HARNACK INEQUALITY FOR A DEGENERATE ELLIPTIC EQUATION

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Abstract. We consider degenerate elliptic equations of the form

$$Lu = a(x,y,z)X_{1,1}u + 2b(x,y,z)X_{1,2}u + c(x,y,z)X_{2,2}u = 0,$$

where $X_{i,j}$ are defined with the Heisenberg vector fields, and the matrix coefficient is uniformly elliptic. We obtain an invariant Harnack's inequality on metric balls for nonnegative solutions under the additional assumption that the ratio between the maximum and minimum eigenvalues of the coefficient matrix is sufficiently close to one. In the paper we prove critical density and double ball estimates. Once this is established, Harnack follows directly from the results from [FGL08].

1. Introduction

Consider $\mathbb{R}^3$*the vector fields given by

$$X_1u = u_x - \frac{y}{2}u_z, \text{ and } X_2u = u_y + \frac{x}{2}u_z,$$

and the symmetrized second derivatives given by

$$X_{i,j}u = \frac{1}{2}(X_iX_ju + X_jX_iu).$$

Denote by Hu the 2×2 matrix $Hu = (X_{i,j}u)$. We consider the equation

$$Lu = a(x,y,z)X_{1,1}u + 2b(x,y,z)X_{1,2}u + c(x,y,z)X_{2,2}u = 0, \tag{1.1}$$

where a, b, c satisfy, there exist positive constants λ and Λ such that

$$\lambda \le a(x,y,z)\xi_1^2 + 2b(x,y,z)\xi_1\xi_2 + c(x,y,z)\xi_2^2 \le \Lambda,$$

for all points $(x,y,z) \in \mathbb{R}^3$ and for all unit vectors (ξ_1,ξ_2). The constants λ, Λ are called ellipticity constants.

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*We work in $\mathbb{R}^3$ or $\mathbb{H}^1$ for simplicity in the notation. The extension to $\mathbb{H}^n$ or $\mathbb{R}^{2n+1}$ is straightforward.

The purpose of this paper is to establish an invariant Harnack's inequality of the form $\sup_B u \leq C \inf_B u$, valid for all nonnegative solutions u to (1.1) and for all balls B in the metric given by the vector fields X_1, X_2, with a constant C depending only on Λ and λ. When L is a standard uniformly elliptic operator, this is the celebrated Harnack's inequality of Krylov and Safonov and its proof depends in a crucial way upon the maximum principle of Aleksandrov, Bakelman and Pucci, see [GT83, Section 9.8] and [Gut01, Theorem 2.1.1]. It is not known if such a principle holds in the Heisenberg group in a form that permits to establish distribution function estimates for super solutions[†] such us [Gut01, Theorem 2.1.1]. As a result, this impedes the ability of extending that method to get Harnack's inequality in this context.

Therefore, we present in this paper a direct proof of these distribution function estimates -critical density estimates- for super solutions using barriers under the additional assumption that the ratio Λ/λ is sufficiently close to one. Once this is proved, Harnack's inequality follows directly from the theory developed in [FGL08].

We mention that the results of [BBLU10], established for a class of equations defined with Hörmander vector fields, yield Harnack's inequality for the operator L when the coefficients a, b, c are Hölder continuous but with a constant depending in addition on their Hölder norms.

The paper is organized as follows. Section 2 contains a few preliminaries. The construction of the barrier and the critical density estimate are established in Section 3. Finally, in Section 4, we prove if a super solution is bigger than one in a ball, then it is larger than a positive universal constant in the ball of double radius. It is indicated at the end of the paper how to obtain Harnack from the results in [FGL08].

For simplicity the results are presented in $\mathbb{H}^1$ and they can be straightforwardly extended to higher dimensions, i.e., for the Heisenberg group $\mathbb{H}^n$ defined for example in [BLU07, Chapter 3] or [CDPT07, Section 2.1.2], and probably also to the setting of Carnot groups.

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[†]See [GM04] for maximum principles of these type on the Heisenberg group.

2. Preliminaries

The Heisenberg group $\mathbb{H}^1$ is identified with $\mathbb{R}^3$ equipped with the multiplication law defined for $p = (x, y, z)$ and $q = (\xi_1, \xi_2, \xi_3)$ by

$$p \circ q = \left(x + \xi_1, y + \xi_2, z + \xi_3 + \frac{1}{2}(x\xi_2 - y\xi_1)\right),$$

and we have $p^{-1} \circ p = (0, 0, 0)$ with $p^{-1} = (-x, -y, -z)$. Let $\mu > 0$ be fixed, and

$$\rho_\mu(x, y, z) = \left((x^2 + y^2)^2 + \mu z^2\right)^{1/4},$$

and

$$d(p, q) = \rho_\mu(q^{-1} \circ p).$$

If we let $w(p) = f(q^{-1} \circ p)$, then

$$(X_i w)(p) = (X_i f)(q^{-1} \circ p).$$

If $p = (x, y, z)$ and $R > 0$, then the ball in metric of $\mathbb{H}^1$ is

$$N_R(p) = \{q : d(p, q) < R\}.$$

We note that the following identity holds:

$$\langle Hu(x, y, z)\xi, \xi\rangle = \langle D^2 u(x, y, z)\bar{\xi}, \bar{\xi}\rangle, \tag{2.2}$$

where $\xi = (\xi_1, \xi_2)$ and $\bar{\xi} = (\xi_1, \xi_2, \frac{1}{2}(x\xi_2 - y\xi_1))$, and $D^2 u$ is the standard Hessian matrix of u. The weak maximum principle easily follows from (2.2).

Lemma 2.1 (Weak maximum principle)**.** *Let Ω be a bounded open subset of $\mathbb{R}^3$ and $u, v \in C(\bar{\Omega}) \bigcap C^2(\Omega)$ such that $u \geq v$ on $\partial\Omega$ and $Lu \leq Lv$ in Ω. Then, $u \geq v$ in Ω.*

3. Critical density estimate

This section contains the proof of the critical density estimate. This is achieved by constructing a barrier, using the weak maximum principle and rescaling. The construction of the barrier is the contents of the following lemma. It is only for the proof of this lemma where we need to assume that the ratio Λ/λ is sufficiently close to one.

Lemma 3.1. *Let $\eta_\epsilon \in C^\infty(\mathbb{R}^3)$ such that $0 \le \eta_\epsilon \le 1$ and such that $\eta_\epsilon(u,v,w) = 0$ for $(u^2+v^2)^2 + \mu\, w^2 < \epsilon^4$ and $\eta_\epsilon(u,v,w) = 1$ for $(u^2+v^2)^2 + \mu\, w^2 > (2\epsilon)^4$. Let Ω be an open set such that $\Omega \subseteq N_1(0,0,0)$ and we consider the function, $p = (x,y,z)$,*

$$h_\epsilon(x,y,z) = (-4/\alpha) \int_\Omega \rho_\mu(q^{-1}\circ p)^{-\alpha} \eta_\epsilon(q^{-1}\circ p)\, dq.$$

Given α and μ positive numbers satisfying

$$\frac{4}{\alpha+3} < \frac{\mu}{8} < \alpha < 4, \tag{3.3}$$

there is a constant $C(\alpha,\mu) > 1$, depending only on α and μ, such that if $\dfrac{\Lambda}{\lambda} \le C(\alpha,\mu)$, then for $(x,y,z) \in \Omega' \Subset \Omega$, we have

$$Lh_\epsilon(x,y,z) \ge C$$

for all $0 < \epsilon \le \dfrac{dist(\Omega',\partial\Omega)}{2}$, and with a positive constant $C = C^*\lambda$ and C^* depending only on α and μ.

Proof. Let F be a smooth function, possibly at the origin, let $p = (x,y,z)$, and

$$h_\epsilon(x,y,z) = \int_\Omega (F\eta_\epsilon)(q^{-1}\circ p)\, dq.$$

Notice that for $(x,y,z) \in N_1(0,0,0)$, we have $N_1(0,0,0) \subseteq N_3(x,y,z)$. We will calculate Lh_ϵ integrating by parts. Let us first calculate $X_{1,1}h_\epsilon$. If $q = (\xi_1,\xi_2,\xi_3)$, then

$$X_1(F\eta_\epsilon(q^{-1}\circ p)) = (F\eta_\epsilon)_u(q^{-1}\circ p) - \frac{1}{2}(y-\xi_2)(F\eta_\epsilon)_w(q^{-1}\circ p) := G(q^{-1}\circ p),$$

and hence, we have

$$\begin{aligned} X_{1,1}h_\epsilon(p) &= \int_\Omega X_1\left(G(q^{-1}\circ p)\right) dq \\ &= \int_{N_3(x,y,z)} X_1\left(G(q^{-1}\circ p)\right) dq - \int_{N_3(x,y,z)\setminus\Omega} X_1\left(G(q^{-1}\circ p)\right) dq \\ &= \int_{N_3(x,y,z)} \left(G_u(q^{-1}\circ p) - \frac{1}{2}(y-\xi_2)G_w(q^{-1}\circ p)\right) dq \\ &\quad - \int_{N_3(x,y,z)\setminus\Omega} X_{1,1}(F\eta_\epsilon)(q^{-1}\circ p)\, dq := A - B. \end{aligned}$$

We change variables in the integral A, setting $u = x - \xi_1, v = y - \xi_2, w = z - \xi_3 + \frac{1}{2}(x\xi_2 - y\xi_1))$ and we get

$$A = \int_{N_3(0,0,0)} \Big(G_u(u,v,w) - \frac{1}{2} v G_w(u,v,w) \Big) du dv dw,$$

Now integrate by parts in A yields

$$\begin{aligned} A &= \int_{\partial N_3(0,0,0)} G(u,v,w) \Big(\eta_u - \frac{v}{2}\eta_w \Big)\, dS \\ &= \int_{\partial N_3(0,0,0)} \Big((F\eta_\epsilon)_u(u,v,w) - \frac{v}{2}(F\eta_\epsilon)_w(u,v,w) \Big) \Big(\eta_u - \frac{v}{2}\eta_w \Big)\, dS, \end{aligned}$$

where $\eta = (\eta_u, \eta_v, \eta_w)$ is the unit outer normal to $\partial N_3(0,0,0)$. Notice that for $(u,v,w) \in \partial N_3(0,0,0)$, we have $((u^2+v^2)^2 + \mu\, w^2 = 3^4 > (2\epsilon)^4$ and so we get

$$\begin{aligned} X_{1,1} h_\epsilon(p) = &\int_{\partial N_3(0,0,0)} \Big(F_u(u,v,w) - \frac{v}{2} F_w(u,v,w) \Big) \Big(\eta_u - \frac{v}{2}\eta_w \Big)\, dS \\ &- \int_{N_3(x,y,z)\setminus\Omega} X_{1,1}(F\eta_\epsilon)(q^{-1}\circ p)\, dq := A_{11}(p) - B_{11}(p). \end{aligned}$$

In exactly the same way we get

$$\begin{aligned} X_{2,2} h_\epsilon(p) = &\int_{\partial N_3(0,0,0)} \Big(F_v(u,v,w) + \frac{u}{2} F_w(u,v,w) \Big) \Big(\eta_v + \frac{u}{2}\eta_w \Big) dS \\ &- \int_{N_3(x,y,z)\setminus\Omega} X_{2,2}(F\eta_\epsilon)(q^{-1}\circ p) dq := A_{22}(p) - B_{22}(p); \end{aligned}$$

and

$$\begin{aligned} X_{1,2} h_\epsilon(p) = \frac{1}{2} &\int_{\partial N_3(0,0,0)} \Big(F_u(u,v,w) - \frac{v}{2} F_w(u,v,w) \Big) \Big(\eta_v + \frac{u}{2}\eta_w \Big) \\ &+ \Big(F_v(u,v,w) + \frac{u}{2} F_w(u,v,w) \Big) \Big(\eta_u - \frac{v}{2}\eta_w \Big)\, dS \\ &- \int_{N_3(x,y,z)\setminus\Omega} X_{1,2}(F\eta_\epsilon)(q^{-1}\circ p)\, dq := A_{12}(p) - B_{12}(p). \end{aligned}$$

We are now going to pick the function F, depending on μ and α, such that

$$a(p)A_{11}(p) + 2b(p)A_{12}(p) + c(p)A_{22}(p) \geq C^*\, \lambda, \qquad \text{for all } p, \tag{3.4}$$

with C^* positive constant depending only on α and μ, and also

$$a(p)B_{11}(p) + 2b(p)B_{12}(p) + c(p)B_{22}(p) \leq 0, \tag{3.5}$$

this one for all $p \in \Omega' \Subset \Omega$ with $\mathrm{dist}(\Omega', \partial\Omega) \geq 2\epsilon$. Let $\Psi(u,v,w) = (u^2+v^2)^2 + \mu\, w^2$ and $F = -\dfrac{4}{\alpha}\Psi^{-\alpha/4}$. Notice that $\eta = \dfrac{D\Psi}{|D\Psi|}$ and $DF = \Psi^{-(\alpha+4)/4} D\Psi$. Let us prove (3.4). We have

$$
\begin{aligned}
&a(p)A_{11}(p) + 2b(p)A_{12}(p) + c(p)A_{22}(p)\\
&= \frac{16}{\alpha^2}\int_{\partial N_3(0,0,0)} \frac{\Psi^{-(\alpha+4)/4}}{|D\Psi|}\left\{a\left(\Psi_u - \frac{v}{2}\Psi_w\right)^2 + 2b\left(\Psi_u - \frac{v}{2}\Psi_w\right)\left(\Psi_v + \frac{u}{2}\Psi_w\right) + c\left(\Psi_v + \frac{u}{2}\Psi_w\right)^2\right\} dS\\
&\geq \frac{16}{\alpha^2}\,\lambda \int_{\partial N_3(0,0,0)} \frac{\Psi^{-(\alpha+4)/4}}{|D\Psi|}\left\{\left(\Psi_u - \frac{v}{2}\Psi_w\right)^2 + \left(\Psi_v + \frac{u}{2}\Psi_w\right)^2\right\} dS\\
&= \frac{16\,\lambda}{\alpha^2\, 3^{\alpha+4}} \int_{\partial N_3(0,0,0)} \frac{1}{|D\Psi|}\left\{\left(\Psi_u - \frac{v}{2}\Psi_w\right)^2 + \left(\Psi_v + \frac{u}{2}\Psi_w\right)^2\right\} dS\\
&= \frac{16\,\lambda}{\alpha^2\, 3^{\alpha+4}} \int_{(u^2+v^2)^2+\mu\, w^2 = 3^4} \frac{(u^2+v^2)\,(16(u^2+v^2)+\mu^2\, w^2)}{\sqrt{16(u^2+v^2)^3 + 4\mu^2\, w^2}}\, dS\\
&\geq \frac{16\,\lambda}{\alpha^2\, 3^{\alpha+4}} \int_{(u^2+v^2)^2+\mu\, w^2 = 3^4} \frac{(u^2+v^2)\,(16(u^2+v^2)+\mu^2\, w^2)}{\sqrt{9\cdot 16(u^2+v^2)^2 + 4\mu^2\, w^2}}\, dS := J.
\end{aligned}
$$

From (3.3), we have that $32/7 < \mu < 32$, and so $9\cdot 16(u^2+v^2)^2 + 4\mu^2\, w^2 \leq C\,((u^2+v^2)^2 + \mu\, w^2)$ and so in the region of integration the denominator is bounded above by $9\sqrt{C}$ and therefore

$$
J \geq C_1\,\lambda \int_{(u^2+v^2)^2+\mu\, w^2 = 3^4} (u^2+v^2)\left(16(u^2+v^2)+\mu^2\, w^2\right) dS := \lambda\, C^*(\mu,\alpha) > 0.
$$

Next we choose μ and α so that (3.5) holds. We let $H(r,z) = (r^4+z^2)^{-\alpha/4}$, here $r^2 = x^2+y^2$. We have

$$
\begin{aligned}
H_r &= -\alpha\, r^3\,(r^4+z^2)^{-(\alpha/4)-1}\\
H_{rr} &= \alpha\, r^2\left((\alpha+1)r^4 - 3z^2\right)(r^4+z^2)^{-(\alpha/4)-2}\\
H_{rz} &= \frac{\alpha}{2}\,(\alpha+4)\, z\, r^3\,(r^4+z^2)^{-(\alpha/4)-2}\\
H_{zz} &= \frac{\alpha}{2}\left(-r^4 + (\alpha+3)\, z^2\right)(r^4+z^2)^{-(\alpha/4)-2}.
\end{aligned}
$$

Let $G(x,y,z) = H(r, \sqrt{\mu}\, z)$ with $\mu > 0$ and $\alpha > 0$ to be adjusted such that

$$
\tilde{L}G(x,y,z) \geq 0, \tag{3.6}
$$

for all $(x,y,z) \neq 0$ with $\tilde{L}G(x,y,z) = aX_{1,1}G(x,y,z) + 2bX_{1,2}G(x,y,z) + cX_{2,2}G(x,y,z)$, and with a,b,c any numbers satisfying the ellipticity condition $\lambda \leq a\xi_1^2 + 2b\xi_1\xi_2 +$

$c\xi_2^2 \leq \Lambda$ for all ξ_1, ξ_2. Notice that $G = \Psi^{-\alpha/4}$ and so $F = -(4/\alpha)G$, therefore this implies that $\tilde{L}F \leq 0$.

We have

$$\begin{aligned}
X_1(X_1G) &= H_{rr}(r_x)^2 + H_r\, r_{xx} - \sqrt{\mu}\, y\, r_x\, H_{zr} + (y/2)^2\mu\, H_{zz} \\
X_2(X_2G) &= H_{rr}(r_y)^2 + H_r\, r_{yy} + \sqrt{\mu}\, x\, r_x\, H_{zr} + (x/2)^2\mu\, H_{zz} \\
X_{1,2}G &= H_{rr}r_xr_y + H_r\, r_{xy} + \sqrt{\mu}\, \frac{xr_x - yr_y}{2}\, H_{zr} - (xy/4)\mu\, H_{zz}.
\end{aligned}$$

A calculation shows that

$$\begin{aligned}
X_{1,1}G &= H_{rr}\frac{x^2}{r^2} + H_r\frac{y^2}{r^3} - \sqrt{\mu}\frac{xy}{r}H_{zr} + \mu\frac{y^2}{4}H_{zz} \\
X_{2,2}G &= H_{rr}\frac{y^2}{r^2} + H_r\frac{x^2}{r^3} + \sqrt{\mu}\frac{xy}{r}H_{zr} + \mu\frac{x^2}{4}H_{zz} \text{ and} \\
X_{1,2}G &= H_{rr}\frac{xy}{r^2} - H_r\frac{xy}{r^3} + \sqrt{\mu}\,\frac{x^2-y^2}{2r}H_{zr} - \mu\frac{xy}{4}H_{zz}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\tilde{L}G =& aX_{1,1}G + 2bX_{1,2}G + cX_{2,2}G \\
=& H_{rr}\left\{a\frac{x^2}{r^2} + 2b\frac{xy}{r^2} + c\frac{y^2}{r^2}\right\} + H_r\left\{a\frac{y^2}{r^3} - 2b\frac{xy}{r^3} + c\frac{x^2}{r^3}\right\} + \mu\, H_{zz}\left\{a\frac{y^2}{4} - 2b\frac{xy}{4} + c\frac{x^2}{4}\right\} \\
&+ \sqrt{\mu}\, H_{rz}\left\{(c-a)\frac{xy}{r} + b\left(\frac{x^2-y^2}{r}\right)\right\}.
\end{aligned}$$

Inserting the values of the derivatives of H yields

$$\begin{aligned}
\tilde{L}G =& \alpha\, r^2\left((\alpha+1)r^4 - 3\mu z^2\right)(r^4+\mu z^2)^{-(\alpha/4)-2}\left\{a\frac{x^2}{r^2} + 2b\frac{xy}{r^2} + c\frac{y^2}{r^2}\right\} \\
&- \alpha\, r^3\,(r^4+\mu z^2)^{-(\alpha/4)-1}\left\{a\frac{y^2}{r^3} - 2b\frac{xy}{r^3} + c\frac{x^2}{r^3}\right\} \\
&+ \frac{\alpha}{2}\,\mu\left(-r^4 + (\alpha+3)\,\mu z^2\right)(r^4+\mu z^2)^{-(\alpha/4)-2}\left\{a\frac{y^2}{4} - 2b\frac{xy}{4} + c\frac{x^2}{4}\right\} \\
&+ \frac{\alpha}{2}\,(\alpha+4)\,\mu\, z\, r^3\,(r^4+\mu z^2)^{-(\alpha/4)-2}\left\{(c-a)\frac{xy}{r} + b\left(\frac{x^2-y^2}{r}\right)\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}(1/\alpha)\,(r^4+\mu z^2)^{(\alpha/4)+2}\,\tilde{L}G &= \Big((\alpha+1)r^4-3\mu z^2\Big)\left\{ax^2+2bxy+cy^2\right\}\\ &\quad-(r^4+\mu z^2)\left\{ay^2-2bxy+cx^2\right\}\\ &\quad+\frac{1}{8}\,\mu\,\Big(-r^4+(\alpha+3)\,\mu z^2\Big)\left\{ay^2-2bxy+cx^2\right\}\\ &\qquad+\frac{1}{2}\,(\alpha+4)\,\mu\,z\,r^2\left\{(c-a)xy+b(x^2-y^2)\right\}:=\Delta.\end{aligned}$$

We have

$$\begin{aligned}\Delta &= r^4\left((\alpha+1)\left\{ax^2+2bxy+cy^2\right\}-\left(1+\frac{\mu}{8}\right)\left\{ay^2-2bxy+cx^2\right\}\right)\\ &\quad+\mu z^2\left(-3\left\{ax^2+2bxy+cy^2\right\}+\left(\frac{\mu}{8}(\alpha+3)-1\right)\left\{ay^2-2bxy+cx^2\right\}\right)\\ &\quad+\frac{1}{2}\,(\alpha+4)\,\mu\,z\,r^2\left\{(c-a)xy+b(x^2-y^2)\right\}\\ &= r^4\,\Delta_1+\mu\,z^2\,\Delta_2+\frac{1}{2}\,(\alpha+4)\,\mu\,z\,r^2\,\Delta_3.\end{aligned}$$

From the ellipticity

$$\Delta_1\geq\left((\alpha+1)\lambda-\left(1+\frac{\mu}{8}\right)\Lambda\right)r^2,$$

if $\alpha+1>0$, $\mu>0$; and also

$$\Delta_2\geq\left(\left(\frac{\mu}{8}(\alpha+3)-1\right)\lambda-3\Lambda\right)r^2,$$

if $\frac{\mu}{8}(\alpha+3)-1>0$. On the other hand, $\mu\,z\,r^2\,\Delta_3\geq-\mu\,r^2\,|z\Delta_3|$. Again by the ellipticity $|\Delta_3|\leq(\Lambda-\lambda)r^2$, and so

$$\mu\,z\,r^2\,\Delta_3\geq-\mu\,|z|\,r^4\,(\Lambda-\lambda).$$

Therefore

$$\begin{aligned}\Delta &\geq\left((\alpha+1)\lambda-\left(1+\frac{\mu}{8}\right)\Lambda\right)r^6+\left(\left(\frac{\mu}{8}(\alpha+3)-1\right)\lambda-3\Lambda\right)r^2\,\mu z^2\\ &\qquad-\frac{1}{2}\,(\alpha+4)\,\mu\,|z|\,r^4(\Lambda-\lambda)\\ &=\left[\left((\alpha+1)\lambda-\left(1+\frac{\mu}{8}\right)\Lambda\right)r^4+\left(\left(\frac{\mu}{8}(\alpha+3)-1\right)\lambda-3\Lambda\right)\mu z^2-\frac{1}{2}\,(\alpha+4)\,\mu\,|z|\,r^2(\Lambda-\lambda)\right]r^2\end{aligned}$$

Set

$$A = (\alpha+1)\lambda - \left(1+\frac{\mu}{8}\right)\Lambda$$
$$B = \left(\left(\frac{\mu}{8}(\alpha+3)-1\right)\lambda - 3\Lambda\right)\mu$$
$$C = \frac{1}{2}(\alpha+4)\,\mu(\Lambda-\lambda).$$

So

$$\Delta \geq \left(Ar^4 + Bz^2 - C|z|r^2\right)r^2.$$

We choose α and μ such that $A, B \geq 0$, so

$$Ar^4 + Bz^2 - C|z|r^2 = \left(\sqrt{A}\,r^2 - \sqrt{B}\,|z|\right)^2 + \left(2\sqrt{AB} - C\right)|z|r^2,$$

and if in addition

$$2\sqrt{AB} - C \geq 0,$$

we get $\Delta \geq 0$. The inequality $2\sqrt{AB} - C \geq 0$ amounts

$$2\sqrt{\left((\alpha+1) - \left(1+\frac{\mu}{8}\right)\frac{\Lambda}{\lambda}\right)\left(\left(\frac{\mu}{8}(\alpha+3)-1\right) - 3\frac{\Lambda}{\lambda}\right)\mu} \geq \frac{1}{2}(\alpha+4)\,\mu\left(\frac{\Lambda}{\lambda}-1\right),$$

that is,

$$2\sqrt{\left((\alpha+1) - \left(1+\frac{\mu}{8}\right)\frac{\Lambda}{\lambda}\right)\left(\left(\frac{\mu}{8}(\alpha+3)-1\right) - 3\frac{\Lambda}{\lambda}\right)} \geq \frac{1}{2}(\alpha+4)\,\sqrt{\mu}\left(\frac{\Lambda}{\lambda}-1\right).$$

Squaring we get

$$16\left((\alpha+1) - \left(1+\frac{\mu}{8}\right)\frac{\Lambda}{\lambda}\right)\left(\left(\frac{\mu}{8}(\alpha+3)-1\right) - 3\frac{\Lambda}{\lambda}\right) \geq (\alpha+4)^2\,\mu\left(\frac{\Lambda}{\lambda}-1\right)^2. \tag{3.7}$$

Notice that if $\Lambda/\lambda = 1$, then (3.7) becomes

$$16\left((\alpha+1) - \left(1+\frac{\mu}{8}\right)\right)\left(\left(\frac{\mu}{8}(\alpha+3)-1\right) - 3\right) \geq 0$$

and if we choose α, μ such that $A > 0$ and $B > 0$, then the left hand side of this inequality is strictly positive and therefore by continuity (3.7) holds when $\frac{\Lambda}{\lambda} \leq C(\alpha,\mu)$ with $1 < C(\alpha,\mu)$ and $C(\alpha,\mu)$ depending only on α and μ. Also $A > 0$ amounts

$$\frac{\Lambda}{\lambda} < \frac{\alpha+1}{1+\frac{\mu}{8}};$$

and $B > 0$ amounts

$$\frac{\Lambda}{\lambda} < \frac{\frac{\mu}{8}(\alpha+3)-1}{3}, \qquad \mu > 0.$$

Since $\Lambda/\lambda \geq 1$, we must have

$$1 < \min\left\{\frac{\alpha+1}{1+\frac{\mu}{8}}, \frac{\frac{\mu}{8}(\alpha+3)-1}{3}\right\},$$

which means

$$\frac{4}{\alpha+3} < \frac{\mu}{8} < \alpha.$$

Notice that the interval $(4/(\alpha+3), \alpha)$ is non empty for each $\alpha > 1$. Since in addition, we need the function F to be integrable around the origin, this means that $\alpha < 4$. Therefore, we first choose α and μ both positive such that

$$\frac{4}{\alpha+3} < \frac{\mu}{8} < \alpha < 4,$$

and next with this choice of α and μ, there is a constant $C(\alpha,\mu) > 1$ such that $\tilde{L}G > 0$ when $\frac{\Lambda}{\lambda} \leq C(\alpha,\mu)$, and therefore we obtain (3.6).

Then applying (3.6) with $a \rightsquigarrow a(p), b \rightsquigarrow b(p), c \rightsquigarrow c(p)$, and $(x,y,z) \rightsquigarrow q^{-1}\circ p$, $p = (x,y,z)$, we obtain that

$$\begin{aligned}
&a(x,y,z)X_{1,1}(G(q^{-1}\circ p)) + 2b(x,y,z)X_{1,2}(G(q^{-1}\circ p)) + c(x,y,z)X_{2,2}(G(q^{-1}\circ p))\\
&= a(x,y,z)X_{1,1}(G)(q^{-1}\circ p) + 2b(x,y,z)X_{1,2}(G)(q^{-1}\circ p) + c(x,y,z)X_{2,2}(G)(q^{-1}\circ p)\\
&\geq 0,
\end{aligned}$$

for all p, q with $q^{-1}\circ p \neq 0$ and where the differentiation of the fields acts in the variable p. Since $F = -(4/\alpha)G$, we obtain

$$a(p)X_{1,1}(F(q^{-1}\circ p)) + 2b(p)X_{1,2}(F(q^{-1}\circ p)) + c(p)X_{2,2}(F(q^{-1}\circ p)) \leq 0.$$

Therefore

$$\begin{aligned}
&a(p)B_{11}(p) + 2b(p)B_{12}(p) + c(p)B_{22}(p)\\
&= \int_{N_3(p)\setminus\Omega} \left\{a(p)X_{1,1}((F\eta_\epsilon)(q^{-1}\circ p)) + 2b(p)X_{1,2}((F\eta_\epsilon)(q^{-1}\circ p)) + c(p)X_{2,2}((F\eta_\epsilon)(q^{-1}\circ p))\right\} dq\\
&= \int_{N_3(p)\setminus\Omega} \left\{a(p)X_{1,1}(F(q^{-1}\circ p)) + 2b(p)X_{1,2}(F(q^{-1}\circ p)) + c(p)X_{2,2}(F(q^{-1}\circ p))\right\} dq \leq 0,
\end{aligned}$$

for $p \in \Omega' \Subset \Omega$ such that $\rho_\mu(q^{-1} \circ p) > (2\epsilon)$ for all $q \in N_3(p) \backslash \Omega$; that is, for all $p \in \Omega'$ with $\mathrm{dist}(\Omega', \partial\Omega) \geq 2\epsilon$, which completes the proof of (3.5). This combined with (3.4) yields the conclusion in the Lemma with $C = C^* \lambda$.

We also state two facts which are easy to prove: First, we have that $h_\epsilon \longrightarrow h$ uniformly in $\mathbb{R}^3$ as $\epsilon \to 0$, and secondly, we have that $h(x, y, z) \geq -\bar{c}\, |\Omega|^{1-(\alpha/4)}$, for all $(x, y, z) \in \mathbb{R}^3$[‡]; where $\bar{c} = \frac{4}{\alpha} \int_{N_1(0,0,0)} ((u^2 + v^2)^2 + \mu\, w^2)^{\frac{-\alpha}{4}} dudvdw$. □

We are now ready to prove the critical density estimate which we first state as follows.

Theorem 3.2. *Let $C(\alpha, \mu)$ be the constant in Lemma 3.1, and assume the ellipticity constants satisfy $\Lambda/\lambda \leq C(\alpha, \mu)$. Suppose $Lu \leq C$ in $N_1(0, 0, 0)$, with $C > 0$, $u \geq 0$ on $\partial N_1(0, 0, 0)$, and $u(x_0, y_0, z_0) \leq -1$ for some $(x_0, y_0, z_0) \in N_1(0, 0, 0)$. Let $\Omega = \{(x, y, z) \in N_1(0, 0, 0) : u(x, y, z) < 0\}$. Then there exists a constant $\delta > 0$ depending only on C and λ such that $|\Omega| \geq \delta$.*

Proof. Notice that Ω is an open set. Let h and h_ϵ be as above, and set $\bar{h} = \frac{2C}{C^*\lambda} h$ and $\bar{h}_\epsilon = \frac{2C}{C^*\lambda} h_\epsilon$. We claim that $\bar{h} \leq u$ in Ω.

Suppose the claim is not true. Since $\bar{h} < 0$ and $u = 0$ on $\partial\Omega$, we have that the set $\Omega' := \{p \in \Omega : \bar{h}(p) > u(p)\} \Subset \Omega$. Let $\Omega'_\epsilon = \{p \in \Omega : \bar{h}_\epsilon(p) > u(p)\}$. Since $\bar{h}_\epsilon \geq \bar{h}$ (we may assume $C \geq 0$), we have $\Omega' \subset \Omega'_\epsilon$. In addition, since $\bar{h}_\epsilon \to \bar{h}$ uniformly, we have $\Omega'_\epsilon \Subset \Omega$ for all ϵ sufficiently small. If $\phi(t)$ is a smooth function such that $\phi(t) = 0$ for $0 \leq t \leq 1$, $\phi(t) = 1$ for $t \geq 2$, and ϕ is monotone increasing in $[0, +\infty)$, then $\eta_\epsilon(u, v, w) = \phi\left(\left(\rho_\mu(u, v, w)/\epsilon\right)^4\right)$ satisfies the hypotheses of Lemma 3.1 and $\eta_{\epsilon'} \geq \eta_\epsilon$ for $\epsilon' \leq \epsilon$, so $\bar{h}_{\epsilon'} \leq \bar{h}_\epsilon$. Hence $\Omega'_{\epsilon'} \subset \Omega'_\epsilon$ when $\epsilon' \leq \epsilon$. Let

$$\beta = \mathrm{dist}\left(\Omega', \mathbb{R}^3 \setminus \Omega\right) = \inf\{\rho_\mu(q^{-1} \circ p) : p \in \Omega',\, q \in \mathbb{R}^3 \setminus \Omega\} (> 0),$$

and

$$\beta_\epsilon = \mathrm{dist}\left(\Omega'_\epsilon, \mathbb{R}^3 \setminus \Omega\right) = \inf\{\rho_\mu(q^{-1} \circ p) : p \in \Omega'_\epsilon,\, q \in \mathbb{R}^3 \setminus \Omega\} (> 0).$$

[‡]Let $p = (x, y, z)$ and choose R such that $|\Omega| = |N_R(p)| = c\, R^4$, so $R = c'|\Omega|^{1/4}$. Then $|\Omega \setminus N_R(p)| = |N_R(p) \setminus \Omega|$. Write $h(p) = \int_{\Omega \cap N_R(p)} d(q, p)^{-\alpha}\, dq + \int_{\Omega \setminus N_R(p)} d(q, p)^{-\alpha}\, dq \leq \int_{\Omega \cap N_R(p)} d(q, p)^{-\alpha}\, dq + R^{-\alpha}|\Omega \setminus N_R(p)| = \int_{\Omega \cap N_R(p)} d(q, p)^{-\alpha}\, dq + R^{-\alpha}|N_R(p) \setminus \Omega| \leq \int_{\Omega \cap N_R(p)} d(q, p)^{-\alpha}\, dq + \int_{N_R(p) \setminus \Omega} d(q, p)^{-\alpha}\, dq = \int_{N_R(p)} d(q, p)^{-\alpha}\, dq = R^{4-\alpha} = c|\Omega|^{1-(\alpha/4)}$.

We have $\beta_\epsilon \le \beta$ and $\beta_\epsilon \le \beta_{\epsilon'}$ for $\epsilon' \le \epsilon$. Now choose $\epsilon_1 > 0$ sufficiently small with $\beta_{\epsilon_1} \le \beta$. Then for all $0 < \epsilon \le \epsilon_1$ we have $\beta_{\epsilon_1} \le \beta_\epsilon \le \beta$. Hence $\beta_\epsilon \ge \dfrac{\beta}{(\beta/\beta_{\epsilon_1})}$ for $0 < \epsilon \le \epsilon_1$. If we choose $\epsilon_0 = \min\left\{\epsilon_1, \dfrac{\beta}{(\beta/\beta_{\epsilon_1})}\right\}$, then $\beta_\epsilon \ge 2\epsilon$ for all $0 < \epsilon \le \epsilon_0$.

Next, for $\epsilon \le \epsilon_0$ consider $m = \min\{u(p) - \bar{h}_\epsilon(p) : p \in \Omega\}$. There exists $p_\epsilon \in \bar{\Omega}$ such that $u(p_\epsilon) - \bar{h}_\epsilon(p_\epsilon) = m$. Since $\Omega'_\epsilon \ne \emptyset$, $m < 0$ and therefore $p_\epsilon \in \Omega'_\epsilon$. Therefore $H(u - \bar{h}_\epsilon)(p_\epsilon) \ge 0$ and then from (2.2) $Lu(p_\epsilon) \ge L\bar{h}_\epsilon(p_\epsilon)$. Applying Lemma 3.1 in Ω'_ϵ we obtain that $L\bar{h}_\epsilon(p_\epsilon) \ge 2C$ which yields a contradiction since by assumption $C \ge Lu(p_\epsilon)$.

In particular, we get $-1 \ge u(x_0, y_0, z_0) \ge \bar{h}(x_0, y_0, z_0) \ge -\frac{2C}{C^*\lambda}\bar{c}\,|\Omega|^{1-(\alpha/4)}$ which proves the result. □

We now introduce a change of variables that preserves the equation: Fix (x_0, y_0, z_0) and $R \in \mathbb{R}$ and let

$$T(x, y, z) = \left(x_0 + Rx, y_0 + Ry, z_0 + R^2 z + \frac{R}{2}(x_0 y - y_0 x)\right). \tag{3.8}$$

Set $\tilde{u}(x, y, z) = u(T(x, y, z))$. We have $X_i(\tilde{u})(x, y, z) = RX_i u(T(x, y, z))$ and $X_{i,j}(\tilde{u})(x, y, z) = R^2 X_{i,j} u(T(x, y, z))$. Set $\tilde{a}(x, y, z) = a(T(x, y, z))$, $\tilde{b}(x, y, z) = b(T(x, y, z))$, and $\tilde{c}(x, y, z) = c(T(x, y, z))$, then $\tilde{L}\tilde{u}(x, y, z) = R^2 Lu(T(x, y, z))$.

Theorem 3.3. *Let $C(\alpha, \mu)$ be the constant in Lemma 3.1, and assume the ellipticity constants satisfy $\Lambda/\lambda \le C(\alpha, \mu)$. There exist positive constants M and ϵ depending only on the ellipticity constants such that for any $u \ge 0$ with $Lu \le 0$ in $N_{2R}(x_0, y_0, z_0)$ and $\inf_{N_R(x_0,y_0,z_0)} u \le 1$ we have that*

$$|\{(x, y, z) \in N_{2R}(x_0, y_0, z_0) : u(x, y, z) \le M\}| \ge \epsilon\, |N_{2R}(x_0, y_0, z_0)|.$$

Proof. We show first that the theorem holds for $R = 1$ and $(x_0, y_0, z_0) = (0, 0, 0)$. Let $v = r^4 + \mu z^2 - 2^4$. We have $v = 0$ on $\partial N_2(0, 0, 0)$, $v \le 1 - 2^4$ on $N_1(0, 0, 0)$, and $Lv \le (16 + \mu^2/2)\,\Lambda\, r^2 \le (64 + 2\,\mu^2)\,\Lambda$ on $N_2(0, 0, 0)$.

Let $w = u + v$ on $N_2(0, 0, 0)$. Then w satisfies the hypothesis of Theorem 3.2 with $C = (64 + 2\,\mu^2)\,\Lambda$. Hence, we have an ϵ depending only on the ellipticity constants such that $|\{(x, y, z) \in N_2(0, 0, 0) : w(x, y, z) < 0\}| \ge \epsilon\, |N_2(0, 0, 0)|$. Thus, $|\{(x, y, z) \in N_2(0, 0, 0) : u(x, y, z) \le 16\}| \ge \epsilon\, |N_2(0, 0, 0)|$.

For the general case, we change variables and apply what we just proved. We define $\tilde{u}(x, y, z) = u(T(x, y, z))$ for $(x, y, z) \in N_2(0, 0, 0)$ and recall we have

$\tilde{L}\tilde{u}(x,y,z) = R^2 Lu(T(x,y,z))$. We have that $\tilde{u}$ satisfies the hypothesis of the theorem with $R = 1$ and $(x_0, y_0, z_0) = (0,0,0)$ and hence $|\{(x,y,z) \in N_2(0,0,0) : \tilde{u}(x,y,z) \leq 16\}| \geq \epsilon$. And hence we conclude that $|\{(x,y,z) \in N_{2R}(x_0,y_0,z_0) : u(x,y,z) \leq 16\}| = |T(\{(x,y,z) \in N_2(0,0,0) : \tilde{u}(x,y,z) \leq 16\})| = R^4|\{(x,y,z) \in N_2(0,0,0) : \tilde{u}(x,y,z) \leq 16\}| \geq R^4\epsilon = \tilde{\epsilon}|N_{2R}(x_0,y_0,z_0)|$. □

4. Passage to the double cylinder

For the next result, it is convenient to switch from the sets N_R to equivalent sets which are cylinders. Indeed, we will use the following family of cylinders: set $r^2 = x^2 + y^2$ and let $C_R(0,0,0) = \{(x,y,z) : \max\{r, |z|^{\frac{1}{2}}\} < R\}$ and $C_R(x_0,y_0,z_0) = T(C_1(0,0,0))$ with T given in (3.8).

We will prove the next theorem using the weak maximum principle.

Theorem 4.1. *There exists a positive constant γ depending only on the ellipticity constants such that if $u \geq 1$ on $C_R(x_0,y_0,z_0)$ and $u \geq 0, Lu \leq 0$ on $C_{3R}(x_0,y_0,z_0)$, then $u \geq \gamma$ on $C_{2R}(x_0,y_0,z_0)$.*

In order to obtain this theorem, we first prove two lemmas.

Lemma 4.2. *Suppose $u \geq 1$ on a two dimensional disk D of radius $\delta \leq 1$ centered on the z axis and contained on a plane containing the z axis. And assume $u \geq 0, Lu \leq 0$ in the cylinder of height Γ with D at its base. Then $u \geq \gamma$ on the the cylinder with base $\frac{1}{2}D$ and height $\frac{1}{2}\Gamma$, where the constant γ depends only on δ, Γ and the ellipticity constants.*

Proof. We will prove first a particular case. The general case will follow by a rotation leaving the z axis fixed.

Suppose $u \geq 1$ on the two dimensional disk $x^2 + z^2 \leq \delta^2, y = 0$ and $u \geq 0, Lu \leq 0$ on the cylinder $x^2 + z^2 \leq \delta^2, 0 \leq y \leq \Gamma$. Then we shall prove that $u \geq \gamma$ on the cylinder $x^2 + z^2 \leq \frac{\delta^2}{4}, 0 \leq y \leq \frac{\Gamma}{2}$, where the constant γ depends only on δ, Γ and the ellipticity constants.

Set $\beta = \frac{\delta^2}{\Gamma}$ and $Q(x,y,z) = \alpha(\delta^2 - x^2 - z^2 - \beta y)$, where α will be chosen momentarily, and consider the function $h(x,y,z) = \dfrac{e^Q - 1}{e^{\alpha\delta^2} - 1}$ in the set $\Omega := \{\delta^2 - x^2 - z^2 \geq \beta y \geq 0\}$. We will choose α so that $Lh \geq 0$ in Ω. Indeed, we have

$$Lh = \frac{e^Q}{e^{\alpha\delta^2} - 1}\{a(X_1Q)^2 + 2bX_1QX_2Q + c(X_2Q)^2 + aX_{1,1}Q + 2bX_{1,2}Q + cX_{2,2}Q\}.$$

Using that $X_1Q = -2\alpha x + \alpha yz$, $X_2Q = -\alpha\beta - \alpha xz$, $X_{1,1}Q = -2\alpha - \frac{\alpha}{2}y^2$, $X_{1,2}Q = \frac{\alpha}{2}xy$, $X_{2,2}Q = -\frac{\alpha}{2}x^2$, and the ellipticity we get that

$$\begin{aligned} Lh &\geq \frac{\alpha e^Q}{e^{\alpha\delta^2}-1}\left\{\lambda\alpha((yz-2x)^2+(\beta+xz)^2) - 2a - \frac{1}{2}(ay^2-2bxy+cx^2)\right\} \\ &\geq \frac{\alpha e^Q}{e^{\alpha\delta^2}-1}\left\{\lambda\alpha((yz-2x)^2+(\beta+xz)^2) - \Lambda\left(2+\frac{1}{2}(x^2+y^2)\right)\right\} \\ &\geq \frac{\alpha e^Q}{e^{\alpha\delta^2}-1}\left\{\lambda\alpha((yz-2x)^2+(\beta+xz)^2) - \Lambda\left(2+\frac{1}{2}\left(\delta^2+\frac{\delta^4}{\beta^2}\right)\right)\right\}. \end{aligned}$$

A calculation shows that $((yz-2x)^2+(\beta+xz)^2) \geq \dfrac{\beta^2}{4}$ for $|z| \leq 1$ and therefore in $\bar{\Omega}$. Hence, we have $Lh \geq \dfrac{\alpha e^Q}{e^{\alpha\delta^2}-1}\left\{\alpha\lambda\frac{\beta^2}{4} - \Lambda\left(\frac{5}{2}+\frac{\delta^4}{\beta^2}\right)\right\}$. Therefore if we choose α such that $\alpha\lambda\dfrac{\delta^4}{\Gamma^2} \geq \Lambda\left(\frac{5}{2}+\Gamma^2\right)$, then $Lh \geq 0$ in Ω. Notice that $\Omega \subseteq x^2+z^2 \leq \delta^2, 0 \leq y \leq \Gamma$ and we have $h \leq 1 \leq u$ on the disk $y = 0, x^2+z^2 \leq \delta^2$ and $h = 0 \leq u$ on $x^2+z^2 \leq \delta^2, \beta y = \delta^2 - x^2 - y^2$. In other words, $h \leq u$ on $\partial\Omega$ and we also have $Lh \geq 0 \geq Lu$ in Ω. By the maximum principle, we have $h \leq u$ in Ω. We let γ be the minimum of h on the set $x^2+z^2 \leq \frac{\delta^2}{4}, 0 \leq y \leq \frac{\Gamma}{2}$. Notice that $\gamma > 0$ depends only on δ, Γ and the ellipticity constants.

For the general case, assume O is a 2×2 matrix such that $OO^T = O^TO = I$. And let $\tilde{u}(x,y,z) = u(O(x,y)^T, z)$. A calculation shows that $OH\tilde{u}(x,y,z)O^T = Hu(O(x,y)^T,z)$. We define the matrix $\tilde{A}(x,y,z) = O^TA(O(x,y)^T,z)O$, and we have

$$\begin{aligned} \tilde{L}\tilde{u}(x,y,z) &= \operatorname{trace}(\tilde{A}(x,y,z)H\tilde{u}(x,y,z)) \\ &= \operatorname{trace}(A(O(x,y)^T,z)Hu(O(x,y)^T,z)) = Lu(O(x,y)^T,z). \end{aligned}$$

Since the matrix $\tilde{A}$ satisfies the ellipticity inequalities with the same constants as A, the general result follows by the particular case. □

The following lemma is similar but requires a different comparison function.

Lemma 4.3. *Set $r^2 = x^2+y^2$, and suppose $u \geq 1$ on the two dimensional disk $r^2 \leq 1, z = z_0$ and $u \geq 0, Lu \leq 0$ in the cylinder $r^2 \leq 1, z_0 - 8 \leq z \leq z_0 + 8$.*

Then $u \geq \gamma$ on the ball $r^2 + (z-z_0)^2 < \beta^2$, where γ and β are small constants that depend only on the ellipticity constants.

Proof. We may assume $z_0 = 0$.

Let $f(x) = -2x^3 + x + 1$, $Q(x, y, z) = \beta f(r^2) - z$ and $h = \dfrac{e^Q - 1}{e^{c\beta} - 1}$ on the set $\Omega := \{Q \geq 0, z \geq 0\}$ where $c = f(1/\sqrt{6})$ is the maximum of f in the interval $[0, 1]$ and β will be chosen momentarily so that $Lh \geq 0$ in Ω. We have $X_1 Q = \beta f'(r^2)2x + \dfrac{y}{2}$, $X_2 Q = \beta f'(r^2)2y - \dfrac{x}{2}$, $X_{1,1}Q = \beta f''(r^2)4x^2 + 2\beta f'(r^2)$, $X_{1,2}Q = 4xy\beta f''(r^2)$ and $X_{2,2}Q = \beta f''(r^2)4y^2 + 2\beta f'(r^2)$. Then we get

$$\begin{aligned} Lh = \frac{e^Q}{e^{c\beta} - 1}\{&4\beta f''(r^2)(ax^2 + 2bxy + cy^2) \\ &+ 2\beta f'(r^2)(a + c) + a\left(\beta f'(r^2)2x + \frac{y}{2}\right)^2 \\ &+ 2b\left(\beta f'(r^2)2x + \frac{y}{2}\right)\left(\beta f'(r^2)2y - \frac{x}{2}\right) + c\left(\beta f'(r^2)2y - \frac{x}{2}\right)^2\}. \end{aligned}$$

Since $f'' \leq 0$ in $[0, 1]$, it follows that

$$Lh \geq \frac{e^Q}{e^{c\beta} - 1}\left\{4\beta\Lambda f''(r^2)r^2 + 2\beta f'(r^2)(a + c) + \lambda r^2\left(4\beta^2(f'(r^2))^2 + \frac{1}{4}\right)\right\}.$$

Let $x = r^2$. For $0 \leq x \leq 6^{-\frac{1}{2}}$, we have $f'(x) \geq 0$ and hence $Lh \geq -48\beta\Lambda x^2 + \dfrac{\lambda x}{4} + 4\lambda x\beta^2(1 - 6x^2)^2 \geq -48\beta\Lambda x^2 + \dfrac{\lambda x}{4} = x\left(\dfrac{\lambda}{4} - 48\beta x\Lambda\right) \geq x\left(\dfrac{\lambda}{4} - 48\beta\Lambda 6^{-\frac{1}{2}}\right) \geq 0$, for β small enough.

If $6^{-\frac{1}{2}} \leq x \leq 1$, then we have $Lh \geq -48\beta\Lambda x^2 - 4\beta\Lambda(6x^2 - 1) + \dfrac{\lambda x}{4} + 4\lambda x\beta^2(1 - 6x^2)^2 \geq \dfrac{\lambda 6^{-\frac{1}{2}}}{4} - \beta\left(48\Lambda x^2 + 4\Lambda(6x^2 - 1)\right) \geq \dfrac{\lambda 6^{-\frac{1}{2}}}{4} - 68\beta\Lambda \geq 0$ for β small enough. Notice that β depends only on the ellipticity constants. We also notice that for β small enough Ω is contained in the cylinder $r \leq 1, 0 \leq z \leq 8$. Just as in the previous lemma we have $Lh \geq Lu$ in Ω, and $h \leq u$ on $\partial\Omega$ and hence, by the maximum principle $u \geq h$ in Ω. A simple calculation shows that the half ball $r^2 + z^2 \leq \frac{\beta^2}{4}, z \geq 0$ is contained in Ω and that $h \geq \gamma > 0$ on this half ball ($\gamma = \dfrac{e^{3\beta/4} - 1}{e^{c\beta} - 1}$). We emphasize that γ depends only on the ellipticity constants.

To show that $u \geq \gamma$ in the lower part of the ball $r^2 + z^2 \leq \beta^2$, we consider the function $Q = \beta f(r^2) + z$ and the corresponding function h and proceed in the same way.

□

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We initially prove the theorem for the case $R = 1$ and $(x_0, y_0, z_0) = (0, 0, 0)$ and prove the general case by changing variables.

Hence, we assume $u \geq 1$ in the cylinder $r \leq 1, |z| \leq 1$ and $u \geq 0, Lu \leq 0$ in the cylinder $r \leq 3, |z| \leq 9$.

We will show $u \geq \gamma$ in the cylinder $r \leq 2, |z| \leq 4$ with γ a universal constant.

First by Lemma 4.3 with $z_0 = 1$, we get $u \geq \gamma_0$ on the ball $r^2 + (z-1)^2 \leq \beta^2$, and similarly with $z_0 = -1$, we get $u \geq \gamma_0$ on the ball $r^2 + (z+1)^2 \leq \beta^2$. Now using Lemma 4.2 on disks of radius β centered on the z axis with $|z| \leq 1$ and $\Gamma = 3$, we get $u \geq \gamma_1$ on the cylinder $r \leq \frac{3}{2}, |z| \leq 1 + \frac{1}{2}\beta$.

The result follows by continuing the same argument a finite number of steps.

To prove the general case we change variables in exactly the same way as in Theorem 3.3. □

Remark 4.4. *With the same hypothesis as in Theorem 4.1 it follows from the proof that for any $\beta < 1$, there exists γ depending on β and the ellipticity constants such that $u \geq \gamma$ on $C_{3\beta R}$.*

Theorem 4.5. *There exist universal constants $\tilde{M}$ and ϵ (in fact, ϵ is as in Theorem 3.3 and $\tilde{M} = \frac{2M}{\gamma^2}$ where M is as in Theorem 3.3 and γ as in Theorem 4.1) such that if $\inf_{C_{2R}(x_0,y_0,z_0)} u \leq 1$, and $u \geq 0, Lu \leq 0$ in $C_{3R}(x_0, y_0, z_0)$ then*

$$|\{(x, y, z) \in C_R(x_0, y_0, z_0) : u(x, y, z) < \tilde{M}\}| \geq \epsilon |C_R(x_0, y_0, z_0)|.$$

Proof. By contradiction, if $|\{(x, y, z) \in C_R(x_0, y_0, z_0) : u(x, y, z) < \tilde{M}\}| < \epsilon |C_R(x_0, y_0, z_0)|$, then $|\{(x, y, z) \in C_R(x_0, y_0, z_0) : \frac{Mu(x,y,z)}{\tilde{M}} < M\}| < \epsilon |C_R(x_0, y_0, z_0)|$, and hence by Theorem 3.3 we must have $\inf_{C_{\frac{R}{2}}(x_0,y_0,z_0)} \frac{Mu}{\tilde{M}} > 1$. It follows by Theorem 4.1 that $\frac{Mu}{\tilde{M}} \geq \gamma$ on $C_R(x_0, y_0, z_0)$, which is $\frac{Mu}{\gamma\tilde{M}} \geq 1$ on $C_R(x_0, y_0, z_0)$ and again by Theorem 4.1 we get $\frac{Mu}{\gamma\tilde{M}} \geq \gamma$ on $C_{2R}(x_0, y_0, z_0)$ which implies that $u \geq 2$ on $C_{2R}(x_0, y_0, z_0)$ contradicting the hypothesis. □

5. Conclusion

By the results of Di Fazio, Gutiérrez, and Lanconelli [FGL08, Theorem 4.7, condition A1,A2 and Theorem 5.1] applied to K_Ω, the class of nonnegative solutions to (1.1), we obtain the following Harnack inequality.

Theorem 5.1. *Suppose that Λ/λ is sufficiently close to one. There exist constants C and η both bigger than 1 depending only on the ellipticity constants such that if $Lu = 0$ and $u \geq 0$ in $C_{\eta R}(x_0, y_0, z_0)$, then*

$$\sup_{C_R(x_0,y_0,z_0)} u \leq C \inf_{C_R(x_0,y_0,z_0)} u.$$

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