THE PARALLEL REFRACTOR

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Dedicated to the memory of Leon Ehrenpreis

1. Introduction

The problem considered in this paper is the following. Suppose we have a domain $\Omega \subset \mathbb{R}^{n-1}$ and a domain Σ contained in an $n-1$ dimensional surface in $\mathbb{R}^n$; Σ is referred to as the target domain or screen to be illuminated (for the practical application, one can think that $n = 3$). Let n_1 and n_2 be the indexes of refraction of two homogeneous and isotropic media I and II, respectively, and suppose that from the region Ω surrounded by medium I, radiation emanates in the direction e_n with intensity $f(x)$ for $x \in \Omega$, and Σ is surrounded by media II. That is, all emanating rays from Ω are collimated. We seek an optical surface $\mathcal{R}$ interface between media I and II, such that all rays refracted by $\mathcal{R}$ into medium II are received at the surface Σ, and the prescribed radiation intensity received at each point $p \in \Sigma$ is $g(p)$. Of course some conditions on the relative position of Σ and and Ω are needed so rays can be refracted to Σ, see conditions (A) and (B) below. Assuming no loss of energy in this process, we have the conservation of energy equation $\int_\Omega f(x)\,dx = \int_\Sigma g(p)\,dp$.

The purpose of this paper is to show the existence of the interface surface $\mathcal{R}$ solving this problem under general conditions on Ω and Σ, and also when g is a Radon measure in D. This implies that one can design a lens refracting a collimated light beam emanating from Ω so that the screen Σ is illuminated in a prescribed way. The lens is bounded by two optical surfaces, the "upper" surface is $\mathcal{R}$ and the "lower" one is a plane perpendicular to e_n.

From the reversibility of the optical paths we obtain that the surface $\mathcal{R}$ refracts radiation emanating from a surface in $\mathbb{R}^n$ into collimated rays hitting Ω. In particular, we construct an optical surface that refracts radiation emanating from a finite number of sources into a beam of collimated rays.

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Our construction uses ideas from [GH09] involving ellipsoids of revolution and where the far field problem is solved when radiation emanates from a source point. However the method used in the present paper is different from the mass transportation methods used in [GH09]. We first solve the case when the target is a finite set of points and then construct the solution in the general case by approximation. An essential fact used is that an ellipsoid of revolution separating media I and II, and of eccentricity related to the indices of refraction of the media, refracts all radiation emanating from a focus into a collimated beam parallel to the axis of the ellipsoid. This is a consequence of the Snell law of refraction written in vector form, see [GH09, Section 2].

Throughout the paper we assume that media II is denser than media I, that is, $\kappa := \dfrac{n_1}{n_2} < 1$. The case when $\kappa > 1$ can be treated in a similar way but the geometry of the surface changes. One needs to use hyperboloids of revolution instead of ellipsoids as it is indicated in detail in [GH09].

2. Definitions and Preliminaries

We work with ellipsoids of the form $|x| = -kx_n + b$ which can be written as

$$\frac{|x'|^2}{\dfrac{b^2}{1-k^2}} + \frac{\left(x_n + \dfrac{kb}{1-k^2}\right)^2}{\dfrac{b^2}{(1-k^2)^2}} = 1,$$

where $x = (x', x_n)$. This is the equation of an ellipsoid of revolution about the x_n-axis with foci $(0,0)$ and $(0, -2\kappa b/(1-\kappa^2))$. If the focus at $(0,0)$ is moved to the point $p = (p', p_n)$, then the corresponding ellipsoid can be written as

$$\frac{|x'-p'|^2}{\dfrac{b^2}{1-k^2}} + \frac{\left(x_n - \left(p_n - \dfrac{kb}{1-k^2}\right)\right)^2}{\dfrac{b^2}{(1-k^2)^2}} = 1; \tag{2.1}$$

let us denote this ellipsoid by $E_{p,b}$.

We consider the lower part of the ellipsoid as the graph of the function $\phi_{p,b}$, that is, we let

$$\phi_{p,b}(x') = p_n - \frac{kb}{1-k^2} - \sqrt{\frac{b^2}{(1-k^2)^2} - \frac{|x'-p'|^2}{1-k^2}}.$$

The reason to look at the lower part of the ellipsoid is that this is the only part that refracts rays parallel to e_n into the point p, see [GH09, Section 2.2]. $\phi_{p,b}(x')$ is defined for $|x' - p'| \leq b/\sqrt{1-\kappa^2}$, that is, on the ball $B_{b/\sqrt{1-\kappa^2}}(p')$.

We fix two constants $0 < C_1 < C_2$ and we consider a target set $\Sigma \subseteq \mathbb{R}^n$ such that

$$\Sigma \subseteq \{(p', p_n) : C_1 \leq p_n \leq C_2\}. \tag{2.2}$$

We also consider a domain $\Omega \subseteq \mathbb{R}^{n-1} = \{(x', x_n) : x_n = 0\}$.
For $p \in \Sigma$, we will consider $\phi_{p,b}$ with $\dfrac{p_n(1-k^2)}{k} \leq b \leq \dfrac{C_2(1-k^2)(1+k)^2}{k^3}$.
We make two assumptions regarding Σ and Ω.

(A) We assume that there exists $0 < \delta < 1$ such that $\Omega \subseteq B_{\delta p_n \sqrt{1-\kappa^2}/k}(p')$ for all $p \in \Sigma$. This hypothesis implies that for all $p \in \Sigma$ and $b \geq \dfrac{p_n(1-k^2)}{k}$, $\phi_{p,b}$ is defined and $\phi_{p,b} \leq 0$ in $\bar{\Omega}$.

(B) This is a visibility condition. Set $M = C_2\left(\dfrac{1+k}{k}\right)^3 - C_1$. We assume that for all $x \in \bar{\Omega} \times [0, -M]$ and for all $m \in S^{n-1}$, the ray $\{x + tm : t > 0\}$ intersects Σ in at most one point.

We remark that the first condition is equivalent to the assumption that there exists $0 < \beta < 1$ such that $\left\langle -e_n, \dfrac{x-p}{|x-p|} \right\rangle \geq \beta$ for all $p \in \Sigma$ and for all $x \in \bar{\Omega}$.

We now define a parallel refractor with respect to Σ and Ω.

Definition 2.1. *We say a function $u : \bar{\Omega} \longrightarrow \mathbb{R}$ is a parallel refractor if for all $\bar{x} \in \bar{\Omega}$, there exists $p \in \Sigma$ and $b \geq \dfrac{p_n(1-k^2)}{k}$ such that $\phi_{p,b}(\bar{x}) = u(\bar{x})$ and $\phi_{p,b}(x') \geq u(x')$ for all $x' \in \bar{\Omega}$. That is, $\phi_{p,b}$ touches u from above at $\bar{x}$ in $\bar{\Omega}$. In this case we say $p \in \mathcal{N}_u(\bar{x})$ or that $\bar{x} \in \mathcal{T}_u(p)$.*

We first notice the following.

Lemma 2.2. *If u is a parallel refractor, then u is Lipschitz in $\bar{\Omega}$.*

Proof. Let $x, \bar{x} \in \bar{\Omega}$ and let $p \in \mathcal{N}_u(\bar{x})$. There exists $b \geq \dfrac{p_n(1-k^2)}{k}$ such that $u(x) \leq \phi_{p,b}(x)$ for all $x \in \bar{\Omega}$ with equality at $\bar{x}$. It follows that

$$\begin{aligned}
u(x) - u(\bar{x}) &\leq \phi_{p,b}(x) - \phi_{p,b}(\bar{x}) \\
&= \sqrt{\frac{b^2}{(1-k^2)^2} - \frac{|x-p'|^2}{1-k^2}} - \sqrt{\frac{b^2}{(1-k^2)^2} - \frac{|\bar{x}-p'|^2}{1-k^2}} \\
&= \frac{|x-p'|^2 - |\bar{x}-p'|^2}{(1-k^2)\left(\sqrt{\frac{b^2}{(1-k^2)^2} - \frac{|x-p'|^2}{1-k^2}} + \sqrt{\frac{b^2}{(1-k^2)^2} - \frac{|\bar{x}-p'|^2}{1-k^2}}\right)} \\
&= \frac{2\left\langle \xi - p', x - \bar{x}\right\rangle}{(1-k^2)\left(\sqrt{\frac{b^2}{(1-k^2)^2} - \frac{|x-p'|^2}{1-k^2}} + \sqrt{\frac{b^2}{(1-k^2)^2} - \frac{|\bar{x}-p'|^2}{1-k^2}}\right)} \\
&\leq \frac{2|\xi - p'||x-\bar{x}|}{(1-k^2)\sqrt{\frac{b^2}{(1-k^2)^2} - \frac{|\bar{x}-p'|^2}{1-k^2}}}
\end{aligned}$$

for some $\xi \in [x, \bar{x}]$. By assumption (A), $x, \bar{x} \in B_{\delta p_n \sqrt{1-\kappa^2}/k}(p') \subseteq B_{\delta b/\sqrt{1-k^2}}(p')$ and hence, we have $|\xi - p'| \leq \dfrac{\delta b}{\sqrt{1-k^2}}$ and also $|\bar{x} - p'|^2 \leq \dfrac{\delta^2 b^2}{1-k^2}$, and therefore, we get $u(x) - u(\bar{x}) \leq \dfrac{2\delta}{(1-k^2)\sqrt{1-\delta}}|x - \bar{x}|$. Interchanging the roles of x and $\bar{x}$ yields the result. □

Definition 2.3. *Given a parallel refractor $u(x)$ for $x \in \Omega$, the refractor mapping of u is the multi-valued map defined for $x_0 \in \overline{\Omega}$ by*

$$\mathcal{N}_u(x_0) = \{p \in \overline{\Sigma} : \ \phi_{p,b} \text{ touches } u \text{ from above at } x_0 \text{ for some } b \geq \frac{p_n(1-k^2)}{k}\}.$$

Given $p \in \overline{\Sigma}$, the tracing mapping of u is defined by

$$\mathcal{T}_u(p) = \mathcal{N}_u^{-1}(p) = \{x \in \overline{\Omega} : \ p \in \mathcal{N}_u(x)\}.$$

The singular set of u is defined by

$$S_u = \{x \in \bar{\Omega} : \text{there exist } p, q \in \Sigma \text{ such that } p \neq q \text{ and } p, q \in \mathcal{N}_u(x)\},$$

and as usual, this set has Lebesgue measure zero, [Gut01, Lemma 1.1.12]. To see this in the present case, we observe first that if $E_{p,b}$ and $E_{\bar{p},\bar{b}}$ are two ellipsoids

given by (2.1) such that $E_{\bar{p},\bar{b}} \subseteq E_{p,b}$, and they touch at some point x, then it follows that $v := \dfrac{x-p}{|x-p|} = \bar{v} := \dfrac{x-\bar{p}}{|x-\bar{p}|}$ and hence $p, \bar{p}$, and x are on a line. Indeed, from the equation of the normals at x we have that $v + \kappa\, e_n = \lambda\,(\bar{v} + \kappa\, e_n)$ for some $\lambda > 0$. So $v = \lambda\bar{v} + (\lambda - 1)\kappa e_n$, taking norms and since $\kappa < 1$ we obtain that $\lambda = 1$ and we are done. This together with Lemma 2.2 and the visibility condition (B) yields that $|S_u| = 0$ as desired. Then as in [GH09, Lemma 3.5], this implies that the class of sets $\mathcal{C} = \{F \subset \overline{\Sigma} : \mathcal{T}_u(F) \text{ is Lebesgue measurable}\}$ is a Borel σ-algebra in $\overline{\Sigma}$.

Given a nonnegative $f \in L^1(\Omega)$, we then obtain as in [GH09, Lemma 3.6] that the set function

$$\mathcal{M}_{u,f}(F) = \int_{\mathcal{T}_u(F)} f\, dx$$

is a finite Borel measure defined on $\mathcal{C}$, which we call it the parallel refractor measure associated with u and f.

Lemma 2.4. *Let $G \subseteq \Sigma$ be open and $\bar{G} \subseteq \Sigma$. Assume $u_m \longrightarrow u$ uniformly in $\bar{\Omega}$, where u_m, u are parallel refractors. Then $\mathcal{T}_u(G) \setminus S_u \subseteq \liminf_{m\to\infty} \mathcal{T}_{u_m}(G)$.*

Proof. Suppose not and let $\bar{x} \in \mathcal{T}_u(G) \setminus S_u$ such that $\bar{x} \notin \liminf_{m\to\infty} \mathcal{T}_{u_m}(G)$. Since $\bar{x} \notin S_u$, there exists a unique $\bar{p} \in \mathcal{N}_u(\bar{x})$, $\bar{p} \in G$, and $u \le \phi_{\bar{p},b}$ in $\bar{\Omega}$ with equality at $\bar{x}$ for some b.

Since $\bar{x} \notin \liminf_{m\to\infty} \mathcal{T}_{u_m}(G)$, there is a subsequence m_k such that $\bar{x} \notin \mathcal{T}_{u_{m_k}}(G)$. Hence, $\bar{x} \notin \mathcal{T}_{u_{m_k}}(q)$ for all $q \in G$ or equivalently, $q \notin \mathcal{N}_{u_{m_k}}(\bar{x})$ for all $q \in G$, and for all m_k's.

Let $p_{m_k} \in \mathcal{N}_{u_{m_k}}(\bar{x})$, then $p_{m_k} \in \Sigma \setminus G$, which is a compact set. Hence, we may assume, passing through a subsequence, that $p_{m_k} \to p_0$, $p_0 \in \Sigma \setminus G$ and we may also assume $b_{m_k} \to b_0$, as $k \to \infty$. But, since $u_m \longrightarrow u$ uniformly in $\bar{\Omega}$, we will have $u \le \phi_{p_0,b_0}$ in $\bar{\Omega}$ with equality at $\bar{x}$. This means that $p_0 \in \mathcal{N}_u(\bar{x})$, but $p_0 \ne \bar{p}$ since $\bar{p} \in G$, a contradiction with the uniqueness of $\bar{p}$. □

3. Main results

We construct in this section the surfaces that refract collimated radiation in a prescribed way.

Lemma 3.1. *Let $p_i \in \Sigma$ be distinct points, $p_i = (p_1^i, \cdots, p_n^i) = (p_i', p_n^i)$, and $b_1, ..., b_N$ be such that $b_i \ge \dfrac{p_n^i(1-k^2)}{k}$, $i = 1, ..., N$, and $\Omega \subseteq \bigcap_{i=1}^N B_{\delta p_n^i \sqrt{1-\kappa^2}/\kappa}(p_i')$*. Define u in Ω by*

$$u(x) = \min_{1\le i\le N} \phi_{p_i,b_i}(x).$$

*This inclusion follows from condition (A).

Then

$$\mathcal{M}_{u,f}(\{p_1,...,p_N\}) = \Sigma_{i=1}^N \mathcal{M}_{u,f}(\{p_i\}) = \int_\Omega f(x)\,dx.$$

Proof. Let $S_i = \{x \in \bar{\Omega} : \exists q \neq p_i, q \in \mathcal{N}_u(x)\}$, $1 \leq i \leq N$, and $S_u = \{x \in \bar{\Omega} : \exists p, q \in \mathcal{N}_u(x),\ q \neq p\}$. We write $\bar{\Omega} = \bigcup_{i=1}^N \mathcal{T}_u(p_i) = \bigcup_{i=1}^N (\mathcal{T}_u(p_i) \setminus S_i) \bigcup \bigcup_{i=1}^N (\mathcal{T}_u(p_i) \cap S_i)$. We have $\bigcup_{i=1}^N (\mathcal{T}_u(p_i) \cap S_i) \subset S_u$ and $(\mathcal{T}_u(p_i) \setminus S_i) \cap (\mathcal{T}_u(p_j) \setminus S_j)$ for $i \neq j$. The result then follows since $|S_i| = 0$, $i = 1,...,N$, and $|S_u| = 0$. □

Lemma 3.2. *Let $p_i \in \Sigma$ be distinct points, $p_i = (p_1^i, \cdots, p_n^i) = (p_i', p_n^i)$, and $b_1,...,b_N$ be such that $b_i \geq \dfrac{p_n^i(1-k^2)}{k}$, $i = 1,...,N$, and $\Omega \subseteq \bigcap_{i=1}^N B_{\delta p_n^i \sqrt{1-\kappa^2}/\kappa}(p_i')$.*

Let $\epsilon > 0$ and define u and u_ϵ in Ω by

$$u(x) = \min_{1\leq i\leq N} \phi_{p_i,b_i}(x), \qquad \text{and} \qquad u_\epsilon(x) = \min\{\phi_{p_1,b_1+\epsilon}(x), \phi_{p_i,b_i}(x) : i = 2,...,N\}.$$

Then $\mathcal{T}_{u_\epsilon}(p_i) \setminus S_{u_\epsilon} \subseteq \mathcal{T}_u(p_i)$ for $i \neq 1$, and $\limsup_{\epsilon\to 0} \mathcal{T}_{u_\epsilon}(p_1) \subseteq \mathcal{T}_u(p_1)$. Similarly, if b_1 is replaced by b_j, then the first conclusion holds for $i \neq j$ and the second for p_j instead of p_1.

Proof. Let $\bar{x} \in \mathcal{T}_{u_\epsilon}(p_i) \setminus S_{u_\epsilon}$, $i \neq 1$, then $u_\epsilon(\bar{x}) = \phi_{p_i,b_i}(\bar{x})$. Since $\phi_{p_1,b_1+\epsilon} \leq \phi_{p_1,b_1}$, we have $u_\epsilon(x) \leq u(x)$ and so $\phi_{p_i,b_i}(\bar{x}) = u(\bar{x})$.

If $\bar{x} \in \limsup_{\epsilon\to 0} \mathcal{T}_{u_\epsilon}(p_1)$, then for all $\epsilon > 0$, there exists $0 < \beta < \epsilon$ such that $\bar{x} \in \mathcal{T}_{u_\beta}(p_1)$. That is, there exists b_β such that $u_\beta(x) \leq \phi_{p_1,b_\beta}(x)$ with equality at $\bar{x}$. Passing through a subsequence $\beta_\beta \to \bar{b} > 0$ as $\beta \to 0$, and so $u(x) \leq \phi_{p_1,\bar{b}}(x)$ with equality at $\bar{x}$, that is, $\bar{x} \in \mathcal{T}_u(p_1)$.

□

We are now in a position to prove the existence theorem when the target is a set of points.

Theorem 3.3. *Let $p_i \in \Sigma$, $i = 1,...,N$ be distinct points as in Lemma 3.2, and $a_i > 0$ such that $\Sigma_{i=1}^N a_i = \int_\Omega f(x)\,dx$.*

Then there exists $u : \bar{\Omega} \to [-M, 0]$ a parallel refractor such that $\mathcal{M}_{u,f}(\{p_i\}) = a_i$ for $i = 1,...,N$ and such that if $E \subseteq \Sigma$ and $E \cap \{p_1,...,p_N\} = \emptyset$, then $\mathcal{M}_{u,f}(E) = 0$.

Proof. For simplicity in the notation we write $\mathcal{M}_u$ instead $\mathcal{M}_{u,f}$.

We say $b = (b_1,...,b_N)$ is admisible if $b_i \geq \dfrac{p_n^i(1-k^2)}{k}$ for $i = 1,...,N$. For each admisible b define

$$u_b(x) = \min_{1\leq i\leq N} \phi_{p_i,b_i}(x),$$

and set

$$\bar{b}_1 = \frac{1-k^2}{k}\left(p_n^1 + \frac{1}{k}\max_{2\le i\le N} p_n^i\right). \tag{3.3}$$

Clearly, $(\bar{b}_1, b_2, \cdots, b_N)$ is admissible when $(b_1, b_2, \cdots, b_N)$ is admissible. Define the set

$$W = \left\{(b_2, ..., b_N) : b_i \ge \frac{p_n^i(1-k^2)}{k},\ \mathcal{M}_{u_b}(\{p_i\}) \le a_i,\ i = 2, ..., N\right\},$$

where u_b is defined with $b = (\bar{b}_1, b_2, ..., b_N)$.

Claim 1: $W \neq \emptyset$.

Indeed, with the choice $b_i = \dfrac{p_n^i(1-k^2)}{k}$, $i = 2, ..., N$, we have that $\max\{\phi_{p_1,\bar{b}_1}(x) : x \in \bar{\Omega}\} \le p_n^1 - \dfrac{\kappa\,\bar{b}}{1-\kappa^2} \le -\dfrac{b_i}{1-\kappa^2} = \phi_{p_i,b_i}(p_i') = \min\{\phi_{p_i,b_i}(x) : x \in B_{b/\sqrt{1-\kappa^2}}(p_i')\}$ for each $i = 2, ..., N$. Therefore, $\phi_{p_1,\bar{b}_1}(x) \le \phi_{p_i,b_i}(x)$ in Ω and hence $u_b(x) = \phi_{p_1,\bar{b}_1}(x)$ for all $x \in \bar{\Omega}$, which implies that $\mathcal{M}_{u_b}(\{p_i\}) = 0$ for $i = 2, ..., N$.

Claim 2: W is bounded

We shall prove that if $b_j \ge \dfrac{(1-k^2)(1+k)^2 C_2}{k^3}$, for some $2 \le j \le N$, where C_2 is the constant in (2.2), then $(b_2, ..., b_N) \notin W$. We have that

$$\begin{aligned} b_j \ge \frac{(1-k^2)(1+k)^2 C_2}{k^3} &= \frac{1-k^2}{k}\left(C_2 + \frac{(1+k)}{k^2}C_2 + \frac{1}{k}C_2\right) \\ &\ge \left(\frac{1-k^2}{k}\right)\left(p_n^j + \frac{(1+k)}{k^2}\max_{2\le i\le N} p_n^i + \frac{1}{k}p_n^1\right), \end{aligned}$$

which implies that

$$\begin{aligned} \max\{\phi_{p_j,b_j}(x) : x \in \bar{\Omega}\} &\le p_n^j - \frac{kb_j}{1-k^2} \\ &\le p_n^1 - \frac{(1+k)\bar{b}_1}{1-k^2} = \phi_{p_1,\bar{b}_1}(p_1') \le \min\{\phi_{p_1,\bar{b}_1}(x) : x \in \bar{\Omega}\}. \end{aligned}$$

Therefore, $u_b(x) = \min_{2\le i\le N} \phi_{p_i,b_i}(x)$, and so $\mathcal{M}_{u_b}(\{p_1\}) = 0$. Suppose by contradiction that $(b_2, ..., b_N) \in W$. Then $\mathcal{M}_{u_b}(\{p_i\}) \le a_i$, for $i = 2, ..., N$. But, by Lemma 3.1 we have $\int_\Omega f(x)\,dx = \mathcal{M}_{u_b}(\{p_1, ..., p_N\}) = \Sigma_{i=1}^N \mathcal{M}_{u_b}(\{p_i\}) = \Sigma_{i=2}^N \mathcal{M}_{u_b}(\{p_i\}) \le \Sigma_{i=2}^N a_i < \int_\Omega f(x)\,dx$, a contradiction.

Claim 3: W is closed

Let $(b_2^m, ..., b_N^m) \in W$ such that $(b_2^m, ..., b_N^m) \to (\bar{b}_2, ..., \bar{b}_N)$ as $m \to \infty$. Set $b_m = (\bar{b}_1, b_2^m, ..., b_N^m)$ and $\bar{b} = (\bar{b}_1, \bar{b}_2, ..., \bar{b}_N)$.

We have that $u_{b_m} \longrightarrow u_{\bar{b}}$ uniformly in $\bar{\Omega}$. We claim that $\mathcal{M}_{u_{\bar{b}}}(\{p_i\}) \le a_i$, for $i = 2, ..., N$. Without loss of generality we may assume $i = 2$. Let G be open in Σ such that $p_2 \in G$ and $p_i \notin G$ for $i \neq 2$. Then $\mathcal{M}_{u_{b_m}}(G) = \mathcal{M}_{u_{b_m}}(\{p_2\}) \le a_2$ for

all m. From Lemma 2.4 we have that $\mathcal{T}_{u_{\bar{b}}}(G) \setminus S_{u_{\bar{b}}} \subseteq \liminf_{m\to\infty} \mathcal{T}_{u_{b_m}}(G)$, and so $\mathcal{M}_{u_{\bar{b}}}(\{p_2\}) \leq \mathcal{M}_{u_{\bar{b}}}(G) = \mathcal{M}_{u_{\bar{b}}}(G \setminus S_{u_{\bar{b}}}) \leq \liminf_{m\to\infty} \mathcal{M}_{u_{b_m}}(G) \leq a_2$ and Claim 3 is proved.

Define the function $\psi : W \longrightarrow [0,\infty)$ by $\psi(b_2, ..., b_N) = b_2 + ... + b_N$. Since W is a compact set ψ attains its maximum at some point $(\bar{b}_2, ..., \bar{b}_N) \in W$. Set $\bar{b} = (\bar{b}_1, \bar{b}_2, ..., \bar{b}_N)$, with $\bar{b}_1$ from (3.3). We shall prove that $\mathcal{M}_{u_{\bar{b}}}(\{p_i\}) = a_i$ for $i = 1, 2, ..., N$.

Since $(\bar{b}_2, ..., \bar{b}_N) \in W$, we have $\mathcal{M}_{u_{\bar{b}}}(\{p_i\}) \leq a_i$ for $i = 2, ..., N$. Suppose that for some $i \geq 2$, $\mathcal{M}_{u_{\bar{b}}}(\{p_i\}) < a_i$, say $\mathcal{M}_{u_{\bar{b}}}(\{p_2\}) < a_2$. Let $\bar{b}_\epsilon = (\bar{b}_1, \bar{b}_2 + \epsilon, ..., \bar{b}_N)$. Then, by the second assertion of Lemma 3.2, $\mathcal{M}_{u_{\bar{b}_\epsilon}}(\{p_2\}) < a_2$ for ϵ sufficiently small. Also from the first assertion of Lemma 3.2, we have $\mathcal{T}_{u_{\bar{b}_\epsilon}}(\{p_i\}) \setminus S_{u_{\bar{b}_\epsilon}} \subseteq \mathcal{T}_{u_{\bar{b}}}(\{p_i\})$ for $i \neq 2$. Therefore $\mathcal{M}_{u_{\bar{b}_\epsilon}}(\{p_i\}) \leq a_i$ for $i = 2, ..., N$ and hence $(\bar{b}_2 + \epsilon, ..., \bar{b}_N) \in W$ contradicting that ψ has a maximum at $(\bar{b}_2, ..., \bar{b}_N)$. Therefore $\mathcal{M}_{u_{\bar{b}}}(\{p_i\}) = a_i$ for $i = 2, ..., N$. By Lemma 3.1 we have $\Sigma_{i=1}^N \mathcal{M}_{u_{\bar{b}}}(\{p_i\}) = \int_\Omega f(x)\,dx = \Sigma_{i=1}^N a_i$, and therefore we get $\mathcal{M}_{u_{\bar{b}}}(\{p_1\}) = a_1$. This proves the claim.

We also notice that if $E \subseteq \Sigma$ such that $E \cap \{p_1, ..., p_N\} = \emptyset$ and $x \in \mathcal{T}_{u_{\bar{b}}}(E)$ then either $x \in \partial\Omega$ or $u_{\bar{b}}$ is not differentiable at x. Since $u_{\bar{b}}$ is Lipschitz in $\bar{\Omega}$ we have that $\mathcal{M}_{u_{\bar{b}}}(E) = 0$.

We also notice that $u_{\bar{b}} \leq 0$ in $\bar{\Omega}$.

Also, recall that from the proof of Claim 2 above, if $b_i \geq \dfrac{(1-k^2)(1+k)^2 C_2}{k^3}$, for some $2 \leq i \leq N$, then $(b_2, ..., b_N) \notin W$. Notice that for such b_i, we have that $\min\{\psi_{p_i,b_i}(x) : x \in \bar{\Omega}\} = p_n^i - \dfrac{(k+1)b_i}{1-k^2} \geq C_1 - \left(\dfrac{k+1}{k}\right)^3 C_2 = -M$, the constant defined in condition (B) at the outset. Hence $u_{\bar{b}} \geq -M$ in $\bar{\Omega}$. □

For the general case when the distribution of energy to receive is given by a measure we have the following.

Theorem 3.4. *Let μ be a Borel measure on Σ and $f \in L^1(\Omega)$ such that $\mu(\Sigma) = \int_\Omega f(x)\,dx$. There exists a function $u : \Omega \longrightarrow [-M, 0]$ that is a parallel refractor and $\mathcal{M}_{u,f} = \mu$.*

Proof. Let $\mu_m \to \mu$ weakly such that $\mu_m = \Sigma_{i=1}^{N_m} a_{i_m} \delta_{p_{i_m}}$ and such that $\Sigma_{i=1}^{N_m} a_{i_m} = \int_\Omega f(x)\,dx$ for all m.

From Theorem 3.3, let u_m be a solution of $\mathcal{M}_{u_m,f} = \mu_m$. From Lemma 2.2, the sequence $\{u_m\}$ is uniformly Lipschitz in $\bar{\Omega}$, and $-M \leq u_m \leq 0$ in $\bar{\Omega}$ for all m. Therefore, there exists a subsequence $u_{m_j} \longrightarrow \bar{u}$ uniformly in $\bar{\Omega}$ and hence $\mu_{m_j} = \mathcal{M}_{u_{m_j},f} \to \mathcal{M}_{\bar{u},f}$ weakly, and also $\mu_{m_j} \to \mu$ weakly. Hence $\mathcal{M}_{\bar{u},f} = \mu$.

□

Lemma 3.5. *Let u_b and $u_{\bar{b}}$ be two solutions as in Theorem 3.3 with $b = (b_1, ..., b_N)$ and $\bar{b} = (\bar{b}_1, ..., \bar{b}_N)$. If $b_1 \leq \bar{b}_1$, then $b_i \leq \bar{b}_i$ for $i = 2, ..., N$. Moreover, if $u_b(x_0) = u_{\bar{b}}(x_0)$ at some $x_0 \in \Omega$, then $u_b \equiv u_{\bar{b}}$.*

Proof. Let $J = \{j : b_j > \bar{b}_j\}$ and $I = \{i : b_i \leq \bar{b}_i\}$. Suppose $J \neq \emptyset$. For $j \in J$ we have $\phi_{b_j,p_j} < \phi_{\bar{b}_j,p_j}$ in $\bar{\Omega}$, and for $i \in I$ we have $\phi_{b_i,p_i} \geq \phi_{\bar{b}_i,p_i}$ in $\bar{\Omega}$.

Fix $j \in J$ and let $\bar{x} \in \mathcal{T}_{u_{\bar{b}}}(p_j)$. It follows that $u_{\bar{b}}(\bar{x}) = \phi_{\bar{b}_j,p_j}(\bar{x})$. And hence $\phi_{\bar{b}_j,p_j}(\bar{x}) \leq \phi_{\bar{b}_i,p_i}(\bar{x})$ for all $i \in I$ which implies that $\phi_{b_j,p_j}(\bar{x}) < \phi_{\bar{b}_j,p_j}(\bar{x}) \leq \phi_{\bar{b}_i,p_i}(\bar{x}) \leq \phi_{b_i,p_i}(\bar{x})$ for all $i \in I$. By continuity, there exists $\epsilon > 0$ such that for all $x \in B_\epsilon(\bar{x})$, $\phi_{b_j,p_j}(x) < \phi_{b_i,p_i}(x)$ for all $i \in I$ and this implies that for $x \in B_\epsilon(\bar{x})$, $u_b(x) = \min\{\phi_{b_j,p_j}(x) : j \in J\}$. This means that $B_\epsilon(\bar{x}) \subseteq \mathcal{T}_{u_b}(\{p_j : j \in J\})$. So we have shown that $\mathcal{T}_{u_{\bar{b}}}(\{p_j : j \in J\}) \subseteq (\mathcal{T}_{u_b}(\{p_j : j \in J\}))^\circ$. Since $\mathcal{T}_{u_{\bar{b}}}(\{p_j : j \in J\})$ is closed, then we obtain that $(\mathcal{T}_{u_b}(\{p_j : j \in J\}))^\circ \setminus \mathcal{T}_{u_{\bar{b}}}(\{p_j : j \in J\})$ is a non empty open set. Since u_b and $u_{\bar{b}}$ are solutions, we have $\int_{\mathcal{T}_{u_{\bar{b}}}(\{p_j : j \in J\})} f(x), dx = \int_{\mathcal{T}_{u_b}(\{p_j : j \in J\})} f(x)\, dx = \Sigma_{j \in J} a_j$, a contradiction.

If $b_1 = \bar{b}_1$, then $b_j = \bar{b}_j$ for all $j > 1$ from the first part, and we are done. We claim that if $b_1 > \bar{b}_1$, then $b_j > \bar{b}_j$ for all $j > 1$. Indeed, if $b_j = \bar{b}_j$ for some $j \neq 1$, then $b_k = \bar{b}_k$ for all $k \neq j$ by the first part, a contradiction. Therefore $u_b(x_0) = \min_{1 \leq i \leq N} \phi_{p_j,b_j}(x_0) < \min_{1 \leq i \leq N} \phi_{p_j,\bar{b}_j}(x_0) = u_{\bar{b}}(x_0)$, a contradiction. □

Theorem 3.6. *There exists a constant $-\beta < 0$ depending on C_1, C_2 and k such that if $x_0 \in \bar{\Omega}$ and $t \leq -\beta$, then there exists a parallel refractor u as in Theorem 3.3 satisfying $u(x_0) = t$.*

Proof. To obtain a solution passing through a given point we can modify the proof of Theorem 3.3 as follows.

We consider $\bar{b}_1 \geq \dfrac{(1-k^2)}{k}\left(p_n^1 + \dfrac{1}{k}\max_{2 \leq i \leq N} p_n^i\right)$ and we assume the visibility condition (B) holds on $\bar{\Omega} \times (-\infty, 0]$.

We claim that for each such $\bar{b}_1$ we can obtain a solution denoted $u_{\bar{b}_1}$ with the property that

$$\begin{aligned}&\frac{(1+k)}{k}p_n^1 + \min_{2 \leq i \leq N} p_n^i - \frac{(1+k)}{k}\max_{2 \leq i \leq N} p_n^i - \frac{(1+k)^2}{k(1-k^2)}\bar{b}_1 \\ &\leq u_{\bar{b}_1}(x) \leq \phi_{p_1,\bar{b}_1}(x)\end{aligned}$$

in $\bar{\Omega}$. This follows just as in the proof of Theorem 3.3 defining the set W in the same way and noticing that if $b_i \geq \dfrac{(1-k^2)}{k}\left(\max_{2 \leq i \leq N} p_n^i - p_n^1 + \dfrac{(1+k)}{1-k^2}\bar{b}_1\right)$, for $i = 2, ..., N$, then $(b_2, ..., b_N) \notin W$. Since the solution is of the form u_b with $b = (\bar{b}_1, b_2, ..., b_N)$ and $(b_2, ..., b_N) \in W$, it follows that $\min_{2 \leq i \leq N} \phi_{p_i,b_i}(x) \leq u_b(x) \leq \phi_{p_1,\bar{b}_1}(x)$,

where $b_i = \dfrac{(1-k^2)}{k}\left(\max_{2\le i\le N} p_{i_n} - p_{1_n} + \dfrac{(1+k)}{1-k^2}\bar{b}_1\right)$ and since $\min_{2\le i\le N}\phi_{p_i,b_i}(x) \ge \dfrac{(1+k)}{k}p_n^1 + \min_{2\le i\le N} p_n^1 - \dfrac{(1+k)}{k}\max_{2\le i\le N} p_{i_n} - \dfrac{(1+k)^2}{k(1-k^2)}\bar{b}_1$ the claim follows.

With $\bar{b}_1 = \dfrac{(1-k^2)}{k}\left(p_n^1 + \dfrac{1}{k}\max_{2\le i\le N} p_n^i\right)$, let $-\beta = \dfrac{(1+k)}{k}p_n^1 + \min_{2\le i\le N} p_n^i - \dfrac{(1+k)}{k}\max_{2\le i\le N} p_n^i - \dfrac{(1+k)^2}{k(1-k^2)}\bar{b}_1$. That is $-\beta = -\dfrac{(1+k)}{k^2}p_n^1 + \min_{2\le i\le N} p_n^i - \dfrac{(1+k)}{k^2}\max_{2\le i\le N} p_n^i$.

Given a point (x_0, t) with $x_0 \in \bar{\Omega}$ and $t \le -\beta$, we use continuity of the solution $u_{\bar{b}_1}$ in the parameter $\bar{b}_1$ to show that for some $\bar{b}_1 \ge \dfrac{(1-k^2)}{k}\left(p_n^1 + \dfrac{1}{k}\max_{2\le i\le N} p_n^i\right)$, we have $u_{\bar{b}_1}(x_0) = t$. Indeed, if $\bar{b}_1 = \dfrac{(1-k^2)}{k}\left(p_n^1 + \dfrac{1}{k}\max_{2\le i\le N} p_n^i\right)$, then $u_{\bar{b}_1}(x_0) \ge -\beta \ge t$; while if $\bar{b}_1$ is large enough, then we will have $u_{\bar{b}_1}(x_0) \le \phi_{p_1,\bar{b}_1}(x_0) \le t$.

□

References

[GH09] C. E. Gutiérrez and Qingbo Huang, *The refractor problem in reshaping light beams*, Arch. Rational Mech. Anal. **193** (2009), no. 2, 423–443.

[Gut01] C. E. Gutiérrez, *The Monge–Ampère equation*, Birkhäuser, Boston, MA, 2001.

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