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Calculus of Variations

An iterative method for generated Jacobian equations

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Abstract The purpose of this paper is to present an iterative scheme to find approximate solutions, to any preset degree of accuracy, for a class of generated Jacobian equations introduced in Trudinger (Discrete Contin Dyn Syst 34(4): 1663–1681 2014). A proof of an upper bound on the number of iteration steps, without assuming any condition of Ma-Trudinger-Wang type, is presented. Applications to optimal mass transport and to the parallel reflector problem in optics are also provided.

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1 Introduction

An iterative scheme for finding approximate solutions to the far-field reflector problem in geometric optics was first developed by Caffarelli et al. [2]. These authors consider the global reflector problem with a smooth source density which allows them to show differentiability of the reflector map, which in turn implies that the method converges in a finite number of steps. In several settings, the global problem does not make physical sense and one needs to deal with non-smooth densities. In particular, the differentiability of the transport map strongly depends on smoothness properties of the domains considered and therefore may not hold in general. However, it is realized in [3] for the one source far-field refractor problem introduced in [6], that the method in [2] can be simplified and extended by showing that the refractor

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map satisfies a one-sided Lipschitz estimate. This does not require any differentiability of the source density and due to internal reflection issues, this problem demands to consider general domains.

We believe this idea is far reaching and it is our purpose here to show that it can be implemented to obtain a numerical scheme in the setting, introduced by Trudinger [12], of generated Jacobian equations. Such a setting contains as special cases the optimal mass transport problem and the parallel reflector and refractor problems in optics (cf. Sect. 6).

We would like to mention the work by Kitagawa [9] who extended the techniques developed in [2] to the discrete mass transport problem. The key assumptions used by Kitagawa include smoothness of the density and the so-called strong Ma, Trudinger, and Wang (MTW) condition on the cost function (cf. [11]).

As discovered by Loeper [10], the strong MTW condition is essential for establishing the smoothness of optimal transport maps between measures satisfying very weak regularity hypotheses. Analogous MTW-type conditions are required in order to establish regularity of solutions to generated Jacobian equations and parallel reflector and refractor surfaces in geometric optics (cf. [1,5,8,11]).

We show in the present paper that the upper bound on the number of iteration steps in our scheme can be obtained by making only generic assumptions on the generating function (cf. Sects. 2, 5). In particular, and more importantly, we do not require any MTW-type condition.

The paper is organized as follows; in Sect. 2, we explain the set up and notation, paying special attention to the semi-discrete problem. We also establish certain monotonicity properties of the discrete transport map that will be used when describing the iterative scheme in Sect. 3. The derivation of the quantitative upper bound for the number of iterations is shown in Sect. 4 [cf. Eq. (4.3)] under the assumption that the transport map satisfies a one-sided Lipschitz estimate. This estimate is established in Sect. 5, Theorem 5.1. The final section of this paper shows applications to the optimal mass transport problem and the parallel reflector problem, which was studied in [8].

2 Preliminaries

Let $\Omega, \Omega', \Omega^* \subset \mathbb{R}^n$ be bounded domains with $\overline{\Omega} \subset \Omega'$, and consider a function $G : \Omega' \times \overline{\Omega^*} \times \mathbb{R}^+ \to \mathbb{R}^+$, which we will call a *generating function* as in [12]. We assume $G = G(x, y, v)$ satisfies the following structural assumptions:

(1) (Regularity) $G(x, y, v)$ is continuously differentiable in v and twice continuously differentiable in x and y for $x \in \Omega'$, $y \in \Omega^*$. Furthermore, for any $\alpha > 0$ we have

$$\sup_{(x,y,v)\in\Omega\times\Omega^*\times[0,\alpha]} |G_x(x, y, v)| < \infty. \tag{2.1}$$

(2) (Monotonicity) $G_v(x, y, v) < 0$ for all $(x, y) \in \Omega \times \Omega^*$.
(3) (Twist) The map $(y, v) \mapsto (G(x, y, v), G_x(x, y, v))$ is injective for each $x \in \Omega$.
(4) (Uniform Convergence Property) For each $y \in \Omega^*$, we have $G(x, y, v) \to \infty$ as $v \to 0^+$ uniformly in $x \in \overline{\Omega}$.

Let μ be an absolutely continuous measure on Ω with density $g \in L^1(\Omega)$, $g > 0$ Lebesgue a.e., and let $\nu = \sum_{i=1}^{N} f_i \delta_{y_i}$ for $y_1, \ldots, y_N \in \Omega^*$ distinct and $f_1, \ldots, f_N > 0$. Suppose furthermore that μ and ν satisfy the mass-balance condition $\mu(\Omega) = \nu(\Omega^*)$; that is

$$\int_{\Omega} g(x)\, dx = \sum_{i=1}^{N} f_i. \tag{2.2}$$

2.1 G-convexity

A function $\phi : \Omega \to \mathbb{R}$ is said to be G-convex if for all $x_0 \in \Omega$, there exists $y_0 \in \Omega^*$ and $\lambda_0 \in \mathbb{R}$ such that $\phi(x) \geq G(x, y_0, \lambda_0)$ with equality at $x = x_0$. The function $G(\cdot, y_0, \lambda_0)$ is said to be a G-support to ϕ at x_0. Given a G-convex function ϕ, we define the G-subdifferential of ϕ to be the set-valued function

$$\partial_G \phi(x_0) = \left\{ y \in \Omega^* : \exists \lambda_0 \in \mathbb{R}^+ \text{ s.t. } G(\cdot, y, \lambda_0) \text{ supports } \phi \text{ at } x_0 \right\}. \tag{2.3}$$

Notice that by (2.1) and Taylor's Formula, all G-convex functions are locally Lipschitz; hence, by Rademacher's Theorem, they are differentiable Lebesgue a.e. on Ω.
The inverse of the G-subdifferential will be denoted

$$(\partial_G \phi)^{-1}(y_0) := \{x \in \Omega : y_0 \in \partial_G \phi(x)\}. \tag{2.4}$$

We define $(\partial_G \phi)^{-1}(F) := \bigcup_{y \in F} (\partial_G \phi)^{-1}(y)$.

Let us note that by the monotonicity condition $G_v < 0$ and the implicit function theorem, it is possible to solve for v in the equation $u = G(x, y, v)$; we write this as $v = \hat{G}(x, y, u)$. Since G is C^2 in (x, y) and continuously differentiable in v, it follows that $\hat{G}(x, y, u)$ is also C^2 in (x, y) and continuously differentiable in u and we have the relationship

$$G[x, y, \hat{G}(x, y, u)] = u. \tag{2.5}$$

We will require the above fact in the following lemma.

Lemma 2.1 *The G-subdifferential $\partial_G \phi$ of a G-convex function $\phi : \Omega \to \mathbb{R}$ is a measurable map from Ω to Ω^* which is single-valued Lebesgue a.e. on Ω.*

Proof Suppose $y_1, y_2 \in \partial_G \phi(x_0)$ for some $x_0 \in \Omega$. Then there exist λ_1, λ_2 such that $\phi(x) \geq G(x, y_i, \lambda_i)$ for each $i = 1, 2$ with equality at $x = x_0$. Suppose ϕ is differentiable at x_0. Then, by the supporting property, we obtain $D\phi(x_0) = D_x G(x_0, y_1, \lambda_1) = D_x G(x_0, y_2, \lambda_2)$. Since $\phi(x_0) = G(x_0, y_1, \lambda_1) = G(x_0, y_2, \lambda_2)$, it follows from (Twist) that $(y_1, \lambda_1) = (y_2, \lambda_2)$. Hence, the set of points where $\partial_G \phi$ is not single-valued is contained in the set of points where ϕ is not differentiable, which has zero Lebesgue measure.

To prove that $\partial_G \phi$ is a measurable map, it suffices to show that if $K \subset \Omega^*$ is compact, then $(\partial_G \phi)^{-1}(K)$ is compact. Since $(\partial_G \phi)^{-1}(K) \subset \Omega$ and Ω is bounded, we only need to show that $(\partial_G \phi)^{-1}(K)$ is closed. So let $x_\ell \in (\partial_G \phi)^{-1}(K)$ be a sequence of points converging to a point $x^* \in \overline{\Omega}$. We will show that $x^* \in (\partial_G \phi)^{-1}(K)$. Since $x_\ell \in (\partial_G \phi)^{-1}(K)$, there exist points $y_\ell \in K$ and positive numbers λ_ℓ such that

$$\phi(x) \geq G(x, y_\ell, \lambda_\ell). \tag{2.6}$$

with equality at $x = x_\ell$. By (2.5), it follows that $\lambda_\ell = \hat{G}(x_\ell, y_\ell, \phi(x_\ell))$. Since K is compact, we may assume that $y_\ell \to y^* \in K$. Therefore, by continuity of $\hat{G}$ and ϕ, we conclude that

$$\lambda_\ell \to \lambda^* := \hat{G}(x^*, y^*, \phi(x^*)). \tag{2.7}$$

By (2.6), we thus conclude that $\phi(x) \geq G(x, y^*, \lambda^*)$ with equality at $x = x^*$. Hence $x^* \in \partial_G \phi(y^*)$, and since $y^* \in K$, it follows that $x^* \in \partial_G \phi(K)$. □

For any G-convex function ϕ and a Radon measure μ on Ω, the push-forward of μ by $\partial_G\phi$, denoted $(\partial_G\phi)_\# \mu$, is the measure on Ω^* defined through the equation

$$\left[(\partial_G\phi)_\# \mu\right](F) := \mu\left[(\partial_G\phi)^{-1}(F)\right] = \int\limits_{(\partial_G\phi)^{-1}(F)} g(x)\,dx, \text{ for all } F \subset \Omega^*. \tag{2.8}$$

By Lemma 2.1, $\partial_G\phi$ is a measurable map from Ω to Ω^* which is single-valued Lebesgue a.e. on Ω. Thus, $(\partial_G\phi)_\# \mu$ is a Radon measure on Ω^*.

Definition 2.2 The G-convex function ϕ is said to be a Brenier solution of the generated Jacobian equation if

$$\left[(\partial_G\phi)_\# \mu\right](F) = \nu(F) \text{ for each Borel set } F \subset \Omega^*.$$

2.2 Basic lemmas

We use the short-hand $\mathbf{b} > 0$ to denote a vector $\mathbf{b} = (b_1, \ldots, b_N) \in \mathbb{R}^N$ with $b_i > 0$ for $1 \le i \le N$. Given $\mathbf{b} > 0$, consider the function

$$\phi_{\mathbf{b}}(x) := \max_{1\le i\le N} G(x, y_i, b_i). \tag{2.9}$$

Also define the functions

$$H_i(\mathbf{b}) := \left[(\partial_G\phi_{\mathbf{b}})_\# \mu\right](y_i) = \int\limits_{(\partial_G\phi_{\mathbf{b}})^{-1}(y_i)} g(x)\,dx, \quad 1 \le i \le N. \tag{2.10}$$

We will need to consider the sets

$$V_{i,j}^{\mathbf{b}} := \left\{x \in \Omega : G(x, y_i, b_i) \ge G(x, y_j, b_j)\right\}, \tag{2.11}$$

and

$$V_i^{\mathbf{b}} := \bigcap_{j\ne i} V_{i,j}^{\mathbf{b}} = \{x \in \Omega : \phi_{\mathbf{b}}(x) = G(x, y_i, b_i)\}. \tag{2.12}$$

Let us begin by proving the following elementary properties of the sets $V_i^{\mathbf{b}}$.

Lemma 2.3 *Fix $i \in \{1, \ldots, N\}$.*

(1) *If $V_i^{\mathbf{b}} \ne \emptyset$, then $V_i^{\mathbf{b}} = (\partial_G\phi_{\mathbf{b}})^{-1}(y_i)$.*
(2) *If $V_i^{\mathbf{b}} = \emptyset$, then $H_i(\mathbf{b}) = 0$.*

Proof (1) If $\hat{x} \in V_i^{\mathbf{b}}$, then $\phi_{\mathbf{b}}(\hat{x}) = G(\hat{x}, y_i, b_i)$. Also by definition of $\phi_{\mathbf{b}}$, we have $G(x, y_i, b_i) \le \phi_{\mathbf{b}}(x)$ for all $x \in \Omega$. Thus, $G(\cdot, y_i, b_i)$ is a G-support to $\phi_{\mathbf{b}}$ at $\hat{x}$, which shows $V_i^{\mathbf{b}} \subset (\partial_G\phi_{\mathbf{b}})^{-1}(y_i)$. To show the reverse inclusion, let $\bar{x} \in (\partial_G\phi_{\mathbf{b}})^{-1}(y_i)$. Then there exists $\lambda \in \mathbb{R}$ such that $G(\cdot, y_i, \lambda)$ is a G-support to $\phi_{\mathbf{b}}$ at $\bar{x}$. We claim $\lambda = b_i$; this would show $\bar{x} \in V_i^{\mathbf{b}}$. By definition of $\phi_{\mathbf{b}}$, we have that

$$G(\bar{x}, y_i, \lambda) = \phi_{\mathbf{b}}(\bar{x}) \ge G(\bar{x}, y_i, b_i).$$

By the monotonicity condition $G_v < 0$, we thus have $\lambda \le b_i$. On the other hand, since $V_i^{\mathbf{b}} \ne \emptyset$ by assumption, there exists $\tilde{x} \in V_i^{\mathbf{b}}$. Thus, since $G(\cdot, y_i, \lambda)$ is a G-support to $\phi_{\mathbf{b}}$, we obtain

$$G(\tilde{x}, y_i, \lambda) \le \phi_{\mathbf{b}}(\tilde{x}) = G(\tilde{x}, y_i, b_i).$$

Again, by the monotonicity condition $G_v < 0$, it follows that $\lambda \geq b_i$. Thus, $\lambda = b_i$ as claimed. Since $\bar{x} \in (\partial_G \phi_{\mathbf{b}})^{-1}(y_i)$ was arbitrary, we conclude that $(\partial_G \phi_{\mathbf{b}})^{-1}(y_i) \subset V_i^{\mathbf{b}}$.
(2) Suppose $V_i^{\mathbf{b}} = \emptyset$, and again consider $\bar{x} \in (\partial_G \phi_{\mathbf{b}})^{-1}(y_i)$. By definition of $\phi_{\mathbf{b}}$, there exists $j \in \{1, \ldots, N\}$ such that $\phi_{\mathbf{b}}(\bar{x}) = G(\bar{x}, y_j, b_j)$. Since $V_i^{\mathbf{b}} = \emptyset$, it follows that $j \neq i$. On the other hand, since $\bar{x} \in (\partial_G \phi_{\mathbf{b}})^{-1}(y_i)$, we know there exists $\bar{\lambda} \in \mathbb{R}$ such that $G(\cdot, y_i, \bar{\lambda})$ is a G-support to $\phi_{\mathbf{b}}$ at $\bar{x}$. In particular, $G(\bar{x}, y_i, \bar{\lambda}) = \phi_{\mathbf{b}}(\bar{x})$.

Suppose now that $\phi_{\mathbf{b}}$ is differentiable at $\bar{x}$. Then $D\phi_{\mathbf{b}}(\bar{x}) = D_x G(\bar{x}, y_j, b_j)$. But since $G(\cdot, y_i, \bar{\lambda})$ is a G-support to $\phi_{\mathbf{b}}$ at $\bar{x}$, we must also have $D\phi_{\mathbf{b}}(\bar{x}) = D_x G(\bar{x}, y_i, \bar{\lambda})$. Thus,

$$G(\bar{x}, y_j, b_j) = G(\bar{x}, y_i, \bar{\lambda}) \text{ and } D_x G(\bar{x}, y_j, b_j) = D_x G(\bar{x}, y_i, \bar{\lambda}). \tag{2.13}$$

This contradicts the assumption (Twist), and so $\bar{x}$ must be a point where $\phi_{\mathbf{b}}$ is not differentiable. Since $\bar{x} \in (\partial_G \phi_{\mathbf{b}})^{-1}(y_i)$ was arbitrary, we conclude that $(\partial_G \phi_{\mathbf{b}})^{-1}(y_i)$ is a subset of the set of points where $\phi_{\mathbf{b}}$ is not differentiable. Since $\phi_{\mathbf{b}}$ is differentiable a.e. on Ω and μ is absolutely continuous on Ω, it follows that $H_i(\mathbf{b}) = \mu((\partial_G \phi_{\mathbf{b}})^{-1}(y_i)) = 0$. □

Corollary 2.4 *Let $1 \leq i \leq N$ and $b_j > 0$ for all $j \neq i$. Then $H_i(\mathbf{b})$ is increasing in b_i and $H_j(\mathbf{b})$ is decreasing in b_i if $j \neq i$. Furthermore,*

$$\lim_{b_i \to 0^+} H_i(\mathbf{b}) = \mu(\Omega) \text{ and } \lim_{b_i \to 0^+} H_j(\mathbf{b}) = 0 \text{ for all } j \neq i. \tag{2.14}$$

Proof If all b_j are positive and fixed for $j \neq i$, then it follows from the definition of $V_{i,j}^{\mathbf{b}}$ and the monotonicity condition $G_v < 0$ that the set $V_i^{\mathbf{b}}$ becomes larger as b_i becomes smaller, while for $j \neq i$, the sets $V_j^{\mathbf{b}}$ become smaller as b_i becomes smaller. Thus, if $\mathbf{b}^{(1)} = (b_1, \ldots, b_i^{(1)}, \ldots b_N)$ and $\mathbf{b}^{(2)} = (b_1, \ldots, b_i^{(2)}, \ldots b_N)$, then $H_i(\mathbf{b}^{(1)}) \geq H_i(\mathbf{b}^{(2)})$ if $b_i^{(1)} \leq b_i^{(2)}$, while $H_j(\mathbf{b}^{(1)}) \leq H_j(\mathbf{b}^{(2)})$ if $b_i^{(1)} \leq b_i^{(2)}$.

To obtain (2.14), note that if b_j are fixed for $j \neq i$, then by the uniform convergence property of G, we have $\lim_{b_i \to 0^+} G(\cdot, y_i, b_i) = \infty$ uniformly on Ω. Thus, there exists $\hat{b_i} > 0$ such that if $0 < b_i \leq \hat{b_i}$, then $G(x, y_i, b_i) > G(x, y_j, b_j)$ for all $x \in \Omega$ and each $j \neq i$. This is equivalent to saying $V_i^{\mathbf{b}} = \Omega$ and $V_j^{\mathbf{b}} = \emptyset$ for each $j \neq i$. By Lemma 2.3, we conclude that $H_i(\mathbf{b}) = \mu(\Omega)$ and $H_j(\mathbf{b}) = 0$ whenever $0 < b_i \leq \hat{b_i}$. This clearly implies (2.14). □

3 Description of the iterative scheme

Let $\epsilon > 0$ be given. We now describe an iterative scheme to find a vector $\hat{\mathbf{b}} > 0$ such that $|H_i(\hat{\mathbf{b}}) - f_i| < \epsilon$ for all $i = 1, \ldots, N$. Fix $\delta := \min\left\{\frac{\epsilon}{N-1}, \frac{f_1}{N}\right\}$. We first initialize $b_2 = \cdots = b_N = 1$. Let $\sigma = \max_{2 \leq i \leq N} \left\{||G(\cdot, y_i, 1)||_{L^\infty(\Omega)}\right\}$. Then, by the uniform convergence property, there exists a positive number $\beta = \beta(\sigma, y_1, \Omega)$ in the interval $(0, 1)$ such that if $b_1 = \beta$, then $G(x, y_1, b_1) > G(x, y_i, 1)$ for each $i = 2, \ldots, N$ and $x \in \Omega$. This way, we have $H_1(\mathbf{b}) = \mu(\Omega)$ and $H_i(\mathbf{b}) = 0$ for each $i = 2, \ldots, N$.

Define the set W_δ by

$$W_\delta := \{\mathbf{b} = (b_1, \cdots, b_N) > 0 : b_1 = \beta \text{ and } H_i(\mathbf{b}) \leq f_i + \delta \text{ for all } i = 2, \ldots, N\}. \tag{3.1}$$

By the discussion above, the vector $(\beta, 1, \ldots, 1) \in W_\delta$, and so $W_\delta \neq \emptyset$. Pick an element $\mathbf{b}^1 = (\beta, b_2, b_3, \cdots, b_N) \in W_\delta$. We will construct a sequence of $N-1$ consecutive vectors

$\mathbf{b}^2, \cdots, \mathbf{b}^N$ associated with $\mathbf{b}^1$ as follows. Notice that since $\mathbf{b}^1 \in W_\delta$, then $H_2(\mathbf{b}^1) \leq f_2 + \delta$. If $f_2 - \delta \leq H_2(\mathbf{b}^1)$, then we set $\mathbf{b}^2 = \mathbf{b}^1$. On the other hand, if $f_2 - \delta > H_2(\mathbf{b}^1)$, then we shall pick $\bar{b} \in (0, b_2)$, such that the vector $\mathbf{b}^2 = (\beta, \bar{b}, b_3, \cdots, b_N) \in W_\delta$ satisfies $f_2 \leq H_2(\mathbf{b}^2)$. In fact, this is possible by Corollary 2.4 because since $f_2 < \mu(\Omega)$, and H_i is continuous in the i-th coordinate [1], we may find $0 < \bar{b} < b_2$ such that $\mathbf{b}^2 := (\beta, \bar{b}, b_3, \ldots, b_N)$ satisfies $H_2(\mathbf{b}^2) \in [f_2, f_2 + \delta)$. Notice that, as a result, if $f_2 - \delta > H_2(\mathbf{b}^1)$, then we must have

$$H_2(\mathbf{b}^2) - H_2(\mathbf{b}^1) > f_2 - (f_2 - \delta) = \delta. \tag{3.2}$$

Also from Corollary 2.4, it follows that by lowering the value of the second coordinate, we preserve the inequalities $H_i(\mathbf{b}^2) \leq f_i + \delta$ for $i = 3, \ldots, N$. Hence, $\mathbf{b}^2 \in W_\delta$.

Next, we test the inequality $f_3 - \delta \leq H_3(\mathbf{b}^2)$. If this holds, we set $\mathbf{b}^3 = \mathbf{b}^2$. Otherwise, we proceed as before decreasing now the value of the component b_3 and find a vector $\mathbf{b}^3 \in W_\delta$ such that $H_3(\mathbf{b}^3) \in [f_3, f_3 + \delta)$. Therefore, when $f_3 - \delta > H_3(\mathbf{b}^2)$, we obtain

$$H_3(\mathbf{b}^3) - H_3(\mathbf{b}^2) > \delta.$$

We continue in this manner, obtaining a group of consecutive vectors $\mathbf{b}^1, \mathbf{b}^2, \cdots, \mathbf{b}^N$, such that to pass from $\mathbf{b}^{j-1}$ to $\mathbf{b}^j$ we only decrease, if necessary, the j-th coordinate of $\mathbf{b}^{j-1}$. In case we need to decrease such a coordinate, it is because $H_j(\mathbf{b}^{j-1}) < f_j - \delta$, and this implies $H_j(\mathbf{b}^j) - H_j(\mathbf{b}^{j-1}) > \delta$.

Let us relabel the group of vectors constructed as $\mathbf{b}^{1,1}, \mathbf{b}^{1,2}, \cdots, \mathbf{b}^{1,N}$. Taking the last vector $\mathbf{b}^{1,N}$ as a first vector we repeat the process above obtaining a second group of N vectors, $\mathbf{b}^{2,1}, \mathbf{b}^{2,2}, \cdots, \mathbf{b}^{2,N}$, i.e., $\mathbf{b}^{2,1} = \mathbf{b}^{1,N}$. Continuing this process ad infinitum, we obtain for each M a group of N vectors $\mathbf{b}^{M,1}, \mathbf{b}^{M,2}, \cdots, \mathbf{b}^{M,N}$ with $\mathbf{b}^{M,1} = \mathbf{b}^{M-1,N}$ and each vector $\mathbf{b}^{M,j}$ is obtained from $\mathbf{b}^{M,j-1}$ by only decreasing, if necessary, its j-th coordinate. At each step i, we will either have $H_{i+1}(\mathbf{b}^{M,i}) \geq f_{i+1} - \delta$, or

$$H_{i+1}(\mathbf{b}^{M,i+1}) - H_{i+1}(\mathbf{b}^{M,i}) > f_{i+1} - (f_{i+1} - \delta) = \delta, \ i = 1, \ldots, N-1. \tag{3.3}$$

If at some step M, all vectors in the M-th group are equal, i.e., $\mathbf{b}^M := \mathbf{b}^{M,1} = \mathbf{b}^{M,2} = \cdots = \mathbf{b}^{M,N}$, then that means $|H_i(\mathbf{b}^M) - f_i| \leq \delta < \epsilon$ for each $i = 2, \ldots, N$. By the choice of δ, and by the mass-balance condition $\mu(\Omega) = \nu(\Omega^*)$, we then also have

$$\begin{aligned}
\left|H_1(\mathbf{b}^M) - f_1\right| &= \left|\mu(\Omega) - \sum_{i=2}^{N} H_i(\mathbf{b}^M) - \nu(\Omega^*) + \sum_{i=2}^{N} f_i\right| \\
&\leq \sum_{i=2}^{N} \left|H_i(\mathbf{b}^M) - f_i\right| \\
&\leq (N-1)\delta < \epsilon.
\end{aligned}$$

Thus, we have obtained an approximation with the desired level of accuracy.

4 Upper bound on number of iteration steps

We now proceed to show that under certain conditions, the iterative scheme described above terminates after a finite number of steps. Let us begin by establishing an important property of the set W_δ.

[1] This follows from Eq. (5.3).

Lemma 4.1 *There exists a positive number* $\Lambda = \Lambda(\beta, \Omega, \Omega^*) < \beta$ *such that the set*

$$\mathcal{B}_\Lambda := \{\mathbf{b} > 0 : b_1 = \beta,\ b_k \geq \Lambda \text{ for } k = 2, \ldots, N\}. \tag{4.1}$$

satisfies $W_\delta \subset \mathcal{B}_\Lambda$ *for each* $\delta > 0$.

Proof By the assumption $\delta \leq \dfrac{f_1}{N}$ and the mass-balance condition (2.2), it follows that for any $\mathbf{b} \in W_\delta$, we have

$$\begin{aligned} 0 &\leq f_1 - N\delta < f_1 - (N-1)\delta \\ &= \nu(\Omega^*) - \sum_{i=2}^{N}(f_i + \delta) \\ &\leq \mu(\Omega) - \sum_{i=2}^{N} H_i(\mathbf{b}) = H_1(\mathbf{b}). \end{aligned}$$

On the other hand, by the uniform convergence property, there exists a positive number $\Lambda = \Lambda(\beta, \Omega, \Omega^*) < \beta$ such that if $0 < b_i < \Lambda$ for any $i \neq 1$, then $G(x, y_i, b_i) > G(x, y_1, \beta)$ for all $x \in \Omega$. Hence, $V_1^{\mathbf{b}} = \emptyset$ and so $H_1(\mathbf{b}) = 0$ by Lemma 2.3, which is a contradiction. □

We will use the result above to establish the upper bound on the number of iteration steps required to achieve a desired level of accuracy. Suppose that we are now at the $(M, i-1)$-st step of the algorithm described above. Then we must decrease $b_i^{M,i-1}$ to $b_i^{M,i}$. We make the ansatz that $H_i(\mathbf{b})$ is Lipschitz on W_δ for each $i = 2, \ldots, N$, with Lipschitz constant L; this estimate will be proved in the next section. Now since $\mathbf{b}^{M,i}$ and $\mathbf{b}^{M,i-1}$ may differ only in the i-th coordinate, we have $\mathbf{b}^{M,i} = \mathbf{b}^{M,i-1} - t\mathbf{e}_i$ where $t = |\mathbf{b}^{M,i} - \mathbf{b}^{M,i-1}| = b_i^{M,i-1} - b_i^{M,i} > 0$. By construction, both the vectors $\mathbf{b}^{M,i}$ and $\mathbf{b}^{M,i-1}$ belong to W_δ. Therefore, $b_i^{M,i-1} - t = b_i^{M,i} \geq \Lambda$. It now follows from (3.3) and Theorem 5.1 that

$$\delta < H_i(\mathbf{b}^{M,i}) - H_i(\mathbf{b}^{M,i-1}) \leq L\left(b_i^{M,i-1} - b_i^{M,i}\right), \tag{4.2}$$

that is, $(b_i^{M,i-1} - b_i^{M,i}) \geq L^{-1}\delta$. Since from Lemma 4.1 only vectors $\mathbf{b}$ with all coordinates bounded below by a positive number are admissible, we conclude that each b_i can only be decreased a finite number of times. As a result, if one has the required Lipschitz estimate for H_i, then the algorithm terminates after a finite number of iterations. Therefore, there is an M_0 such that the vectors $\mathbf{b}^{M_0,1} \cdots \mathbf{b}^{M_0,N}$ are all equal and the desired approximation is achieved.

Let us estimate M_0. Fix a coordinate j and notice that in each group k the j-th coordinate of any vector in the group can decrease at most only once and only when passing from a vector from a vector $\mathbf{b}^{k,j-1}$ to the vector $\mathbf{b}^{k,j}$. The j-th coordinate of all vectors are at most $\mathbf{b}_j^{1,1}$, the j-th coordinate of the initial vector $\mathbf{b}^{1,1}$. By Lemma 4.1, the j-th coordinate, $j \geq 2$, of each vector is greater than Λ. Thus, the change in the j-th coordinate from a vector in the first group to the M_0-th group is at most $\mathbf{b}_j^{1,1} - \Lambda$. On the other hand, if the j-th coordinate needs to be decreased in each group, then from (4.2) it must be reduced by at least δ/L. Having M_0 groups, the total decrease of passing from the first group to the M_0-th group is then at least $M_0\,\dfrac{\delta}{L}$; this, in turn, must be smaller than the total possible change of the j-th

coordinate, and so $M_0 \dfrac{\delta}{L} \leq \mathbf{b}_j^{1,1} - \Lambda$. Since this must hold for all the coordinates $2 \leq j \leq N$, we obtain the bound

$$M_0 \leq \frac{L}{\delta} \max_{2 \leq j \leq N} \left(\mathbf{b}_j^{1,1} - \Lambda \right), \tag{4.3}$$

where L is the constant in the Lipschitz estimate (5.3).

5 Lipschitz estimate for H_i

Let $\mathbf{b} = (b_1, \cdots, b_N) > 0$ and let $\mathbf{e}_i$ be the unit direction in the i-th coordinate (fixed from here onward). For $0 < t < b_i$, define $\mathbf{b}^t = \mathbf{b} - t\mathbf{e}_i$. For any $j \neq i$, consider the sets $V_{i,j}^{\mathbf{b}}$ and $V_{i,j}^{\mathbf{b}^t}$. It is immediate that, since $t > 0$, $V_{i,j}^{\mathbf{b}} \subset V_{i,j}^{\mathbf{b}^t}$ for all $j \neq i$. Hence $V_i^{\mathbf{b}} \subset V_i^{\mathbf{b}^t}$. Since $\mathbf{b} > 0$ is fixed in this section, we adopt the short-hand

$$V_{i,j} := V_{i,j}^{\mathbf{b}}, \quad V_{i,j}^t := V_{i,j}^{\mathbf{b}^t}, \quad V_i := V_i^{\mathbf{b}}, \quad V_i^t := V_i^{\mathbf{b}^t}.$$

It will be necessary to consider the functions

$$\mathcal{G}_{ij}(x, v, b) := G(x, y_j, b) - G(x, y_i, v) \tag{5.1}$$

for $1 \leq i, j \leq N, i \neq j$. We will assume that for all $x \in \Omega$, there exists a constant $\lambda > 0$ such that

$$\inf_{\Lambda \leq b, v \leq 1} |D_x \mathcal{G}_{ij}(x, v, b)| \geq \lambda > 0. \tag{5.2}$$

where Λ is the constant in Lemma 4.1. This can be interpreted as a quantitative (Twist) condition; we will show that this condition is satisfied in the applications we have in mind (cf. Sect. 6).

Our goal is to prove the following one-sided Lipschitz estimate for H_i. In the statement of the theorem, $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ denote n-dimensional Lebesgue measure and $n-1$-dimensional Hausdorff measure respectively.

Theorem 5.1 *Let $i \in \{1, \ldots, N\}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary $\partial\Omega$, and $g \in L^\infty(\Omega)$. Let G be a generating function satisfying the assumptions outlined in Sects. 2 and (5.2). Then for $b_i > \Lambda$ and $0 < t \leq b_i - \Lambda$, we have the one-sided Lipschitz estimate*

$$0 \leq H_i(\mathbf{b}^t) - H_i(\mathbf{b}) \leq \frac{C}{\lambda}(N-1)||g||_{L^\infty(\Omega)} \left(\mathcal{H}^{n-1}(\partial\Omega) + K\mathcal{L}^n(\Omega) \right) t, \tag{5.3}$$

where $K = K\left(\lambda, \|D_x G\|_{L^\infty(\Omega)}, \|D_x^2 G\|_{L^\infty(\Omega)}\right)$ is a positive constant, λ is the constant in (5.2) and $C = \sup_{x \in \Omega, \Lambda \leq b \leq 1} |G_v(x, y_i, b)|$.

Proof: First, by Lemma 2.3 and Corollary 2.4, we have

$$\begin{aligned} 0 \leq H_i(\mathbf{b}^t) - H_i(\mathbf{b}) &= \mu(V_i^t) - \mu(V_i) \\ &= \mu(V_i^t \backslash V_i) \\ &= \int_{V_i^t \backslash V_i} g(x)\, dx. \end{aligned}$$

Using (2.12) we obtain

$$\begin{aligned} V_i^t \backslash V_i &= \Omega \cap \left\{ \left\{ \cap_{j \neq i} V_{i,j}^t \right\} \setminus \left\{ \cap_{r \neq i} V_{i,r} \right\} \right\} \\ &= \Omega \cap \left\{ \left\{ \cap_{j \neq i} V_{i,j}^t \right\} \cap \left\{ \left(\cap_{r \neq i} V_{i,r} \right)^c \right\} \right\} \\ &= \Omega \cap \left\{ \left\{ \cap_{j \neq i} V_{i,j}^t \right\} \cap \left\{ \cup_{r \neq i} V_{i,r}^c \right\} \right\} \\ &= \Omega \cap \left\{ \cup_{r \neq i} \left\{ V_{i,r}^c \cap \left\{ \cap_{j \neq i} V_{i,j}^t \right\} \right\} \right\} \\ &\subset \Omega \cap \left\{ \cup_{r \neq i} \left\{ V_{i,r}^c \cap V_{i,r}^t \right\} \right\} \\ &\subset \bigcup_{r \neq i} \left(V_{i,r}^t \setminus V_{i,r} \right). \end{aligned}$$

Hence,

$$V_i^t \backslash V_i \subset \bigcup_{j \neq i} \left(V_{i,j}^t \backslash V_{i,j} \right). \tag{5.4}$$

It follows that

$$\begin{aligned} 0 \leq H_i(\mathbf{b}^t) - H_i(\mathbf{b}) &= \int_{V_i^t \backslash V_i} g(x)\, dx \\ &\leq ||g||_{L^\infty(\Omega)} \mathcal{L}^n \left(\bigcup_{j \neq i} \left(V_{i,j}^t \backslash V_{i,j} \right) \right) \\ &\leq ||g||_{L^\infty(\Omega)} \sum_{j \neq i} \mathcal{L}^n \left(V_{i,j}^t \backslash V_{i,j} \right). \end{aligned} \tag{5.5}$$

Remark 5.2 It suffices to assume $g \in L^p(\Omega)$ for some $1 < p \leq \infty$. By applying Holder's inequality in the argument above, one then obtains a Holder estimate for H_i, with Holder exponent $\frac{1}{p'}$, where p' is the Holder conjugate of p. This is sufficient for the algorithm to terminate in a finite number of steps, as the argument at the end of Sect. 4 still applies.

We proceed to estimate $\mathcal{L}^n \left(V_{i,j}^t \backslash V_{i,j} \right)$ for $j \neq i$. Notice that, by definition of $V_{i,j}$,

$$\begin{aligned} V_{i,j}^t \backslash V_{i,j} &= \left\{ x \in \Omega : G(x, y_i, b_i) < G(x, y_j, b_j) \leq G(x, y_i, b_i - t) \right\} \\ &= \left\{ x \in \Omega : 0 < G(x, y_j, b_j) - G(x, y_i, b_i) \leq G(x, y_i, b_i - t) - G(x, y_i, b_i) \right\}. \end{aligned}$$

By the mean value theorem,

$$G(x, y_i, b_i - t) - G(x, y_i, b_i) = G_v(x, y_i, \xi_i) \cdot (-t) \leq \sup_{x \in \Omega, \Lambda \leq b \leq 1} |G_v(x, y_i, b)| \cdot t \leq Ct. \tag{5.6}$$

Thus,

$$V_{i,j}^t \backslash V_{i,j} \subset \left\{ x \in \Omega : 0 < \mathcal{G}_{ij}(x, b_i, b_j) \leq Ct \right\}. \tag{5.7}$$

The set $\left\{ x \in \Omega : 0 < \mathcal{G}_{ij}(x, b_i, b_j) \leq Ct \right\}$ is the region contained between two level sets of the function $\mathcal{G}_{ij}$. Let us assume, for now, that this region is non-empty. We now estimate the measure of this set. Let us first recall the co-area formula (cf. [4], Section 3.4.2, Theorem 1).

Proposition 5.3 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, and $\Sigma \subset \mathbb{R}^n$ measurable. Then*

$$\int_{\Sigma} |D\psi(x)|\, dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\Sigma \cap \psi^{-1}(s))\, ds, \tag{5.8}$$

where $\mathcal{H}^{n-1}(\cdot)$ denotes $n-1$-dimensional Hausdorff measure.

This has the following simple corollary.

Corollary 5.4 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, with $\inf_{\mathbb{R}^n} |D\psi| \geq \lambda > 0$. Let $-\infty \leq a \leq b \leq \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded set. Then*

$$\mathcal{L}^n(\{x \in \Omega : a \leq \psi(x) \leq b\}) \leq \frac{1}{\lambda} \int_a^b \mathcal{H}^{n-1}\left(\Omega \cap \psi^{-1}(s)\right) ds. \tag{5.9}$$

By the assumption (5.2) and the corollary above applied to the functions $\mathcal{G}_{ij}(x)$, we conclude that

$$\mathcal{L}^n\left(\left\{x \in \Omega : 0 < \mathcal{G}_{ij}(x, b_i, b_j) \leq Ct\right\}\right) \leq \frac{1}{\lambda} \int_0^{Ct} \mathcal{H}^{n-1}\left(\Omega \cap \mathcal{G}_{ij}^{-1}(s)\right) ds. \tag{5.10}$$

We now show that there exists a positive number $\alpha > 0$, depending only on derivatives of the generating function G, such that $\mathcal{H}^{n-1}\left(\Omega \cap \mathcal{G}_{ij}^{-1}(s)\right) \leq \alpha < \infty$ for each s in the range of $\mathcal{G}_{ij}$. For this, we need the following proposition.

Proposition 5.5 *Suppose $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain and $\psi \in C^2(\mathbb{R}^n)$ satisfies $\min_{x \in \overline{\Omega}} |D\psi(x)| \geq \lambda > 0$ and $\|D\psi\|_{L^\infty(\Omega)}, \|D^2\psi\|_{L^\infty(\Omega)}$ are both finite. For any $s \in \mathrm{Range}(\psi)$, let $\Gamma_s = \{x \in \Omega : \psi(x) = s\}$. Then there exists a constant $K = K\left(\lambda, \|D\psi\|_{L^\infty(\Omega)}, \|D^2\psi\|_{L^\infty(\Omega)}\right)$ such that*

$$\mathcal{H}^{n-1}(\Gamma_s) \leq \mathcal{H}^{n-1}(\partial\Omega) + K\, \mathcal{L}^n(\Omega). \tag{5.11}$$

Proof We may assume, without loss of generality, that $s = 0$. We write

$$E := \Omega \setminus \{x \in \Omega : \psi(x) = 0\} = \bigcup_{j=1}^{\infty} E_j,$$

where E_j are the connected components of E. First observe that, $\overline{E_j} \cap \partial\Omega \neq \emptyset$ for all j. Because if $\overline{E_j} \subset \Omega$, then $\psi = 0$ on ∂E_j and so ψ attains an extreme at some $x^* \in E_j$; this implies $D\psi(x^*) = 0$, which is impossible. The boundary of each E_j is a piecewise C^1, $(n-1)$-dimensional surface intersecting the boundary. We write

$$\partial E_j = \gamma_j \cup \sigma_j,$$

where $\psi = 0$ on γ_j and $\sigma_j \subset \partial\Omega$. Now consider the vector field

$$F(x) = \frac{D\psi(x)}{|D\psi(x)|}, \qquad x \in \overline{\Omega}.$$

Along γ_j, either $F(x)$ or $-F(x)$ is the unit outer normal to E_j, with the plus or minus sign depending on j. So assume without loss of generality that $F(x)$ is the outer unit normal to E_j along γ_j. By the divergence theorem

$$\int_{E_j} \operatorname{div} F(x)\, dx = \int_{\partial E_j} F(x) \cdot \nu(x)\, d\sigma(x),$$

where ν is the outer unit normal to ∂E_j. Thus

$$\begin{aligned}\int_{\partial E_j} F(x) \cdot \nu(x)\, d\sigma(x) &= \int_{\gamma_j \cup \sigma_j} F(x) \cdot \nu(x)\, d\sigma(x)\\ &= \int_{\gamma_j} \nu(x) \cdot \nu(x)\, d\sigma(x) + \int_{\sigma_j} F(x) \cdot \nu(x)\, d\sigma(x)\\ &= \mathcal{H}^{n-1}(\gamma_j) + \int_{\sigma_j} F(x) \cdot \nu(x)\, d\sigma(x) = \mathcal{H}^{n-1}(\gamma_j) + B_j.\end{aligned}$$

Since $|B_j| \leq \mathcal{H}^{n-1}(\sigma_j)$, we obtain

$$\mathcal{H}^{n-1}(\gamma_j) \leq \int_{E_j} \operatorname{div} F(x)\, dx + \mathcal{H}^{n-1}(\sigma_j) \leq \int_{E_j} |\operatorname{div} F(x)|\, dx + \mathcal{H}^{n-1}(\sigma_j) \qquad j = 1, 2, \ldots.$$

Thus,

$$\begin{aligned}\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\gamma_j) &\leq \sum_{j=1}^{\infty} \int_{E_j} |\operatorname{div} F(x)|\, dx + \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\sigma_j)\\ &\leq \int_{\Omega} |\operatorname{div} F(x)|\, dx + \mathcal{H}^{n-1}(\partial\Omega),\end{aligned}$$

since the sets E_j are disjoint up to a set of $\mathcal{L}^n$ measure zero, and the sets σ_j are disjoint up to a set of $\mathcal{H}^{n-1}$ measure zero. Now, since $|D\psi(x)| \geq \lambda > 0$, we have

$$|\operatorname{div} F(x)| = \frac{1}{|D\psi(x)|^3} \left| |D\psi(x)|^2 \Delta\psi(x) - \left\langle D\psi(x), D^2\psi(x) D\psi(x) \right\rangle \right| \leq K$$

with $K = K\left(\lambda, \|D\psi\|_{L^\infty(\Omega)}, \|D^2\psi\|_{L^\infty(\Omega)}\right)$. Therefore we obtain

$$\mathcal{H}^{n-1}\left(\{x \in \Omega : \psi(x) = 0\}\right) \leq K\mathcal{L}^n(\Omega) + \mathcal{H}^{n-1}(\partial\Omega). \tag{5.12}$$

□

We can now complete the proof of Theorem 5.1. The assumption (5.2) allows us to apply Proposition 5.5 to the function $\psi = \mathcal{G}_{ij}(x, b_i, b_j)$, and so (5.10) implies

$$\mathcal{L}^n\left(\left\{x \in \Omega : 0 < \mathcal{G}_{ij}(x, b_i, b_j) \leq Ct\right\}\right) \leq \frac{Ct}{\lambda}(\mathcal{H}^{n-1}(\partial\Omega) + K\mathcal{L}^n(\Omega)). \tag{5.13}$$

Theorem 5.1 then follows from (5.5) and (5.7). □

6 Applications

6.1 Optimal mass transport

As mentioned in the introduction, the optimal transport problem falls within the framework of generated Jacobian equations. Specifically, given bounded domains $\Omega, \Omega^* \subset \mathbb{R}^n$, the problem of transporting a Radon measure μ on Ω to a Radon measure ν on Ω^* while minimizing the cost $c(x, y)$ is represented by the generating function $G(x, y, v) = c(x, y) - \log(v)$, $v > 0$. We will assume the cost $c : \overline{\Omega} \times \overline{\Omega^*} \to \mathbb{R}$ satisfies the following conditions:

(1) (Regularity) $c(x, y)$ is twice continuously differentiable in x and y for $x \in \Omega$ and $y \in \Omega^*$.
(2) (Twist) For each $x_0 \in \overline{\Omega}$, the map $y \mapsto D_x c(x_0, y)$ is injective.

Clearly, the generating function $G(x, y, v) = c(x, y) - \log(v)$ satisfies $G_v(x, y, v) = -\dfrac{1}{v} < 0$ and $G_x(x, y, v) = D_x c(x, y)$. Thus it satisfies (Monotonicity) and (Twist) from Sect. 2. Due to the presence of the $\log(v)$ term, G also satisfies the uniform convergence property whenever Ω, Ω^* are bounded. Furthermore,

$$\sup_{x \in \Omega, \Lambda \le v \le 1} |G_v(x, y, b)| \le \frac{1}{\Lambda}. \tag{6.1}$$

Finally, we check that $\mathcal{G}_{ij}(x, v, b) := G(x, y_j, b) - G(x, y_i, v)$ satisfies (5.2). This follows from the fact that $D_x G_{ij}(x, v, b) = D_x c(x, y_j) - D_x c(x, y_i)$, so by (Twist) above, $D_x G_{ij}(x, v, b)$ never vanishes on Ω for all $v, b \in [\Lambda, 1]$. Hence there exists $\lambda > 0$ such that $\mathcal{G}_{ij}(x, v, b)$ satisfies (5.2).

6.2 The parallel reflector problem

The parallel reflector and refractor problems were studied in [7] and [8]. The setup of the parallel reflector problem is as follows. Let $\Omega, \Omega^* \subset \mathbb{R}^n$ be bounded domains which will represent the source and target domains respectively. Let μ and ν be Radon measures on Ω and Ω^* respectively which represent source and target intensities of radiation. Light beams emanate in the vertical direction from Ω, strike a surface $\Sigma \subset \mathbb{R}^{n+1}$ and are then reflected onto Ω^*. The map $\Phi : \Omega \to \Omega^*$ which sends points from the source to the target via the path taken by a light ray is called reflector map. The parallel reflector problem is to determine the reflecting surface Σ and the reflector map Φ from the domains Ω and Ω^* and the measures μ and ν.

The reflector problem described above can be cast as a generalized Jacobian equation with generating function

$$G(x, y, v) := \frac{1}{2v} - \frac{v}{2}|x - y|^2. \tag{6.2}$$

In this case, the reflector surface Σ is the graph of a G-convex function $u : \Omega \to \mathbb{R}$ while the reflector map is the G-subdifferential of u. Let us check that the generating function above satisfies the assumptions required to establish finite step convergence of the algorithm. For this we will need to assume the following visibility condition:

Visibility Condition: For any two points $y_i, y_j \in \Omega^*$ distinct, the line containing both the points y_i and y_j does not intersect Ω.

Note that this is equivalent to saying any point $x \in \Omega$ and any pair of points $y_i, y_j \in \Omega^*$ are not collinear.

- Clearly G is smooth in (x, y) and locally continuously differentiable in v for $v > 0$. Also, $G_x(x, y, v) = v(y - x)$ and so for any $v \in [0, \alpha]$, we have

$$\sup_{(x,y,v)\in\Omega\times\Omega^*\times[0,\alpha]} |G_x(x, y, v)| < \infty. \tag{6.3}$$

- To establish (Twist), suppose there exists (y_1, v_1) and (y_2, v_2) distinct such that $G(\bar{x}, y_1, v_1) = G(\bar{x}, y_2, v_2)$ and $D_x G(\bar{x}, y_1, v_1) = D_x G(\bar{x}, y_2, v_2)$ for some $\bar{x} \in \Omega$. Then $v_1(y_1 - \bar{x}) = v_2(y_2 - \bar{x})$, which violates the visibility condition if $y_1 \neq y_2$. Hence $y_1 = y_2$. But this implies $v_1 = v_2$, and so $(y_1, v_1) = (y_2, v_2)$.
- The uniform convergence property holds due to the $\dfrac{1}{2v}$ and $\dfrac{v}{2}$ terms in G and the fact that Ω and Ω^* are bounded.
- Obviously, $G_v = -\dfrac{1}{2v^2} - \dfrac{1}{2}|x - y|^2 < 0$. Also, since Ω and Ω^* are bounded domains, we find that

$$\begin{aligned}\sup_{(x,y)\in\Omega\times\Omega^*,\Lambda\le b\le 1} |G_v(x, y, b)| &= \sup_{(x,y)\in\Omega\times\Omega^*,\Lambda\le b\le 1} \left\{\frac{1}{2b^2} + \frac{1}{2}|x - y|^2\right\} \\ &\le \frac{1}{2\Lambda^2} + \frac{1}{2}\mathrm{diam}(\Omega \cup \Omega^*)^2.\end{aligned}$$

- Finally, we verify that $\mathcal{G}_{ij}(x, v, b) := G(x, y_j, b) - G(x, y_i, v)$ satisfies (5.2) for some $\lambda > 0$. Let us fix i and j and suppose for contradiction that there exists a sequence of points x_k and a sequence of numbers $b_k, v_k \in [\Lambda, 1]$ such that $|D_x \mathcal{G}_{ij}(x_k, v_k, b_k)| \to 0$. Since the domain Ω and the interval $[\Lambda, 1]$ are compact, we may assume that $x_k \to x^* \in \Omega$ and $(b_k, v_k) \to (b^*, v^*) \in [\Lambda, 1]$. By the continuity of $D_x G$, we conclude that $D_x \mathcal{G}_{ij}(x_k, v_k, b_k) \to D_x \mathcal{G}_{ij}(x^*, v^*, b^*) = 0$. However, for the parallel reflector problem, an easy calculation shows

$$D_x \mathcal{G}_{ij}(x, v, b) = b(y_j - x) - v(y_i - x). \tag{6.4}$$

  In particular, if $D_x \mathcal{G}_{ij}(x^*, v^*, b^*) = 0$ for $(b^*, v^*) \neq 0$, then the vectors $y_j - x$ and $y_i - x$ are parallel, which violates the visibility condition. Hence, there exists some $\lambda > 0$ depending only on Ω and the interval $[\Lambda, M]$ such that (5.2) is satisfied.

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References

1. Abedin, F., Gutiérrez, C.E., Tralli, G.: $C^{1,\alpha}$ regularity for the parallel refractor. Nonlinear Anal. **142**, 1–25 (2016)
2. Caffarelli, L.A., Kochengin, S.A., Oliker, V.: On the numerical solution of the problem of reflector design with given far-field scattering data. Contemp. Math. **226**, 13–32 (1999)
3. De Leo, R., Gutiérrez, C.E., Mawi, H.: On the numerical solution of the far field refractor problem. Nonlinear Anal. **157**, 123–145 (2017)
4. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions. CRC Press, Boca Raton (1992)
5. Guillen, N., Kitagawa, J.: Pointwise estimates and regularity in geometric optics and other generated Jacobian Equations. Comm. Pure Appl. Math. **70**(6), 1146–1220 (2017)
6. Gutiérrez, C.E., Huang, Qingbo: The refractor problem in reshaping light beams. Arch. Ration. Mech. Anal. **193**(2), 423–443 (2009)
7. Gutiérrez, C.E., Tournier, F.: The parallel refractor. Dev. Math. **28**, 325–334 (2013)

8. Gutiérrez, C.E., Tournier, F.: Regularity for the near field parallel refractor and reflector problems. Calc. Var. PDEs **54**(1), 917–949 (2015)
9. Kitagawa, J.: An iterative scheme for solving the optimal transportation problem. Calc. Var. PDEs **51**(1–2), 243–263 (2014)
10. Loeper, G.: On the regularity of solutions of optimal transportation problems. Acta Math. **202**, 241–283 (2009)
11. Ma, X.-N., Trudinger, N., Wang, X.-J.: Regularity of potential functions of the optimal transportation problem. Arch. Ration. Mech. Anal. **177**(2), 151–183 (2005)
12. Trudinger, N.S.: On the local theory of prescribed Jacobian equations. Discrete Contin. Dyn. Syst. **34**(4), 1663–1681 (2014)