λ-homogeneous of degree $1 - \alpha_j$ and so it is bounded above on every closed subset of $\mathbb{R}^n \setminus \{0\}$ since $1 - \alpha_j \leq 0$. We now claim that we can choose $M > 0$ such that if $d(\eta) \geq M$, then

$$d(\gamma(t)) \geq 1 \quad \forall t \in [0, 1]. \tag{2.11}$$

From (2.8), (2.9) and (2.10) it follows that

$$\begin{aligned} d(\xi \circ \eta) - d(\eta) &\leq C_1 \sum_{j=1}^{N} \sup_{0 \leq t \leq 1} |Z_j(d(\gamma(t)))| \\ &\leq C_1 \sum_{j=1}^{N} \sup_{d(z \geq 1} |Z_j(d(z))| := C < \infty. \end{aligned}$$

Then we are left with the proof of (2.11). From the quasi triangular inequality of d, we obtain

$$d(\gamma(t)) \geq \frac{1}{K} d(\eta) - d(\gamma(t), \eta). \tag{2.12}$$

On the other hand,

$$d(\gamma(t), \eta) = d(\eta^{-1} \circ \gamma(t)) = d(\exp(tY)(0)) = d\left(\exp\left(t \sum_{j=1}^{N} u_j(\xi) Z_j\right)(0)\right).$$

Then from (2.10), there exists a positive constant C_2 independent of ξ and t such that $d(\gamma(t), \eta) \leq C_2$. Then from (2.12), $d(\gamma(t)) \geq \frac{M}{K} - C_2$ for all $t \in [0, 1]$ if $d(\eta) \geq M$. If we choose $M = K(1 + C_2)$, then (2.11) follows and the proof of the remark is complete.

2.3 Continuity of the measure

In Theorem 3.3 we will assume a property of d given in the following definition.

Definition 2.6 (Ring condition) The doubling quasi metric space (Y, d, μ) satisfies the ring condition if there exists a nonnegative function $\omega(\epsilon)$ such that $\omega(\epsilon) \to 0$ as $\epsilon \to 0^+$ and

$$\mu(B(x, r) \setminus B(x, (1 - \epsilon)r)) \leq \omega(\epsilon)\, \mu(B(x, r)),$$

for every ball $B(x, r)$ and all ϵ sufficiently small.

The quasi metric space in Remark 2.5 is an important example of space satisfying the ring condition. Indeed, from (2.5) we get

$$|B(x, r) \setminus B(x, (1 - \epsilon)r)| = w_Q\, r^Q\, (1 - (1 - \epsilon)^Q) \leq Q\, \epsilon\, |B(x, r)|,$$

for all $r > 0$ and $0 < \epsilon < 1$.

In Theorem 3.4 we will use the continuity of the measure of the balls with respect to its radius. This property, which is weaker than the ring condition, holds true in metric spaces with the segment property accordingly with the following definition.

Definition 2.7 (Segment property) The metric space (Y, d) has the segment property, if for any $x, y \in Y$ there exists a d-continuous curve $\gamma : [0, 1] \to Y$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$d(x, y) = d(x, \gamma(t)) + d(\gamma(t), y), \quad \forall t \in [0, 1].$$

The following lemma, in the context of Carnot-Carathéodory spaces, is proved in [10]. However, we notice that the proof in that paper only relies on general metric properties. Here we explicitly show the proof for the reader's convenience.

Lemma 2.8 *Let (Y, d, μ) be a doubling metric space satisfying the segment property, Then, for each $x \in Y$, the function $r \mapsto \mu(B(x, r))$ is continuous.*

Proof We have

$$\lim_{\varrho \to r^-} \mu(B_\varrho) = \mu(B_r), \quad \lim_{\varrho \to r^+} \mu(B_\varrho) = \mu(\overline{B_r}),$$

and so to prove the lemma it is enough to show that $\mu(\partial B_r) = 0$. Suppose by contradiction that $\mu(\partial B_r) > 0$. By Lebesgue's differentiation theorem [16, Theorem 1.8], we have $\lim_{R \to 0} \fint_{B_R(y)} \chi_{B_r}(x)\, \mathrm{d}\mu(x) = 0$ for a.e. $y \in \partial B_r$ in the measure μ, and therefore if we show that

$$\frac{\mu(B_r \cap B_R(y))}{\mu(B_R(y))} \geq c > 0 \tag{2.13}$$

for all $y \in \partial B_r$ and for all R sufficiently small, we get a contradiction. Fix $y \in \partial B_r$, let x_0 be the center of B_r, and by the segment property let γ be a d-continuous curve joining x_0 and y such that $d(x_0, y) = d(x_0, z) + d(z, y)$ for all $z \in \gamma$. Picking $z \in \gamma$ such that $d(z, y) = R/2$, we get that $B_{R/2}(z) \subset B_r \cap B_R(y)$ for $R < r$, and $B_R(y) \subset B_{3R/2}(z)$. Therefore (2.13) follows from the doubling property. □

Important examples of metric spaces satisfying the segment property are the *Carnot-Carathéodory spaces* related to families of vector fields in $\mathbb{R}^N$. Let $X_1, \ldots, X_m$ be a system of vector fields defined in an open set $Y \subset \mathbb{R}^N$ with locally Lipschitz continuous coefficients. A piecewise C^1 curve $\gamma : [0, T] \to Y$ is called X-subunit if whenever $\gamma'(t)$ exists one has

$$\gamma'(t) = \sum_{j=1}^{m} c_j(t) X_j(\gamma(t)), \quad \text{with} \sum_{j=1}^{m} c_j(t)^2 \leq 1. \tag{2.14}$$

The X-subunit length of γ is by definition $l_S(\gamma) = T$. Given $x, y \in Y$, $\Phi(x, y)$ denotes the collection of all sub-unit curves connecting x and y. We assume $\Phi(x, y) \neq 0$ for all $x, y \in \mathbb{R}^N$. Then the function

$$d(x, y) := \inf \{l_S(\gamma) \ : \ \gamma \in \Phi(x, y)\}$$

is a metric in Y, usually called the *Carnot-Carathéodory distance, or control distance*, related to the system X of vector fields. If the function $(x, y) \mapsto d(x, y)$ is continuous with respect to the Euclidean topology, then the metric space (Y, d) satisfies the segment property, see [13, Lemma 3.7] and also [11], where this property was established for the first time in a non-Euclidean Carnot-Carathéodory space.

2.4 Reverse doubling

The following reverse doubling condition will be crucially used in the paper:

$$\mu(B(x,r)) \leq \delta\, \mu(B(x,2r)), \tag{2.15}$$

for some $0 < \delta < 1$. This inequality holds true, with a δ independent of r, if some "contraction" of the ring $B(x,2r) \setminus B(x,r)$ is non empty. More precisely, we have the following proposition.

Proposition 2.9 *Let (Y,d,μ) be a doubling quasi metric Hölder space. If there exist constants η,θ with $1 < \eta < 2\theta < 2$ such that $B(x,2\theta r) \setminus B(x,\eta r) \neq \emptyset$, then (2.15) holds with a constant $\delta \in (0,1)$ only depending on η,θ,K,C_D and the constants β and α in (2.3).*

Proof Let $y \in B(x,2\theta r) \setminus B(x,\eta r)$. We claim that there exists a constant $\sigma = \sigma(\beta,\alpha) < 1$ such that $B(y,\sigma r) \subset B(x,2\theta r) \setminus B(x,\eta r)$. This inclusion implies by doubling that

$$\mu(B(x,2r)) \geq \mu(B(x,r)) + \mu(B(y,\sigma r)) \geq \mu(B(x,r)) + \frac{1}{C_D}\left(\frac{\sigma}{4K}\right)^Q \mu(B(x,2r)).$$

Thus, (2.15) follows letting $\delta = 1 - \frac{1}{C_D}\left(\frac{\sigma}{4K}\right)^Q$. So we are left with the proof of the claim. Let $z \in B(y,\sigma r)$. From (2.3) we have

$$\begin{aligned} d(z,x) &\geq d(x,y) - \beta\, d(z,y)^\alpha\, (d(z,y) + d(x,y))^{1-\alpha} \\ &\geq \eta r - \beta(\sigma r)^\alpha(\sigma r + 2r)^{1-\alpha} = \left(\eta - \beta\sigma^\alpha(\sigma+2)^{1-\alpha}\right) r. \end{aligned}$$

Thus, if we choose $\sigma > 0$ such that $\beta\sigma^\alpha(\sigma+2)^{1-\alpha} < \eta - 1$, we obtain $B(y,\sigma r) \cap B(x,r) = \emptyset$. Analogously,

$$\begin{aligned} d(z,y) &\leq d(x,y) + \beta\, d(z,y)^\alpha\, (d(z,y) + d(x,y))^{1-\alpha} \\ &\leq 2\theta r + \beta(\sigma r)^\alpha(\sigma r + 2r)^{1-\alpha} = \left(2\theta + \beta\sigma^\alpha(\sigma+2)^{1-\alpha}\right) r. \end{aligned}$$

Thus, choosing $\sigma > 0$ satisfying, together with the previous condition, that $\beta\sigma^\alpha(\sigma+2)^{1-\alpha} < 2(1-\theta)$, we also obtain $B(y,\sigma r) \subset B(x,2r) \setminus B(x,r)$ and the claim is proved. □

An interesting example of a metric satisfying the hypotheses of Proposition 2.9 is the Carnot-Carathéodory distance d related to the family of locally Lipschitz continuous vector fields $X_1,\ldots,X_m$ in a connected open set $Y \subset \mathbb{R}^N$. More precisely, if d is a continuous with respect to the Euclidean distance, we have $\partial B(x,r) \neq \emptyset$ for every r sufficiently small.

Proposition 2.10 *Let $K \subset Y$ be compact, $K \neq Y$. Then there exists $\rho = \rho(K,Y) > 0$ such that $\partial B(x,r) \neq \emptyset$ for every $x \in K$ and $0 < r < \rho$.*

The proof of the proposition requires the following lemma whose elementary proof can be found in [17, Proposition 11.2].

Lemma 2.11 *Let $K \subset Y$ be compact. Then there exists a positive constant $M = M(K, Y)$ such that*

$$|x - y| \leq M\, d(x, y), \quad \forall x, y \in K,$$

where $|x - y|$ is the Euclidean distance in $\mathbb{R}^N$.

Proof of Proposition 2.10 Let $y \in Y \setminus K$ and let M be the constant given in the previous lemma and related to $K \cup \{y\}$. Let $\rho > 0$ be such that $|x - y| \geq M\rho$ for all $x \in K$. Thus, by Lemma 2.11, $d(x, y) \geq \rho$ for all $x \in K$. We now fix a point $x \in K$ and choose a continuous curve $\gamma : [0, 1] \to Y$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The function $t \mapsto d(t) := d(x, \gamma(t))$ is continuous and satisfies $d(0) = 0$ and $d(1) \geq \rho$. Then given $0 < r < \rho$ there exists $t \in (0, 1)$ such that $d(x, \gamma(t)) = r$ which completes the proof of the proposition. □

3 Covering theorems

In this section we prove some covering results that are important for our later developments and have independent interest. The main results are Lemma 3.1 and Theorems 3.3 and 3.4. The covering argument given in Lemma 3.1 is a measure theoretic result inspired from [14], [15, Theorem 6.3.3] and [7] and adapted to metric balls. It is known that the Besicovitch covering lemma is not true for general metric spaces, see e.g. [19]. [3] Lemma 3.1 is a suitable variant of such lemma in our setting.

Lemma 3.1 (ϵ-Besicovitch) *Let (Y, d) be quasi metric Hölder space of homogeneous type. Let A be a set contained in the ball $B(z_0, R)$, and suppose that for each $a \in A$ a ball $B(a, r)$ is given such that r is bounded by a fix number M. Let us denote by $\mathcal{F}$ the family of all these balls. Then there exists a countable subfamily of $\mathcal{F}$, $\{B(a_k, r_k)\}_{k=1}^{\infty}$, with the following properties*:

(i) $A \subset \cup_{k=1}^{\infty} B(a_k, r_k)$;

(ii) $a_k \notin \cup_{j<k} B(a_j, r_j) \quad \forall k \geq 2$;

(iii) *The family $\{B(a_k, r_k/\alpha)\}_{k=1}^{\infty}$ is disjoint, where $\alpha = \dfrac{7K}{3}$, and K is the constant in the quasi triangle inequality for d*;

(iv) *For all $0 < \epsilon < 1$ the family*

$$\mathcal{F}_\epsilon = \{B(a_k, (1 - \epsilon)r_k)\}_{k=1}^{\infty}$$

has bounded overlaps, namely

$$\sum_{k=1}^{\infty} \chi_{B(a_k,(1-\epsilon)r_k)}(x) \leq C \log \frac{1}{\epsilon},$$

[3] For counterexamples to the Besicovitch covering lemma in Carnot groups see [24].

where C depends only on K and the constants α and β in Definition 2.3 and χ_E denotes the characteristic function of the set E.

Proof We may assume $\alpha_1 \equiv M = \sup\{r : B(a,r) \in \mathcal{F}\}$. Let

$$\mathcal{F}_0 = \left\{ B(a,r) : \frac{M}{2} < r \leq M,\ B(a,r) \in \mathcal{F} \right\},$$

and

$$A_0 = \{a : B(a,r) \in \mathcal{F}_0\}.$$

Pick $B(a_1, r_1) \in \mathcal{F}_0$ such that $r_1 > \frac{3}{4}M$. Then either $A_0 \setminus B(a_1, r_1) = \emptyset$ or $A_0 \setminus B(a_1, r_1) \neq \emptyset$. In the first case, $A_0 \subset B(a_1, r_1)$ and we stop. In the second case, the set

$$\{r : B(a,r) \in \mathcal{F}_0 \text{ and } a \in A_0 \setminus B(a_1, r_1)\}$$

is not empty and let α_2 denote its supremum. Pick r_2 in this set such that $\alpha_2 \geq r_2 > \frac{3}{4}\alpha_2$ and let $B(a_2, r_2)$ be the corresponding ball. We then have $a_2 \notin B(a_1, r_1)$ and $r_1 > \frac{3}{4}\alpha_1 \geq \frac{3}{4}\alpha_2 \geq \frac{3}{4}r_2$. Again, we have either $A_0 \setminus (B(a_1, r_1) \cup B(a_2, r_2)) = \emptyset$ or $A_0 \setminus (B(a_1, r_1) \cup B(a_2, r_2)) \neq \emptyset$. In the first case, we have $A_0 \subset B(a_1, r_1) \cup B(a_2, r_2)$ and we stop. In the second case, we continue the process. In general, for the jth-step we pick r_j such that $\alpha_j \geq r_j > \frac{3}{4}\alpha_j$ where

$$\alpha_j = \sup \left\{ r : B(a,r) \in \mathcal{F}_0 \ \text{ and } \ a \in A_0 \setminus \bigcup_{i<j} B(a_i, r_i) \right\},$$

and select $B(a_j, r_j)$. We have $r_j \leq \frac{4}{3}r_i$ for $j > i$, since

$$r_j \leq \alpha_j \leq \alpha_i < \frac{4}{3}r_i\,. \tag{3.1}$$

Continuing in this way we construct a family, at this point possibly infinite, which we denote by

$$\mathcal{F}_0' = \{B(a_k^0, r_k^0)\}_{k=1}^{\infty}$$

with

$$a_j^0 \in A_0 \setminus \bigcup_{i<j} B(a_i^0, r_i^0).$$

We now claim that the balls $B(a_i^0, r_i^0/\alpha)$, are disjoint. In fact, if $z \in B(a_i^0, r_i^0/\alpha) \cap B(a_j^0, r_j^0/\alpha)$ and $i < j$, then from (3.1)

$$r_i^0 < d(a_i^0, a_j^0) \leq K\,(d(a_i^0, z) + d(z, a_j^0)) < K\left(\frac{r_i^0}{\alpha} + \frac{r_j^0}{\alpha}\right) \leq \frac{7K}{3\alpha} r_i^0.$$

Now $d(a_i^0, a_j^0) \geq \dfrac{M}{2}$ and since $a_i^0 \in B(z_0, R)$, it follows from [9, p. 68,Lemma (1.1)] that the number of balls in $\mathcal{F}_0'$ is bounded by a constant $N(R, M)$. This implies that the union of all members in $\mathcal{F}_0'$ covers A_0. The family $\mathcal{F}_0'$ will be called the first generation and denoted by $\mathcal{F}_0' = \{B(a_i^0, r_i^0)\}_{i \in I_0}$.

We now proceed with the construction of the second generation.

Let us consider the family

$$\mathcal{F}_1 = \left\{B(a, r) : \frac{M}{4} < r \leq \frac{M}{2},\ B(a, r) \in \mathcal{F}\right\}.$$

Let

$$A_1 = \left\{a : B(a, r) \in \mathcal{F}_1 \text{ and } a \notin \bigcup_{i \in I_0} B(a_i^0, r_i^0)\right\}.$$

We repeat the construction above for the set A_1 obtaining a family of balls denoted by

$$\mathcal{F}_1' = \{B(a_i^1, r_i^1)\}_{i \in I_1}$$

that we call the second generation. We continue this process and in the kth-step we consider the family

$$\mathcal{F}_k = \left\{B(a, r) : \frac{M}{2^{k+1}} < r \leq \frac{M}{2^k},\ B(a, r) \in \mathcal{F}\right\}$$

and the set

$$A_k = \left\{a : B(a, r) \in \mathcal{F}_k \text{ and } a \notin \bigcup \text{balls previously selected}\right\}.$$

As before we obtain a family of balls denoted by

$$\mathcal{F}_k' = \{B(a_i^k, r_i^k)\}_{i \in I_k}.$$

By construction, each family $\mathcal{F}_k'$ has a finite number of members whose union covers A_k, each ball $B(a_i^k, r_i^k) \in \mathcal{F}_k'$ has the property that

$$\frac{M}{2^{k+1}} < r_i^k \leq \frac{M}{2^k},$$

and the balls $\{B(a_i^k, r_i^k/\alpha)\}_i$ are disjoint. In addition, and again by [9, p. 68, Lemma (1.1)] the number of balls is finite and bounded by a number $N(R, M, k)$.

We claim that the collection of all the balls in all generations $\mathcal{F}'_k$, $k \geq 1$, is the family that satisfies the conclusion.

To show (i), let $a \in A$. Then there exists a ball $B(a, r) \in \mathcal{F}_k$ for some k. If $a \notin \bigcup_{i,l;l<k} B(a_i^l, r_i^l)$, then $a \in A_k$ and therefore $a \in B(a_j^k, r_j^k)$ for some j since $\mathcal{F}'_k$ covers A_k.

Since each generation $\mathcal{F}'_k$ has a finite number of members by relabelling the indices of all of the members of all generations $\mathcal{F}'_k$ we obtain (ii).

In order to show (iii), let $a_i \neq a_j$. If $B(a_i, r_i)$ and $B(a_j, r_j)$ belong to the same generation, then (iii) was already proved. On the other hand, suppose $B(a_i, r_i)$ belongs to $\mathcal{F}'_k$ and $B(a_j, r_j)$ belongs to $\mathcal{F}'_{k+p}$ for some $p \geq 1$. Then, by construction, $a_j \notin B(a_i, r_i)$ and

$$r_j \leq \frac{M}{2^{k+p}} \leq \frac{M}{2^{k+1}} < r_i \,.$$

This implies our claim, otherwise

$$r_i \leq d(a_i, a_j) \leq K \left(\frac{r_i}{\alpha} + \frac{r_j}{\alpha}\right) \leq \left(\frac{2K}{\alpha}\right) r_i,$$

a contradiction by the choice of α.

To show the bounded overlapping property (iv), we first prove that balls in each generation $\mathcal{F}'_k$ enjoy this property. Suppose that

$$z \in B\big(a_{j_1}^k, r_{j_1}^k\big) \cap \cdots \cap B\big(a_{j_m}^k, r_{j_m}^k\big),$$

with $B(a_{j_i}^k, r_{j_i}^k) \in \mathcal{F}'_k$. To simplify notations we set $a_{j_i}^k = a_i$, $r_{j_i} = t_i$, $1 \leq i \leq m$. We have

$$\frac{M}{2^{k+1}} < t_i \leq \frac{M}{2^k} \,,$$

and

$$a_i \in B(z, t_i) \subset B\left(z, \frac{M}{2^k}\right).$$

Since $d(a_i, a_j) > t_i \geq M/2^{k+1}$ if $i \neq j$, it follows from the homogeneity of the space that $m \leq N$.

We now estimate the overlapping of balls belonging to different generations, and for this we shrink the balls selected. Let $0 < \epsilon < 1$ and

$$z \in \bigcap_i B\left(a_{m_i}^{e_i}, (1-\epsilon) r_{m_i}^{e_i}\right) \,, \tag{3.2}$$

where $e_1 < e_2 < \cdots < e_i < \cdots$, $M2^{-(e_i+1)} < r_{m_i}^{e_i} \le M2^{-e_i}$. For simplicity in the notation we set $a_i = a_{m_i}^{e_i}$ and $r_i = r_{m_i}^{e_i}$. We may assume ϵ is sufficiently small because otherwise, by (iii) the balls are disjoint. Fix i and let $j > i$, we shall measure the gap between e_j and e_i. Since $z \in B\left(a_j, (1-\epsilon)r_j\right) \cap B\left(a_i, (1-\epsilon)r_i\right)$, and $a_j \notin B(a_i, r_i)$, we obtain from (2.3) that

$$r_i < d(a_i, a_j) \le d(a_i, z) + \beta\, d(a_j, z)^\alpha \left(d(a_j, a_i) + d(a_j, z)\right)^{1-\alpha}.$$

On the other hand, by the quasi triangle inequality

$$d(a_i, a_j) \le K(d(a_i, z) + d(z, a_j)) \le K(1-\epsilon)(r_i + r_j) < 2Kr_i,$$

since $r_i > r_j$, and inserting this in the previous inequality we get

$$r_i \le (1-\epsilon)r_i + \beta r_j^\alpha (2Kr_i)^{1-\alpha}.$$

Thus

$$\epsilon\, r_i^\alpha < C\, r_j^\alpha,$$

and so $\dfrac{\epsilon}{2^{(e_i+1)\alpha}} < C\,\dfrac{1}{2^{e_j\alpha}}$. This implies that $e_j - e_i < c\,\log_2\left(\dfrac{1}{\epsilon}\right)$ and the proof of the lemma is complete. □

Remark 3.2 Under the assumptions of Lemma 3.1, there exists a disjoint subfamily $\{B(a_k, r_k)\}_{k=1}^\infty$ such that the family $\{B(a_k, \alpha r_k)\}_{k=1}^\infty$ covers A, that is, we get the conclusion in Vitali's lemma. Indeed, if $\mathcal{F}$ is the family in the ϵ-Besicovitch Lemma, and $\mathcal{F}^* = \{B(a, \alpha r) \ : \ B(a, r) \in \mathcal{F}\}$, then applying Lemma 3.1 to $\mathcal{F}^*$ yields the remark.

We are now in a position to prove the following selection theorem of importance in the following sections.

Theorem 3.3 *Let (Y, d, μ) be a doubling quasi metric Hölder space satisfying the ring condition in Definition 2.6 with $\omega(\epsilon) = o((\log(1/\epsilon))^{-2})$ as $\epsilon \to 0^+$, that is,*

$$\mu\left(B(x,t) \setminus B(x, (1-\epsilon)t)\right) \le \omega(\epsilon)\, \mu(B(x,t)), \tag{3.3}$$

for all balls $B(x,t)$ and for all ϵ sufficiently small.

If $E \subset B_{R_0}(z)$ is a μ-measurable set with $\mu(E) > 0$, $0 < \delta < 1$, and $\mu(B_{R_0}(z)) < \delta\, \mu(B_{2R_0}(z))$, then there is a constant $0 < c(\delta) < 1$, depending only on δ and not on E, and a family of balls $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ with $r_j \le 3KR_0$ satisfying

(i) *all x_j are density points*[4] *of E with respect to μ;*

[4] $x \in Y$ is a density point for the set E if $\dfrac{\mu(B_r(x) \cap E)}{\mu(B_r(x))} \to 1$ as $r \to 0$.

(ii) $E \subset \cup_{j=1}^{\infty} B_j$ *a.e. in the measure* μ;

(iii) $\dfrac{\mu(B_j \cap E)}{\mu(B_j)} = \delta$ *for* $j = 1, 2, \ldots$;

(iv) $\mu(E) \leq c(\delta)\, \mu\left(\bigcup_{j=1}^{\infty} B_j\right)$.

Here K *denotes the constant in the quasi-triangle inequality of* d.

Proof Let E_0 be the set of density points of E with respect to μ. We first notice that from [16, Theorem 1.8], $\dfrac{\mu(B_r(x) \cap E)}{\mu(B_r(x))} \to \chi_E(x)$ for a.e. $x \in Y$ in the measure μ, and since $\mu(E) > 0$, we get that $E_0 \neq \emptyset$. In addition, we claim that $\mu(E_0) = \mu(E)$. Indeed, there exists $F \subset Y$ with $\mu(F) = 0$ such that $\dfrac{\mu(B_r(x) \cap E)}{\mu(B_r(x))} \to \chi_E(x)$ for each $x \in Y \setminus F$. Hence $E \cap F^c \subset E_0$ and so $\mu(E) \leq \mu(E_0)$. Also $E_0 \cap F^c \subset E$ and consequently $\mu(E_0) \leq \mu(E)$ which proves the claim.

Let $0 < \delta < 1$ such that $\mu(B_{R_0}(z)) < \delta\, \mu(B_{2R_0}(z))$, and for each $x \in E_0$ let

$$r_x = \sup I(x) \equiv \sup\left\{r : \frac{\mu(B_r(x) \cap E)}{\mu(B_r(x))} \geq \delta\right\}.$$

Since x is a density point of E, the set $I(x)$ is non empty. Moreover, $I(x) \subset (0, 3KR_0]$. Indeed, $x \in B_{R_0}(z)$ and $r \geq 3KR_0$, then

$$\frac{\mu(B_r(x) \cap E)}{\mu(B_r(x))} \leq \frac{\mu(E)}{\mu(B_r(x))} \leq \frac{\mu(B_{R_0}(z))}{\mu(B_{3KR_0}(x))} \leq \frac{\mu(B_{R_0}(z))}{\mu(B_{(2R_0}(z))} < \delta,$$

and consequently $I(x) \subseteq (0, MR_0]$. It follows from (3.3) that the function $r \to \mu(B_r(x))$ is continuous. From this fact it follows that the function $r \to \mu(B_r(x) \cap E)$ is also continuous for each fix x and each measurable set E because $(B_r(x) \cap E) \setminus (B_s(x) \cap E) = (B_r(x) \setminus B_s(x)) \cap E \subset B_r(x) \setminus B_s(x)$ for $s < r$. Therefore from the definition of r_x and these continuity properties we get that

$$\frac{\mu(B_{r_x}(x) \cap E)}{\mu(B_{r_x}(x))} = \delta.$$

For each $x \in E_0$ consider the ball $B(x, r_x)$. Since (Y, d, μ) is a doubling metric space, then it is of homogeneous type and d is Lipschitz continuous. Therefore applying Lemma 3.1 to this family of balls we obtain a countable subfamily of balls $\{B_k\}_{k=1}^{\infty}$, $B_k = B(x_k, r_{x_k})$, satisfying (i)–(iv) in that lemma. We obviously obtain that (i), (iii) hold, and $E_0 \subset \cup_{j=1}^{\infty} B_j$. It is enough to prove (iv) with $\mu(E_0)$ in place of $\mu(E)$. Define the overlapping function with respect to the balls B_j

$$f(x) = \begin{cases} \#\{k : x \in B_k\} & \text{if } x \in \cup_j B_j \\ 1 & \text{otherwise,} \end{cases}$$

and the overlapping function with respect to the shrunken balls $B_j^\epsilon = B(x_j, (1-\epsilon)r_{x_j})$

$$f_\epsilon(x) = \begin{cases} \#\left\{k \,:\, x \in B_k^\epsilon\right\} & \text{if } x \in \cup_j B_j^\epsilon \\ 1 & \text{otherwise,} \end{cases}$$

where ϵ is a small number that will be chosen at the end. Notice that,

$$\chi_{\cup B_k}(x) = \frac{1}{f(x)} \sum_k \chi_{B_k}(x)\,.$$

Obviously, $1 \le f_\epsilon(x) \le f(x)$. We have

$$\begin{aligned} \mu(E_0) = \mu(\cup B_k \cap E_0) &= \int \frac{1}{f(x)} \sum_k \chi_{B_k \cap E_0}(x)\,\mathrm{d}\mu(x) \\ &\le \int \frac{1}{f_\epsilon(x)} \sum_k \chi_{B_k \cap E_0}(x)\,\mathrm{d}\mu(x). \end{aligned} \tag{3.4}$$

Let us write

$$\int \frac{\chi_{B_k}(x)}{f_\epsilon(x)}\,\mathrm{d}\mu(x) = \int \frac{\chi_{B_k \cap E_0}(x)}{f_\epsilon(x)}\,\mathrm{d}\mu(x) + \int \frac{\chi_{B_k \setminus E_0}(x)}{f_\epsilon(x)}\,\mathrm{d}\mu(x). \tag{3.5}$$

Since $\mu(E) = \mu(E_0)$, we have $\mu(E_0 \cap E^c) = 0$. Consequently,

$$\begin{aligned} \mu(B_k) &= \mu(B_k \cap E_0 \cap E) + \mu(B_k \cap E_0 \cap E^c) + \mu(B_k \setminus E_0) \\ &\le \mu(B_k \cap E) + \mu(B_k \setminus E_0) \\ &= \delta\,\mu(B_k) + \mu(B_k \setminus E_0), \end{aligned}$$

and so

$$\mu(B_k \setminus E_0) \ge (1-\delta)\mu(B_k).$$

Thus

$$\begin{aligned} \int \frac{\chi_{B_k \setminus E_0}(x)}{f_\epsilon(x)}\,\mathrm{d}\mu(x) &\ge \frac{1}{K\ \log(1/\epsilon)} \int \chi_{B_k \setminus E_0}(x)\,\mathrm{d}\mu(x) = \frac{1}{K\ \log(1/\epsilon)} \mu(B_k \setminus E_0) \\ &\ge \frac{1-\delta}{K\ \log(1/\epsilon)} \mu(B_k) = \frac{1-\delta}{K\ \log(1/\epsilon)} \int \chi_{B_k}(x)\,\mathrm{d}\mu(x) \\ &\ge \frac{1-\delta}{K\ \log(1/\epsilon)} \int \frac{\chi_{B_k}(x)}{f_\epsilon(x)}\,\mathrm{d}\mu(x). \end{aligned}$$

Inserting this inequality in (3.5) yields

$$\int \frac{\chi_{B_k \cap E_0}(x)}{f_\epsilon(x)}\,\mathrm{d}\mu(x) \le \left(1 - \frac{1-\delta}{K\ \log(1/\epsilon)}\right) \int \frac{\chi_{B_k}(x)}{f_\epsilon(x)}\,\mathrm{d}\mu(x),$$

and consequently from (3.4)

$$\mu\left(\cup B_k \cap E_0\right) \leq \left(1 - \frac{1-\delta}{K\ \log(1/\epsilon)}\right) \sum_k \int \frac{\chi_{B_k}(x)}{f_\epsilon(x)}\, \mathrm{d}\mu(x)\,. \tag{3.6}$$

We next estimate each term in the last sum. We write

$$\begin{aligned}
\int \frac{\chi_{B_k}(x)}{f_\epsilon(x)}\, \mathrm{d}\mu(x) &= \int \frac{1}{f_\epsilon(x)} \left\{\chi_{B_k}(x) - \chi_{B_k^\epsilon}(x)\right\} \mathrm{d}\mu(x) + \int \frac{\chi_{B_k^\epsilon}(x)}{f_\epsilon(x)}\, \mathrm{d}\mu(x) \\
&\leq \mu(B_k) - \mu(B_k^\epsilon) + \int \frac{\chi_{B_k^\epsilon}(x)}{f_\epsilon(x)}\, \mathrm{d}\mu(x) \\
&\leq \frac{\omega(\epsilon)}{1-\omega(\epsilon)} \mu(B_k^\epsilon) + \int \frac{\chi_{B_k^\epsilon}(x)}{f_\epsilon(x)}\, \mathrm{d}\mu(x) \quad \text{by (3.3)} \\
&= \frac{\omega(\epsilon)}{1-\omega(\epsilon)} K\ \log(1/\epsilon) \int \frac{\chi_{B_k^\epsilon}(x)}{K\ \log(1/\epsilon)}\, \mathrm{d}\mu(x) + \int \frac{\chi_{B_k^\epsilon}(x)}{f_\epsilon(x)}\, \mathrm{d}\mu(x) \\
&\leq \left(1 + \frac{\omega(\epsilon)}{1-\omega(\epsilon)} K\ \log(1/\epsilon)\right) \int \frac{\chi_{B_k^\epsilon}(x)}{f_\epsilon(x)}\, \mathrm{d}\mu(x).
\end{aligned}$$

Using this estimate in (3.6) yields the inequality

$$\mu(E_0) = \mu(\cup B_k \cap E_0) \leq h(\epsilon)\, \mu(\cup B_k^\epsilon), \tag{3.7}$$

with

$$h(\epsilon) = \left(1 - \frac{1-\delta}{K\ \log(1/\epsilon)}\right) \left(1 + \frac{\omega(\epsilon)}{1-\omega(\epsilon)} K\ \log(1/\epsilon)\right).$$

There exists $0 < \epsilon_0 = \epsilon_0(\delta, \omega, K) < 1$ such that the function $h(\epsilon)$ is strictly less than 1 in the interval $(0, \epsilon_0]$. Indeed, this follows writing $h(\epsilon) = 1 + \gamma(\epsilon)$ where

$$\gamma(\epsilon) = (K\ \log(1/\epsilon) - (1-\delta))\, \frac{\omega(\epsilon)}{1-\omega(\epsilon)} - \frac{1-\delta}{K\ \log(1/\epsilon)},$$

and it is clear that $\gamma(\epsilon) < 0$ for all ϵ sufficiently small since $\omega(\epsilon) = o((\log(1/\epsilon))^{-2})$ as $\epsilon \to 0^+$. The desired result then follows from (3.7) picking $\epsilon > 0$ sufficiently small. □

Without assuming (3.3), we obtain the following result with a little weaker conclusion than in Theorem 3.3.

Theorem 3.4 *Let (Y, d, μ) be a doubling quasi metric Hölder space, and assume that the function $r \to \mu(B_r(x))$ is continuous.*

If $E \subset B_{R_0}(z)$ is a μ-measurable set with $\mu(E) > 0$, $0 < \delta < 1$, and $\mu(B_{R_0}(z)) < \delta\, \mu(B_{2R_0}(z))$, then there are constants $C_\delta > 0$, $0 < c(\delta) < 1$, depending only on δ and not on E, and a family of balls $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ with $r_j \leq C_\delta R_0$ satisfying

(i) *all* x_j *are density points of* E *with respect to* μ;
(ii) $E \subset \cup_{j=1}^{\infty} B_j$ *a.e. in the measure* μ;
(iii) $\dfrac{\delta}{C_D} \leq \dfrac{\mu(B_j \cap E)}{\mu(B_j)} < \delta$ *for* $j = 1, 2, \ldots$;
(iv) $\mu(E) \leq c(\delta)\, \mu\left(\bigcup_{j=1}^{\infty} B_j\right)$.

Proof The proof is similar to that of Theorem 3.3. We select balls $B(x, r_x)$, $x \in E_0$, as in that Theorem and we enlarge them in the following way. Let ϵ be a small positive number. From [7, Lemma 2] there exists a constant $c > 0$ such that for each ball $B(x, r)$ there exists $t = t(\epsilon, r)$ such that $1 + \epsilon \leq t \leq 2$, and

$$\mu(B(x, tr)) - \mu(B(x, (t - \epsilon)r)) \leq c\,\epsilon\,\mu(B(x, tr)). \tag{3.8}$$

Set $r'_x = tr_x$ and consider the family of balls $\mathcal{F} = \{B(x, r'_x)\}_{x \in E_0}$. We have

$$B(x, r_x) \subset B(x, (t - \epsilon)r_x) \subset B\left(x, \left(1 - \frac{\epsilon}{2}\right) r'_x\right). \tag{3.9}$$

Since $r'_x > r_x$, by definition of r_x in Theorem 3.3 we have that $\frac{\mu(B_{r'_x}(x) \cap E)}{\mu(B_{r'_x}(x))} < \delta$. And by doubling, $\frac{\mu(B_{r'_x}(x) \cap E)}{\mu(B_{r'_x}(x))} \geq \frac{\delta}{c_D}$. Applying Lemma 3.1 to the family $\mathcal{F}$, we get a countable sub family $\{B(x_k, r'_k)\}$ that covers E_0 and satisfies the conclusion of that Lemma. If we set $B_k = B(x_k, r'_k)$ and $B_k^{\epsilon} = B\left(x_k, \left(1 - \frac{\epsilon}{2}\right) r'_k\right)$, then combining (3.8) and (3.9) we get that

$$\mu(B_k) - \mu(B_k^{\epsilon}) \leq c\,\epsilon\,\mu(B_k). \tag{3.10}$$

We now proceed as in the proof of Theorem 3.3 and using inequality (3.10) in place of (3.3), we obtain (3.7) with the function $h(\epsilon)$ defined with $\omega(\epsilon) = c\,\epsilon$. This completes the proof. □

4 Critical density, double ball property and power decay

In this section (Y, d, μ) is a doubling quasi metric Hölder space. We begin with the following two definitions. Let $\Omega \subseteqq Y$ be open. We shall denote by $\mathbb{K}_\Omega$ a family of μ-measurable functions with domain contained in Ω. If $u \in \mathbb{K}_\Omega$ and its domain contains a set $A \subset \Omega$, we write $u \in \mathbb{K}_\Omega(A)$.

Definition 4.1 (Critical density) Let $0 < \epsilon < 1$. We say that $\mathbb{K}_\Omega$ satisfies the ϵ critical density property if there exists a constant $c = c(\epsilon) > 0$ such that for every ball $B_{2R}(x_0) \subset \Omega$ and for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with

$$\mu(\{x \in B_R(x_0) : u(x) \geq 1\}) \geq \epsilon\,\mu(B_R(x_0)),$$

we have

$$\inf_{B_{R/2}(x_0)} u \geq c.$$

Definition 4.2 (Double ball property) We say that $\mathbb{K}_\Omega$ satisfies the double ball property if there exists a positive constant γ such that for every $B_{2R}(x_0) \subset \Omega$ and every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with $\inf_{B_{R/2}(x_0)} u \geq 1$ we have $\inf_{B_R(x_0)} u \geq \gamma$.

We point out that we do not identify two functions that differ on a set of μ measure zero. We also notice that if $\mathbb{K}_\Omega$ satisfies the ϵ_0 critical density property, then $\mathbb{K}_\Omega$ satisfies the ϵ critical density for any $\epsilon > \epsilon_0$.

The ϵ critical density and the double ball properties are in general independent but if the critical density holds for ϵ sufficiently small, then the double ball property also holds as the following proposition shows.

Proposition 4.3 *If c_D is the doubling constant of μ and $\mathbb{K}_\Omega$ satisfies the ϵ critical density for some $0 < \epsilon < 1/c_D^2$, then $\mathbb{K}_\Omega$ satisfies the double ball property.*

Proof Let c be the constant in the ϵ critical density property, and let $u \in \mathbb{K}_\Omega(B_{2R})$ with $B_{2R}(x_0) \subset \Omega$ and such that $\inf_{B_{R/2}} u \geq 1$. We prove that $\inf_{B_R} u \geq c$. Suppose by contradiction that $\inf_{B_R} u < c$. Then

$$\mu(\{x \in B_{2R} : u(x) \geq 1\}) < \epsilon\, \mu(B_{2R}),$$

and since $B_{R/2} \subseteq \{x \in B_{2R} : u(x) \geq 1\}$, from the doubling condition we get that

$$\mu(B_{R/2}) \leq \epsilon \mu(B_{2R}) \leq \epsilon c_D^2 \mu(B_{R/2}),$$

a contradiction. □

As a consequence of the critical density and the double ball properties we get the following proposition.

Proposition 4.4 *Assume that $\mathbb{K}_\Omega$ satisfies the double ball property and the critical density property for some $0 < \epsilon < 1$, and $\mathbb{K}_\Omega$ is closed under multiplications by positive constants. Then there exists a structural constant $M_0 > 1$ depending on ϵ such that for any $\alpha > 0$ and any $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with*

$$\mu(\{x \in B_R(x_0) : u(x) \geq \alpha\}) \geq \epsilon\, \mu(B_R(x_0)),$$

we have

$$\inf_{B_R(x_0)} u \geq \alpha / M_0.$$

Proof We have $\frac{u}{\alpha} \in \mathbb{K}_\Omega(B_{2R}(x_0))$ for $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$. If c is the constant in the ϵ critical density, then $\inf_{B_{R/2}} \frac{u(x)}{\alpha} \geq c$. Now $\frac{u(x)}{\alpha c} \in \mathbb{K}_\Omega(B_{2R}(x_0))$ and so by double ball property we have $\inf_{B_R} \frac{u(x)}{\alpha c} \geq \gamma$ and the proposition follows with $M_0 = (c\,\gamma)^{-1}$. □

From the previous proposition we get the following lemma about the size of the metric balls. It will be used very soon.

Lemma 4.5 *Let $\mathbb{K}_\Omega$ satisfy the double ball property and the critical density for some $0 < \epsilon < 1$, and assume $\mathbb{K}_\Omega$ is closed under positive multiplications. There exist structural positive constants σ, M_1, θ such that if $u \in \mathbb{K}_\Omega(B_{\theta R}(x_0))$ with $\inf_{B_R(x_0)} u \leq 1$, $\alpha > 0$, $\varrho < 2KR$ and $y \in B_R(x_0)$ are such that*

$$\mu(\{x \in B_\varrho(y) : u(x) > \alpha\}) \geq \epsilon\, \mu(B_\varrho(y)), \tag{4.11}$$

then $\varrho \leq \left(\frac{M_1}{\alpha}\right)^\sigma R$. Here K is the constant in the quasi triangle inequality, $M_1 = (4K)^{1/\sigma} M_0$, where $M_0 = M_0(\epsilon)$ is the constant in Proposition 4.4, $\theta = K(1+8K)$ and $\sigma = \log 2/\log \gamma^{-1}$.

Proof From (4.11) and Proposition 4.4, we get

$$\inf_{B_\varrho(y)} u \geq \alpha/M_0,$$

if $B_{2\rho}(y) \subset B_{\theta R}(x_0)$. Then from the double ball property

$$\inf_{B_{2\varrho}(y)} u \geq \frac{\gamma\,\alpha}{M_0},$$

if $B_{4\rho}(y) \subset B_{\theta R}(x_0)$. Iterating this inequality yields

$$\inf_{B_{2^p\varrho}(y)} u \geq \frac{\gamma^p\,\alpha}{M_0},$$

if $B_{2^{p+1}\rho}(y) \subset B_{\theta R}(x_0)$. Let us now choose $p \in \mathbb{N}$ such that $2^{p-1} \leq \frac{2KR}{\rho} \leq 2^p$. Since $y \in B_R(x_0)$, it follows that $B_{2^{p+1}\rho}(y) \subset B_{\theta R}(x_0)$ and $B_R(x_0) \subset B_{2^p\rho}(y)$. It follows that

$$1 \geq \inf_{B_R(x_0)} u \geq \inf_{B_{2^p\varrho}(y)} u \geq \frac{\gamma^p\,\alpha}{M_0}.$$

Since $\gamma^\sigma = 1/2$, from the choice of p we get the lemma. □

Definition 4.6 The family $\mathbb{K}_\Omega$ satisfies the power decay property if there exist constants $M, \eta > 1$ and $0 \le \gamma < 1$ such that for each $u \in \mathbb{K}_\Omega(B_{\eta R}(x_0))$ with $\inf_{B_r(x_0)} u \le 1$ we have

$$\mu(\{x \in B_{r/2}(x_0) : u(x) > M^k\}) \le \gamma^k \mu(B_{r/2}(x_0)), \quad k = 1, 2, \ldots.$$

We now show that families of functions satisfying the critical density and double ball properties satisfy the power decay property.

Theorem 4.7 (Power decay) *Let (Y, d, μ) be a doubling quasi-metric Hölder space and let $\Omega \subset Y$ be open and such that $\mu(B_r(x)) \le \delta\mu(B_{2r}(x))$ for a suitable $0 \le \delta < 1$ and for every ball $B_{2r}(x) \subset \Omega$. Suppose the set $\mathbb{K}_\Omega$ is closed under multiplication by positive constants and satisfies either one of the following set of conditions*:

(A) (A1) *$\mathbb{K}_\Omega$ satisfies the double ball property and the ϵ-critical density property for some $0 < \epsilon < 1$.*
(A2) *(Y, d, μ) satisfies the ring condition (3.3) with $\omega(s) = o((\log \frac{1}{\epsilon})^2)$ as s goes to zero.*

or

(B) (B1) *$\mathbb{K}_\Omega$ satisfies the ϵ-critical density property for some $0 < \epsilon < 1/C_D^2$.*
(B2) *The function $r \mapsto \mu(B_r(x))$ is continuous for each $x \in Y$.*

Then, the family $\mathbb{K}_\Omega$ satisfies the power decay property.

Proof We first assume (A1) and (A2). Let $0 < \epsilon < 1$ be the critical density parameter of $\mathbb{K}_\Omega$. Taking the maximum between ϵ and δ we may assume that $\epsilon \ge \delta$. Let M and η be positive numbers that will be determined soon, and set

$$E_k = \{x \in B_{\eta r} : u(x) \ge M^k\}, \quad k = 1, 2, \ldots.$$

We will construct a family of balls $B_k = B_{t_k}(x_0)$ with $t_0 = r > t_1 > t_2 > \cdots$, and

$$\mu(B_{k+1} \cap E_{k+2}) \le c(\epsilon)\, \mu(B_k \cap E_{k+1}), \quad c(\epsilon) < 1, \quad k = 0, 1, 2, \ldots. \tag{4.12}$$

We first prove (4.12) for $k = 0$. Using Theorem 3.3, let $\mathcal{F}_1 = \{B(x_k, r_k)\}$ be the covering of $B_1 \cap E_2$ at level ϵ, where the x_k's are density points of $B_1 \cap E_2$, and

$$\epsilon = \frac{\mu(B(x_k, r_k) \cap B_1 \cap E_2)}{\mu(B(x_k, r_k))} \le \frac{\mu(B(x_k, r_k) \cap E_2)}{\mu(B(x_k, r_k))}. \tag{4.13}$$

Moreover, $r_k \le 3Kt_1 \le 3Kr$ for every k. If we construct $t_1 < r$ such that

$$B(x_k, r_k) \subset B_0 \cap E_1, \quad k = 1, 2, \ldots, \tag{4.14}$$

then by Theorem 3.3(iv) we obtain (4.12) for $k = 0$. To show (4.14), we first prove that $B(x_k, r_k) \subset E_1$ for all k. This is a consequence of Proposition 4.4. We first notice that, if $\eta > K(6K + 1)$ then $B(x_k, 2r_k) \subset B_{\eta r}(x_0)$ so that $u \in \mathbb{K}_\Omega(B(x_k, 2r_k)$. From (4.13)

and Proposition 4.4 it follows that $\inf_{B(x_k,r_k)} u \geq M^2/M_0$, and if we pick $M \geq M_0$ we then get that $B(x_k, r_k) \subset E_1$.

We next show that $r_k \leq 2Kr$ for all k. Suppose by contradiction that $r_j > 2Kr$ for some j. Then $B(x_j, 2Kr) \subset B(x_j, r_j)$. Since $\inf_{B(x_j,r_j)} u \geq M^2/M_0$ and $M \geq M_0$, we obtain that $\inf_{B(x_j,r_j)} u > 1$ and so $\inf_{B(x_j,2r)} u > 1$. In addition, since x_j is a density point of $B_1 \cap E_2$, it follows that $x_j \in \overline{B(x_0, t_1)}$. Then, since $t_1 < r$, $B(x_0, r) \subset B(x_j, 2Kr)$ and consequently, $\inf_{B(x_0,r)} u > 1$, a contradiction.

From the second inequality in (4.13), we can apply Lemma 4.5 with $y \rightsquigarrow x_k$, $\varrho \rightsquigarrow r_k$, $\alpha \rightsquigarrow M^2$, and we then obtain

$$r_k \leq \left(\frac{M_1}{M^2}\right)^{\sigma} r.$$

We want to explicitly remark that choosing $\eta > \theta$, where θ is the constant in Lemma 4.5, we have $u \in \mathbb{K}_\Omega(B_{\theta r}(x_0))$. Now, if $z \in B(x_k, r_k)$, inequality (2.3) yields

$$d(z, x_0) \leq d(x_k, x_0) + \beta(d(x_k, z))^\alpha \, (d(x_k, x_0) + d(x_k, z))^{1-\alpha} \tag{4.15}$$

$$\leq t_1 + \beta \left(\frac{M_1}{M^2}\right)^{\sigma\alpha} r^\alpha \left(t_1 + \left(\frac{M_1}{M^2}\right)^{\sigma} r\right)^{1-\alpha} \tag{4.16}$$

and picking $t_1 = T_1 r$ we get

$$d(z, x_0) \leq (T_1 + \beta_1 q^2) r,$$

where we have set $\beta_1 = \beta \, M_1^{\sigma\alpha}(1 + M_1^\sigma)^{1-\alpha}$, and $q = 1/M^\sigma$. Let us now choose $T_1 = 3/4$ and M big enough such that $\beta_1 q^2 < 1/4$. It follows that $T_1 + \beta_1 q^2 < 1$ and so $B(x_k, r_k) \subset B_r(x_0)$.Thus (4.14) is proved.

We next construct t_2 and prove (4.12) for $k = 1$. Again, by Theorem 3.3, let now $\mathcal{F}_2 = \{B(x_k, r_k)\}$ be the covering of $B_2 \cap E_3$ at level ϵ, where the x_k's are density points of $B_2 \cap E_3$ and

$$\epsilon = \frac{\mu(B(x_k, r_k) \cap B_2 \cap E_3)}{\mu(B(x_k, r_k))} \leq \frac{\mu(B(x_k, r_k) \cap E_3)}{\mu(B(x_k, r_k))}. \tag{4.17}$$

Moreover $r_k \leq 3Kt_2 \leq 3Kr$ for every k. As before, if we construct $t_2 < t_1$ such that

$$B(x_k, r_k) \subset B_1 \cap E_2, \quad k = 1, 2, \ldots, \tag{4.18}$$

then by Theorem 3.3 we obtain (4.12) for $k = 1$. We first show that $B(x_k, r_k) \subset E_2$ for all k. Since $\eta > K(6K+1)$ we have $B(x_k, 2r_k) \subset B_{\eta r}(x_0)$ so that $u \in \mathbb{K}_\Omega(B(x_k, 2r_k)$. From (4.17) and Proposition 4.4 we get that $\inf_{B(x_k,r_k)} u \geq M^3/M_0$ and if we pick $M^3/M_0 \geq M^2$ we get $B(x_k, r_k) \subset E_2$. As before we show that $r_k \leq 2Kr$ for all k. Suppose by contradiction that $r_j > 2Kr$ for some j. Then $B(x_j, 2Kr) \subset B(x_j, r_j)$. Since $\inf_{B(x_j,r_j)} u \geq M^3/M_0$ and $M \geq M_0$, we obtain that $\inf_{B(x_j,r_j)} u > 1$ and so $\inf_{B(x_j,2r)} u > 1$. In addition, since x_j is a density point of $B_1 \cap E_2$, it follows that

$x_j \in \overline{B(x_0, t_2)}$. Then, since $t_2 < t_1 < r$, $B(x_0, r) \subset B(x_j, 2Kr)$ and consequently, $\inf_{B(x_0,r)} u > 1$, a contradiction. From the second inequality in (4.17), we can apply Lemma 4.5 with $y \rightsquigarrow x_k$, $\varrho \rightsquigarrow r_k$, $\alpha \rightsquigarrow M^3$, and we then obtain

$$r_k \leq \left(\frac{M_1}{M^3}\right)^{\sigma} r.$$

If $z \in B(x_k, r_k)$, then from inequality (2.3) we get

$$d(z, x_0) \leq d(x_k, x_0) + \beta(d(x_k, z))^{\alpha} \left(d(x_k, x_0) + d(x_k, z)\right)^{1-\alpha} \tag{4.19}$$

$$\leq t_2 + \beta \left(\frac{M_1}{M^3}\right)^{\sigma\alpha} r^{\alpha} \left(t_2 + \left(\frac{M_1}{M^3}\right)^{\sigma} r\right)^{1-\alpha} \tag{4.20}$$

and letting $t_2 = T_2 r$ we get

$$d(z, x_0) \leq (T_2 + \beta_1 q^3) r.$$

Let us now choose T_2 such that $T_2 + \beta_1 q^3 = T_1$, that is, $T_2 = T_1 - \beta_1 q^3$. Since $x_k \in B(x_0, t_2)$, it follows that $B(x_k, r_k) \subset B(x_0, t_1)$ and so (4.18) is proved. Continuing in this way we construct $t_k = T_k r$, with

$$T_k = T_1 - \beta_1 q^3 \sum_{j=o}^{k-1} q^j, \quad q = \frac{1}{M^{\sigma}}$$

and M big enough to obtain

$$\beta_1 q^3 \sum_{j=o}^{\infty} q^j < \frac{1}{4}.$$

Therefore, (4.12) holds and we have

$$\begin{aligned} \mu(\{x \in Br/2(x_0) : u(x) > M^{k+2}\}) &\leq (c(\epsilon))^{k+1}\ \mu(B_r(x_0)) \\ &\leq c(\epsilon))^{k=1}\ C_D\, \mu(B_{r/2}(x_0)), \end{aligned}$$

for every $k = 0, 1, 2, \ldots$. Now, if we choose a positive integer k_0 such that $(c(\epsilon))^{k_0} C_D < 1$, by replacing M with M^{1-k_0}, the last inequalities give

$$\mu(\{x \in Br/2(x_0) : u(x) > M^k\}) \leq (\gamma)^k\, \mu(B_{r/2}(x_0)) \quad \gamma = c(\epsilon)\,.$$

This completes the proof under assumption (A). The proof under assumptions (B1) and (B2) follows as before using Theorem 3.4 instead of Theorem 3.3, and with (4.13) replaced by

$$\epsilon/C \leq \frac{\mu(B(x_k, r_k) \cap B_1 \cap E_2)}{\mu(B(x_k, r_k))} \leq \frac{\mu(B(x_k, r_k) \cap E_2)}{\mu(B(x_k, r_k))},$$

and with (4.17) similarly replaced. □

Remark 4.8 If in addition to the assumption (B1) in Theorem 4.7, $1/C_D^2 \le 1/2$, then the functions in the class $\mathbb{K}_\Omega$ are locally Hölder continuous. Indeed, it is a standard fact that the ϵ-critical density with $\epsilon \le 1/2$ implies the local Hölder continuity, see [18, Sect. 4.3].

5 Power decay and abstract Harnack inequality

Given any quasi metric doubling space, in this section we prove that families of functions satisfying the power decay property satisfy Harnack's inequality. More precisely we have the following theorem.

Theorem 5.1 *Let (Y, d, μ) be a doubling quasi metric space. Suppose that the family of functions $\mathbb{K}_\Omega$ satisfies the power decay property in Definition 4.6 and assume in addition that $\mathbb{K}_\Omega$ is closed under multiplications by positive constants and if $u \in \mathbb{K}_\Omega(B_r(x_0))$ satisfies $u \le \lambda$ in $B_r(x_0)$, then $\lambda - u \in \mathbb{K}_\Omega(B_r(x_0))(x_0)$.*

There exists a positive constant c independent of u, R and x_0 such that if $u \in \mathbb{K}_\Omega(B_{2\eta_0 R}(x_0))$ is nonnegative and locally bounded, then

$$\sup_{B_R(x_0)} u \le c \inf_{B_R(x_0)} u\,,$$

where $\eta_0 = K(2K\eta + 1)$ and η is the constant in Definition 4.6.

We first remark that to prove this theorem we can and do assume that d is a Hölder quasi distance. Indeed, by Macías and Segovia [23, Theorem 2], every quasi metric has an equivalent Hölder quasi metric. On the other hand, it easy to see that if a family of functions satisfies the power decay property, or the Harnack inequality, with respect to a quasi distance d, then it satisfies the same properties with respect to any quasi distance d' equivalent to d.

In order to prove Theorem 5.1 we need the following result.

Theorem 5.2 *Let $u \in \mathbb{K}_\Omega(B_{2R\eta}(z_0))$ be such that $\inf_{B_{2R}(x_0)} \le 1$, and assume $\mathbb{K}_\Omega$ satisfies that the assumptions of Theorem 5.1. Then, there exist a structural constant c such that if $x_0 \in B_R(z_0)$ and $k \ge 2$ are such that $u(x_0) \ge M^k$ and $B_\rho(x_0) \subset B_R(z_0)$, $\rho = c\gamma^{k/Q} R$, then*

$$\sup_{B_\rho(x_0)} u \ge u(x_0)\left(1 + \frac{1}{M}\right), \tag{5.21}$$

where M, η and γ are the constants in the Definition 4.6 of power decay and Q is the exponent at the right hand side of (2.1).

Proof From the power decay property of $\mathbb{K}_\Omega$ we have that

$$\mu(A_1) = \mu(\{x \in B_R(z_0) : u(x) \ge M^{k-1}\}) \le \gamma^{k-1}\mu(B_R(z_0))\,.$$

Suppose by contradiction that (5.21) is not true. Then the function

$$w(x) = \frac{u(x_0)(1+\frac{1}{M}) - u(x)}{\frac{u(x_0)}{M}} = M + 1 - \frac{M}{u(x_0)} u(x)\,,$$

belongs to $\mathbb{K}_\Omega(B_\rho(x_0))$ and $w(x_0) = 1$. Then $\inf_{B_{2\rho/\eta}(x_0)} w(x) \le 1$ and we can apply the power decay property obtaining

$$\mu(A_2) = \mu(\{x \in B_{\rho/2\eta}(x_0) : w(x) \ge M\}) \le \gamma\, \mu(B_{\rho/2\eta}(x_0))\,.$$

We claim that $B_{\rho/2\eta}(x_0) \subset A_1 \cup A_2$. Indeed, $B_{\rho/2\eta}(x_0) \subset B_\rho(x_0) \subset B_R(z_0)$ and if $x \notin A_1, x \in B_{\rho/2\eta}(x_0)$, then $u(x) < M^{k-1}$ so $w(x) > M + 1 - \dfrac{M^k}{u(x_0)} \ge M$ and hence $x \in A_2$.

Consequently, $\mu(B_{\rho/2\eta}(x_0)) \le \mu(A_1) + \mu(A_2) \le \gamma^{k-1}\mu(B_R(z_0)) + \gamma\, \mu(B_{\rho/2\eta R}(x_0))$. Since $B_R(z_0) \subset B_{2KR}(x_0)$ by the doubling property (2.1) we have

$$\mu(B_R(z_0)) \le \mu(B_{2KR}(x_0)) \le C_D \left(\frac{4KR\eta}{\rho}\right)^Q \mu(B_{\rho/2\eta}(x_0))\,.$$

By using this estimate in the previous inequality and keeping in mind that $\mu(B_{\rho/2\eta}(x_0)) > 0$,

$$\begin{aligned}\mu(B_{\rho/2\eta}(x_0)) &\le \mu(A_1 \cup A_2) \le \mu(A_1) + \mu(A_2) \le \gamma^{k-1}\mu(B_R(z_0)) + \gamma\mu(B_{\rho/2\eta}(x_0)) \\ &\le \left(\gamma^{k-1} C_D \left(\frac{4KR\eta}{\rho}\right)^Q + \gamma\right)\mu(B_{\rho/2\eta}(x_0))\,,\end{aligned}$$

and so we get

$$1 - \gamma \le \gamma^{k-1} C_D \left(\frac{4KR\eta}{\rho}\right)^Q = C_D \left(\frac{4K\eta}{c}\right)^Q \frac{\gamma^{k-1}}{\gamma^k} = \frac{C_D}{\gamma}\left(\frac{4K\eta}{c}\right)^Q\,,$$

which is equivalent to $c_0 := \frac{C_D}{\gamma(1-\gamma)}\left(\dfrac{4K\eta}{c}\right)^Q \ge 1$. Then we obtain a contradiction if we choose c such that $c_0 < 1$. □

Proof of Theorem 5.1 It suffices to prove that $\sup_{B_R(x_0)} u \le C\lambda$ for every $\lambda > \inf_{B_R(x_0)} u$ (notice $\lambda > 0$ since $u \ge 0$).

By dividing u by λ we can assume $\inf_{B_R(x_0)} u < 1$ and it will be enough to prove that $\sup_{B_R(x_0)} u \le C$. We shall prove that for each ball $B_R = B_R(z)$, with $z \in B_R(x_0)$,

$$u(x) \le C\left(\frac{R}{R - d(x,z)}\right)^{\delta/\alpha}, \quad \forall x \in B_R(z), \tag{5.22}$$

for a structural constants δ and α the exponent in the Hölder property of d. It will follow, by taking $x = z$ in (5.22) that $u(x) \leq C$ as we wanted to prove. To prove (5.22), pick $\delta > 0$ such that $\frac{1}{M} = \gamma^{\delta/Q}$, and let

$$D := \sup_{x \in B_R(z)} u(x) f(x, R) \tag{5.23}$$

where we have set

$$f(x, R) = \left(\frac{R - d(x, z)}{R} \right)^{\delta/\alpha} .$$

Assume $D > 0$—otherwise there is nothing to prove—and let $0 < D^* < D$. Our goal is to show that D^* can be bounded above by a structural constant. Pick $x_* \in B_R(z)$ such that $D^* < u(x_*) f(x_*, R)$ and let k be an integer such that

$$M^k \leq u(x_*) < M^{k+1},$$

where M is the constant in the power decay. Let k_0 be a structural constant that will be fixed later. If $k \leq k_0$, then

$$D^* < M^{k_0+1} f(x_*, R) \leq M^{k_0+1}$$

and we are done. Let us then assume that $k > k_0$. We have

$$f(x_*, R) > \frac{D^*}{M^{k+1}} = \frac{D^*}{M} \gamma^{k\delta/Q} = \frac{D^*}{M} \left(c \frac{\rho}{R} \right)^{\delta} , \tag{5.24}$$

with ρ and c given by Theorem 5.2.

Case 1 $\dfrac{D^*}{M} c^{\delta} < \beta_*^{\frac{\delta}{\alpha}}$, where β_* is a structural constant that will be fixed later. In this case D^* is bounded above by a structural constant, and we are done again.

Case 2 $\dfrac{D^*}{M} c^{\delta} \geq \beta_*^{\frac{\delta}{\alpha}}$.

Then from (5.24) and the definition of f we get

$$d(x_*, z) < R - \beta_* R^{1-\alpha} \rho^{\alpha} . \tag{5.25}$$

This inequality implies the inclusion

$$B_\rho(x_*) \subset B_R(z) . \tag{5.26}$$

Indeed, if $y \in B_\rho(x_*)$, by the Hölder property (2.4) of d, we have

$$\begin{aligned} d(y, z) &\leq d(z, x_*) + \beta (d(x_*, y))^{\alpha} \left(d(x_*, y) + d(z, x_*) \right)^{1-\alpha} \\ &\leq R - \beta_* R^{1-\alpha} \rho^{\alpha} + \beta \rho^{\alpha} \left(\rho + R \right)^{1-\alpha} . \end{aligned}$$

Then, if we choose k_0 such that

$$\left(1+\frac{\rho}{R}\right)^{1-\alpha}=\left(1+c\,\gamma^{\frac{k}{Q}}\right)^{1-\alpha}<2\,, \tag{5.27}$$

for every $k \geq k_0$ (note: k_0 can be chosen as a structural constant), we have

$$d(y,z)\leq R-\beta_* R^{1-\alpha}\rho^{\alpha}+2\beta R^{1-\alpha}\rho^{\alpha}<R$$

if $\beta_* > 2\beta$. Hence, with this choice of β_*, inclusion (5.26) holds true. On the other hand, since $z \in B_R(x_0)$, we have $B_{2KR}(z) \supset B_R(x_0)$, and so $\inf_{B_{2KR}(z)} u \leq \inf_{B_R(x_0)} u < 1$. We remark that $u \in \mathbb{K}_\Omega(B_{2KR\eta}(z))$ since $B_{2KR\eta}(z) \subset B_{\eta_0 R}(x_0)$. In addition we have $u(x_*) \geq M^k$, so that we can apply Theorem 5.2 with R, z_0 and x_0 replaced by KR, z and x_*, respectively.

Then

$$\sup_{B_\rho(x_*)} u \geq u(x_*)\left(1+\frac{1}{M}\right)>\frac{D^*}{f(x_*,R)}\left(1+\frac{1}{M}\right). \tag{5.28}$$

By keeping in mind the inclusion (5.26), we also have

$$\sup_{B_\rho(x_*)} u \leq D \sup_{y\in B_\rho(x_*)}\frac{1}{f(y,R)}=\frac{D}{f(x_*,R)}\sup_{y\in B_\rho(x_*)}\frac{f(x_*,R)}{f(y,R)}. \tag{5.29}$$

By using again the Hölder property (2.4) of d and denoting by $d_* := d(z, x_*)$, for any $y \in B_\rho(x_*)$, we have

$$\begin{aligned}\left(\frac{f(x_*,R)}{f(y,R)}\right)^{\frac{\alpha}{\delta}} &= \frac{R-d_*}{R-d(y,z)}\\ &\leq \frac{R-d_*}{R-\left(d_*+\beta\rho^{\alpha}(d_*+\rho)^{1-\alpha}\right)}\\ &\leq \frac{1}{1-\dfrac{\beta\rho^{\alpha}(R+\rho)^{1-\alpha}}{\beta_*\rho^{\alpha}R^{1-\alpha}}} \quad \text{by (5.25)}\\ &\leq \frac{1}{1-\frac{2\beta}{\beta_*}} \quad \text{by (5.27).}\end{aligned}$$

As a consequence, from (5.29) and (5.28) we obtain

$$1+\frac{1}{M}\leq\frac{D}{D^*}\left(\frac{\beta_*}{\beta*-2\beta}\right)^{\frac{\delta}{\alpha}}.$$

Letting $D^* \to D$ we get

$$1 + \frac{1}{M} \leq \left(\frac{\beta_*}{\beta* - 2\beta} \right)^{\frac{\delta}{\alpha}}$$

and we reach a contradiction by choosing

$$\beta* > 2\beta \left(1 - \left(1 + \frac{1}{M} \right)^{-\frac{\alpha}{\delta}} \right)^{-1}.$$

We stress that this choice implies $\beta* > 2\beta$ as previously assumed. The proof is complete. □

From the Harnack inequality in the previous theorem a Hölder regularity result readily follows.

Theorem 5.3 *Suppose $\mathbb{K}$ satisfies, together with the hypotheses of Theorem 5.1 the following one: for any ball $B_R(x_0)$, if $u \in \mathbb{K}(B_R(x_0))$ satisfies $u \geq \lambda$ in $B_R(x_0)$, then $u - \lambda \in \mathbb{K}(B_R(x_0))$. There exist positive constants C and α independent of u, R and x_0 such that if $u \in \mathbb{K}(B_{\frac{2R}{\eta}}(x_0))$, then*

$$|u(x) - u(y)| \leq C \left(\frac{d(x, y)}{R} \right)^{\alpha} \sup_{B_R(x_0)} |u|, \quad \forall x, y \in B_R(x_0).$$

Proof For every $\rho \leq R$ define

$$M(\rho) := \sup_{B\rho(x_0)} u, \quad m(\rho) := \inf_{B\rho(x_0)} u, \quad \omega(\rho) := M(\rho) - m(\rho)$$

From the assumptions on $\mathbb{K}$ it follows that the functions $M(\rho) - u$ and $u - m(\rho)$ belong to $\mathbb{K}(B_\rho(x_0))$, so that, by Theorem 5.1, we have

$$M(\rho) - m(\theta\rho) \leq (M(\rho) - M(\theta\rho)), \quad M(\theta\rho) - m(\rho) \leq m(\theta\rho) - m(\theta\rho)$$

where we have set $\theta := \frac{\eta}{2}$. Summing up these inequalities we obtain

$$\omega(\rho) + \omega(\theta\rho) \leq 2C(\omega(\rho) - \omega(\theta\rho))$$

hence

$$\omega(\theta\rho) \leq \frac{2C - 1}{2C + 1} \omega(\rho).$$

It is a standard fact that this inequality implies

$$\omega(\rho) \leq \theta \left(\frac{\rho}{\theta R} \right)^{\alpha} \omega(R)$$

where $\alpha = \frac{\log \frac{2C-1}{2C+1}}{\log \theta}$. Then, since $\omega(R) \leq 2 \sup_{B_R(x_0)} |u|$, the theorem follows. □

6 Another approach to prove Harnack inequality

In this section we give an alternative proof of Harnack's inequality. Our starting point is the following theorem which is a slight variant of Theorem 5.2.

Theorem 6.1 *Assume $\mathbb{K}_\Omega$ satisfies that the assumptions of Theorem 5.1. Let $u \in \mathbb{K}_\Omega(B_{4K\eta R}(z_0))$ be such that $\inf_{B_{4KR}(z_0)} u \leq 1$. If $x_0 \in B_R(z_0)$ and $\varrho < 2R$ satisfy*

(i) $u(x_0) > \nu^{j-1} M$, *with* $\nu = \dfrac{M}{M - \frac{1}{2}}$;

(ii) $\mu(B_\varrho(x_0)) > c\nu^{-\delta j}\mu(B_R(z_0))$ *for some* $j \in \mathbb{N}$,

then

$$\sup_{B_\rho(x_0)} u > \nu^j M,$$

where M, η and γ are the constants in the Definition 4.6 of power decay, K is the constant in the quasi triangle inequality for d; $\delta = -\dfrac{\log \gamma}{\log M}$ and $c = C_D(2K)^Q \frac{2^\delta}{1-\gamma}$.

Proof Suppose by contradiction that $\sup_{B_\rho(x_0)} u \leq \nu^j M$, and let

$$w(x) = \frac{\nu^j M - u(x)}{\nu^j M - \nu^{j-1} M}.$$

We have $w \geq 0$ in $B_\rho(x_0)$ and $w(x_0) \leq 1$. Let

$$A_1 = \{x \in B_{2KR}(z_0) : u(x) > \nu^j M/2\}, \quad A_2 = \{x \in B_{\rho/2}(x_0) : w(x) > M\}.$$

From the quasi triangle inequality and since $\varrho < 2R$ we get that $B_{\rho/2}(x_0) \subset B_{2KR}(z_0)$. Moreover, if $x_1 \in B_{\rho/2}(x_0)$ with $u(x_1) \leq \nu^j M/2$, then by definition of ν, $w(x_1) \geq M$. Therefore $B_{\rho/2}(x_0) \subset A_1 \cup A_2$, so that

$$\mu(B_{\rho/2}(x_0)) \leq \mu(A_1) + \mu(A_2). \tag{6.30}$$

Let $k \in \mathbb{N} \cup \{0\}$ be such that $M^k > \dfrac{\nu^j}{2} \geq M^{k-1}$. Then

$$A_1 \subseteq \{x \in B_{2KR}(z_0) \ : \ u(x) > M^k\}.$$

Since $\inf_{B_{4KR}(z_0)} u \leq 1$, from the power decay we get $\mu(A_1) \leq \gamma^k \mu(B_{2KR}(z_0))$. On the other hand $\gamma^k = M^{-k\delta} < 2^\delta \nu^{-\delta j}$, and by doubling property

$$\mu(B_{2KR}(z_0)) \leq C_D(2K)^Q \mu(B_R(z_0)).$$

Therefore

$$\mu(A_1) \le C_D(2K)^Q 2^\delta \nu^{-\delta j} \mu(B_R(z_0)) .$$

Again using the power decay property (notice that $\inf_{B_\varrho(x_0)} w \le w(x_0) \le 1$ and $u \in \mathbb{K}_\Omega(B_{\eta\varrho(x_0)})$ since $B_{\eta\varrho(x_0)} \subset B_{4K\eta R(z_0)}$) we get

$$\mu(A_2) \le \gamma \mu(B_{\varrho/2}(x_0)) .$$

Summing up, from (6.30) we get

$$(1-\gamma)\mu(B_{\varrho/2}(x_0)) \le C_D(2K)^Q 2^\delta \nu^{-\delta j} \mu(B_R(z_0)) ,$$

in contradiction with assumption (ii). □

Second proof of Theorem 5.1 We first note that if $x_0 \in B_t(z)$, then by quasi triangle inequality $B_t(z) \subseteq B_{2Kt}(x_0)$ and therefore, by the doubling property we get

$$\mu(B_\varrho(x_0)) \ge C_Q \left(\frac{\varrho}{t}\right)^Q \mu(B_t(z)), \quad \varrho < t\,, C_Q = \frac{1}{C_D(2K)^Q} . \tag{6.31}$$

We also remark that it suffices to prove that $\sup_{B_R(x_0)} u \le c\lambda$ for all $\lambda > \inf_{B_R(x_0)} u$. So we may assume $\inf_{B_R(x_0)} u < 1$ and prove that

$$\sup_{B_R(x_0)} u \le \nu^{m-1} M \tag{6.32}$$

for sufficiently large m (independent of u).

Let us now put

$$\varrho_j := C_1 \nu^{-j\frac{\delta}{Q}} R, \quad j = 1, 2, \ldots, \tag{6.33}$$

where C_1 is such that $C_Q \left(\frac{C_1}{2}\right)^Q \ge C$, being C constant in Theorem 6.1.

Suppose inequality (6.32) is false. Then there exists $x_m \in B_R(x_0)$ such that $u(x_m) > \nu^{m-1} M$. Inequality (6.31) and the definition of ϱ_m give

$$\begin{aligned}\mu(B_{\varrho_m}(x_m)) &\ge C_Q \left(\frac{\varrho_m}{R}\right)^Q \mu(B_R(x_0)) = C_Q C_1^Q \nu^{-m\delta} \mu(B_R(x_0)) \\ &= C \nu^{-m\delta} \mu(B_R(x_0)) .\end{aligned}$$

Then we can use Theorem 6.1, by replacing $j \mapsto m$, $z_0 \mapsto x_0$, $x_0 \mapsto x_m$, $\varrho \mapsto \varrho_m$. We stress that we can assume m so large that $\varrho_m < 2R$. Thus

$$\sup_{B_{\varrho_m}(x_m)} > \nu^m M .$$

Hence there exists $x_{m+1} \in B_{\varrho_m}(x_m)$ such that $u(x_{m+1}) > \nu^m M$. Using (6.31) and the definition of ϱ_j, we get

$$\begin{aligned}\mu(B_{\varrho_{m+1}}(x_{m+1})) &\geq C_Q \left(\frac{\varrho_{m+1}}{2R}\right)^Q \mu(B_{2R}(x_0)) \\ &= C_Q \left(\frac{C_1}{2}\right)^Q \nu^{-(m+1)\delta} \mu(B_{2R}(x_0)) \geq c\nu^{-(m+1)\delta} \mu(B_{2R}(x_0)) \,. \end{aligned} \tag{6.34}$$

Moreover $\inf_{B_{8KR}(x_0)} u \leq \inf_{B_R(x_0)} u < 1$, and so applying again Theorem 6.1 we obtain $\sup_{B_{\varrho_{m+1}}(x_{m+1})} u > \nu^{m+1} M$. So there exists $x_{m+2} \in B_{\varrho_{m+1}}(x_{m+1})$ such that $u(x_{m+2}) > \nu^{m+1} M$. Continuing now in this way we construct a sequence $\{x_{m+k+1}\}_{k\in\mathbb{N}}$ with $u(x_{m+k+1}) > \nu^{m+k} M$, for all $k \in \mathbb{N}_0$.

We shall prove that $x_{m+j} \in B_{2R}(x_0)$ for all j.

Indeed, by the Hölder continuity of the quasi distance we get

$$\begin{aligned}\left|d(x_0, x_{m+j}) - d(x_0, x_{m+j+1})\right| \leq \beta\, d^\alpha(x_{m+j+1}, x_{m+j}) \, &\big(d(x_0, x_{m+j}) \\ &+ d(x_0, x_{m+j+1})\big)^{1-\alpha} \,.\end{aligned}$$

We set $d_j := d(x_0, x_{m+j})$. Then

$$d_{j+1} \leq d_j + \beta \nu^{-(m+j)} R^\alpha \left(d_j + d_{j+1}\right)^{1-\alpha} .$$

Setting $D_j = d_j R$ we rewrite it in the following way

$$D_{j+1} \leq D_j + \beta \nu^{-(m+j)} \left(D_j + D_{j+1}\right)^{1-\alpha} .$$

To get our goal we show that our sequence is bounded above by an absolute constant.

We argue inductively. Since $D_0 = 1$

$$D_1 \leq 1 + \beta \nu^{-m} \left(1 + D_1\right)^{1-\alpha} \leq 1 + \beta \nu^{-m} \left(1 + D_1\right),$$

and then

$$D_1 \leq \frac{1 + \beta \nu^{-m}}{1 - \beta \nu^{-m}} .$$

We now bound D_2:

$$\begin{aligned} D_2 &\leq D_1 + \beta \nu^{-(m+1)} \left(D_2 + D_1\right)^{1-\alpha} \\ &\leq \frac{1 + \beta \nu^{-m}}{1 - \beta \nu^{-m}} + \beta \nu^{-(m+1)} \left(D_2 + \frac{1 + \beta \nu^{-m}}{1 - \beta \nu^{-m}}\right)^{1-\alpha} \\ &\leq \frac{1 + \beta \nu^{-m}}{1 - \beta \nu^{-m}} + \beta \nu^{-(m+1)} \left(D_2 + \frac{1 + \beta \nu^{-m}}{1 - \beta \nu^{-m}}\right), \end{aligned}$$

and then

$$D_2 \leq \frac{1+\beta\nu^{-m}}{1-\beta\nu^{-m}} \frac{1+\beta\nu^{-(m+1)}}{1-\beta\nu^{-(m+1)}}.$$

After k steps we get

$$D_{k+1} \leq \prod_{j=0}^{k} \frac{1+\beta\nu^{-(m+j)}}{1-\beta\nu^{-(m+j)}}, \quad \forall k \in \mathbb{N}_0,$$

and then, by choosing m sufficiently large,

$$D_{k+1} \leq \prod_{j=0}^{+\infty} \frac{1+\beta\nu^{-(m+j)}}{1-\beta\nu^{-(m+j)}} \leq 2.$$

So we have that all the points $\{x_{m+j}\}$ are contained in the same ball $B_{2R}(x_0)$ and then

$$\sup_{B_{2R}(x_0)} u \geq u(x_{m+j+1}) > \nu^{m+j} M, \quad \forall j \in \mathbb{N}_0,$$

which contradicts the local boundedness of u. □

7 Applications to X-elliptic operators

Let $X = (X_1, \ldots, X_m)$ be a family of vector fields, with locally Lipschitz continuous coefficients, defined in an open set $Y \subset \mathbb{R}^N$.

We will consider "degenerate" operators which are elliptic with respect to the given system X of vector fields.

Definition 7.1 Let $(a_{i,j})_{i,j=1,\ldots,N}$ be a symmetric matrix with measurable entries. We say that the operator

$$L \equiv \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) \tag{7.35}$$

is uniformly X–elliptic in an open set $\Omega \subset Y$ if there exist positive constants λ, Λ such that

$$\begin{aligned} \lambda \sum_{i=1}^{m} |\langle X_i(x), \xi\rangle|^2 &\leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \\ &\leq \Lambda \sum_{i=1}^{m} |\langle X_i(x), \xi\rangle|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega. \end{aligned} \tag{7.36}$$

Regarding the given system of vector fields, we shall assume that the Carnot-Carathéodory distance d related to X is well defined and *continuous* with respect to the Euclidean topology. Moreover, we assume (Y, d, μ) is a doubling metric space, where $\mathrm{d}\mu = \mathrm{d}x$ denotes the Lebesgue measure. We also assume the following

(P) (Poincaré inequality): There exists a positive constant C such that

$$\left(\fint_{B_R} (u - u_R)^2 \mathrm{d}x \right)^{1/2} \leq C\, R \left(\fint_{B_R} |Xu|^2 \, \mathrm{d}x \right)^{1/2} \quad \forall u \in C_0^1(B_R),$$

and for every d-ball with $B_R = B_R(x_0)$, $x_0 \in Y$, $R > 0$.

Given a set E we denote $u_E \equiv \fint_E u$. If E is a metric ball B_r we put $u_r = u_{B_r}$. Moreover, Xu denotes the X- gradient of u, i.e.

$$Xu = (X_1 u, \ldots, X_m u)$$

where, if $X = (c_j^1, \ldots, c_j^N)$,

$$X_j u = \sum_{k=1}^{N} c_j^k \partial_{x_k} u \,, \quad j = 1, \ldots, m \,.$$

We define the (generalized) Sobolev space $W^1(\Omega, X)$ related to the family X and the open set Ω, as follows

$$W^1(\Omega, X) = \{u \in L^2(\Omega) : |Xu| \in L^2(\Omega)\} \,.$$

We recall that $W^1(\Omega, X)$ is the closure of $\{u \in C^1(\Omega) : u \in L^2(\Omega) : |Xu| \in L^2(\Omega)\}$, with respect to the norm $u \to \|u\|_{L^2(\Omega)} + \|Xu\|_{L^2(\Omega)}$, see [13]. We say that $u \in W^1_{\mathrm{loc}}(\Omega, X)$ if $u\varphi \in W^1(\Omega, X)$ for any $\varphi \in C_0^1(\Omega)$.

Let us now consider the bilinear form

$$\mathcal{L}(u, \varphi) = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x)\, u_{x_i} \varphi_{x_j} \, \mathrm{d}x$$

defined in $C_0^1(\Omega) \times C_0^1(\Omega)$. Since

$$|\mathcal{L}(u, \varphi)| \leq \Lambda \|Xu\| \, \|X\varphi\|, \quad \forall u, \varphi \in C_0^1(\Omega)$$

$\mathcal{L}$ can be extended to $W^1_{\mathrm{loc}}(\Omega, X) \times C_0^1(\Omega)$.

Definition 7.2 (weak solutions) We say that a function $u \in L^2(\Omega)$ is a weak subsolution (supersolution) to $Lu = 0$ in Ω, if $Xu \in L^2(\Omega)$ and

$$\mathcal{L}(u, \varphi) \leq (\geq) 0, \tag{7.37}$$

for all $\varphi \in C_0^1(\Omega)$, $\varphi \geq 0$. We say u is a solution if it is both a sub and a super-solution.

We want to show that the set of nonnegative solutions to X-elliptic equations satisfies the critical density property in Definition 4.1 for all $0 < \epsilon < 1$. First of all we point out that weak subsolutions to X-elliptic equations are (essentially) locally bounded. Indeed, the following theorem hold true, see [12, inequality (4.12)].

Theorem 7.3 (Local boundedness) *Let $u \geq 0$ be a weak subsolution of $Lu = 0$ in Ω and let $B_{2R}(x_0)$ be such that $B_{2R}(x_0) \subset \Omega$. Then there exists a constant c such that*

$$\operatorname{esssup}_{B_R(x_0)} u \leq c \left(\fint_{B_{2R}(x_0)} u^2 \, \mathrm{d}x \right)^{1/2} . \tag{7.38}$$

It is a standard fact that if u is a weak solution to $Lu = 0$ in Ω then $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ are (non negative) sub solutions (see [12]). Then, $u = u^+ - u^-$ is essentially locally bounded. As a consequence, there exists a locally bounded function $\hat{u} \in W^1_{\mathrm{loc}}(\Omega, X)$ such that $L\hat{u} = 0$ and $\hat{u} = u$ a.e. in Ω. From now on we will identify u with $\hat{u}$.

We also need the following Fabes Lemma-type, an easy consequence of the Poincaré inequality (**P**) (compare also with [20]).

Lemma 7.4 (Fabes Lemma) *Let $v \in W^1_{\mathrm{loc}}(\Omega, X)$ and assume the metric ball $B_R(x_0) \subset \Omega$ Let $0 < \varepsilon \leq 1$ and suppose that*

$$|E| \equiv |\{x \in B_R(x_0) : v(x) = 0\}| \geq \varepsilon \, |B_R| \, .$$

Then there exists $C = C_\varepsilon \geq 0$ such that

$$\fint_{B_R} |v|^2 \, \mathrm{d}x \leq C \, R^2 \fint_{B_R} |Xv|^2 \, \mathrm{d}x \, .$$

Proof Set $B = B_R$. We have

$$\begin{aligned} |v(x)| = |v(x) - v_E| &\leq |v(x) - v_B| + |v_B - v_E| \\ &\leq |v(x) - v_B| + \frac{|B|}{|E|} \fint_B |v - v_B| \, \mathrm{d}x \\ &\leq |v(x) - v_B| + \frac{1}{\varepsilon} \fint_B |v - v_B| \, \mathrm{d}x. \end{aligned}$$

Squaring both sides and taking averages over B yields

$$\fint_B |v(x)|^2 \, \mathrm{d}x \leq 2 \left(1 + \frac{1}{\varepsilon^2} \right) \fint_B |v - v_B|^2 \, \mathrm{d}x.$$

The lemma now follows from Poincaré inequality. □

The critical density is a consequence of the previous results.

Theorem 7.5 (Critical density for all $0 < \epsilon < 1$) *Let $B_{2R}(x_0) \subset \Omega$ be a metric ball. Given $0 < \epsilon < 1$, there exists $c = c(\epsilon) > 0$ such that for any nonnegative supersolution to $Lu = 0$ in the ball $B_{2R}(x_0)$ satisfying*

$$|\{x \in B_R(x_0) : u(x) \geq 1\}| \geq \epsilon\, |B_R(x_0)|,$$

we have

$$\inf_{B_{R/2}(x_0)} u \geq c.$$

Proof From [12, inequality (4.16)] we get

$$\int_{B_R(x_0)} |X \log u|^2 \, \mathrm{d}x \leq c \frac{|B_R(x_0)|}{R^2},$$

where c is a structural constant.[5] Consider now $g(t) = \max\{-\log t, 0\}, \ t > 0$. It can be proved in a standard way that $v := g(u)$ is a subsolution of $Lu = 0$. Then by Theorem 7.3

$$\sup_{B_{R/2}(x_0)} v \leq c \left(\fint_{B_R(x_o)} g^2(u) \, \mathrm{d}x \right)^{1/2}. \tag{7.39}$$

On the other hand,

$$|\{x \in B_R(x_0) : v(x)) = 0\}| = |\{x \in B_R(x_0) : u(x) \geq 1\}| \geq \varepsilon |B_R(x_0)|\,.$$

Then Lemma 7.4 implies,

$$\fint_{B_R(x_0)} |v(x))|^2 \, \mathrm{d}x \leq cR^2 \fint_{B_R(x_0)} |Xv(x))|^2 \, \mathrm{d}x \leq cR^2 \fint_{B_{R(x_0)}} |X \log(u(x))|^2 \, \mathrm{d}x.$$

This combined with (7.39) yields

$$\sup_{B_{R/2}(x_0)} v \leq cR \left(\fint_{B_R(x_0)} |X \log(u(x))|^2 \, \mathrm{d}x \right)^{1/2} \leq c$$

[5] We will call structural constant to any constant depending only on the vector fields, the ellipticity constants λ, Λ, the doubling constant and the constant in the Poincaré inequality **(P)**. They will always be denoted by c, even if they are different.

and so

$$u(x) \geq e^{-c}, \quad \forall x \in B_{R/2}.$$

□

Let us assume now that the open set Ω is such that

$$\mu(B_r(x)) < \delta\mu(B_{2r}(x)), \tag{7.40}$$

for every d-ball $B_{2r}(x) \subset \Omega$. This assumption is not restrictive because we are interested in local properties of the solutions and (7.40) holds if the d-diameter of Ω is sufficiently small, see Propositions 2.9 and 2.10. Let us now define

$$\mathbb{K}_\Omega := \left\{ u \in W^1_{\text{loc}}(A) \ : \ A \subset \Omega, A \text{ open} : Lu = 0 \text{ in } A, u \in L^\infty_{\text{loc}}, \ u \geq 0 \right\}.$$

Obviously if $u \in \mathbb{K}_\Omega(A)$ and $\lambda_1 \leq u \leq \lambda_2$ then $\lambda_2 - u$ and $u - \lambda_1$ belong to $\mathbb{K}_\Omega(A)$.

Thus also keeping in mind Lemma 2.8 and the fact that d has the segment property, Theorems 4.7-B, 5.1 and 5.3, applied to the family $\mathbb{K}_\Omega(A)$, we get the following result.

Theorem 7.6 (Harnack inequality for X-elliptic operators) *Let $u \in W^1_{\text{loc}}(\Omega, X)$ be a non negative solution to $Lu = 0$ in Ω. There exist structural constants $c, \theta > 1$ such that*

$$\sup_{B_R(x_0)} u \leq c \inf_{B_R(x_0)} u,$$

for every d-ball such that $B_{\theta R}(x_0) \subset \Omega$.

Remark 7.7 Since the metric balls are relatively compact and connected, a standard covering argument shows that the previous theorem holds for any constant θ bigger than one.

Theorem 7.8 (Hölder continuity for X-harmonic functions) *Let $u \in W^1_{\text{loc}}(\Omega, X)$ be a solution to $Lu = 0$ in Ω. There exists structural positive constants c and $0 < \alpha \leq 1$, such that*

$$|u(x) - u(y)| \leq c \left(\frac{d(x, y)}{R} \right)^\alpha \sup_{B_R(x_0)} |u|,$$

for every ball $B_R(x_0) \subseteq \Omega$.

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References

1. Aimar, H., Forzani, L., Toledano, R.: Holder regularity of solutions of PDE's: a geometrical view. Commun. Partial Differ. Equ. **26**, 1145–1173 (2002)
2. Aimar, H.: Singular integral and approximate identities on spaces of homogeneous type. Trans. Am. Math. Soc. **292**(1), 135–153 (1985)
3. Biroli, M., Mosco, U.: Sobolev inequalities on homogeneous spaces. Pot. Anal. **4**, 311–324 (1995)
4. Buckley, S.M.: Inequalities of John–Nirenberg type in doubling spaces. J. Anal. Math. **79**, 215–240 (1999)
5. Caffarelli, L.A.: Interior a priori estimates for solutions of fully nonlinear elliptic equations. Ann. Math. **130**, 189–213 (1989)
6. Caffarelli, L.A., Cabré, X.: Fully nonlinear elliptic equations, volume 43 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1995
7. Caffarelli, L.A., Gutiérrez, C.E.: Real analysis related to the Monge–Ampère equation. Trans. Am. Math. Soc. **348**(3), 1075–1092 (1996)
8. Caffarelli, L.A., Gutiérrez, C.E.: Properties of the solutions of the linearized Monge–Ampère equation. Am. J. Math. **119**(2), 423–465 (1997)
9. Coifman, R., Weiss, G.: Analyse Harmonique Non-commutative Sur Certains Espaces Homogenes, volume 242 of Lecture Notes in Mathematics. Springer, Berlin-New York, 1971
10. Danielli, D., Garofalo, N., M-Nhieu, D.: Trace inequality for Carnot-Carathéodory distance. Ann. Scuola Norm. Sup. Pisa **27**, 195–152 (1998)
11. Franchi, B., Lanconelli, E.: Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10**(4), 523–541 (1983)
12. Gutiérrez, C.E., Lanconelli, E.: Maximum principle, Harnack inequality, and Liouville theorems for X-elliptic operators. Commun. Partial Differ. Equ. **28**(11–12), 1883–1862 (2003)
13. Garofalo, N., M-Nhieu, D.: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. Commun. Pure Appl. Math. **49**, 1081–1144 (1996)
14. Gutiérrez, C.E.: A covering lemma of Besicovitch type on the Heisenberg group H^n. unpublished manuscript, 1998
15. Gutiérrez, C.E.: The Monge–Ampère equation. Birkhäuser, Boston (2001)
16. Heinonen, J.: Lectures on Analysis on Metric Spaces. Springer, Berlin-New York (2001)
17. Hajlasz, P., Koskela, P.: Sobolev met Poincaré. In Memoirs of the American Mathematical Society **145**, AMS (2000)
18. Han, Q., Lin, F-H.: Elliptic partial differential equations. In: Courant Lecture Notes in Mathematics, volume 1. Courant Institute (1997)
19. Korányi, A., Reimann, H.M.: Foundations for the theory of quasiconformal mappings on the Heisenberg group. Adv. Math. **111**(1), 1–87 (1995)
20. Kinnunen, J., Shanmugalingam, N.: Regularity of quasi-minimizers on metric spaces. Manuscr. Math. **105**, 401–423 (2001)
21. Mosco, U.: Energy functionals on certain fractal structures. J. Convex Anal. **9**(2), 581–600 (2002)
22. Mosco, U.: Harnack inequalities on recurrent metric fractals. Proc. Steklov Inst. Math **326**, 490–495 (2002)
23. Macías, R.A., Segovia, C.: Lipschitz functions on spaces of homogeneous type. Adv. Math. **33**, 257–270 (1979)
24. Rigot, S.: Counter example to the Besicovitch covering property for some Carnot groups equipped with their Carnot Carath eodory metric. Math. Z. **248**(4), 827–848 (2004)