REFRACTION PROBLEMS IN GEOMETRIC OPTICS
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1. Introduction

These are notes expanding the material of a mini course I taught on Monge-Ampère type equations and geometric optics in June 2012 at Cetraro. The purpose of these lectures was to explain basic facts about the Monge-Ampère equation and the application of these ideas to pose and solve problems in geometric optics concerning refraction with input and output energies. These problems are basically of two types: the far field and the near field. In the far field case the goal is to send radiation into a set of directions and in the second is to send radiation to a specific target set. The far field case can be treated with optimal transportation methods, Section 3, but the near field does not. Instead in this case, a method based on the Minkowski method in convex geometry can be used. In fact, with this method one can treat both far and near fields problems. The method is illustrated in Section 4 in the far field case. Refraction and reflection occur simultaneously, in other words, if an incident ray is refracted there is always a percentage of the incident ray that is internally reflected. The fractions of energy refracted and reflected are given by the Fresnel formulas. Beginning with Maxwell's equations this is developed and explained in Section 5. The application to the refraction problem with loss of energy is described in Subsection 5.13.

These notes include the some of my work with Qingbo Huang and Henok Mawi, and the relevant references are [GH09], [GH08], and [GH13]. The equations describing the solutions to these problems are Monge-Ampère type equations whose derivation is included in [GH08], and [GH13].

The notes are organized as follows. Section 2 contains the Snell law of refraction in vector form. This is simple but essential for the developments of the results. In particular, we introduce in this section the notion of surfaces having the uniform refraction property used later to give the definition of refractors. Section 3 contains basic facts about optimal mass transport and its application to solve the refractor problem in the far field case. Section 4 introduces a method to solve the far field refractor problem that has its roots in the Minkowski method. Section 5 contains a detailed development of Maxwell's equations, the so called boundary conditions explaining the propagation across two different materials, and its application to deduce the Fresnel formulas. These formulas are written in a convenient form that is used finally to propose, in Subsection 5.13, a model for the refractor problem with loss of energy in internal reflection.

It is a pleasure to thank the CIME for their support in the organization of this series of lectures and in particular, to Elvira Mascolo and Pietro Zecca for their great support and help that made this series possible.

2. Snell's law of refraction

2.1. **In vector form.** Suppose Γ is a surface in $\mathbb{R}^3$ that separates two media I and II that are homogeneous and isotropic. Let v_1 and v_2 be the velocities of propagation of light in the media I and II respectively. The index of refraction of the medium I is by definition $n_1 = c/v_1$, where c is the velocity of propagation of light in the

vacuum, and similarly $n_2 = c/v_2$. If a ray of light* having direction $x \in S^2$ and traveling through the medium I hits Γ at the point P, then this ray is refracted in the direction $m \in S^2$ through the medium II according with the Snell law in vector form:

$$n_1(x \times \nu) = n_2(m \times \nu), \tag{2.1}$$

where ν is the unit normal to the surface to Γ at P going towards the medium II. The derivation of this formula is in Subsection 2.2. This has several consequences:

(a) the vectors x, m, ν are all on the same plane (called plane of incidence);
(b) the well known Snell law in scalar form

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where θ_1 is the angle between x and ν (the angle of incidence), θ_2 the angle between m and ν (the angle of refraction),

From (2.1), $(n_1 x - n_2 m) \times \nu = 0$, which means that the vector $n_1 x - n_2 m$ is parallel to the normal vector ν. If we set $\kappa = n_2/n_1$, then

$$x - \kappa\, m = \lambda \nu, \tag{2.2}$$

for some $\lambda \in \mathbb{R}$. Taking dot products we get $\lambda = \cos\theta_1 - \kappa\cos\theta_2$, $\cos\theta_1 = x \cdot \nu > 0$, and $\cos\theta_2 = m \cdot \nu = \sqrt{1 - \kappa^{-2}[1 - (x \cdot \nu)^2]}$, so

$$\lambda = x \cdot \nu - \kappa\, \sqrt{1 - \kappa^{-2}(1 - (x \cdot \nu)^2}. \tag{2.3}$$

Refraction behaves differently for $\kappa < 1$ and for $\kappa > 1$.

2.1.1. $\kappa < 1$. This means $v_1 < v_2$, and so waves propagate in medium II faster than in medium I, or equivalently, medium I is denser than medium II. In this case the refracted rays tend to bent away from the normal, that is the case for example, when medium I is glass and medium II is air. Indeed, from the scalar Snell law, $\sin\theta_1 = \kappa \sin\theta_2 < \sin\theta_2$ and so $\theta_1 < \theta_2$. For this reason, the maximum angle of refraction θ_2 is $\pi/2$ which, from the Snell law in scalar form, is achieved when $\sin\theta_1 = n_2/n_1 = \kappa$. So there cannot be refraction when the incidence angle θ_1 is beyond this critical value, that is, we must have $0 \le \theta_1 \le \theta_c = \arcsin\kappa$.† Once again from the Snell law in scalar form,

$$\theta_2 - \theta_1 = \arcsin(\kappa^{-1}\sin\theta_1) - \theta_1 \tag{2.4}$$

and it is easy to verify that this quantity is strictly increasing for $\theta_1 \in [0,\ \theta_c]$, and therefore $0 \le \theta_2 - \theta_1 \le \dfrac{\pi}{2} - \theta_c$. We then have $x \cdot m = \cos(\theta_2 - \theta_1) \ge \cos(\pi/2 - \theta_c) = \kappa$, and therefore obtain the following physical constraint for refraction:

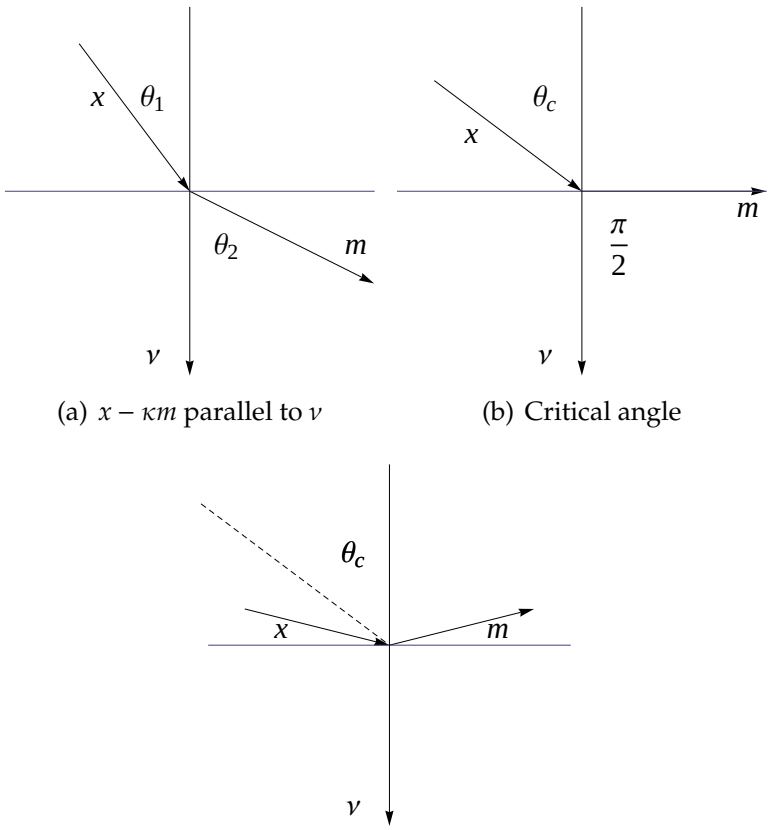
if $\kappa = n_2/n_1 < 1$ and a ray of direction x through medium I
is refracted into medium II in the direction m, then $m \cdot x \ge \kappa$. (2.5)

*Since the refraction angle depends on the frequency of the radiation, we assume radiation is monochromatic.

†If $\theta_1 > \theta_c$, then the phenomenon of total internal reflection occurs, see Figure 1(c).

Notice also that in this case $\lambda > 0$ in (2.3).

Conversely, given $x, m \in S^2$ with $x \cdot m \geq \kappa$ and $\kappa < 1$, it follows from (2.4) that there exists a hyperplane refracting any ray through medium I with direction x into a ray of direction m in medium II.



(a) $x - \kappa m$ parallel to ν (b) Critical angle

(c) Internal reflection

FIGURE 1. Snell's law $\kappa < 1$, e.g., glass to air

2.1.2. $\kappa > 1$. In this case, waves propagate in medium I faster than in medium II, and the refracted rays tend to bent towards the normal. By the Snell law, the maximum angle of refraction denoted by θ_c^* is achieved when $\theta_1 = \pi/2$, and $\theta_c^* = \arcsin(1/\kappa)$. Once again from the Snell law in scalar form

$$\theta_1 - \theta_2 = \arcsin(\kappa \sin \theta_2) - \theta_2 \tag{2.6}$$

which is strictly increasing for $\theta_2 \in [0,\ \theta_c^*]$, and $0 \leq \theta_1 - \theta_2 \leq \dfrac{\pi}{2} - \theta_c^*$. We therefore obtain the following physical constraint for the case $\kappa > 1$:

if a ray with direction x traveling through medium I
is refracted into a ray in medium II with direction m, then $m \cdot x \geq 1/\kappa$. (2.7)

Notice also that in this case $\lambda < 0$ in (2.3).

On the other hand, by (2.6), if $x, m \in S^2$ with $x \cdot m \geq 1/\kappa$ and $\kappa > 1$, then there exists a hyperplane refracting any ray of direction x through medium I into a ray with direction m in medium II.

We summarize the above discussion on the physical constraints of refraction in the following lemma.

Lemma 2.1. *Let n_1 and n_2 be the indices of refraction of two media I and II, respectively, and $\kappa = n_2/n_1$. Then a light ray in medium I with direction $x \in S^2$ is refracted by some surface into a light ray with direction $m \in S^2$ in medium II if and only if $m \cdot x \geq \kappa$, when $\kappa < 1$; and if and only if $m \cdot x \geq 1/\kappa$, when $\kappa > 1$.*

2.1.3. $\kappa = 1$. This corresponds to reflection. It means

$$x - m = \lambda\, \nu. \tag{2.8}$$

Taking dot products with x and then with m yields $1 - m \cdot x = \lambda\, x \cdot \nu$ and $x \cdot m - 1 = \lambda\, m \cdot \nu$, then $x \cdot \nu = -m \cdot \nu$. Also taking dot product with x in (2.8) then yields $\lambda = 2\, x \cdot \nu$. Therefore

$$m = x - 2(x \cdot \nu)\nu.$$

2.2. **Derivation of the Snell law.** At time t, $\psi(x, y, z, t) = 0$ denotes a surface that separates the part of the space that is at rest with the part of the space that is disturbed by the electric and magnetic fields. For each t fixed the surface defined by $\psi(x, y, z, t) = 0$ is called a *wave front*. The *light rays* are the orthogonal trajectories to the wave fronts. We assume that $\psi_t \neq 0$ and so we can solve $\psi(x, y, z, t) = 0$ in t obtaining that

$$\phi(x, y, z) = c\, t,$$

where c is the speed of light in vacuum. Therefore, it t runs, then we get that the wave fronts are the level sets of the function $\phi(x, y, z)$.

Let us assume that the wave fronts travel in an homogenous and isotropic medium I with refractive index $n_1 (= c/v_1)$, v_1 is the speed of propagation in medium I. This wave front is transmitted to another homogeneous and isotropic medium II having refractive index n_2. Let Σ be the surface in 3-d separating the media I and II, and suppose it is given by the equations $x = f(\xi, \eta)$, $y = g(\xi, \eta)$ and $z = h(\xi, \eta)$. Let $\phi_1(x, y, z) = c\, t$ be the wave front in medium I and $\phi_2(x, y, z) = c\, t$ be the wave front, to be determined, in medium II. On the surface Σ the two wave fronts agree, that is,

$$\phi_1\left(f(\xi, \eta), g(\xi, \eta), h(\xi, \eta)\right) = \phi_2\left(f(\xi, \eta), g(\xi, \eta), h(\xi, \eta)\right).$$

Differentiating this equation with respect to ξ and η yields

$$\left(\frac{\partial \phi_1}{\partial x} - \frac{\partial \phi_2}{\partial x}\right) f_\xi + \left(\frac{\partial \phi_1}{\partial y} - \frac{\partial \phi_2}{\partial y}\right) g_\xi + \left(\frac{\partial \phi_1}{\partial z} - \frac{\partial \phi_2}{\partial z}\right) h_\xi = 0$$

$$\left(\frac{\partial \phi_1}{\partial x} - \frac{\partial \phi_2}{\partial x}\right) f_\eta + \left(\frac{\partial \phi_1}{\partial y} - \frac{\partial \phi_2}{\partial y}\right) g_\eta + \left(\frac{\partial \phi_1}{\partial z} - \frac{\partial \phi_2}{\partial z}\right) h_\eta = 0.$$

This means that the vector $D\phi_1 - D\phi_2$ is perpendicular to both vectors (f_ξ, g_ξ, h_ξ) and (f_η, g_η, h_η), and therefore it is normal to the surface Σ. Let ν be the outer normal at the surface Σ. Then we have

$$\Big(D\phi_1 - D\phi_2\Big) = \lambda\,\nu, \tag{2.9}$$

for some scalar λ. A light ray $\gamma_1(t)$ in medium I has constant speed v_1 and a light ray $\gamma_2(t)$ in II constant speed v_2. Say $\gamma_1(t)$ is the light ray in medium I and $\gamma_2(t)$ the light ray in medium II. So we have $\phi_1(\gamma_1(t)) = c\,t$ and $\phi_2(\gamma_2(t)) = c\,t$. Differentiating with respect to t yields $D\phi_i(\gamma_i(t)) \cdot \gamma_1'(t) = c$, $i = 1, 2$. Let θ_i be the angle between the vectors $D\phi_i(\gamma_i(t)), \gamma_1'(t)$. Since the rays are orthogonal trajectories we get that $\theta_i = 0$ for $i = 1, 2$. If γ_i is parametrized so that $|\gamma_i'(t)| = v_i$, we then obtain that

$$|D\phi_i(\gamma_i(t))| = \frac{c}{v_i} = n_i.$$

If we let $x = \dfrac{D\phi_1(\gamma_1(t))}{|D\phi_1(\gamma_1(t))|}$ and $m = \dfrac{D\phi_2(\gamma_2(t))}{|D\phi_2(\gamma_2(t))|}$, we then obtain from (2.9) that

$$n_1\,x - n_2\,m = \lambda\,\nu$$

which is equivalent to (2.1).

2.3. **Surfaces with the uniform refracting property: far field case.** Let $m \in S^2$ be fixed, and we ask the following: if rays of light emanate from the origin inside medium I, what is the surface Γ, interface of the media I and II, that refracts all these rays into rays parallel to m?

Suppose Γ is parameterized by the polar representation $\rho(x)x$ where $\rho > 0$ and $x \in S^2$. Consider a curve on Γ given by $r(t) = \rho(x(t))\,x(t)$ for $x(t) \in S^2$. According to (2.2), the tangent vector $r'(t)$ to Γ satisfies $r'(t) \cdot (x(t) - \kappa\,m) = 0$. That is, $([\rho(x(t))]'x(t) + \rho(x(t))x'(t)) \cdot (x(t) - \kappa\,m) = 0$, which yields $(\rho(x(t))(1 - \kappa m \cdot x(t)))' = 0$. Therefore

$$\rho(x) = \frac{b}{1 - \kappa\,m \cdot x} \tag{2.10}$$

for $x \in S^2$ and for some $b \in \mathbb{R}$. To understand the surface given by (2.10), we distinguish two cases $\kappa < 1$ and $\kappa > 1$.

2.3.1. *Case* $\kappa = 1$. When $\kappa = 1$ we see this is a paraboloid. Indeed, let $m = -e_n$, then a point $X = \rho(x)x$ is on the surface (2.10) if $|X| = b - x_n$. The distance from X to the plane $x_n = b$ is $b - x_n$, and the distance from X to 0 is $|X|$. So this is a paraboloid with focus at 0, directrix plane $x_n = b$ and axis in the direction $-e_n$.

2.3.2. *Case* $\kappa < 1$. For $b > 0$, we will see that the surface Γ given by (2.10) is an ellipsoid of revolution about the axis of direction m. Suppose for simplicity that $m = e_n$, the nth-coordinate vector. If $y = (y', y_n) \in \mathbb{R}^n$ is a point on Γ, then $y = \rho(x)x$

with $x = y/|y|$. From (2.10), $|y| - \kappa\, y_n = b$, that is, $|y'|^2 + y_n^2 = (\kappa\, y_n + b)^2$ which yields $|y'|^2 + (1-\kappa^2)y_n^2 - 2\kappa b y_n = b^2$. This surface Γ can be written in the form

$$\frac{|y'|^2}{\left(\dfrac{b}{\sqrt{1-\kappa^2}}\right)^2} + \frac{\left(y_n - \dfrac{\kappa b}{1-\kappa^2}\right)^2}{\left(\dfrac{b}{1-\kappa^2}\right)^2} = 1 \tag{2.11}$$

which is an ellipsoid of revolution about the y_n axis with foci $(0,0)$ and $(0, 2\kappa b/(1-\kappa^2))$. Since $|y| = \kappa y_n + b$ and the physical constraint for refraction (2.5), $\dfrac{y}{|y|} \cdot e_n \geq \kappa$ is equivalent to $y_n \geq \dfrac{\kappa b}{1-\kappa^2}$. That is, for refraction to occur y must be in the upper part of the ellipsoid (2.11); we denote this semi-ellipsoid by $E(e_n, b)$, see Figure 2. To verify that $E(e_n, b)$ has the uniform refracting property, that is, it refracts any ray emanating from the origin in the direction e_n, we check that (2.2) holds at each point. Indeed, if $y \in E(e_n, b)$, then $\left(\dfrac{y}{|y|} - \kappa e_n\right)\dfrac{y}{|y|} \geq 1 - \kappa > 0$, and $\left(\dfrac{y}{|y|} - \kappa e_n\right) \cdot e_n \geq 0$, and so $\dfrac{y}{|y|} - \kappa e_n$ is an outward normal to $E(e_n, b)$ at y.

Rotating the coordinates, it is easy to see that the surface given by (2.10) with $\kappa < 1$ and $b > 0$ is an ellipsoid of revolution about the axis of direction m with foci 0 and $\dfrac{2\kappa b}{1-\kappa^2} m$. Moreover, the semi-ellipsoid $E(m, b)$ given by

$$E(m,b) = \{\rho(x)x : \rho(x) = \frac{b}{1 - \kappa\, m \cdot x},\ x \in S^{n-1},\ x \cdot m \geq \kappa\}, \tag{2.12}$$

has the uniform refracting property, any ray emanating from the origin O is refracted in the direction m.

2.3.3. *Case* $\kappa > 1$. Due to the physical constraint of refraction (2.7), we must have $b < 0$ in (2.10). Define for $b > 0$

$$H(m,b) = \{\rho(x)x : \rho(x) = \frac{b}{\kappa\, m \cdot x - 1},\ x \in S^{n-1},\ x \cdot m \geq 1/\kappa\}. \tag{2.13}$$

We claim that $H(m, b)$ is the sheet with opening in direction m of a hyperboloid of revolution of two sheets about the axis of direction m. To prove the claim, set for simplicity $m = e_n$. If $y = (y', y_n) \in H(e_n, b)$, then $y = \rho(x)x$ with $x = y/|y|$. From (2.13), $\kappa\, y_n - |y| = b$, and therefore $|y'|^2 + y_n^2 = (\kappa\, y_n - b)^2$ which yields $|y'|^2 - (\kappa^2 - 1)\left[\left(y_n - \dfrac{\kappa b}{\kappa^2 - 1}\right)^2 - \left(\dfrac{\kappa b}{\kappa^2 - 1}\right)^2\right] = b^2$. Thus, any point y on $H(e_n, b)$

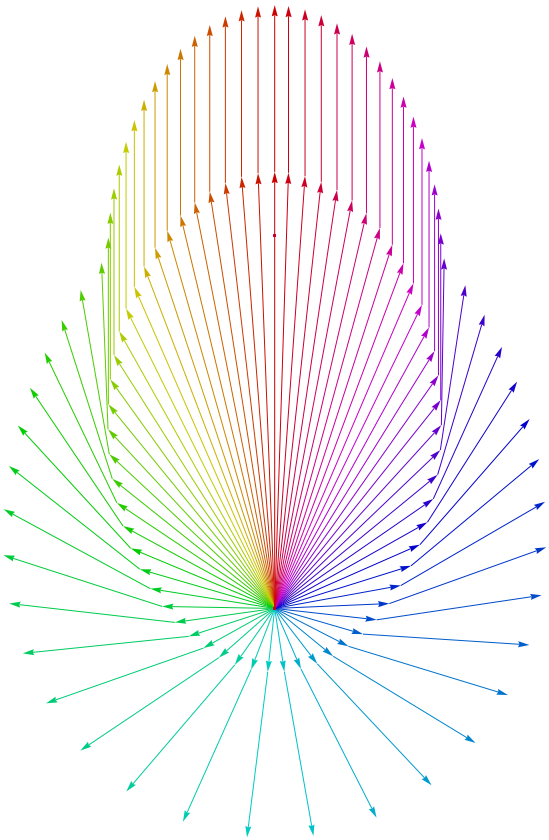


FIGURE 2. Only half of the ellipse refracts in the direction $m = e_n$

satisfies the equation

$$\frac{\left(y_n - \dfrac{\kappa b}{\kappa^2 - 1}\right)^2}{\left(\dfrac{b}{\kappa^2 - 1}\right)^2} - \frac{|y'|^2}{\left(\dfrac{b}{\sqrt{\kappa^2 - 1}}\right)^2} = 1 \tag{2.14}$$

which represents a hyperboloid of revolution of two sheets about the y_n axis with foci $(0, 0)$ and $(0, 2\kappa b/(\kappa^2 - 1))$. Moreover, the upper sheet of this hyperboloid of revolution is given by

$$y_n = \frac{\kappa b}{\kappa^2 - 1} + \frac{b}{\kappa^2 - 1}\sqrt{1 + \frac{|y'|^2}{\left(b/\sqrt{\kappa^2 - 1}\right)^2}}$$

and satisfies $\kappa y_n - b > 0$, and hence has polar equation $\rho(x) = \dfrac{b}{\kappa\, e_n \cdot x - 1}$. Similarly, the lower sheet satisfies $\kappa y_n - b < 0$ and has polar equation $\rho(x) = \dfrac{b}{\kappa\, e_n \cdot x + 1}$. For a general m, by a rotation, we obtain that $H(m, b)$ is the sheet with opening in direction m of a hyperboloid of revolution of two sheets about the axis of direction m with foci $(0, 0)$ and $\dfrac{2\kappa b}{\kappa^2 - 1} m$.

Notice that the focus $(0, 0)$ is outside the region enclosed by $H(m, b)$ and the focus $\dfrac{2\kappa b}{\kappa^2 - 1} m$ is inside that region. The vector $\kappa m - \dfrac{y}{|y|}$ is an inward normal to $H(m, b)$ at y, because by (2.13)

$$\begin{aligned}\left(\kappa m - \frac{y}{|y|}\right) \cdot \left(\frac{2\kappa b}{\kappa^2 - 1} m - y\right) &\geq \frac{2\kappa^2 b}{\kappa^2 - 1} - \frac{2\kappa b}{\kappa^2 - 1} - \kappa m \cdot y + |y| \\ &= \frac{2\kappa b}{\kappa + 1} - b = \frac{b(\kappa - 1)}{\kappa + 1} > 0.\end{aligned}$$

Clearly, $\left(\kappa m - \dfrac{y}{|y|}\right) \cdot m \geq \kappa - 1$ and $\left(\kappa m - \dfrac{y}{|y|}\right) \cdot \dfrac{y}{|y|} > 0$. Therefore, $H(m, b)$ satisfies the uniform refraction property.

We remark that one has to use $H(-e_n, b)$ to uniformly refract in the direction $-e_n$, and due to the physical constraint (2.7), the lower sheet of the hyperboloid of equation (2.14) cannot refract in the direction $-e_n$.

From the above discussion, we have proved the following.

Lemma 2.2. *Let n_1 and n_2 be the indexes of refraction of two media I and II, respectively, and $\kappa = n_2/n_1$. Assume that the origin O is inside medium I, and $E(m, b), H(m, b)$ are defined by* (2.12) *and* (2.13)*, respectively. We have:*

(i) *If $\kappa < 1$ and $E(m, b)$ is the interface of media I and II, then $E(m, b)$ refracts all rays emitted from O into rays in medium II with direction m.*
(ii) *If $\kappa > 1$ and $H(m, b)$ separates media I and II, then $H(m, b)$ refracts all rays emitted from O into rays in medium II with direction m.*

2.4. **Uniform refraction: near field case.** The question we ask is: given a point O inside medium I and a point P inside medium II, find an interface surface $\mathcal{S}$ between media I and II that refracts all rays emanating from the point O into the point P. Suppose O is the origin, and let $X(t)$ be a curve on $\mathcal{S}$. By the Snell law of refraction the tangent vector $X'(t)$ satisfies

$$X'(t) \cdot \left(\frac{X(t)}{|X(t)|} - \kappa \frac{P - X(t)}{|P - X(t)|}\right) = 0.$$

That is,

$$|X(t)|' + \kappa |P - X(t)|' = 0.$$

Therefore $\mathcal{S}$ is the Cartesian oval

$$|X| + \kappa|X - P| = b. \tag{2.15}$$

Since $f(X) = |X| + \kappa|X - P|$ is a convex function, the oval is a convex set.

We need to find and analyze the polar equation of the oval. Write $X = \rho(x)x$ with $x \in S^{n-1}$. Then writing $\kappa|\rho(x)x - P| = b - \rho(x)$, squaring this quantity and solving the quadratic equation yields

$$\rho(x) = \frac{(b - \kappa^2 x \cdot P) \pm \sqrt{(b - \kappa^2 x \cdot P)^2 - (1 - \kappa^2)(b^2 - \kappa^2|P|^2)}}{1 - \kappa^2}. \tag{2.16}$$

Set

$$\Delta(t) = (b - \kappa^2 t)^2 - (1 - \kappa^2)(b^2 - \kappa^2|P|^2). \tag{2.17}$$

2.5. **Case** $0 < \kappa < 1$**.** We have

$$\Delta(x \cdot P) > \kappa^2(x \cdot P - b)^2, \qquad \text{if } |x \cdot P| < |P|. \tag{2.18}$$

If $b \geq |P|$, then O and P are inside or on the oval, and so the oval cannot refract rays to P. If the oval is non empty, then $\kappa|P| \leq b$. In case $\kappa|P| = b$, the oval reduces to the point O. The only interesting case is then $\kappa|P| < b < |P|$. From the equation of the oval we get that $\rho(x) \leq b$. So we now should decide which values $\pm$ to take in the definition of $\rho(x)$. Let ρ_+ and ρ_- be the corresponding ρ's. We claim that $\rho_+(x) > b$ and $\rho_-(x) \leq b$. Indeed,

$$\begin{aligned}
\rho_+(x) &= \frac{(b - \kappa^2 x \cdot P) + \sqrt{\Delta(x \cdot P)}}{1 - \kappa^2} \\
&\geq \frac{(b - \kappa^2 x \cdot P) + \kappa|b - x \cdot P|}{1 - \kappa^2} \\
&= b + \frac{\kappa^2(b - x \cdot P) + \kappa|b - x \cdot P|}{1 - \kappa^2} \\
&\geq b.
\end{aligned}$$

The equality $\rho_+(x) = b$ holds only if $|x \cdot P| = |P|$ and $b = x \cdot P$. So $\rho_+(x) > b$ if $\kappa|P| < b < |P|$. Similarly,

$$\begin{aligned}
\rho_-(x) &= \frac{(b - \kappa^2 x \cdot P) - \sqrt{\Delta(x \cdot P)}}{1 - \kappa^2} \\
&\leq \frac{(b - \kappa^2 x \cdot P) - \kappa|b - x \cdot P|}{1 - \kappa^2} \\
&= b + \frac{\kappa^2(b - x \cdot P) - \kappa|b - x \cdot P|}{1 - \kappa^2} \\
&\leq b.
\end{aligned}$$

So the claim is proved. Therefore the polar equation of the oval is then given by

$$h(x, P, b) = \rho_-(x) = \frac{(b - \kappa^2 x \cdot P) - \sqrt{\Delta(x \cdot P)}}{1 - \kappa^2}. \tag{2.19}$$

From the physical constraint for refraction we must have $x\cdot\left(\dfrac{P-h(x,P,b)x}{|P-h(x,P,b)x|}\right)\geq\kappa$, and from the equation of the oval we then get that to have refraction we need

$$x\cdot P\geq b. \tag{2.20}$$

Concluding with the case $\kappa<1$, given $P\in\mathbb{R}^n$ and $\kappa|P|<b<|P|$, keeping in mind (2.19) and (2.20), a refracting oval is the set

$$\mathcal{O}(P,b)=\{h(x,P,b)\,x : x\in S^{n-1}, x\cdot P\geq b\}$$

where

$$h(x,P,b)=\frac{(b-\kappa^2x\cdot P)-\sqrt{(b-\kappa^2x\cdot P)^2-(1-\kappa^2)(b^2-\kappa^2|P|^2)}}{1-\kappa^2}.$$

(a) $|X|+2/3|X-P|=1.4-1.9$, $P=(2,0)$ (b) $|X|+2/3|X-P|=1.7$, $P=(2,0)$

FIGURE 3. Cartesian ovals $\kappa<1$, e.g., glass to air

Remark 2.3. If $|P|\to\infty$, then the oval converges to an ellipsoid which is the surface having the uniform refraction property in the far field case, see Subsection

2.3.2. In fact, if $m = P/|P|$ and $b = \kappa|P| + C$ with C positive constant we have

$$\begin{aligned} h(x,P,b) &= \frac{b^2 - \kappa^2|P|^2}{b - \kappa^2 x\cdot P + \sqrt{\Delta(x\cdot P)}} \\ &= \frac{C(2\kappa|P| + C)}{(\kappa|P| - \kappa^2 x\cdot m|P| + C) + \sqrt{(\kappa|P| - \kappa^2 x\cdot m|P| + C)^2 - (1-\kappa^2)C(2\kappa|P| + C)}} \\ &\to \frac{2\kappa C}{(\kappa - \kappa^2 x\cdot m) + \sqrt{(\kappa - \kappa^2 x\cdot m)^2}} = \frac{C}{1 - \kappa x\cdot m} \end{aligned}$$

as $|P| \to \infty$.

2.6. **Case $\kappa > 1$.** In this case we must have $|P| \le b$, and in case $b = |P|$ the oval reduces to the point P. Also $b < \kappa|P|$, since otherwise the points $0, P$ are inside the oval or 0 is on the oval, and therefore there cannot be refraction if $b \ge \kappa|P|$. So to have refraction we must have $|P| < b < \kappa|P|$ and so the point P is inside the oval and 0 is outside the oval.

Rewriting ρ in (2.16) we get that

$$\rho_\pm(x) = \frac{(\kappa^2 x\cdot P - b) \pm \sqrt{(\kappa^2 x\cdot P - b)^2 - (\kappa^2-1)(\kappa^2|P|^2 - b^2)}}{\kappa^2 - 1}$$

for $\Delta(x\cdot P) \ge 0$ which amounts $x\cdot P \ge \dfrac{b + \sqrt{(\kappa^2-1)(\kappa^2|P|^2 - b^2)}}{\kappa^2}$. Notice that $\rho_\pm(x) < 0$ for $\kappa^2 x\cdot P - b < 0$. We have that $\rho_-(x) \le \rho_+(x) \le \dfrac{(\kappa^2|P| - b) + \sqrt{\Delta(|P|)}}{\kappa^2 - 1} = \dfrac{\kappa|P| + b}{\kappa + 1} < b$. To have refraction, by the physical constraint we need to have $x\cdot \dfrac{P - x\rho_\pm(x)}{|P - x\rho_\pm(x)|} \ge 1/\kappa$, which is equivalent to $\kappa^2 x\cdot P - b \ge (\kappa^2-1)\rho_\pm(x)$. Therefore, the physical constraint is satisfied only by ρ_-.

Therefore if $\kappa > 1$, refraction only occurs when $|P| < b < \kappa|P|$, and the refracting piece of the oval is then given by

$$\mathcal{O}(P,b) = \left\{ h(x,P,b)x : x\cdot P \ge \frac{b + \sqrt{(\kappa^2-1)(\kappa^2|P|^2 - b^2)}}{\kappa^2} \right\} \tag{2.21}$$

with

$$h(x,P,b) = \rho_-(x) = \frac{(\kappa^2 x\cdot P - b) - \sqrt{(\kappa^2 x\cdot P - b)^2 - (\kappa^2-1)(\kappa^2|P|^2 - b^2)}}{\kappa^2 - 1}. \tag{2.22}$$

Remark 2.4. If $|P| \to \infty$, then the oval $\mathcal{O}(P,b)$ converges to the semi hyperboloid appearing in the far field refraction problem when $\kappa > 1$, see Subsection 2.3.3. Indeed, let $m = \dfrac{P}{|P|} \in S^{n-1}$ and $b = \kappa|P| - a$ with $a > 0$ a constant. For $x \in \Gamma(P,b)$ we

have

$$\frac{b+\sqrt{(\kappa^2-1)(\kappa^2|P|^2-b^2)}}{\kappa^2|P|}=\frac{\kappa|P|-a+\sqrt{(\kappa^2-1)(\kappa^2|P|^2-(\kappa|P|-a)^2)}}{\kappa^2|P|}\to\frac{1}{\kappa}$$

as $|P|\to\infty$. On the other hand, if $x\cdot m>1/\kappa$, we get

$$h(x,P,b)=\frac{a(2\kappa|P|-a)}{(\kappa^2|P|x\cdot m-\kappa|P|+a)+\sqrt{(\kappa^2|P|x\cdot m-\kappa|P|+a)^2-(\kappa^2-1)a(2\kappa|P|-a)}}$$
$$\to\frac{a2\kappa}{\kappa^2x\cdot m-\kappa+\sqrt{(\kappa^2x\cdot m-\kappa)^2}}=\frac{a}{\kappa x\cdot m-1},$$

as $|P|\to\infty$.

(a) $|X|+3/2|X-P|=2.9-2.4$, $P=(2,0)$ (b) $|X|+3/2|X-P|=2.7$, $P=(2,0)$

FIGURE 4. Cartesian ovals $\kappa>1$, e.g., air to glass

3. Optimal mass transportation

Let D, D^* be two domains on S^{n-1} or domains in a manifold (D might be contained in one manifold and D^* in another) with $|\partial D|=0$.

Let $\mathcal{N}$ be a multi-valued mapping from $\overline{D}$ onto $\overline{D^*}$ such that $\mathcal{N}(x)$ is single-valued a.e. on $\overline{D}$. We also assume that $|\{x\in\bar{D}:\mathcal{N}(x)=\emptyset\}|=0$. For $F\subset\overline{D^*}$, we set

$$\mathcal{T}(F)=\mathcal{N}^{-1}(F)=\{x\in\overline{D}:\mathcal{N}(x)\cap F\neq\emptyset\}.$$

We say $\mathcal{N}$ is measurable if $\mathcal{T}(F)$ is Lebesgue measurable for any Borel set $F\subset\overline{D^*}$.

For example, if $\mathcal{N} = \partial u$ with u convex, then $\mathcal{N}$ is measurable. Because for u convex we have $m \in \partial u(x_0)$ iff $x_0 \in \partial u^*(m)$, where u^* is the Legendre transformation of u (see exercises). Therefore $(\partial u)^{-1}(F) = \partial u^*(F)$, and since $\partial u^*(F)$ is Lebesgue measurable for each Borel set F, we get that ∂u is measurable.

Suppose $g \in L^1(D)$ is nonnegative and Γ on $\overline{D^*}$ is a finite Radon measure satisfying the conservation condition

$$\int_{\overline{D}} g(x)\,dx = \Gamma(\overline{D^*}) > 0. \tag{3.1}$$

Notice that the set function $\mu(F) = \int_{\mathcal{T}(F)} g(x)\,dx$ is Borel measure because $\mathcal{N}$ is single valued a.e., and measurable. Indeed, if S is the set of measure zero such that $\mathcal{N}(x)$ is single valued for all $x \in \bar{D} \setminus S$, and F_j is a sequence of disjoint Borel sets, then $|\mathcal{T}(F_i) \cap \mathcal{T}(F_j)| = 0$ for $i \neq j$, then μ is σ-additive. Therefore μ is a finite Borel measure and so is regular.

We say $\mathcal{N}$ is measure preserving from $g(x)dx$ to Γ if for any Borel $F \subset \overline{D^*}$

$$\int_{\mathcal{T}(F)} g(x)\,dx = \Gamma(F). \tag{3.2}$$

Lemma 3.1. *$\mathcal{N}$ is a measure preserving mapping from $g(x)dx$ to Γ if and only if for any $v \in C(\overline{D^*})$*

$$\int_{\overline{D}} v(\mathcal{N}(x))g(x)\,dx = \int_{\overline{D^*}} v(m)\,d\Gamma(m). \tag{3.3}$$

We remark that $v(\mathcal{N}(x))$ is well defined for $x \in \overline{D} \setminus S$ where $\mathcal{N}(x)$ is single-valued on $\overline{D} \setminus S$ and $|S| = 0$, and $\int_{\overline{D}} v(\mathcal{N}(x))g(x)\,dx$ is understood as $\int_{\overline{D}\setminus S} v(\mathcal{N}(x))g(x)\,dx$.

Proof. Let $\mathcal{N}$ be a measure preserving mapping. To show (3.3), it suffices to prove it for $v = \chi_F$, the characteristic function of a Borel set F, because for each v continuous there exists a sequence of simple functions converging uniformly to v. It is easy to verify that $\chi_{\mathcal{T}(F)}(x) = \chi_F(\mathcal{N}(x))$ for $x \in \overline{D} \setminus S$. Therefore by (3.2)

$$\int_{\overline{D^*}} \chi_F(m)\,d\Gamma = \int_{\mathcal{T}(F)\cap(\overline{D}\setminus S)} g\,dx = \int_{\overline{D}\setminus S} \chi_F(\mathcal{N}(x))g(x)\,dx.$$

To prove the converse, assume that (3.3) holds. We will show that for any Borel set $E \subset \overline{D^*}$

$$\int_{\mathcal{T}(E)} g\,dx \leq \Gamma(E). \tag{3.4}$$

Indeed, let us first assume that $E = G$ is open, then given a compact set $K \subset G$, choose $v \in C(\overline{D^*})$ such that $0 \leq v \leq 1$, $v = 1$ on K, and $v = 0$ outside G. By (3.3), one gets

$$\int_{\mathcal{T}(K)} g(x)\,dx \leq \int_{\overline{D}} v(\mathcal{N}(x))g(x)\,dx \leq \Gamma(G),$$

for each compact $K \subset G$. Since μ is regular, (3.4) follows for E open. For a general Borel set $E \subset \overline{D^*}$, since Γ is also regular, given $\epsilon > 0$ there exists G open $E \subset G$ with $\Gamma(G \setminus E) < \epsilon$. Then

$$\int_{\mathcal{T}(E)} g(x)\,dx \le \int_{\mathcal{T}(G)} g(x)\,dx \le \Gamma(G) = \Gamma(E) + \Gamma(G \setminus E) < \Gamma(E) + \epsilon$$

and so (3.4) follows.

We next prove that equality holds in (3.4).

First notice that

$$\{x \in \bar{D} : \mathcal{N}(x) \neq \emptyset\} \cap (\mathcal{T}(F))^c \subset \mathcal{T}(F^c),$$

for any set $F \subset \bar{D}^*$. Then applying (3.4) to $\overline{D^*} \setminus F$ with F Borel set yields

$$\int_{\{x\in\bar{D}:\mathcal{N}(x)\neq\emptyset\}\cap(\mathcal{T}(F))^c} g(x)\,dx \le \int_{\mathcal{T}(F^c)} g(x)\,dx \le \Gamma(\overline{D^*} \setminus F) = \Gamma(\overline{D^*}) - \Gamma(F).$$

Since $|\{x \in \bar{D} : \mathcal{N}(x) = \emptyset\}| = 0$, we have

$$\int_{\{x\in\bar{D}:\mathcal{N}(x)\neq\emptyset\}\cap(\mathcal{T}(F))^c} g(x)\,dx = \int_{(\mathcal{T}(F))^c} g(x)\,dx = \int_D g(x)\,dx - \int_{\mathcal{T}(F)} g(x)\,dx.$$

So from the conservation condition (3.1), we obtain the reverse inequality in (3.4). □

Consider the general cost function $c(x, m) \in Lip(\overline{D} \times \overline{D^*})$, the space of Lipschitz functions on $\overline{D} \times \overline{D^*}$, and *the set of admissible functions*

$$\mathcal{K} = \{(u, v) : u \in C(\overline{D}), v \in C(\overline{D^*}), u(x) + v(m) \le c(x, m),\ \forall x \in D, \forall m \in D^*\}.$$

Define the *dual functional* I for $(u, v) \in C(\overline{D}) \times C(\overline{D^*})$

$$I(u, v) = \int_D u(x)g(x)\,dx + \int_{\overline{D^*}} v(m)\,d\Gamma,$$

and define the *c- and c*- transforms*

$$u^c(m) = \inf_{x\in\overline{D}} [c(x, m) - u(x)], \quad m \in \overline{D^*}; \qquad v_c(x) = \inf_{m\in\overline{D^*}} [c(x, m) - v(m)], \quad x \in \overline{D}.$$

Definition 3.2. *A function $\phi \in C(\overline{D})$ is c-concave if for $x_0 \in \overline{D}$, there exist $m_0 \in \overline{D^*}$ and $b \in \mathbb{R}$ such that $\phi(x) \le c(x, m_0) - b$ on $\overline{D}$ with equality at $x = x_0$.*

Obviously v_c is c-concave for any $v \in C(\overline{D^*})$. We collect the following properties:

(1) For any $u \in C(\overline{D})$ and $v \in C(\overline{D^*})$, $v_c \in Lip(\overline{D})$ and $u^c \in Lip(\overline{D^*})$ with Lipschitz constants bounded uniformly by the Lipschitz constant of c. Indeed, (x_0 the point where the minimum is attained)

$$\begin{aligned} u^c(m_1) - u^c(m_2) &\le u^c(m_1) - (c(x_0, m_2) - u(x_0)) \\ &\le c(x_0, m_1) - u(x_0) - c(x_0, m_2) + u(x_0) \le K|m_1 - m_2|. \end{aligned}$$

(2) If $(u, v) \in \mathcal{K}$, then $v(m) \le u^c(m)$ and $u(x) \le v_c(x)$. Also $(v_c, v), (u, u^c) \in \mathcal{K}$.

(3) ϕ is c-concave iff $\phi = (\phi^c)_c$.

Indeed, if $\phi(x) \le c(x, m_0) - b$ on $\overline{D}$ and the equality holds at $x = x_0$, then $b = \phi^c(m_0)$. So $\phi(x_0) = c(x_0, m_0) - \phi^c(m_0)$ which yields $\phi(x_0) \ge (\phi^c)_c(x_0)$. On the other hand, from the definitions of c and c^* transforms we always have that $(\phi^c)_c \ge \phi$ for any ϕ.

Definition 3.3. *Given a function $\phi(x)$, the c-normal mapping of ϕ is defined by*

$$\mathcal{N}_{c,\phi}(x) = \{m \in \overline{D^*} : \phi(x) + \phi^c(m) = c(x, m)\}, \qquad \text{for } x \in \overline{D},$$

and $\mathcal{T}_{c,\phi}(m) = \mathcal{N}_{c,\phi}^{-1}(m) = \{x \in \overline{D} : m \in \mathcal{N}_{c,\phi}(x)\}$.

We assume that the cost function $c(x, m)$ satisfies the following:

(3.5) For any c-concave function ϕ, $\mathcal{N}_{c,\phi}(x)$ is single-valued a.e. on $\overline{D}$ and $\mathcal{N}_{c,\phi}$ is Lebesgue measurable.

Notice that if $c(x, m) = x \cdot m$, then $\mathcal{N}_{c,\phi}(x) = \partial^*\phi(x)$, where $\partial^*\phi$ is the super-differential of ϕ

$$\partial^*\phi(x) = \{m \in \mathbb{R}^n : \phi(y) \le \phi(x) + m \cdot (y - x) \,\forall\, y \in \Omega\},$$

and we have $\partial^*\phi(x) = -\partial(-\phi)(x)$.

Lemma 3.4. *Suppose that $c(x, m)$ satisfies the assumption* (3.5). *Then*

(i) *If ϕ is c-concave and $\mathcal{N}_{c,\phi}$ is measure preserving from $g(x)dx$ to Γ, then (ϕ, ϕ^c) is a maximizer of $I(u, v)$ in $\mathcal{K}$.*

(ii) *If $\phi(x)$ is c-concave and (ϕ, ϕ^c) maximizes $I(u, v)$ in $\mathcal{K}$, then $\mathcal{N}_{c,\phi}$ is measure preserving from $g(x)dx$ to Γ.*

Proof. First prove (i). Given $(u, v) \in \mathcal{K}$, obviously

$$u(x) + v(\mathcal{N}_{c,\phi}(x)) \le c(x, \mathcal{N}_{c,\phi}(x)) = \phi(x) + \phi^c(\mathcal{N}_{c,\phi}(x)), \qquad \text{a.e. } x \text{ on } \overline{D}.$$

Integrating the above inequality with respect to $g\,dx$ yields

$$\int_{\overline{D}} ug\,dx + \int_{\overline{D}} v(\mathcal{N}_{c,\phi}(x))g(x)\,dx \le \int_{\overline{D}} \phi g\,dx + \int_{\overline{D}} \phi^c(\mathcal{N}_{c,\phi}(x))g(x)\,dx.$$

By Lemma 3.1, it yields $I(u, v) \le I(\phi, \phi^c)$ and from (2) above the conclusion follows.

To prove (ii), let $\psi = \phi^c$, and for $v \in C(\overline{D^*})$, let $\psi_\theta(m) = \psi(m) + \theta\, v(m)$ where $-\epsilon_0 < \theta \le \epsilon_0$ with ϵ_0 small, and let $\phi_\theta = (\psi_\theta)_c$. We shall prove that

$$\lim_{\theta \to 0} \frac{I(\phi_\theta, \psi_\theta) - I(\phi, \psi)}{\theta} = \int_{\overline{D}} -v(\mathcal{N}_{c,\phi}(x))\, g\,dx + \int_{\overline{D^*}} v(m)\,d\Gamma. \tag{3.6}$$

Since $(\phi_\theta, \psi_\theta) \in \mathcal{K}$, we have $I(\phi_\theta, \psi_\theta) \le I(\phi, \psi)$ for $-\epsilon_0 < \theta \le \epsilon_0$, and hence the existence of the limit (3.6) implies it must be zero. Therefore the measure preserving property of $N_{c,\phi}$ follows from Lemma 3.1.

To prove (3.6) we write

$$\frac{I(\phi_\theta, \psi_\theta) - I(\phi, \psi)}{\theta} = \int_{\overline{D}} \frac{\phi_\theta - \phi}{\theta}\, g\,dx + \int_{\overline{D^*}} v(m)\,d\Gamma.$$

By Lebesgue dominated convergence theorem, to show (3.6), it is enough to show that $\dfrac{\phi_\theta(x)-\phi(x)}{\theta}$ is uniformly bounded, and $\dfrac{\phi_\theta(x)-\phi(x)}{\theta} \to -v(\mathcal{N}_{c,\phi}(x))$ for all $x \in D \setminus S$, where $\mathcal{N}_{c,\phi}(x)$ is single-valued on $D \setminus S$ and $|S| = 0$. Let us first prove the uniform boundedness. Fix $x \in \bar{D}$, we have by continuity that $\phi_\theta(x) = c(x, m_\theta) - \psi_\theta(m_\theta)$ for some $m_\theta \in \overline{D^*}$. Since ϕ is c-concave there exists $m_1 \in \overline{D^*}$ and $b \in \mathbb{R}$ such that $\phi(y) \le c(y, m_1) - b$ for all $y \in \bar{D}$ with equality when $y = x$. This implies that $b = \phi^c(m_1)$ and so $\phi(x) = c(x, m_1) - \psi(m_1)$. Hence

$$\begin{aligned}\phi_\theta(x) - \phi(x) &= c(x, m_\theta) - \psi(m_\theta) - \theta\, v(m_\theta) - \phi(x)\\ &\ge \psi_c(x) - \theta\, v(m_\theta) - \phi(x) = (\phi^c)_c(x) - \theta\, v(m_\theta) - \phi(x) \ge -\theta\, v(m_\theta),\end{aligned}$$

by (3) above. We also have

$$\begin{aligned}\phi_\theta(x) - \phi(x) &= \phi_\theta(x) - c(x, m_1) + \psi(m_1) = \phi_\theta(x) - c(x, m_1) + \psi_\theta(m_1) - \theta\, v(m_1)\\ &\le \phi_\theta(x) - (\psi_\theta)_c(x) - \theta\, v(m_1) = -\theta\, v(m_1).\end{aligned}$$

Then we get

$$-\theta\, v(m_\theta) \le \phi_\theta(x) - \phi(x) \le -\theta\, v(m_1).$$

Moreover, if $x \in \bar{D} \setminus S$, then $m_1 = \mathcal{N}_{c,\phi}(x)$ since $\psi = \phi^c$. To finish the proof, we show that m_θ converges to m_1 as $\theta \to 0$. Otherwise, there exists a sequence m_{θ_k} such that $m_{\theta_k} \to m_\infty \ne m_1$. So $\phi(x) = \lim_{\theta\to 0} \phi_\theta(x) = c(x, m_\infty) - \psi(m_\infty)$, which yields $m_\infty \in \mathcal{N}_{c,\phi}(x)$. We then get $m_1 = m_\infty$, a contradiction. The proof is complete. □

Lemma 3.5. *There exists a c-concave ϕ such that*

$$I(\phi, \phi^c) = \sup\{I(u, v) : (u, v) \in \mathcal{K}\}.$$

Proof. Let

$$I_0 = \sup\{I(u, v) : (u, v) \in \mathcal{K}\},$$

and let $(u_k, v_k) \in \mathcal{K}$ be a sequence such that $I(u_k, v_k) \to I_0$. Set $\bar{u}_k = (v_k)_c$ and $\bar{v}_k = (\bar{u}_k)^c$. From property (2) above, $(\bar{u}_k, \bar{v}_k) \in \mathcal{K}$, $u_k \le \bar{u}_k$, $v_k \le \bar{v}_k$, and so $I(\bar{u}_k, \bar{v}_k) \to I_0$. Let $c_k = \min_{\overline{D}} \bar{u}_k$ and define

$$u_k^\sharp = \bar{u}_k - c_k, \qquad v_k^\sharp = \bar{v}_k + c_k.$$

Obviously $(u_k^\sharp, v_k^\sharp) \in \mathcal{K}$ and by the mass conservation condition on $g dx$ and Γ, equation (3.2), $I(\bar{u}_k, \bar{v}_k) = I(u_k^\sharp, v_k^\sharp)$. Since $\bar{u}_k$ are uniformly Lipschitz, $u_k^\sharp$ are uniformly bounded. In addition, $v_k^\sharp = (\bar{u}_k)^c + c_k = (u_k^\sharp)^c$ and consequently $v_k^\sharp$ are also uniformly bounded. By Arzelá-Ascoli's theorem, $(u_k^\sharp, v_k^\sharp)$ contains a subsequence converging uniformly to (ϕ, ψ) on $\overline{D} \times \overline{D^*}$. We then obtain that $(\phi, \psi) \in \mathcal{K}$ and $I_0 = \sup\{I(u, v) : (u, v) \in \mathcal{K}\} = I(\phi, \psi)$. Notice that this shows in particular that the supremum of I over $\mathcal{K}$ is finite. From property (2) above, $(\psi_c, (\psi_c)^c)$ is the sought maximizer of $I(u, v)$, and ψ_c is c-concave. □

Lemma 3.6. *Suppose that $c(x,m)$ satisfies the assumption* (3.5). *Let (ϕ,ϕ^c) with ϕ c-concave be a maximizer of $I(u,v)$ in $\mathcal{K}$. Then $\inf_{s\in\mathcal{S}}\int_{\overline{D}} c(x,s(x))g(x)\,dx$ is attained at $s=\mathcal{N}_{c,\phi}$, where $\mathcal{S}$ is the class of measure preserving mappings from $g(x)dx$ to Γ. Moreover*

$$\inf_{s\in\mathcal{S}}\int_{\overline{D}} c(x,s(x))g(x)\,dx = \sup\{I(u,v):(u,v)\in\mathcal{K}\}. \tag{3.7}$$

Proof. Let $\psi=\phi^c$. For $s\in\mathcal{S}$, we have

$$\begin{aligned}\int_{\overline{D}} c(x,s(x))g(x)\,dx &\geq \int_{\overline{D}}\Big(\phi(x)+\psi(s(x))\Big)g(x)\,dx\\ &= \int_{\overline{D}}\phi(x)g(x)\,dx+\int_{\overline{D}}\psi(s(x))g(x)\,dx\\ &= \int_{\overline{D}}\phi(x)g(x)\,dx+\int_{\overline{D^*}}\psi(m)\,d\Gamma = I(\phi,\psi)\\ &= \int_{\overline{D}}\Big(\phi(x)+\psi(\mathcal{N}_{c,\phi}(x))\Big)g(x)\,dx, \text{ from Lemma 3.4(ii)}\\ &= \int_{\overline{D}} c(x,\mathcal{N}_{c,\phi}(x))g(x)\,dx.\end{aligned}$$

□

Obviously, for any c-concave function ϕ, $\mathcal{N}_{c,\phi}$ has the following converging property (C): if $m_k\in\mathcal{N}_{c,\phi}(x_k)$, $x_k\longrightarrow x_0$ and $m_k\longrightarrow m_0$, then $m_0\in\mathcal{N}_{c,\phi}(x_0)$.

Lemma 3.7. *Assume that $c(x,m)$ satisfies the assumption* (3.5) *and that $\int_G g\,dx>0$ for any open $G\subset D$. Then the minimizing mapping of $\inf_{s\in\mathcal{S}}\int_{\overline{D}} c(x,s(x))g(x)\,dx$ is unique in the class of measure preserving mappings from $g(x)dx$ to Γ with the converging property (C).*

Proof. From Lemmas 3.5 and 3.6, let $\mathcal{N}_{c,\phi}$ be a minimizing mapping associated with a maximizer (ϕ,ϕ^c) of $I(u,v)$ with ϕ c-concave. Suppose that $\mathcal{N}_0$ is another minimizing mapping with the converging property (C). Clearly

$$\begin{aligned}&\int_{\overline{D}}\Big(c(x,\mathcal{N}_0(x))-\phi(x)-\phi^c(\mathcal{N}_0(x))\Big)g(x)\,dx\\ &=\inf_{s\in\mathcal{S}}\int_{\overline{D}} c(x,s(x))g(x)\,dx-\left(\int_{\overline{D}}\phi(x)g(x)\,dx+\int_{\overline{D^*}}\phi^c(m)\,d\Gamma\right)=0,\end{aligned}$$

and since $\phi(x)+\phi^c(\mathcal{N}_0(x))\leq c(x,\mathcal{N}_0(x))$, it follows that $\phi(x)+\phi^c(\mathcal{N}_0(x))=c(x,\mathcal{N}_0(x))$ on the set $\{x\in D: g(x)>0\}$ which is dense in D. Hence from (3.5) and the converging property (C), we get $\mathcal{N}_0(x)=\mathcal{N}_{c,\phi}(x)$ a.e. on D. □

We remark from the above proof that if $g(x)>0$ on D, then the minimizing mapping of $\inf_{s\in\mathcal{S}}\int_{\overline{D}} c(x,s(x))g(x)\,dx$ is unique in the class of measure preserving mappings from $g(x)dx$ to Γ.

3.1. **Application to the refractor problem $\kappa < 1$.** Let n_1 and n_2 be the indexes of refraction of two homogeneous and isotropic media I and II, respectively. Suppose that from a point O inside medium I light emanates with intensity $f(x)$ for $x \in \Omega$. We want to construct a refracting surface $\mathcal{R}$ parameterized as $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, separating media I and II, and such that all rays refracted by $\mathcal{R}$ into medium II have directions in Ω^* and the prescribed illumination intensity received in the direction $m \in \Omega^*$ is $f^*(m)$.

We first introduce the notions of refractor mapping and measure, and weak solution. In the next section we then convert the refractor problem into an optimal mass transport problem from $\overline{\Omega}$ to $\overline{\Omega^*}$ with the cost function $\log \dfrac{1}{1-\kappa\, x\cdot m}$ and establish existence and uniqueness of weak solutions.

Let Ω, Ω^* be two domains on S^{n-1}, the illumination intensity of the emitting beam is given by nonnegative $f(x) \in L^1(\overline{\Omega})$, and the prescribed illumination intensity of the refracted beam is given by a nonnegative Radon measure μ on $\overline{\Omega^*}$. Throughout this section, we assume that $|\partial\Omega| = 0$ and the physical constraint

$$\inf_{x\in\Omega, m\in\Omega^*} x\cdot m \geq \kappa. \tag{3.1}$$

We further suppose that the total energy conservation

$$\int_\Omega f(x)\,dx = \mu(\overline{\Omega^*}) > 0, \tag{3.2}$$

and for any open set $G \subset \Omega$

$$\int_G f(x)\,dx > 0, \tag{3.3}$$

where dx denotes the surface measure on S^{n-1}.

3.2. **Refractor measure and weak solutions.** We begin with the notions of refractor and supporting semi-ellipsoid.

Definition 3.8. *A surface $\mathcal{R}$ parameterized by $\rho(x)x$ with $\rho \in C(\overline{\Omega})$ is a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ for the case $\kappa < 1$ (often simply called as refractor in this section) if for any $x_0 \in \overline{\Omega}$ there exists a semi-ellipsoid $E(m,b)$ with $m \in \overline{\Omega^*}$ such that $\rho(x_0) = \dfrac{b}{1-\kappa\, m\cdot x_0}$ and $\rho(x) \leq \dfrac{b}{1-\kappa\, m\cdot x}$ for all $x \in \overline{\Omega}$. Such $E(m,b)$ is called a supporting semi-ellipsoid of $\mathcal{R}$ at the point $\rho(x_0)x_0$.*

From the definition, any refractor is globally Lipschitz on $\overline{\Omega}$.

Definition 3.9. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined by for $x_0 \in \overline{\Omega}$*

$$\mathcal{N}_{\mathcal{R}}(x_0) = \{m \in \overline{\Omega^*} :\ E(m,b) \text{ supports } \mathcal{R} \text{ at } \rho(x_0)x_0 \text{ for some } b > 0\}.$$

Given $m_0 \in \overline{\Omega^*}$, the tracing mapping of $\mathcal{R}$ is defined by

$$\mathcal{T}_{\mathcal{R}}(m_0) = \mathcal{N}_{\mathcal{R}}^{-1}(m_0) = \{x \in \overline{\Omega} : \ m_0 \in \mathcal{N}_{\mathcal{R}}(x)\}.$$

Definition 3.10. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the Legendre transform of $\mathcal{R}$ is defined by*

$$\mathcal{R}^* = \{\rho^*(m)m : \ \rho^*(m) = \inf_{x\in\overline{\Omega}} \frac{1}{\rho(x)(1-\kappa\, x\cdot m)}, \ m \in \overline{\Omega^*}\}.$$

We now give some basic properties of Legendre transforms.

Lemma 3.11. *Let $\mathcal{R}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$. Then*

(i) *$\mathcal{R}^*$ is a refractor from $\overline{\Omega^*}$ to $\overline{\Omega}$.*
(ii) *$\mathcal{R}^{**} = (\mathcal{R}^*)^* = \mathcal{R}$.*
(iii) *If $x_0 \in \overline{\Omega}$ and $m_0 \in \overline{\Omega^*}$, then $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$ iff $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$.*

Proof. Given $m_0 \in \overline{\Omega^*}$, $\rho(x)(1-\kappa x\cdot m_0)$ must attain the maximum over $\overline{\Omega}$ at some $x_0 \in \overline{\Omega}$. Then $\rho^*(m_0) = 1/[\rho(x_0)(1-\kappa x_0\cdot m_0)]$. We always have

$$\rho^*(m) = \inf_{x\in\overline{\Omega}} \frac{1}{\rho(x)(1-\kappa m\cdot x)} \le \frac{1}{\rho(x_0)(1-\kappa x_0\cdot m)}, \qquad \forall m \in \overline{\Omega^*}. \tag{3.4}$$

Hence $E(x_0, 1/\rho(x_0))$ is a supporting semi-ellipsoid to $\mathcal{R}^*$ at $\rho^*(m_0)m_0$. Thus, (i) is proved.

To prove (ii), from the definitions of Legendre transform and refractor mapping we have

$$\rho(x_0)\,\rho^*(m_0) = \frac{1}{1-\kappa m_0\cdot x_0} \quad \text{for } m_0 \in \mathcal{N}_{\mathcal{R}}(x_0). \tag{3.5}$$

For $x_0 \in \overline{\Omega}$, there exists $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$ and so from (3.5) $\rho^*(m_0) = \dfrac{1/\rho(x_0)}{1-\kappa x_0\cdot m_0}$. By (3.4), $\rho^*(m)(1-kx_0\cdot m)$ attains the maximum $1/\rho(x_0)$ at m_0. Thus,

$$\rho^{**}(x_0) = \inf_{m\in\overline{\Omega^*}} \frac{1}{\rho^*(m)(1-kx_0\cdot m)} = \frac{1}{\rho(x_0)^{-1}}.$$

To prove (iii), we get from the proof of (ii) that if $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$, then the semi-ellipsoid $E(x_0, 1/\rho(x_0))$ supports $\mathcal{R}^*$ at $\rho^*(m_0)m_0$ and so $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$. On the other hand, if $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$, we get that $m_0 \in \mathcal{N}_{\mathcal{R}^{**}}(x_0)$, and since $\mathcal{R}^{**} = \mathcal{R}$, $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$. □

The next two lemmas discuss the refractor measure.

Lemma 3.12. *$\mathcal{C} = \{F \subset \overline{\Omega^*} : \mathcal{T}_{\mathcal{R}}(F)$ is Lebesgue measurable$\}$ is a σ-algebra containing all Borel sets in $\overline{\Omega^*}$.*

Proof. Obviously, $\mathcal{T}_{\mathcal{R}}(\emptyset) = \emptyset$ and $\mathcal{T}_{\mathcal{R}}(\overline{\Omega^*}) = \overline{\Omega}$. Since $\mathcal{T}_{\mathcal{R}}(\cup_{i=1}^{\infty} F_i) = \cup_{i=1}^{\infty} \mathcal{T}_{\mathcal{R}}(F_i)$, $\mathcal{C}$ is closed under countable unions. Clearly for $F \subset \overline{\Omega^*}$

$$\begin{aligned}
\mathcal{T}_{\mathcal{R}}(F^c) &= \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F^c \neq \emptyset\} \\
&= \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F = \emptyset\} \cup \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F^c \neq \emptyset,\ \mathcal{N}_{\mathcal{R}}(x) \cap F \neq \emptyset\} \\
&= [\mathcal{T}_{\mathcal{R}}(F)]^c \cup [\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)]. \qquad (3.6)
\end{aligned}$$

If $x \in \mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F) \cap \Omega$, then $\mathcal{R}$ parameterized by ρ has two distinct supporting semi-ellipsoids $E(m_1, b_1)$ and $E(m_2, b_2)$ at $\rho(x)x$. We show that $\rho(x)x$ is a singular point of $\mathcal{R}$. Otherwise, if $\mathcal{R}$ has the tangent hyperplane Π at $\rho(x)x$, then Π must coincide both with the tangent hyperplane of $E(m_1, b_1)$ and that of $E(m_2, b_2)$ at $\rho(x)x$. It follows from the Snell law that $m_1 = m_2$. Therefore, the area measure of $\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)$ is 0. So $\mathcal{C}$ is closed under complements, and we have proved that $\mathcal{C}$ is a σ-algebra.

To prove that $\mathcal{C}$ contains all Borel subsets, it suffices to show that $\mathcal{T}_{\mathcal{R}}(K)$ is compact if $K \subset \overline{\Omega^*}$ is compact. Let $x_i \in \mathcal{T}_{\mathcal{R}}(K)$ for $i \geq 1$. There exists $m_i \in \mathcal{N}_{\mathcal{R}}(x_i) \cap K$. Let $E(m_i, b_i)$ be the supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_i)x_i$. We have

$$\rho(x)(1 - \kappa m_i \cdot x) \leq b_i \qquad \text{for } x \in \overline{\Omega}, \tag{3.7}$$

where the equality in (3.7) occurs at $x = x_i$. Assume that $a_1 \leq \rho(x) \leq a_2$ on $\overline{\Omega}$ for some constants $a_2 \geq a_1 > 0$. By (3.7) and (3.1), $a_1(1 - \kappa) \leq b_i \leq a_2(1 - \kappa^2)$. Assume through subsequence that $x_i \longrightarrow x_0$, $m_i \longrightarrow m_0 \in K$, $b_i \longrightarrow b_0$, as $i \longrightarrow \infty$. By taking limit in (3.7), one obtains that the semi-ellipsoid $E(m_0, b_0)$ supports $\mathcal{R}$ at $\rho(x_0)x_0$ and $x_0 \in \mathcal{T}_{\mathcal{R}}(m_0)$. This proves $\mathcal{T}_{\mathcal{R}}(K)$ is compact.

To show that $\mathcal{C}$ is closed by complements, it is enough to notice the formula that

$$\mathcal{T}_{\mathcal{R}}(\Omega^* \setminus F) = [\mathcal{T}_{\mathcal{R}}(\Omega^*) \setminus \mathcal{T}_{\mathcal{R}}(F)] \cup [\mathcal{T}_{\mathcal{R}}(\Omega^* \setminus F) \cap \mathcal{T}_{\mathcal{R}}(F)].$$

□

Lemma 3.13. *Given a nonnegative $f \in L^1(\overline{\Omega})$, the set function*

$$\mathcal{M}_{\mathcal{R},f}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f\, dx$$

is a finite Borel measure defined on $\mathcal{C}$ and is called the refractor measure associated with $\mathcal{R}$ and f.

Proof. Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{C}$. Let $H_1 = \mathcal{T}_{\mathcal{R}}(F_1)$, and $H_k = \mathcal{T}_{\mathcal{R}}(F_k) \setminus \cup_{i=1}^{k-1} \mathcal{T}_{\mathcal{R}}(F_i)$, for $k \geq 2$. Since $H_i \cap H_j = \emptyset$ for $i \neq j$ and $\cup_{k=1}^{\infty} H_k = \cup_{k=1}^{\infty} \mathcal{T}_{\mathcal{R}}(F_k)$, it is easy to get

$$\mathcal{M}_{\mathcal{R},f}(\cup_{k=1}^{\infty} F_k) = \int_{\cup_{k=1}^{\infty} H_k} f\, dx = \sum_{k=1}^{\infty} \int_{H_k} f\, dx.$$

Observe that $\mathcal{T}_{\mathcal{R}}(F_k) \setminus H_k = \mathcal{T}_{\mathcal{R}}(F_k) \cap (\cup_{i=1}^{k-1}\mathcal{T}_{\mathcal{R}}(F_i))$ is a subset of the singular set of $\mathcal{R}$ and has area measure 0 for $k \geq 2$. Therefore, $\int_{H_k} f\,dx = \mathcal{M}_{\mathcal{R},f}(F_k)$ and the σ-additivity of $\mathcal{M}_{\mathcal{R},f}$ follows. □

The notion of weak solutions is introduced through the conservation of energy.

Definition 3.14. *A refractor $\mathcal{R}$ is a weak solution of the refractor problem for the case $\kappa < 1$ with emitting illumination intensity $f(x)$ on $\overline{\Omega}$ and prescribed refracted illumination intensity μ on $\overline{\Omega^*}$ if for any Borel set $F \subset \overline{\Omega^*}$*

$$\mathcal{M}_{\mathcal{R},f}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f\,dx = \mu(F). \tag{3.8}$$

3.3. **Solution of the refractor problem.** We introduce the cost

$$c(x,m) = \log \frac{1}{1-\kappa\, x\cdot m}$$

for $x \in \Omega$ and $m \in \Omega^*$ where we assume $\Omega \cdot \Omega^* \geq \kappa$. From Definitions 3.2 and 3.8, $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ is a refractor iff $\log\rho$ is c-concave. Using Definitions 3.3 and 3.9 we get that

$$\mathcal{N}_{c,\phi}(x) = \mathcal{N}_{\mathcal{R}}(x), \qquad \mathcal{R} = \{\rho(x)x : x \in \Omega\}, \qquad \rho(z) = e^{\phi(z)}.$$

Furthermore, $\log\rho^* = (\log\rho)^c$, $\log\rho = (\log\rho^*)_c$ by Remark (3) after Definition 3.2, and $\mathcal{N}_{\mathcal{R}}(x_0) = \mathcal{N}_{c,\log\rho}(x_0)$ by (3.5). By the Snell law and Lemma 3.12, $c(x,m)$ satisfies (3.5). From the definitions, $\mathcal{R}$ is a weak solution of the refractor problem iff $\log\rho$ is c-concave and $\mathcal{N}_{c,\log\rho}$ is a measure preserving mapping from $f(x)dx$ to μ.

By Lemma 3.5, there exists a c-concave $\phi(x)$ such that (ϕ,ϕ^c) maximizes

$$I(u,v) = \int_{\overline{\Omega}} uf\,dx + \int_{\overline{\Omega^*}} v\,d\mu(m)$$

in $\mathcal{K} = \{(u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega^*}) : u(x) + v(m) \leq c(x,m),\ \text{for } x \in \overline{\Omega}, m \in \overline{\Omega^*}\}$. Then by Lemma 3.4, $\mathcal{N}_{c,\phi}(x)$ is a measure preserving mapping from $f dx$ to μ. Therefore, $\mathcal{R} = \{e^{\phi(x)}x : x \in \overline{\Omega}\}$ is a weak solution of the refractor problem.

It remains to prove the uniqueness of solutions up to dilations. Let $\mathcal{R}_i = \{\rho_i(x)x : x \in \overline{\Omega}\}$, $i = 1,2$, be two weak solutions of the refractor problem. Obviously, $\mathcal{N}_{c,\log\rho_i}$ have the converging property (C) stated before Lemma 3.7. It follows from Lemmas 3.4, 3.6 and 3.7 that $\mathcal{N}_{c,\log\rho_1}(x) = \mathcal{N}_{c,\log\rho_2}(x)$ a.e. on Ω. That is, $\mathcal{N}_{\mathcal{R}_1}(x) = \mathcal{N}_{\mathcal{R}_2}(x)$ a.e. on Ω. From the Snell law $\nu_i(x) = \dfrac{x - \kappa\mathcal{N}_{\mathcal{R}_i}(x)}{|x - \kappa\mathcal{N}_{\mathcal{R}_i}(x)|}$ is the unit normal to $\mathcal{R}_i$ towards medium II at $\rho_i(x)x$ where $\mathcal{R}_i$ is differentiable. So $\nu_1(x) = \nu_2(x)$ a.e. and consequently $\rho_1(x) = C\,\rho_2(x)$ for some $C > 0$.

4. Solution to the refractor problem for $\kappa < 1$ with the Minkowski method

In this section we solve the refractor problem using a method having its roots in the Minkowski method from convex analysis, [Sch93, Section 7.1]. In addition, to the definitions and lemmas from Subsection 3.2, we have the following.

Lemma 4.1. *We have for a refractor $\mathcal{R}$ that*

(i) $[\mathcal{T}_{\mathcal{R}}(F)]^c \subset \mathcal{T}_{\mathcal{R}}(F^c)$ *for all $F \subset \overline{\Omega^*}$, with equality except for a set of measure zero.*

(ii) *The set $\mathcal{C} = \{F \subset \overline{\Omega^*} : \mathcal{T}_{\mathcal{R}}(F)$ is Lebesgue measurable$\}$ is a σ-algebra containing all Borel sets in $\overline{\Omega^*}$.*

Lemma 4.2. *Let $\mathcal{R}_j = \{\rho_j(x)x : x \in \overline{\Omega}\}$, $j \geq 1$ be refractors from $\overline{\Omega}$ to $\overline{\Omega^*}$. Suppose that $0 < a_1 \leq \rho_j \leq a_2$ and $\rho_j \to \rho$ uniformly on $\overline{\Omega}$. Then:*

(i) *$\mathcal{R} := \{\rho(x)x : x \in \overline{\Omega}\}$ is a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$.*

(ii) *For any compact set $K \subset \overline{\Omega^*}$*

$$\limsup_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(K) \subset \mathcal{T}_{\mathcal{R}}(K).$$

(iii) *For any open set $G \subset \overline{\Omega^*}$,*

$$\mathcal{T}_{\mathcal{R}}(G) \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S,$$

where S is the singular set of $\mathcal{R}$.

Proof. (i) Obviously $\rho \in C(\overline{\Omega})$ and $\rho > 0$. Fix $x_o \in \overline{\Omega}$. Then there exist $m_j \in \overline{\Omega^*}$ and $b_j > 0$ such that $E(m_j, b_j)$ supports $\mathcal{R}_j$ at $\rho(x_o)x_o$ and thus

$$\rho_j(x_o) = \frac{b_j}{1 - \kappa m_j \cdot x_o} \quad \text{and} \quad \rho_j(x) \leq \frac{b_j}{1 - \kappa m_j \cdot x}$$

for all $x \in \overline{\Omega}$. Consequently

$$\frac{b_j}{1 - \kappa m_j \cdot x_o} \leq a_2 \quad \text{and} \quad a_1 \leq \frac{b_j}{1 - \kappa m_j \cdot x}$$

for all j and therefore

$$a_1(1 - \kappa) \leq b_j \leq a_2$$

for all j. If need be by passing to a subsequence we obtain m_o and b_o such that $m_j \to m_o \in \overline{\Omega^*}$ and $b_j \to b_o$. We claim $E(m_o, b_o)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$. Indeed

$$\rho(x_o) = \lim_j \rho_j(x_o) = \lim_j \frac{b_j}{1 - \kappa m_j \cdot x_o} = \frac{b_o}{1 - \kappa m_o \cdot x_o}$$

and

$$\rho(x) = \lim_j \rho_j(x) \leq \lim_j \frac{b_j}{1 - \kappa m_j \cdot x} = \frac{b_o}{1 - \kappa m_o \cdot x}$$

for all $x \in \overline{\Omega}$. Thus $\mathcal{R}$ is a refractor.

(ii) Let $x_o \in \limsup \mathcal{T}_{\mathcal{R}_j}(K)$. Without loss of generality assume that $x_o \in \mathcal{T}_{\mathcal{R}_j}(K)$ for all $j \geq 1$. Then there exist $m_j \in \mathcal{N}_{\mathcal{R}_j}(x_o) \cap K$ and b_j such that

$$\rho_j(x_o) = \frac{b_j}{1 - \kappa m_j \cdot x_o} \qquad \text{and} \qquad \rho_j(x) \leq \frac{b_j}{1 - \kappa m_j \cdot x}$$

for all $x \in \overline{\Omega}$. As in the proof of (i) we may assume that $m_j \to m_o \in K$ and $b_j \to b_o$ and conclude that $E(m_o, b_o)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$, proving that $x_o \in \mathcal{T}_{\mathcal{R}}(m_o)$. Hence $x_o \in \mathcal{T}_{\mathcal{R}}(K)$.

(iii) Let G be an open subset of $\overline{\Omega^*}$. By (ii) $\limsup \mathcal{T}_{\mathcal{R}_j}(G^c) \subset \mathcal{T}_{\mathcal{R}}(G^c)$ as G^c is compact. Also

$$\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c \subset \limsup_{j\to\infty}\{[\mathcal{T}_{\mathcal{R}_j}(G)]^c \cup [\mathcal{T}_{\mathcal{R}_j}(G) \cap \mathcal{T}_{\mathcal{R}_j}(G^c)]\} \tag{4.1}$$

and by Lemma 4.1 the right hand side of (4.1) is equal to $\limsup_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G^c)$. By (ii) we will then have

$$\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c \subset \mathcal{T}_{\mathcal{R}}(G^c) = \{[\mathcal{T}_{\mathcal{R}}(G)]^c \cup [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]\}.$$

Taking complements we obtain

$$\{\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c\}^c \supset [\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c.$$

Consequently

$$\liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \supset [\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c$$

and thus

$$[[\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c] \cup S \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S.$$

But $\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c) \subset S$. Thus

$$\mathcal{T}_{\mathcal{R}}(G) \subset \mathcal{T}_{\mathcal{R}}(G) \cup S \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S$$

as required. □

Remark 4.3 (Invariance by dilations)**.** Suppose that $\mathcal{R}$ is a refractor weak solution in the sense of Definition 3.14 with intensities f, μ and defined by $\rho(x)x$ for $x \in \bar{\Omega}$. Then for each $\alpha > 0$, the refractor $\alpha\mathcal{R}$ defined by $\alpha\,\rho(x)x$ for $x \in \bar{\Omega}$ is a weak solution in the sense of Definition 3.14 with the same intensities. In fact, $E(m, b)$ is a supporting ellipsoid to $\mathcal{R}$ at the point y if and only if $E(m, \alpha\, b)$ is a supporting ellipsoid to $\alpha\mathcal{R}$ at the point y. This means that $\mathcal{T}_{\mathcal{R}}(m) = \mathcal{T}_{\alpha\mathcal{R}}(m)$ for each $m \in \bar{\Omega}^*$.

4.1. **Existence of solutions in the discrete case.**

Theorem 4.4. *Let $f \in L^1(\Omega)$ with $\inf_\Omega f > 0$, $g_1, \cdots, g_N$ positive numbers, $m_1, \cdots, m_N \in \Omega^*$ distinct points, $N \geq 2$, with $x \cdot m_j \geq \kappa$ for all $x \in \Omega$ and $1 \leq j \leq N$. Let $\mu = \sum_{j=1}^N g_j \delta_{m_j}$, and assume the conservation of energy condition*

$$\int_\Omega f(x)\, dx = \mu(\Omega^*). \tag{4.2}$$

Then there exists a refractor $\mathcal{R}$ such that

(a) $\bar{\Omega} = \cup_{j=1}^N \mathcal{T}_{\mathcal{R}}(m_j)$,

(b) $\int_{\mathcal{T}_{\mathcal{R}}(m_j)} f(x)\, dx = g_j$ *for* $1 \leq j \leq N$.

To prove the theorem, we prove first a sequence of lemmas.

Lemma 4.5. *Let*

$$W = \left\{ b = (1, b_2, \cdots, b_N) : b_j > 0,\ \mathcal{M}_{\mathcal{R}(b),f}(m_j) = \int_{\mathcal{T}_{\mathcal{R}(b)}(m_j)} f(x)\, dx \leq g_j,\ j = 2, \cdots, N \right\}, \tag{4.3}$$

where

$$\rho(x) = \mathcal{R}(b)(x) = \min_{1 \leq j \leq N} \frac{b_i}{1 - \kappa\, x \cdot m_j}. \tag{4.4}$$

Then, with the assumptions of Theorem 4.4, we have

(a) $W \neq \emptyset$

(b) *if* $b = (1, b_2, \cdots, b_N) \in W$, *then* $b_j > \dfrac{1}{1+\kappa}$ *for* $j = 2, \cdots, N$.

Proof. (a) If for some $j \neq 1$, the semi-ellipsoid $E(m_j, b)$ supports $\mathcal{R}(b)$ at some $x \in \Omega$, then $\rho(z) \leq \dfrac{b}{1 - \kappa\, z \cdot m_j}$ for all $z \in \Omega$, and $\rho(x) = \dfrac{b}{1 - \kappa\, x \cdot m_j}$. Since $x \cdot m_j \geq \kappa$, we have

$$\frac{b}{1-\kappa^2} \leq \frac{b}{1 - \kappa\, x \cdot m_j} = \rho(x) \leq \frac{1}{1 - \kappa\, x \cdot m_1} \leq \frac{1}{1-\kappa},$$

and so $b \leq 1 + \kappa$. Therefore, if $b_i > 1 + \kappa$ for $2 \leq i \leq N$, then $E(m_i, b_i)$ cannot be a supporting ellipsoid to $\mathcal{R}(b)$ at any $x \in \Omega$. On the other hand, if $x \in \mathcal{T}_{\mathcal{R}}(m_j)$, then $m_j \in \mathcal{N}_{\mathcal{R}}(x)$ and if x is not a singular point of $\mathcal{R}(b)$ there is a unique ellipsoid $E(m_j, b)$ supporting $\mathcal{R}$ at x. But from the definition of $\mathcal{R}$ there is an ellipsoid $E(m_k, b_k)$ that supports $\mathcal{R}$ at x, and so $E(m_k, b_k) = E(m_j, b)$, i.e., $k = j$ and $b = b_j$. Consequently the set $\mathcal{T}_{\mathcal{R}}(m_j)$ is contained in the set of singular points and therefore has measure zero. So $\mathcal{M}_{\mathcal{R}(b),f}(m_j) = 0 < g_j$ for $j = 2, \cdots, N$ and so any point $b = (1, b_2, \cdots, b_N) \in W$ as long as $b_i > 1 + \kappa$ for $i = 2, \cdots, N$.

Claim 2. For each $b \in W$, $\mathcal{T}_{\mathcal{R}(b)}(m_1) \cap \left(\cup_{i=2}^N \mathcal{T}_{\mathcal{R}(b)}(m_i)\right)^c \neq \emptyset$.

Otherwise, $\mathcal{T}_{\mathcal{R}(b)}(m_1) \subset \cup_{i=2}^N \mathcal{T}_{\mathcal{R}(b)}(m_i)$ which means that each point in $\mathcal{T}_{\mathcal{R}(b)}(m_1)$ is singular, and therefore $|\mathcal{T}_{\mathcal{R}(b)}(m_1)| = 0$. This contradicts Claim 1, since $g_1 > 0$.

Therefore, if $b \in W$, then we can pick $x_0 \in \mathcal{T}_{\mathcal{R}(b)}(m_1) \cap \left(\cup_{i=2}^N \mathcal{T}_{\mathcal{R}(b)}(m_i)\right)^c$ and so

$$\rho(x_0) = \frac{1}{1-\kappa\, x_0 \cdot m_1} < \frac{b_i}{1-\kappa\, x_0 \cdot m_i}, \qquad i = 2, \cdots, N$$

so

$$b_i > \frac{1-\kappa\, x_0 \cdot m_i}{1-\kappa\, x_0 \cdot m_1} \geq \frac{1-\kappa\, x_0 \cdot m_i}{1-\kappa^2} \geq \frac{1-\kappa}{1-\kappa^2} = \frac{1}{1+\kappa}.$$

□

Lemma 4.6. *If $b_j = (b_1^j, \cdots, b_N^j) \to b_0 = (b_1^0, \cdots, b_N^0)$ as $j \to \infty$, then $\rho_j = \mathcal{R}(b_j) \to \rho_0 = \mathcal{R}(b_0)$ uniformly in $\bar{\Omega}$ as $j \to \infty$.*

Proof. Given $y \in \bar{\Omega}$, there exists $1 \leq \ell \leq N$ such that $\rho_0(y) = \dfrac{b_\ell^0}{1-\kappa\, y \cdot m_\ell}$. Hence

$$\rho_j(y) - \rho_0(y) \leq \frac{b_\ell^j}{1-\kappa\, y \cdot m_\ell} - \frac{b_\ell^0}{1-\kappa\, y \cdot m_\ell} \leq \frac{|b_\ell^j - b_\ell^0|}{1-\kappa\, y \cdot m_\ell} \leq \frac{|b_\ell^j - b_\ell^0|}{1-\kappa} \to 0,$$

as $j \to \infty$. □

Lemma 4.7. *Let $\delta > 0$ and the region $R_\delta = \{(1, b_2, \cdots, b_N) : b_j \geq \delta, 2 \leq j \leq N\}$. The functions $G_{\mathcal{R}(b)}(m_i) := \mathcal{M}_{\mathcal{R}(b)}(m_i)$ are continuous for $b \in R_\delta$ for $i = 1, 2, \cdots, N$.*

Proof. Let $b_j = (1, b_2^j, \cdots, b_N^j) \in R_\delta$ with $b_j \to b_0$ as $j \to \infty$. By Lemma 4.6, $\rho_j \to \rho_0$ uniformly in $\bar{\Omega}$. Given $x \in \bar{\Omega}$, we have $\rho_j(x) = \dfrac{b_\ell^j}{1-\kappa\, x \cdot m_\ell}$ for some $1 \leq \ell \leq N$ and so $\rho_j(x) \geq \dfrac{\min\{1,\delta\}}{1+\kappa}$. On the other hand, $\rho_j(x) = \min\limits_{1\leq \ell \leq N} \dfrac{b_\ell^j}{1-\kappa\, x \cdot m_\ell} \leq \dfrac{1}{1-\kappa\, x \cdot m_1} \leq \dfrac{1}{1-\kappa}$. Therefore

$$\frac{\min\{1,\delta\}}{1+\kappa} \leq \rho_j(x) \leq \frac{1}{1-\kappa}$$

for all $x \in \bar{\Omega}$ and for all j.

Let us fix $1 \leq i \leq N$. Let $G \subset \bar{\Omega}^*$ be a neighborhood of m_i such that $m_\ell \notin G$ for $\ell \neq i$. If $x_0 \in \mathcal{T}_{\mathcal{R}(b_j)}(G)$ and x_0 is not a singular point, then there exists a unique $m \in G$ and $b > 0$ such that $\rho_j(x_0) = \dfrac{b}{1-\kappa\, x_0 \cdot m}$ and $\rho_j(z) \leq \dfrac{b}{1-\kappa\, z \cdot m}$ for all $x \in \bar{\Omega}$. From the definition of $\mathcal{R}(b_j)$ and since x_0 is not singular, $m = m_\ell$. Since $m \in G$, we get $m = m_i$. Therefore

$$\mathcal{T}_{\mathcal{R}(b_j)}(G) \subset \mathcal{T}_{\mathcal{R}(b_j)}(m_i) \cup S,$$

where S is the set of singular points. By Lemma 4.2

$$\mathcal{T}_{\mathcal{R}(b_0)}(G) \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}(b_j)}(G) \cup S,$$

and we therefore obtain

$$\mathcal{T}_{\mathcal{R}(b_0)}(G) \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}(b_j)}(m_i) \cup S.$$

Thus

$$\int_{\mathcal{T}_{\mathcal{R}(b_0)}(m_i)} f(x)\,dx \le \int_{\mathcal{T}_{\mathcal{R}(b_0)}(G)} f(x)\,dx \le \int_{\liminf_{j\to\infty}\mathcal{T}_{\mathcal{R}(b_j)}(m_i)} f(x)\,dx$$
$$\le \liminf_{j\to\infty}\int_{\mathcal{T}_{\mathcal{R}(b_j)}(m_i)} f(x)\,dx \qquad \text{by Fatou.}$$

We next prove that

$$\limsup_{j\to\infty}\int_{\mathcal{T}_{\mathcal{R}(b_j)}(m_i)} f(x)\,dx \le \int_{\mathcal{T}_{\mathcal{R}(b_0)}(m_i)} f(x)\,dx.$$

By Lemma 4.2

$$\limsup_{j\to\infty}\mathcal{T}_{\mathcal{R}(b_j)}(K) \subset \mathcal{T}_{\mathcal{R}(b_0)}(K)$$

for each K compact. Hence

$$\int_{\limsup_{j\to\infty}\mathcal{T}_{\mathcal{R}(b_j)}(m_i)} f(x)\,dx \le \int_{\mathcal{T}_{\mathcal{R}(b_0)}(m_i)} f(x)\,dx.$$

By reverse Fatou we have

$$\limsup_{j\to\infty}\int_{\mathcal{T}_{\mathcal{R}(b_j)}(m_i)} f(x)\,dx \le \int_{\limsup_{j\to\infty}\mathcal{T}_{\mathcal{R}(b_j)}(m_i)} f(x)\,dx$$

and therefore the lemma is proved. □

Proof of Theorem 4.4. Fix $\tilde{b} = (1, \tilde{b}_2, \cdots, \tilde{b}_N) \in W$ and let

$$\tilde{W} = \{b = (1, b_2, \cdots, b_N) \in W : b_j \le \tilde{b}_j,\ j = 2, \cdots, N\}.$$

$\tilde{W}$ is compact. Let $d : \tilde{W} \to \mathbb{R}$ be given by $d(b) = 1 + b_2 + \cdots + b_N$; d attains its minimum value in $\tilde{W}$ at a point $b^* = (1, b_2^*, \cdots, b_N^*)$ (notice that the minimum is strictly positive by Lemma 4.5(b)). We prove that $\mathcal{R}(b^*)$ is the refractor that solves the problem. By conservation of energy it is enough to show that $\int_{\mathcal{T}_{\mathcal{R}(b^*)}(m_j)} f(x)\,dx = g_j$ for $j = 2, \cdots, N$. Since $b^* \in W$, we have $\int_{\mathcal{T}_{\mathcal{R}(b^*)}(m_j)} f(x)\,dx \le g_j$ for $j = 2, \cdots, N$. Suppose by contradiction that this inequality is strict for some j, suppose for example that

$$\int_{\mathcal{T}_{\mathcal{R}(b^*)}(m_2)} f(x)\,dx < g_2. \tag{4.5}$$

Let $0 < \lambda < 1$ and $b_\lambda^* = (1, \lambda b_2^*, b_3^*, \cdots, b_N^*)$.

We claim that

$$\mathcal{T}_{\mathcal{R}(b_\lambda^*)}(m_i) \setminus \text{set of measure zero} \subset \mathcal{T}_{\mathcal{R}(b^*)}(m_i) \tag{4.6}$$

for $i = 3, 4, \cdots, N$ and all $0 < \lambda < 1$. Indeed, if $x_0 \in \mathcal{T}_{\mathcal{R}(b_\lambda^*)}(m_i)$ is not a singular point of $\mathcal{R}(b_\lambda^*)$, then there is a unique ellipsoid $\dfrac{a}{1 - \kappa\, x \cdot m_i}$ that supports $\mathcal{R}(b_\lambda^*)$ at x_0

for some $a > 0$. Since $\mathcal{R}(b^*_\lambda)(x) = \min_{1\le i\le N} \dfrac{(b^*_\lambda)_i}{1-\kappa\, x\cdot m_i}$, there exists $1 \le j \le N$ such that $\mathcal{R}(b^*_\lambda)(x_0) = \dfrac{(b^*_\lambda)_j}{1-\kappa\, x\cdot m_j}$. That is, the ellipsoid $\dfrac{(b^*_\lambda)_j}{1-\kappa\, x\cdot m_j}$ supports $\mathcal{R}(b^*_\lambda)$ at x_0. Therefore, $\dfrac{a}{1-\kappa\, x\cdot m_i} = \dfrac{(b^*_\lambda)_j}{1-\kappa\, x\cdot m_j}$ implying $j = i$ and so $a = (b^*_\lambda)_i = b^*_i$. Since $\dfrac{b^*_i}{1-\kappa\, x\cdot m_i} \ge \mathcal{R}(b^*)(x) \ge \mathcal{R}(b^*_\lambda)(x)$ for all x, it follows that $\dfrac{b^*_i}{1-\kappa\, x\cdot m_i}$ is a supporting ellipsoid to $\mathcal{R}(b^*)$ at x_0. This proves the claim.

This implies that

$$\int_{\mathcal{T}_{\mathcal{R}(b^*_\lambda)}(m_i)} f(x)\,dx \le \int_{\mathcal{T}_{\mathcal{R}(b^*)}(m_i)} f(x)\,dx \le g_i$$

for $i = 3, 4, \cdots, N$ and all $0 < \lambda < 1$.

Finally from Lemma 4.7, inequality (4.5) holds for all λ sufficiently close to one, and therefore the point $b^*_\lambda \in \tilde{W}$ for all λ close to one. This is a contradiction because $d(b^*_\lambda) < d(b^*)$. □

Remark 4.8. Notice that the solution in Theorem 4.4 has the form given by formula (4.4), where $b_1 = 1$ and $(1, b_2, \cdots, b_N) \in W$. So from Lemma 4.5(b), we have $b_i > 1/(1+\kappa)$ for $i = 2, \cdots, N$. This implies that $\inf_{\bar\Omega} \rho(x) = \alpha > 0$.

4.2. **Solution in the general case.**

Lemma 4.9. *Let $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ such that $\inf_{x\in\overline{\Omega}} \rho(x) = 1$. Then there is a constant C, depending only on κ, such that*

$$\sup_{x\in\overline{\Omega}} \rho(x) \le C.$$

Proof. Suppose $\inf_{x\in\overline{\Omega}} \rho(x)$ is attained at $x_o \in \overline{\Omega}$, and let $E(m, b)$ be a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_0)x_0$. Then

$$1 = \rho(x_0) = \frac{b}{1-\kappa\, m\cdot x_0} \quad \text{and} \quad \rho(x) \le \frac{b}{1-\kappa\, m\cdot x} \quad \forall x \in \overline{\Omega}.$$

From the first equation we get that $b \le 1+\kappa$, and using this in the inequality we obtain

$$\rho(x) \le \frac{1+\kappa}{1-\kappa^2} \quad \text{for all } x \in \overline{\Omega}$$

which proves the lemma. □

Theorem 4.10. *Let $f \in L^1(\overline{\Omega})$ with $\inf_{\overline{\Omega}} f > 0$, and let μ be a Radon measure on $\overline{\Omega^*}$, such that*

$$\int_{\overline{\Omega}} f(x)\,dx = \mu(\overline{\Omega^*})$$

Then there exists a weak solution $\mathcal{R}$ of the refractor problem for the case $\kappa < 1$, with emitting illumination intensity f and prescribed refracted illumination intensity μ.

Proof. Fix $l \in \mathbf{N}, l \geq 2$. Partition $\overline{\Omega^*}$ into a finite number of disjoint Borel subsets $\omega_1^l, \ldots, \omega_{k_l}^l$ such that $\operatorname{diam}(\omega_i^l) \leq \dfrac{1}{l}$. Choose points $m_i^l \in \omega_i^l$ and define a measure on $\overline{\Omega^*}$

$$\mu_l = \sum_{i=1}^{k_l} \mu(\omega_i^l)\delta_{m_i^l}.$$

Then

$$\mu_l(\overline{\Omega^*}) = \sum_{i=1}^{k_l} \mu(\omega_i^l) = \mu(\overline{\Omega^*}) = \int_{\overline{\Omega^*}} f(x)\, dx.$$

If $h \in C(\overline{\Omega^*})$, then

$$\begin{aligned}
\int_{\overline{\Omega^*}} h\, d\mu_l - \int_{\overline{\Omega^*}} h\, d\mu &= \sum_{i=1}^{k_l} \left(\int_{\omega_i^l} h(x)\, d\mu_l - \int_{\omega_i^l} h(x)\, d\mu \right) \\
&= \sum_{i=1}^{k_l} \left(\int_{\omega_i^l} h(m_i^l) d\mu - \int_{\omega_i^l} h(x) d\mu \right) \\
&= \sum_{i=1}^{k_l} \int_{\omega_i^l} (h(m_i^l) - h(x)) d\mu.
\end{aligned}$$

Since $h \in C(\overline{\Omega^*})$ and $\operatorname{diam}(\omega_i^l) < \dfrac{1}{l}$, we obtain that

$$\int_{\overline{\Omega^*}} h\, d\mu_l \to \int_{\overline{\Omega^*}} h\, d\mu \quad \text{as } l \to \infty$$

and hence μ_l *converges weakly* to μ.

By Theorem 4.4, let $\mathcal{R}_l = \{\rho_l(x)x : x \in \overline{\Omega}\}$ be the solution corresponding to μ_l, that is,

$$\mathcal{M}_{\mathcal{R}_l, f}(\omega) = \mu_l(\omega)$$

for every Borel subset ω of $\overline{\Omega^*}$. Notice that from Remark 4.8, $\inf_{\bar{\Omega}} \rho_l(x) = \alpha_l > 0$. In view of Remark 4.3, the refractor defined by the function $\dfrac{\rho_l(x)}{\alpha_l}$ solves the same problem and $\inf_{\bar{\Omega}} \dfrac{\rho_l(x)}{\alpha_l} = 1$. So by normalizing ρ_l, we may assume that $\inf_{\overline{\Omega}} \rho_l(x) = 1$. Then by Lemma (4.9) there exists a uniform bound $C = C(\kappa)$ such that

$$\sup_{x \in \overline{\Omega}} \rho_l(x) \leq C \quad \text{for all } l \geq 1.$$

Also if $x_o, x_1 \in \overline{\Omega}$ and $E(m_o, b_o)$ is a supporting semi ellipsoid to $\mathcal{R}_l$ at $\rho_l(x_o)x_o$ then for $x_1 \in \overline{\Omega}$ we have

$$\rho_l(x_1) - \rho_l(x_o) \leq \frac{b_o}{1 - \kappa\, m_o \cdot x_1} - \frac{b_o}{1 - \kappa\, m_o \cdot x_o} = \frac{\kappa\, b_0\, m_o \cdot (x_1 - x_0)}{(1 - \kappa m_o \cdot x_1)(1 - \kappa m_o \cdot x_o)}$$
$$\leq \frac{\kappa\, b_0\, |m_o|\, |x_1 - x_0|}{(1 - \kappa m_o \cdot x_1)(1 - \kappa m_o \cdot x_o)} \leq C\, \frac{\kappa}{1 - \kappa} |x_1 - x_o|.$$

By changing the roles of x_o and x_1 we conclude that

$$|\rho_l(x_1) - \rho_l(x_o)| \leq C \frac{\kappa}{1 - \kappa} |x_1 - x_o| \text{ for all } l \geq 1.$$

Thus $\{\rho_l : l \geq 1\}$ is an equicontinuous family which is bounded uniformly. Then by *Arzelà - Ascoli* Theorem, if need be by taking subsequence, we have that $\rho_l \to \rho$ uniformly on $\overline{\Omega}$. By Lemma 4.2(i), $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ is a refractor.

We claim that $\mathcal{M}_{\mathcal{R}_l,f}$ converges weakly to $\mathcal{M}_{\mathcal{R},f}$. Indeed, if F is any closed subset of $\overline{\Omega^*}$ then by Lemma 4.2(ii) and reverse Fatou we have

$$\limsup_{l\to\infty} \mathcal{M}_{\mathcal{R}_l,f}(F) \leq \int_{\limsup_{l\to\infty} \mathcal{T}_{\mathcal{R}_l}(F)} f(x)\, dx \leq \int_{\mathcal{T}_{\mathcal{R}}(F)} f(x)\, dx = \mathcal{M}_{\mathcal{R},f}(F).$$

Moreover for any open set $G \subset \overline{\Omega^*}$ we claim that

$$\mathcal{M}_{\mathcal{R}_l,f}(G) = \int_{\mathcal{T}_{\mathcal{R}}(G)} f(x)\, dx \leq \liminf_{\ell\to\infty} \mathcal{M}_{\mathcal{R}_l,f}(G). \tag{4.7}$$

Indeed, from Lemma 4.2(iii) we have

$$\mathcal{M}_{\mathcal{R},f}(G) = \int_{\mathcal{T}_{\mathcal{R}}(G)} f(x)\, dx \leq \int_{\liminf_{\ell\to\infty} \mathcal{T}_{\mathcal{R}_\ell}(G)} f(x)\, dx$$
$$= \int_{\bar{\Omega}^*} \liminf_{\ell\to\infty} \chi_{\mathcal{T}_{\mathcal{R}_\ell}(G)}(x)\, f(x) dx$$
$$\leq \liminf_{\ell\to\infty} \int_{\bar{\Omega}^*} \chi_{\mathcal{T}_{\mathcal{R}_\ell}(G)}(x)\, f(x)\, dx = \liminf_{\ell\to\infty} \mathcal{M}_{\mathcal{R}_l,f}(G),$$

by Fatou's lemma. Consequently $\mathcal{M}_{\mathcal{R}_l,f} \to \mathcal{M}_{\mathcal{R},f}$ weakly.

Since $\mathcal{M}_{\mathcal{R}_l,f}(\omega) = \mu_l(\omega)$ for each Borel set ω, it follows by uniqueness of the weak limit that $\mathcal{M}_{\mathcal{R},f} = \mu$.

□

4.3. **Uniqueness discrete case.** With the method used in this section we show uniqueness when the target measure is discrete. Notice that uniqueness (up to dilations) was already proved with the mass transport approach in Subsection 3.3. But the method we describe here is applicable to the near field problem which is not a mass transport problem.

Recall that $(b_1, \cdots, b_N)$ are positive numbers, $m_1, \cdots, m_N$ are different points in the sphere S^{n-1}, and $\Omega \subset S^{n-1}$ with $\inf_{x\in\bar{\Omega}, 1\le j\le N} x \cdot m_j \ge \kappa$. We let

$$\rho(x) = \min_{1\le i\le N} \frac{b_i}{1 - \kappa x \cdot m_i},$$

and let $\mathcal{R} = \mathcal{R}(\mathbf{b}) = \{\rho(x)x : x \in \bar{\Omega}\}$.

Lemma 4.11. *Suppose that the set $\mathcal{T}_{\mathcal{R}}(m_j)$ has positive measure. If $x_0 \in \mathcal{T}_{\mathcal{R}}(m_j)$, then the semi-ellipsoid $E(m_j, b_j)$ supports $\mathcal{R}$ at the point x_0.*

Proof. We have $m_j \in \mathcal{N}_{\mathcal{R}}(x_0)$, that is, there exists a supporting semi-ellipsoid $E(m_j, b)$ to $\mathcal{R}$ at the point x_0 for some $b > 0$. We claim that $b = b_j$, and therefore $\dfrac{b_j}{1 - \kappa m_j \cdot x}$ supports $\mathcal{R}$ at x_0. Since $E(m_j, b)$ supports $\mathcal{R}$, we have $\rho(x) \le \dfrac{b}{1 - \kappa x \cdot m_j}$ for all $x \in \bar{\Omega}$ with equality at $x = x_0$. Hence $\dfrac{b}{1 - \kappa x_0 \cdot m_j} \le \dfrac{b_j}{1 - \kappa x_0 \cdot m_j}$, and so $b \le b_j$. If $b = b_j$ we are done. If $b < b_j$, then

$$\rho(x) \le \frac{b}{1 - \kappa x \cdot m_j} < \frac{b_j}{1 - \kappa x \cdot m_j}, \quad \forall x \in \bar{\Omega},$$

so $\rho(x) = \min_{k\ne j} \dfrac{b_k}{1 - \kappa m_k \cdot x}$, and therefore $\mathcal{R}$ cannot refract in the direction m_j (except on a set of directions with measure zero) and so $\mathcal{T}_{\mathcal{R}}(m_j)$ has measure zero, a contradiction. □

Lemma 4.12. *Let $\mathcal{R}_b, \mathcal{R}_{b^*}$ be two solutions from Theorem 4.4, with $b = (b_1, \cdots, b_N)$, and $b^* = (b_1^*, \cdots, b_N^*)$. Assume that $f > 0$ a.e. in Ω. If $b_1^* \le b_1$, then $b_i^* \le b_i$ for all $1 \le i \le N$. In particular, if $b_1^* = b_1$, then $b_i^* = b_i$ for all $1 \le i \le N$.*

Proof. Let $J = \{j : b_j < b_j^*\}$ and $I = \{i : b_i \ge b_i^*\}$. Suppose by contradiction that $J \ne \emptyset$. We have $I \ne \emptyset$ since $1 \in I$. Fix $j \in J$, we have $\dfrac{b_j}{1 - \kappa z \cdot m_j} < \dfrac{b_j^*}{1 - \kappa z \cdot m_j}$ for all $z \in \bar{\Omega}$ since $b_j < b_j^*$. And also $\dfrac{b_i^*}{1 - \kappa z \cdot m_i} \le \dfrac{b_i}{1 - \kappa z \cdot m_i}$ for all $i \in I$ and all $z \in \bar{\Omega}$. Fix $j \in J$ and let $x \in \mathcal{T}_{\mathcal{R}_{b^*}}(m_j)$. Since $\mathcal{R}_{b^*}$ is a solution to the discrete problem and $g_i > 0$ for all $1 \le i \le N$, we have that $\mathcal{T}_{\mathcal{R}_{b^*}}(m_j)$ has positive measure. So from Lemma 4.11, the ellipsoid $\dfrac{b_j^*}{1 - \kappa m_j \cdot z}$ supports $\mathcal{R}_{b^*}$ at the point x. Since the function defining $\mathcal{R}_{b^*}$ is given by $\rho^*(z) = \min_{1\le i\le N} \dfrac{b_i^*}{1 - \kappa m_i \cdot z}$, and $\rho^*(x) = \dfrac{b_j^*}{1 - \kappa m_j \cdot x}$, we therefore obtain

$$\frac{b_j}{1 - \kappa m_j \cdot x} < \frac{b_j^*}{1 - \kappa m_j \cdot x} \le \frac{b_i^*}{1 - \kappa m_i \cdot x} \le \frac{b_i}{1 - \kappa m_i \cdot x}, \quad \forall i \in I.$$

Hence by continuity, there exists N_x an open neighborhood of x such that

$$\frac{b_j}{1-\kappa m_j\cdot y} < \frac{b_i}{1-\kappa m_i\cdot y} \quad \forall i\in I \quad \forall y\in N_x.$$

Since the function defining $\mathcal{R}_b$ is $\rho(z)=\min_{1\le i\le N}\dfrac{b_i}{1-\kappa m_i\cdot z}$, we get for $y\in N_x$ that $\rho(y)=\min_{j\in J}\dfrac{b_j}{1-\kappa m_j\cdot y}$, that is, $\rho(y)=\dfrac{b_{j'}}{1-\kappa m_{j'}\cdot y}$ for some $j'\in J$ (depending also on y) which means that $\dfrac{b_{j'}}{1-\kappa m_{j'}\cdot y}$ is a supporting ellipsoid to $\mathcal{R}_b$ at y. Therefore

$$N_x\subset \mathcal{T}_{\mathcal{R}_b}\left(\cup_{j\in J}m_j\right).$$

We then have that every point $x\in \mathcal{T}_{\mathcal{R}_{b^*}}\left(\cup_{j\in J}m_j\right)$ has a neighborhood contained in $\mathcal{T}_{\mathcal{R}_b}\left(\cup_{j\in J}m_j\right)$, that is,

$$\mathcal{T}_{\mathcal{R}_{b^*}}\left(\cup_{j\in J}m_j\right)\subset\left(\mathcal{T}_{\mathcal{R}_b}\left(\cup_{j\in J}m_j\right)\right)^\circ\neq\overline{\Omega}. \tag{4.8}$$

Since $\overline{\Omega}$ is connected and $\mathcal{T}_{\mathcal{R}_{b^*}}\left(\cup_{j\in J}P_j\right)$ is closed, we get that $\left(\mathcal{T}_{\mathcal{R}_b}\left(\cup_{j\in J}m_j\right)\right)^\circ\setminus\mathcal{T}_{\mathcal{R}_{b^*}}\left(\cup_{j\in J}m_j\right)$ is a non empty open set. This is a contradiction with the fact that

$$\int_{\mathcal{T}_{\mathcal{R}_b}\left(\cup_{j\in J}m_j\right)}f(x)\,dx=\sum_{j\in J}f_j=\int_{\mathcal{T}_{\mathcal{R}_{b^*}}\left(\cup_{j\in J}m_j\right)}f(x)\,dx,$$

since $f>0$ a.e..

□

From the lemma we deduce the uniqueness up to dilations in the discrete case. Let $\lambda>0$. Notice that if $E(m,b)$ is a supporting ellipsoid to the refractor $\mathcal{R}_\lambda$ with defining function $\lambda\rho(x)$ at x_0 if and only if $E(m,b/\lambda)$ is a supporting ellipsoid to the refractor $\mathcal{R}$ with defining function $\rho(x)$ at the point x_0. This implies that $\mathcal{N}_{\mathcal{R}_\lambda}(x_0)=\mathcal{N}_{\mathcal{R}}(x_0)$, and consequently $\mathcal{T}_{\mathcal{R}_\lambda}(m)=\mathcal{T}_{\mathcal{R}}(m)$. Therefore, if $\mathcal{R}$ is a refractor solving the problem in Theorem 4.4, then $\mathcal{R}_\lambda$ solves the same problem for any $\lambda>0$. We now prove the uniqueness. Suppose $\mathcal{R}_b$ and $\mathcal{R}_{b^*}$ are two solutions as in Theorem 4.4. Pick λ such that $\lambda b_1=b_1^*$. The refractor $\mathcal{R}_{\lambda b}$ is also a solution to Theorem 4.4, and by Lemma 4.12 we obtain that $\lambda b_i=b_i^*$ for all i. This means that $\mathcal{R}_b$ and $\mathcal{R}_{b^*}$ are multiples one of each other and we obtain the uniqueness.

5. Maxwell's equations

The *electromagnetic field* (EM) is a physical field produced by electrically charged objects. It extends indefinitely throughout space and describes the electromagnetic interaction. The field propagates by electromagnetic radiation; in order of increasing energy (decreasing wavelength) electromagnetic radiation comprises: radio waves, microwaves, infrared, visible light, ultraviolet, X-rays, and gamma

rays. The field (EM) can be viewed as the combination of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{H}$, that is, these are three-dimensional vector fields that have a value defined at every point of space and time: $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{r}, t)$, where $\mathbf{r}$ represents a point in 3-d space $\mathbf{r} = (x, y, z)$. The electric field is produced by stationary charges, and the magnetic field by moving charges (currents); these two are often described as the sources of the field. The way in which $\mathbf{E}$ and $\mathbf{H}$ interact is described by Maxwell's equations (see [Som52, p. 22]):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \text{Gauss's law} \tag{5.1}$$

$$\nabla \cdot \mathbf{H} = 0, \quad \text{Gauss's law for magnetism} \tag{5.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \quad \text{Faraday's law} \tag{5.3}$$

$$\nabla \times \mathbf{H} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \text{Ampère-Maxwell's law.} \tag{5.4}$$

Here

$\nabla = (\partial_x, \partial_y, \partial_z)$	the gradient
$\rho = \rho(\mathbf{r}, t)$	charge density
ϵ_0	permittivity of free space
$\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$	current density vector
μ_0	permeability of free space

We have $c = 1/\sqrt{\epsilon_0 \mu_0}$, the speed of light in vacuum.

5.1. **General case.** In several situations is necessary to consider a medium where the magnetic permeability $\mu = \mu(x, y, z)$[‡] and the electric permittivity $\epsilon = \epsilon(x, y, z)$[§] are not constants. This is the case when the physical properties of the medium change from point to point, in particular, this happens in geometric optics when materials of different refractive indices are considered. In such case the Maxwell equations have the form:

$$\nabla \times \mathbf{E} = -\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t}, \tag{5.5}$$

$$\nabla \times \mathbf{H} = \frac{2\pi}{c} \sigma \mathbf{E} + \frac{\epsilon}{c} \frac{\partial \mathbf{E}}{\partial t} \tag{5.6}$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 4\pi\rho \tag{5.7}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0, \tag{5.8}$$

c being the speed of light in vacuum. Recall that substances for which $\sigma \neq 0$ are conductors and if σ is negligibly small, the substances are called insulators or

[‡]For values of μ for different substances see `http://en.wikipedia.org/wiki/Permeability_(electromagnetism)#Values_for_some_common_materials`.

[§]For relative permittivity of some substances see `http://en.wikipedia.org/wiki/Relative_permittivity`.

dielectrics, see [BW59][Section 1.1.2]. Under certain assumptions on the field and the physical set up we have that $\mathbf{J} = \sigma\mathbf{E}$, see [BW59][Section 1.1.2, formula (9)].

It is important to notice that these equations are written in Gaussian units, and the Maxwell equations in the first section written in SI units.

5.2. **Maxwell equations in integral form.** Points in $\mathbb{R}^4$ are denoted by (x, y, z, t), and suppose $D \subset \mathbb{R}^4$ is a domain for which the divergence theorem holds, for example, the boundary is piecewise smooth, that is, a finite union of C^1 surfaces. For a point $P = (x, y, z, t)$ on the boundary ∂D, the unit outer normal at P is denoted by $\nu = (\nu_x, \nu_y, \nu_z, \nu_t)$. From equation (5.6)

$$\nabla \times \mathbf{H} - \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{2\pi}{c}\sigma\mathbf{E}. \tag{5.9}$$

Recall we assume $\epsilon = \epsilon(x, y, z)$, and we want to derive an integral form of the last equation that does not require differentiability of the fields. In order to do that, we initially assume the fields are smooth and applying the divergence theorem we will obtain formulas independent of the derivatives of the fields. Set $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)$. We have

$$\begin{aligned}
&\int_D \nabla \times \mathbf{H}\, dxdydzdt \\
&= \mathbf{i} \int_D (\partial_y \mathbf{H}_3 - \partial_z \mathbf{H}_2) - \mathbf{j} \int_D (\partial_x \mathbf{H}_3 - \partial_z \mathbf{H}_1) + \mathbf{k} \int_D (\partial_x \mathbf{H}_2 - \partial_y \mathbf{H}_1) \\
&= \mathbf{i} \int_D \operatorname{div}\,(0, \mathbf{H}_3, -\mathbf{H}_2, 0) - \mathbf{j} \int_D \operatorname{div}\,(\mathbf{H}_3, 0, -\mathbf{H}_1, 0) + \mathbf{k} \int_D \operatorname{div}\,(\mathbf{H}_2, -\mathbf{H}_1, 0, 0) \\
&= \mathbf{i} \int_{\partial D} (0, \mathbf{H}_3, -\mathbf{H}_2, 0) \cdot (\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma \\
&\qquad - \mathbf{j} \int_{\partial D} (\mathbf{H}_3, 0, -\mathbf{H}_1, 0) \cdot (\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma + \mathbf{k} \int_{\partial D} (\mathbf{H}_2, -\mathbf{H}_1, 0, 0) \cdot (\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)\, d\sigma.
\end{aligned}$$

So integrating (5.9) over D yields

$$\begin{aligned}
&\int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)\, d\sigma - \int_D \frac{\epsilon}{c}\mathbf{E}_t\, dxdydzdt \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)\, d\sigma - \int_D \left(\frac{\epsilon}{c}\mathbf{E}\right)_t dxdydzdt \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)\, d\sigma - \int_{\partial D} \frac{\epsilon}{c}\mathbf{E}\,\nu_t\, d\sigma \\
&= \int_{\partial D} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3) - \frac{\epsilon}{c}\mathbf{E}\,\nu_t\right) d\sigma = \int_D \frac{2\pi}{c}\sigma\mathbf{E}\, dxdydzdt.
\end{aligned}$$

Therefore the surface integral

$$\int_{\partial D}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3)-\frac{\epsilon}{c}\mathbf{E}\,\nu_t\right)d\sigma=\int_D\frac{2\pi}{c}\sigma\mathbf{E}\,dxdydzdt, \tag{5.10}$$

for each closed hyper-surface ∂D in $\mathbb{R}^4$. Proceeding in the same way with equation (5.5) we obtain that the surface integral

$$\int_{\partial D}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)+\frac{\mu}{c}\mathbf{H}\,\nu_t\right)d\sigma=0, \tag{5.11}$$

for each closed hyper-surface ∂D in $\mathbb{R}^4$.

Concerning equations (5.7) and (5.8), proceeding in the same way as before we obtain that

$$\int_{\partial D}\epsilon\mathbf{E}\cdot\nu\,d\sigma=4\pi\int_D\rho\,dxdydzdt \tag{5.12}$$

$$\int_{\partial D}\mu\mathbf{H}\cdot\nu\,d\sigma=0, \tag{5.13}$$

for each domain $D\subset\mathbb{R}^4$ for which the divergence theorem holds. These formulas make sense as long as the fields $\mathbf{E},\mathbf{H}$ and the coefficients μ and ϵ are piecewise continuous over ∂D and bounded. Equations (5.10), (5.11), (5.12), and (5.13) are Maxwell's equations in integral form.

5.3. **Boundary conditions at a surface of discontinuity.** Let us consider a point $P_0=(x_0,y_0,z_0,t_0)$, a hyper-surface Γ_0 passing through P_0 and suppose that the fields $\mathbf{H}$ and $\mathbf{E}$, solutions to the Maxwell equations in integral form, as well as the functions ϵ and μ, are discontinuous on Γ_0. Suppose that all these quantities are defined locally around P_0 say in the 4-dimensional ball $B_R(P_0)$. This situation is typical when we have two media with different indices of refraction and the surface Γ_0 is the one separating the two media. The surface Γ_0 divides the open ball $B_R(P_0)$ into two open pieces: B_R^+ and B_R^-. In order to make sense of the integrals we assume the surface Γ_0 is C^1, the fields $\mathbf{E}$ and $\mathbf{H}$, and ϵ and μ are bounded in $B_R(P_0)$, and all continuous on $B_R(P_0)\setminus\Gamma_0$. We assume also that for each $Q\in\Gamma_0\cap B_R(P_0)$ the following limits exist and are finite:

$$\lim_{P\to Q,P\in B_R^+}\mathbf{E}(P)=\mathbf{E}^+(Q),\qquad \lim_{P\to Q,P\in B_R^+}\mathbf{H}(P)=\mathbf{H}^+(Q)$$
$$\lim_{P\to Q,P\in B_R^-}\mathbf{E}(P)=\mathbf{E}^-(Q),\qquad \lim_{P\to Q,P\in B_R^-}\mathbf{H}(P)=\mathbf{H}^-(Q),$$

and similar quantities for ϵ and μ. Let us call $\Gamma_+(R)$ the boundary of B_R^+, and $\Gamma_-(R)$ the boundary of B_R^-. If we let $\mathbf{E}^+(Q)=\mathbf{E}(Q)$ for $Q\in B_R^+$ and $\mathbf{E}^+(Q)=\lim_{P\to Q,P\in B_R^+}\mathbf{E}(P)$ for $Q\in\Gamma_0\cap B_R$, and similarly $\mathbf{H}^+$, μ^+, and ϵ^+, then all these functions are continuous in B_R^+. In a similar way, we define $\mathbf{E}^-,\mathbf{H}^-,\mu_-$ and ϵ_- in B_R^- that are continuous in

B_R^-. Hence applying (5.10) with $D = B_R^\pm$ yields

(5.14)
$$\int_{\Gamma_+(R)\cup(\Gamma_0\cap B_R(P_0))} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^+,\mathbf{H}_2^+,\mathbf{H}_3^+) - \frac{\epsilon^+}{c}\mathbf{E}^+\,\nu_t\right) d\sigma = \int_{B_R^+} \frac{2\pi}{c}\sigma\mathbf{E}^+\,dxdydzdt,$$

and

(5.15)
$$\int_{\Gamma_-(R)\cup(\Gamma_0\cap B_R(P_0))} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^-,\mathbf{H}_2^-,\mathbf{H}_3^-) - \frac{\epsilon^-}{c}\mathbf{E}^-\,\nu_t\right) d\sigma = \int_{B_R^-} \frac{2\pi}{c}\sigma\mathbf{E}^-\,dxdydzdt.$$

Now

$$\begin{aligned}
(5.16)\qquad &\int_{\Gamma_+(R)\cup(\Gamma_0\cap B_R(P_0))} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^+,\mathbf{H}_2^+,\mathbf{H}_3^+) - \frac{\epsilon^+}{c}\mathbf{E}^+\,\nu_t\right) d\sigma \\
&= \int_{\Gamma_+(R)} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3) - \frac{\epsilon}{c}\mathbf{E}\,\nu_t\right) d\sigma \\
&\quad + \int_{\Gamma_0\cap B_R(P_0)} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^+,\mathbf{H}_2^+,\mathbf{H}_3^+) - \frac{\epsilon^+}{c}\mathbf{E}^+\,\nu_t\right) d\sigma,
\end{aligned}$$

where in the integral over $\Gamma_0 \cap B_R(P_0)$, $\nu := (\nu_x,\nu_y,\nu_z,\nu_t)$ is the downward unit normal to Γ_0; and

$$\begin{aligned}
(5.17)\qquad &\int_{\Gamma_-(R)\cup(\Gamma_0\cap B_R(P_0))} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^-,\mathbf{H}_2^-,\mathbf{H}_3^-) - \frac{\epsilon^-}{c}\mathbf{E}^-\,\nu_t\right) d\sigma \\
&= \int_{\Gamma_-(R)} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3) - \frac{\epsilon}{c}\mathbf{E}\,\nu_t\right) d\sigma \\
&\quad + \int_{\Gamma_0\cap B_R(P_0)} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^-,\mathbf{H}_2^-,\mathbf{H}_3^-) - \frac{\epsilon^-}{c}\mathbf{E}^-\,\nu_t\right) d\sigma,
\end{aligned}$$

where in the integral over $\Gamma_0 \cap B_R(P_0)$, ν is the upward unit normal to Γ_0. Adding (5.16) and (5.17), and using (5.14) and (5.15) yields

$$\begin{aligned}
&\int_{B_R^+} \frac{2\pi}{c}\sigma\mathbf{E}^+\,dxdydzdt + \int_{B_R^-} \frac{2\pi}{c}\sigma\mathbf{E}^-\,dxdydzdt \\
&= \int_{\Gamma_+(R)} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3) - \frac{\epsilon}{c}\mathbf{E}\,\nu_t\right) d\sigma + \int_{\Gamma_-(R)} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3) - \frac{\epsilon}{c}\mathbf{E}\,\nu_t\right) d\sigma \\
&+ \int_{\Gamma_0\cap B_R(P_0)} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}^+ - \mathbf{H}^-) - \frac{1}{c}\left(\epsilon^+\,\mathbf{E}^+ - \epsilon^-\,\mathbf{E}^-\right)\nu_t\right) d\sigma.
\end{aligned}$$

On the other hand, applying (5.10) with $D = B_R$ yields

$$\int_{\Gamma_+(R)\cup\Gamma_-(R)} \left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3) - \frac{\epsilon}{c}\mathbf{E}\,\nu_t\right) d\sigma = \int_{B_R} \frac{2\pi}{c}\sigma\mathbf{E}\,dxdydzdt.$$

Since the field $\mathbf{E}$ is discontinuous only on Γ_0, which is a set of measure zero, we therefore obtain

$$\int_{\Gamma_0 \cap B_R(P_0)} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{H}^+ - \mathbf{H}^-) - \frac{1}{c} \left(\epsilon^+ \mathbf{E}^+ - \epsilon^- \mathbf{E}^- \right) \nu_t \right) d\sigma = 0$$

for all R sufficiently small. Now letting $R \to 0$ we obtain the following equation valid at P_0

$$(\nu_x, \nu_y, \nu_z) \times (\mathbf{H}^+ - \mathbf{H}^-) - \frac{1}{c}(\epsilon^+ \mathbf{E}^+ - \epsilon^- \mathbf{E}^-)\, \nu_t = 0, \tag{5.18}$$

where ν is the normal to the interface Γ_0 at the point P_0.

Suppose the interface is independent of time and is given by a function $\phi(x, y, z) = 0$, then the normal at a point is $\nu = (\phi_x, \phi_y, \phi_z, 0)$, therefore equation (5.18) becomes

$$\nabla\phi \times (\mathbf{H}^+ - \mathbf{H}^-) = 0.$$

We can write $\mathbf{H}^\pm = \mathbf{H}^\pm_{\text{tan}} + \mathbf{H}^\pm_{\text{perp}}$, where $\mathbf{H}^\pm_{\text{perp}}$ is the component in the direction of the normal $\nabla\phi$ and $\mathbf{H}^\pm_{\text{tan}}$ is the component perpendicular to the normal. We have $\nabla\phi \times \mathbf{H}^\pm = \nabla\phi \times \mathbf{H}^\pm_{\text{tan}} + \nabla\phi \times \mathbf{H}^\pm_{\text{perp}} = \nabla\phi \times \mathbf{H}^\pm_{\text{tan}}$. So

$$0 = \nabla\phi \times (\mathbf{H}^+ - \mathbf{H}^-) = \nabla\phi \times (\mathbf{H}^+_{\text{tan}} - \mathbf{H}^-_{\text{tan}}) = |\nabla\phi|\, |\mathbf{H}^+_{\text{tan}} - \mathbf{H}^-_{\text{tan}}|,$$

since the vectors are perpendicular. So if $\nabla\phi \neq 0$, the obtain the important relation that

$$\mathbf{H}^+_{\text{tan}} - \mathbf{H}^-_{\text{tan}} = 0,$$

that is, *the tangential components of the magnetic field are continuous across the boundary*.

Proceeding in the same manner this time with (5.11) yields the equation

$$(\nu_x, \nu_y, \nu_z) \times (\mathbf{E}^+ - \mathbf{E}^-) + \frac{1}{c}(\mu^+ \mathbf{H}^+ - \mu^- \mathbf{H}^-)\, \nu_t = 0, \tag{5.19}$$

where ν is the normal to the interface Γ_0 at the point P_0. If the interface is independent of t, proceeding exactly as before, we obtain

$$\nabla\phi \times (\mathbf{E}^+ - \mathbf{E}^-) = 0,$$

and

$$\mathbf{E}^+_{\text{tan}} - \mathbf{E}^-_{\text{tan}} = 0,$$

that is, also *the tangential components of the electric field are continuous across the boundary*.

In regard to equations (5.12) and (5.13), we obtain similarly that

$$(\epsilon^+ \mathbf{E}^+ - \epsilon^- \mathbf{E}^-) \cdot \nu = 0, \text{ and } (\mu^+ \mathbf{H}^+ - \mu^- \mathbf{H}^-) \cdot \nu = 0.$$

Since $\mathbf{H}^\pm \cdot \nu = \mathbf{H}^\pm_{\text{perp}} \cdot \nu$, and similarly for $\mathbf{E}$, assuming $\phi = \phi(x, y, z)$ yields

$$0 = (\epsilon^+ \mathbf{E}^+_{\text{perp}} - \epsilon^- \mathbf{E}^-_{\text{perp}}) \cdot \nabla\phi = |\epsilon^+ \mathbf{E}^+_{\text{perp}} - \epsilon^- \mathbf{E}^-_{\text{perp}}|\, |\nabla\phi|,$$

and

$$0 = (\mu^+ \mathbf{H}^+_{\text{perp}} - \mu^- \mathbf{H}^-_{\text{perp}}) \cdot \nabla\phi = |\mu^+ \mathbf{H}^+_{\text{perp}} - \mu^- \mathbf{H}^-_{\text{perp}}|\, |\nabla\phi|,$$

and therefore

$$|\epsilon^+\mathbf{E}^+_{\text{perp}} - \epsilon^-\mathbf{E}^-_{\text{perp}}| = |\mu^+\mathbf{H}^+_{\text{perp}} - \mu^-\mathbf{H}^-_{\text{perp}}| = 0.$$

Therefore *the perpendicular components of the fields* $\epsilon\mathbf{E}$ *and* $\mu\mathbf{H}$ *are continuous across the interface.*

5.4. **Maxwell's equations in the absence of charges.** This is the case when $\rho = 0$ and $\mathbf{J} = 0$. So the equations become

$$\nabla \cdot \mathbf{E} = 0, \tag{5.20}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{5.21}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \tag{5.22}$$

$$\nabla \times \mathbf{H} = \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}, \tag{5.23}$$

5.5. **The wave equation.** Recall the following formula from vector analysis for a vector $\mathbf{A} = \mathbf{A}(x, y, z) = (\mathbf{A}_x(x, y, z), \mathbf{A}_y(x, y, z), \mathbf{A}_z(x, y, z))$:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A}. \tag{5.24}$$

Denote $\nabla \cdot \nabla = \nabla^2$, the Laplacian, and so

$$\nabla^2\mathbf{A} = \left(\frac{\partial^2\mathbf{A}_x}{\partial x^2} + \frac{\partial^2\mathbf{A}_x}{\partial y^2} + \frac{\partial^2\mathbf{A}_x}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2\mathbf{A}_y}{\partial x^2} + \frac{\partial^2\mathbf{A}_y}{\partial y^2} + \frac{\partial^2\mathbf{A}_y}{\partial z^2}\right)\mathbf{j} + \left(\frac{\partial^2\mathbf{A}_z}{\partial x^2} + \frac{\partial^2\mathbf{A}_z}{\partial y^2} + \frac{\partial^2\mathbf{A}_z}{\partial z^2}\right)\mathbf{k}.$$

From Faraday's law and Ampère's law

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial(\nabla \times \mathbf{H})}{\partial t} = -\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

and so from formula (5.24) we obtain that $\mathbf{E}$ satisfies the wave equation

$$\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} = \nabla^2\mathbf{E}.$$

Proceeding in the same manner for $\mathbf{H}$ we obtain

$$\epsilon_0\mu_0\frac{\partial^2\mathbf{H}}{\partial t^2} = \nabla^2\mathbf{H}.$$

That is, both the electric and magnetic fields satisfy the wave equation. We have from physical considerations that

$$v = c = \frac{1}{\sqrt{\epsilon_0\mu_0}},$$

c being the speed of propagation of light in free space, which in this case is the velocity v of propagation. If free space is changed by a material with other values

of μ_0 and ϵ_0, the velocity v represent the speed of propagation of waves in this material.[¶]

5.6. **Dispersion equation.** Suppose $\mathbf{E}$ and $\mathbf{H}$ solve the Maxwell equations (5.5), (5.6), (5.7), (5.8) with $\rho = 0$, $\sigma = 0$, and ϵ and μ constants. Then

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\mu}{c}\nabla \times \mathbf{H}_t = -\frac{\mu}{c}(\nabla \times \mathbf{H})_t = -\frac{\epsilon\mu}{c^2}\mathbf{E}_{tt}.$$

On the other hand, from (5.24), $\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2\mathbf{E}$ and so

$$\nabla^2\mathbf{E} - \frac{\epsilon\mu}{c^2}\mathbf{E}_{tt} = 0,$$

and similarly

$$\nabla^2\mathbf{H} - \frac{\epsilon\mu}{c^2}\mathbf{H}_{tt} = 0.$$

If $\mathbf{E} = A\cos(\mathbf{r}\cdot\mathbf{k} + \omega t)$, then we obtain the dispersion equation

$$|\mathbf{k}|^2 = \epsilon\mu\left(\frac{\omega}{c}\right)^2. \tag{5.25}$$

5.7. **Plane waves.** Let $\mathbf{s}$ be a unit vector. Any solution to the wave equation

$$\frac{1}{v^2}\partial_t^2 V = \nabla^2 V,$$

of the form $V(\mathbf{r}, t) = F(\mathbf{r}\cdot\mathbf{s}, t)$ is called a *plane wave*, since at each time t, V is constant on each plane of the form $\mathbf{r}\cdot\mathbf{s}$=constant. That is, for each t the vector $V(\mathbf{r}, t)$ is the same on each plane $\mathbf{r}\cdot\mathbf{s}$=constant. The plane wave propagates in the direction $\mathbf{s}$. It can be proved that any solution to the wave equation of this form can be written as

$$V(\mathbf{r}, t) = V_1(\mathbf{r}\cdot\mathbf{s} - vt) + V_2(\mathbf{r}\cdot\mathbf{s} + vt)$$

where V_1, V_2 are arbitrary functions, see [BW59][Section 1.3.1]. Since the fields $\mathbf{E}$ and $\mathbf{H}$ both satisfy the wave equation, it is then natural to consider the case when

$$\mathbf{E} = \mathbf{E}(\mathbf{r}\cdot\mathbf{s} - vt), \quad \mathbf{H} = \mathbf{H}(\mathbf{r}\cdot\mathbf{s} - vt),$$

that is, $\mathbf{E}$ and $\mathbf{H}$ are functions of the scalar variable $\mathbf{r}\cdot\mathbf{s} - vt$. We have

$$\frac{\partial\mathbf{E}}{\partial t} = -v\mathbf{E}', \text{ and } \nabla \times \mathbf{E} = s \times \mathbf{E}';$$

[¶]The relative permittivity is ϵ/ϵ_0 and the relative permeability is μ/μ_0; the index of refraction is defined by $n = \sqrt{\epsilon_r\mu_r}$. The velocity of propagation $v = \frac{1}{\sqrt{\epsilon\mu}}$. Since $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$, we get that $n = c/v$.

and similarly for $\mathbf{H}$ under the assumption that $\mathbf{J} = 0$. Thus, from the Faraday and Ampère laws, and since $v^2 = \dfrac{1}{\epsilon_0\mu_0}$, we obtain the equations

$$s \times \mathbf{E}' = v\mathbf{H}'$$
$$s \times \mathbf{H}' = -\frac{1}{v}\mathbf{E}'.$$

Since s is a constant vector $s \times \mathbf{E}' = (s \times \mathbf{E})'$, and so the equations are

$$(s \times \mathbf{E})' = v\mathbf{H}'$$
$$(s \times \mathbf{H})' = -\frac{1}{v}\mathbf{E}'.$$

Integrating these equations and taking constants of integration zero (which amounts to neglect constant fields), we obtain the very important equations relating the electric and magnetic fields

$$\mathbf{E} = -v(s \times \mathbf{H}) \tag{5.26}$$

$$\mathbf{H} = \frac{1}{v}(s \times \mathbf{E}). \tag{5.27}$$

This shows that $\mathbf{s} \cdot \mathbf{E} = \mathbf{s} \cdot \mathbf{H} = \mathbf{0}$, that means, the electric and magnetic field are always *perpendicular* to the direction of propagation $\mathbf{s}$. In addition, $\mathbf{E} \cdot \mathbf{H} = v(s \times \mathbf{H}) \cdot \mathbf{H} = 0$, that is, $\mathbf{E}$ and $\mathbf{H}$ are always perpendicular. We also obtain taking absolute values that

$$|\mathbf{E}| = v|\mathbf{H}|.$$

5.8. **Fresnel formulas.** We consider plane waves whose components have the form

$$a\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}}{v}\right) + \delta\right) = a\cos\left(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta\right),$$

that is, $\mathbf{k} = \dfrac{\omega}{v}\mathbf{s}$ and a, δ are real numbers. The quantity $\omega t - \mathbf{k}\cdot\mathbf{r} + \delta$ is called the *phase*, and a is called the *amplitude*.

Let $\mathbf{s}^i$ be the direction (unit) of an incident plane wave traveling for a while in media I with velocity of propagation v_1 that hits, at a point P, a boundary Γ between I and another media II where the velocity of propagation is v_2 (I and II are also called dielectrics as they are materials with zero conductivity, that is $\sigma = 0$ and so the current density vector $\mathbf{J} = 0$, see Subsection 5.1). Then the wave splits into two waves: a *transmitted wave* propagating in media II and a *reflected wave* propagated back into media I. We shall assume that these two waves are also plane. The plane determined by v and $\mathbf{s}^i$ is called the *incidence plane*.

It is important to remark that for our analysis, we will choose a local system of coordinates around the point P. Indeed, we are going to write all fields as functions of $\mathbf{r}$ (position) and t, with $\mathbf{r}$ close to zero, such that the coordinates of P in this system are $\mathbf{r} = 0$. In particular, the fields will be calculated near the point P.

We choose a rectangular right-hand system of coordinates x, y, z such that the normal ν is on the z-axis, and the x and y axes are on the plane perpendicular to ν and in such a way that the vector $\mathbf{s}^i$ lies on the xz-plane. This means that the tangent plane to Γ at P is the xy-plane; in particular, $P = (0, 0, 0)$. So we assume that

$$\mathbf{s}^i = \sin\theta_i \mathbf{i} + \cos\theta_i \mathbf{k}$$

that is, $\mathbf{s}^i$ lives on the xz-plane and so the direction of propagation is perpendicular to the y-axis and θ_i is the angle between the normal vector ν to the boundary at P (the z-axis) and the incident direction $\mathbf{s}^i$ (as usual $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit coordinate vectors).

The electric field corresponding to this incident field is

$$\mathbf{E}^i(\mathbf{r}, t) = (-I_\parallel \cos\theta_i, I_\perp, I_\parallel \sin\theta_i) \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right) = \mathbf{E}_0^i \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right). \tag{5.28}$$

Notice that $\mathbf{E}$ has this form because, as is was proved in Subsection 5.7, $\mathbf{E}$ is always perpendicular to the direction of propagation $\mathbf{s}^i$. Notice also that the field $\mathbf{E}^i$ has a component that is perpendicular to the plane of incidence and a component that is parallel to this plane, indeed, we write

$$\mathbf{E}_\perp^i = I_\perp \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right)\mathbf{j},$$

and

$$\mathbf{E}_\parallel^i = (-I_\parallel \cos\theta_i\, \mathbf{i} + I_\parallel \sin\theta_i\, \mathbf{k}) \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right).$$

Also notice that

$$|\mathbf{E}^i|^2 = \left(I_\parallel^2 + I_\perp^2\right)\cos^2\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right)$$

From (5.27), the magnetic field is then

$$\mathbf{H}^i(\mathbf{r}, t) = \frac{1}{v_1}(-I_\perp \cos\theta_i, -I_\parallel, I_\perp \sin\theta_i) \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right) = \mathbf{H}_0^i \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right).$$

Let us now introduce $\mathbf{s}^t$, the direction of propagation of the transmitted wave, and θ_t the angle between the normal ν and $\mathbf{s}^t$, and similarly, $\mathbf{s}^r$ is the direction of propagation of the reflected wave and θ_r is the angle between the normal ν and $\mathbf{s}^r$. We have from the Snell law that $\mathbf{s}^r = \sin\theta_r \mathbf{i} + \cos\theta_r \mathbf{k} = \sin\theta_i \mathbf{i} - \cos\theta_i \mathbf{k}$. Then the electric and magnetic fields corresponding to transmission are

(5.29)

$$\mathbf{E}^t(\mathbf{r}, t) = (-T_\parallel \cos\theta_t, T_\perp, T_\parallel \sin\theta_t) \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right) = \mathbf{E}_0^t \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right)$$

$$\mathbf{H}^t(\mathbf{r}, t) = \frac{1}{v_2}(-T_\perp \cos\theta_t, -T_\parallel, T_\perp \sin\theta_t) \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right) = \mathbf{H}_0^t \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right);$$

and similarly the fields corresponding to reflection are

(5.30)

$$\mathbf{E}^r(\mathbf{r},t) = (-R_\parallel \cos\theta_r, R_\perp, R_\parallel \sin\theta_r)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right) = \mathbf{E}_0^r \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right)$$

$$\mathbf{H}^r(\mathbf{r},t) = \frac{1}{v_1}(-R_\perp \cos\theta_r, -R_\parallel, R_\perp \sin\theta_r)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right) = \mathbf{H}_0^r \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right).$$

Recall that from Snell's law all vectors $\mathbf{s}^i, \mathbf{s}^t, \mathbf{s}^r$ and ν all live on the same plane, that is, the xz-plane. Each of the electric and magnetic fields can be decomposed uniquely as a sum of a component in the direction of the normal (normal component) or on the z-axis, plus another component perpendicular to the normal (tangential component) or on the xy-plane. From the integral form of Maxwell's equations, as it shown in Subsection 5.3, the tangential components of $\mathbf{E}$ (and also of $\mathbf{H}$ if $\mathbf{J} = 0$) at the interface are continuous (see also [BW59][Section 1.1.3, formula (23)]). Since the electric field on medium 1 near Γ equals $\mathbf{E}^i + \mathbf{E}^r$, we get $\mathbf{E}^i_{\tan} + \mathbf{E}^r_{\tan} = \mathbf{E}^t_{\tan}$ on Γ, since $\mathbf{E}^t_{\tan}$ is the transmitted electric field in media 2 near Γ. From the configuration we have, we can write $\mathbf{E}^i = \mathbf{E}^i_{\text{normal}}\mathbf{k} + \mathbf{E}^i_{\tan}$, and so $\mathbf{k}\times\mathbf{E}^i = \mathbf{k}\times\mathbf{E}^i_{\tan}$. Similarly, $\mathbf{k}\times\mathbf{E}^r = \mathbf{k}\times\mathbf{E}^r_{\tan}$ and $\mathbf{k}\times\mathbf{E}^t = \mathbf{k}\times\mathbf{E}^t_{\tan}$. So $\mathbf{k}\times\mathbf{E}^i + \mathbf{k}\times\mathbf{E}^r = \mathbf{k}\times\mathbf{E}^t$. Then

$$\mathbf{k}\times\mathbf{E}_0^i\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right) + \mathbf{k}\times\mathbf{E}_0^r\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right) = \mathbf{k}\times\mathbf{E}_0^t\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right),$$

for all $\mathbf{r}$ close to zero and all t. The interface point P is $\mathbf{r} = (0,0,0)$, so in particular, we obtain

$$\mathbf{k}\times\mathbf{E}_0^i\cos(\omega t) + \mathbf{k}\times\mathbf{E}_0^r\cos(\omega t) = \mathbf{k}\times\mathbf{E}_0^t\cos(\omega t)$$

for all t. Eliminating the cosines we get

$$\mathbf{k}\times\mathbf{E}_0^i + \mathbf{k}\times\mathbf{E}_0^r = \mathbf{k}\times\mathbf{E}_0^t. \tag{5.31}$$

Since we are assuming the current density vector $\mathbf{J} = 0$, as it was mentioned earlier, the tangential component of the magnetic field is also continuous across the interface. So as before with the electric field, we have $\mathbf{H}^i_{\tan} + \mathbf{H}^r_{\tan} = \mathbf{H}^t_{\tan}$ on Γ, and so

$$\mathbf{k}\times\mathbf{H}_0^i + \mathbf{k}\times\mathbf{H}_0^r = \mathbf{k}\times\mathbf{H}_0^t. \tag{5.32}$$

From (5.31) we obtain the equations

$$I_\perp + R_\perp = T_\perp, \qquad \cos\theta_i(I_\parallel - R_\parallel) = \cos\theta_t T_\parallel;$$

and from (5.32) we obtain

$$\frac{I_\parallel}{v_1} + \frac{R_\parallel}{v_1} = \frac{T_\parallel}{v_2}, \qquad \cos\theta_i\left(\frac{I_\perp}{v_1} - \frac{R_\perp}{v_1}\right) = \cos\theta_t\frac{T_\perp}{v_2}.$$

We have $n_1 = c/v_1$ and $n_2 = c/v_2$ so solving the last two sets of equations yields

$$\begin{aligned}
T_\parallel &= \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} I_\parallel \\
T_\perp &= \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} I_\perp \\
R_\parallel &= \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} I_\parallel \\
R_\perp &= \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} I_\perp.
\end{aligned}$$

These are the *Fresnel equations* expressing the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave.

5.9. **Rewriting the Fresnel Equations.** We will replace $\mathbf{s^i}$ by x and $\mathbf{s^t}$ by m, and we also set $\kappa = n_2/n_1$. Recall ν is the normal to the interface. We have $\cos\theta_i = x \cdot \nu$ and $\cos\theta_t = m \cdot \nu$. In addition, from the Snell law $x - \kappa m = \lambda \nu$, so the Fresnel equations have the form

$$\begin{aligned}
T_\parallel &= \frac{2\,x \cdot \nu}{\kappa\, x \cdot \nu + m \cdot \nu} I_\parallel = \frac{2\,x \cdot \nu}{(\kappa\, x + m) \cdot \nu} I_\parallel = \frac{2\,x \cdot (x - \kappa\, m)}{(\kappa\, x + m) \cdot (x - \kappa\, m)} I_\parallel \\
T_\perp &= \frac{2\,x \cdot \nu}{x \cdot \nu + \kappa\, m \cdot \nu} I_\perp = \frac{2\,x \cdot \nu}{(x + \kappa\, m) \cdot \nu} I_\perp = \frac{2\,x \cdot (x - \kappa\, m)}{(x + \kappa\, m) \cdot (x - \kappa\, m)} I_\perp \\
R_\parallel &= \frac{\kappa\, x \cdot \nu - m \cdot \nu}{\kappa\, x \cdot \nu + m \cdot \nu} I_\parallel = \frac{(\kappa\, x - m) \cdot \nu}{(\kappa\, x + m) \cdot \nu} I_\parallel = \frac{(\kappa\, x - m) \cdot (x - \kappa\, m)}{(\kappa\, x + m) \cdot (x - \kappa\, m)} I_\parallel \\
R_\perp &= \frac{x \cdot \nu - \kappa\, m \cdot \nu}{x \cdot \nu + \kappa\, m \cdot \nu} I_\perp = \frac{(x - \kappa\, m) \cdot \nu}{(x + \kappa\, m) \cdot \nu} I_\perp = \frac{(x - \kappa\, m) \cdot (x - \kappa\, m)}{(x + \kappa\, m) \cdot (x - \kappa\, m)} I_\perp.
\end{aligned}$$

Notice that the denominators of the perpendicular components are the same and likewise for the parallel components.

5.10. **The Poynting vector.** It is defined by

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H},$$

where c is the speed of light in free space. The vector $\mathbf{S}$ represents the flux of energy through a surface. Suppose dA is the area of a surface element at a point P and let ν be the normal at P. Then the flux of energy through dA at the point P is given by

$$dF = \mathbf{S} \cdot \nu\, dA.$$

From (5.27) we get that

$$\mathbf{S} = \frac{c}{4\pi v} \mathbf{E} \times (\mathbf{s} \times \mathbf{E}) = \frac{n}{4\pi} |\mathbf{E}|^2 \mathbf{s}.$$

Using the form of the incident wave from the previous section, the amount of energy J^i flowing through a unit area of the boundary per second at P, of the incident wave $\mathbf{E}^i$ given in (5.28), is then

$$J^i = |\mathbf{S}^i| \cos\theta_i = \frac{n_1}{4\pi}|\mathbf{E}_0^i|^2 \cos\theta_i.^{\|}$$

Similarly, the amount of energy in the reflected and transmitted waves (also given in the previous section) leaving a unit area of the boundary per second at P are given by

$$J^r = |\mathbf{S}^r| \cos\theta_i = \frac{n_1}{4\pi}|\mathbf{E}_0^r|^2 \cos\theta_i$$
$$J^t = |\mathbf{S}^t| \cos\theta_t = \frac{n_2}{4\pi}|\mathbf{E}_0^t|^2 \cos\theta_t.$$

The reflection and transmission coefficients are defined by

$$\mathcal{R} = \frac{J^r}{J^i} = \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2, \text{ and } \mathcal{T} = \frac{J^t}{J^i} = \frac{n_2 \cos\theta_t}{n_1 \cos\theta_i}\left(\frac{|\mathbf{E}_0^t|}{|\mathbf{E}_0^i|}\right)^2. \tag{5.33}$$

By conservation of energy or by direct verification $\mathcal{R} + \mathcal{T} = 1$.

5.11. **Polarization.** Polarization is a property of the field that describes the orientation of their oscillations. Since the electric vector is assumed a plane wave and as we showed it is perpendicular to the direction of propagation $\mathbf{s}$, then for each $\mathbf{r}$ in the plane $\mathbf{r}\cdot\mathbf{s} = c$ and t fixed, the vector $\mathbf{E}(\mathbf{r}, t)$ is constant. We visualize $\mathbf{E}(\mathbf{r}, t)$ as a vector with origin at the intersection of the direction $\mathbf{s}$ with the plane $\mathbf{r}\cdot\mathbf{s} = c$. That is, as a vector with origin at the point $(\mathbf{r}\cdot\mathbf{s})\mathbf{s}$ and terminal point $(\mathbf{r}\cdot\mathbf{s})\mathbf{s} + \mathbf{E}(\mathbf{r}, t)$. Then t is fixed and $\mathbf{r}$ runs over all space, the end point of this vector describes a curve in 3-d. If we now move t, this curve is shifted (and keeps the same shape) by changing the phase because of the presence of ωt in the cos function. So when $\mathbf{r}\cdot\mathbf{s} = c$ and t moves the vector $\mathbf{E}(\mathbf{r}, t)$ describes a curve in the plane $\mathbf{r}\cdot\mathbf{s} = c$. If this curve is an ellipse we say that the wave is *elliptically polarized* and when the ellipse is a circle we say the wave is *circularly polarized*, and if the the ellipse degenerates to a segment we say the wave is *linearly polarized*. If the wave describing the incident field has components that have different phases, then this changes the sense of circulation and inclination of the ellipse (for elliptically polarized light), see [BW59][Section 1.4.2]. See `http://en.wikipedia.org/wiki/Polarization` for pictures.

$^{\|}$Notice that from (5.28), the value of the field $\mathbf{E}^i$ at P is $\mathbf{E}^i(0, t) = \mathbf{E}_0^i \cos(\omega t)$. Hence we actually get $J^i = \frac{n_1}{4\pi}|\mathbf{E}_0^i|^2 \cos\theta_i \cos^2(\omega t)$. Similarly, from (5.29), the value of the field $\mathbf{E}^t$ at P is $\mathbf{E}^t(0, t) = \mathbf{E}_0^t \cos(\omega t)$, and from (5.30), the value of the field $\mathbf{E}^r$ at P is $\mathbf{E}^r(0, t) = \mathbf{E}_0^r \cos(\omega t)$. Therefore we have $J^r = \frac{n_1}{4\pi}|\mathbf{E}_0^r|^2 \cos\theta_i \cos^2(\omega t)$, and $J^t = \frac{n_2}{4\pi}|\mathbf{E}_0^t|^2 \cos\theta_t \cos^2(\omega t)$. So we get formulas (5.33) because the factor $\cos^2(\omega t)$ cancels out.

Suppose for example that the wave is linearly polarized perpendicularly to the plane of incidence. That is, $I_\parallel = 0$. Then from Fresnel equations $T_\parallel = R_\parallel = 0$ and

$$\mathcal{R} = \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2 = \left(\frac{R_\perp}{I_\perp}\right)^2 = \left(\frac{|x - \kappa\, m|^2}{1-\kappa^2}\right)^2,$$

and

$$\begin{aligned}\mathcal{T} &= \kappa \frac{m\cdot v}{x\cdot v}\left(\frac{T_\perp}{I_\perp}\right)^2 = \kappa \frac{m\cdot(x-\kappa\, m)}{x\cdot(x-\kappa\, m)}\left(\frac{2\,x\cdot(x-\kappa\, m)}{(x+\kappa\, m)\cdot(x-\kappa\, m)}\right)^2 \\ &= \frac{4\kappa}{(1-\kappa^2)^2}\,(m\cdot(x-\kappa\, m))\,(x\cdot(x-\kappa\, m))\,.\end{aligned}$$

For the case when no polarization is assumed, that is, radiation has no particular preference for the direction in which it vibrates, we have from Fresnel's equations that

$$|\mathbf{E}_0^r|^2 = R_\parallel^2 + R_\perp^2 = \left[\frac{(\kappa\, x - m)\cdot(x-\kappa\, m)}{(\kappa\, x + m)\cdot(x-\kappa\, m)}\right]^2 I_\parallel^2 + \left[\frac{(x-\kappa\, m)\cdot(x-\kappa\, m)}{(x+\kappa\, m)\cdot(x-\kappa\, m)}\right]^2 I_\perp^2,$$

and so

$$\begin{aligned}\mathcal{R} &= \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2 = \frac{R_\parallel^2 + R_\perp^2}{I_\parallel^2 + I_\perp^2} \\ &= \left[\frac{(\kappa\, x - m)\cdot(x-\kappa\, m)}{(\kappa\, x + m)\cdot(x-\kappa\, m)}\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[\frac{(x-\kappa\, m)\cdot(x-\kappa\, m)}{(x+\kappa\, m)\cdot(x-\kappa\, m)}\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2} \\ &= \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{x\cdot m} - (1+\kappa^2)\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[1 - 2\kappa\, x\cdot m + \kappa^2\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\right)\end{aligned}$$

which is a function only of $x\cdot m$. In principle the coefficients $I_\parallel$ and $I_\perp$ might depend on the direction x, in other words, for each direction x we would have a wave that changes its amplitude with the direction of propagation. The energy of the incident wave would be $f(x) = |\mathbf{E}_0^i|^2 = I_\parallel(x)^2 + I_\perp(x)^2$. Notice that if the incidence is normal, that is, $x = m$, then $\mathcal{R} = \left(\frac{1-\kappa}{1+\kappa}\right)^2$ which shows that even for radiation normal to the interface we lose energy by reflection. For example, if we go from air to glass, $n_1 = 1$ and $n_2 = 1.5$, we have $\kappa = 1.5$ so $\mathcal{R} = .04$, which means that 4% of the energy is lost in internal reflection.

5.12. **Estimation of the Fresnel coefficients.** For later purposes we need to estimate the function

$$\phi(s) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{s} - (1+\kappa^2)\right]^2 \alpha + \left[1 - 2\kappa\, s + \kappa^2\right]^2 \beta\right), \tag{5.34}$$

have $0 \le \alpha, \beta \le 1$ and $\alpha + \beta = 1$. Set

$$g(t) = \left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2, \qquad h(t) = \left[1 - 2\kappa\, t + \kappa^2\right]^2,$$

so $\phi(t) = \dfrac{1}{(1-\kappa^2)^2}\,(g(t)\,\alpha + h(t)\,\beta)$.

Case $\kappa < 1$. Suppose $\kappa + \epsilon \le t \le 1$. We have $g'(t) = -4\kappa\left[\dfrac{2\kappa}{t} - (1+\kappa^2)\right]\dfrac{1}{t^2}$, so $g'(t) > 0$ for $t > \dfrac{2\kappa}{1+\kappa^2}$, and $g'(t) < 0$ for $t < \dfrac{2\kappa}{1+\kappa^2}$. Since $\kappa < 1$, we have $\kappa + \epsilon < \dfrac{2\kappa}{1+\kappa^2} < 1$ for $\epsilon > 0$ small. Therefore, g decreases in the interval $[\kappa+\epsilon, \dfrac{2\kappa}{1+\kappa^2}]$, and g increases in the interval $[\dfrac{2\kappa}{1+\kappa^2}, 1]$. Hence

$$\max_{[\kappa+\epsilon,1]} g(t) = \max\{g(\kappa+\epsilon), g(1)\}.$$

We have that $g(1) = (1-\kappa)^4$, and $g(\kappa+\epsilon) > g(1)$ for ϵ small, so

$$\max_{[\kappa+\epsilon,1]} g(t) = g(\kappa+\epsilon).$$

On the other hand, $h'(t) = -4\kappa\,[1 - 2\kappa\, t + \kappa^2]$, and so $h'(t) > 0$ for $t > \dfrac{1+\kappa^2}{2\kappa}$ and $h'(t) < 0$ for $t < \dfrac{1+\kappa^2}{2\kappa}$. Since $\dfrac{1+\kappa^2}{2\kappa} > 1$, the function h is decreasing in the interval $[\kappa+\epsilon, 1]$ and so

$$\max_{[\kappa+\epsilon,1]} h(t) = h(\kappa+\epsilon).$$

Therefore we obtain that

$$\max_{[\kappa+\epsilon,1]} \phi(t) \le \frac{1}{(1-\kappa^2)^2}\left(\alpha\, g(\kappa+\epsilon) + \beta\, h(\kappa+\epsilon)\right).$$

It is easy to see that

$$g(\kappa+\epsilon) < (1-\kappa^2)^2, \quad \text{and} \quad h(\kappa+\epsilon) < (1-\kappa^2)^2$$

and so we obtain the bound

$$\max_{[\kappa+\epsilon,1]} \phi(t) \le C_\epsilon < 1,$$

with

$$C_\epsilon = \max\left\{\frac{g(\kappa+\epsilon)}{(1-\kappa^2)^2}, \frac{h(\kappa+\epsilon)}{(1-\kappa^2)^2}\right\} \tag{5.35}$$

independent of α and β. We notice also that $C_\epsilon \to 1$ as $\epsilon \to 0^+$, and $C_\epsilon \to \left(\dfrac{1-\kappa}{1+\kappa}\right)^2$ as $\epsilon \to (1-\kappa)^-$. Also notice that the function ϕ in (5.34) is in general not decreasing in the interval $[\kappa+\epsilon, 1]$, that is, one can choose α close to one and β close to zero with $\alpha + \beta = 1$, so that this is the case.

Case $\kappa > 1$. For ϵ small we have $\frac{1}{\kappa} + \epsilon < \frac{2\kappa}{1+\kappa^2} < 1$, so as before, g decreases in the interval $[\frac{1}{\kappa} + \epsilon, \frac{2\kappa}{1+\kappa^2}]$, and g increases in the interval $[\frac{2\kappa}{1+\kappa^2}, 1]$. Hence

$$\max_{[(1/\kappa)+\epsilon,1]} g(t) = \max\left\{g\left(\frac{1}{\kappa} + \epsilon\right), g(1)\right\}.$$

Since now $\kappa > 1$ we have that $g(1) < g\left(\frac{1}{\kappa} + \epsilon\right)$, for ϵ small, and so

$$\max_{[(1/\kappa)+\epsilon,1]} g(t) = g((1/\kappa) + \epsilon).$$

Since we always have $\frac{1+\kappa^2}{2\kappa} > 1$, the function h is decreasing in the interval $[(1/\kappa) + \epsilon, 1]$ and so

$$\max_{[(1/\kappa)+\epsilon,1]} h(t) = h((1/\kappa) + \epsilon).$$

Therefore we obtain that

$$\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \leq \frac{1}{(1-\kappa^2)^2}\left(\alpha\, g((1/\kappa) + \epsilon) + \beta\, h((1/\kappa) + \epsilon)\right).$$

It is clear that $g((1/\kappa) + \epsilon) < (1-\kappa^2)^2$, and $h((1/\kappa) + \epsilon) < (1-\kappa^2)^2$ when $0 < \epsilon < 1 - (1/\kappa)$. So we obtain the bound

$$\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \leq C_\epsilon < 1,$$

with

$$C_\epsilon = \max\left\{\frac{g((1/\kappa) + \epsilon)}{(1-\kappa^2)^2}, \frac{h((1/\kappa) + \epsilon)}{(1-\kappa^2)^2}\right\} \tag{5.36}$$

independent of α and β.

5.13. **Application to the far field refractor problem with loss of energy.** We have then seen that when radiation strikes a surface separating two homogeneous media I and II with different refractive indices, part of the radiation is transmitted through media II and another part is reflected back into media I. In fact, from Subsection 5.11 the percentage of internally reflected energy can be conveniently written for our purposes as

$$r(x) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{x\cdot m} - (1+\kappa^2)\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[1 - 2\kappa\, x\cdot m + \kappa^2\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\right)$$

where $\kappa = n_2/n_1$. Therefore the percentage of energy transmitted is $t(x) = 1 - r(x)$. Here $I_\perp$ and $I_\parallel$ are the coefficients of the amplitude of the incident wave, which might depend on x in a continuous way. It is important to notice that from Snell's law, $x - \kappa m = \lambda\, \nu$, where ν is unit normal to the surface at the striking point and $\lambda > 0$. This implies that the function $r(x)$ is a function only depending on x and the normal ν.

We propose the following new model that takes into account the splitting of energy. Suppose we have $f \in L^1(\Omega)$ and $g \in L^1(\Omega^*)$, both Ω, Ω^* are domains in the sphere in $\mathbb{R}^3$, the space with physical significance for our problem. The question is to find a surface $\mathcal{R}$ parameterized by $\{\rho(x)x : x \in \Omega\}$ that separates media I and II such that each ray emanating from a point source, the origin, in the direction $x \in \Omega$ with intensity $f(x)$ is refracted into a direction $m \in \Omega^*$ and received with intensity $g(m)$. From the Fresnel formulas a surface $\mathcal{R}$ is only able to transmit in the direction x an amount of energy equal to

$$f(x)t_{\mathcal{R}}(x)$$

where $t_{\mathcal{R}} = 1 - r_{\mathcal{R}}$, since the amount $f(x)r_{\mathcal{R}}(x)$ is reflected back. As we said, the function $t_{\mathcal{R}}(x)$ depends of course on the surface $\mathcal{R}$ but only through x and the unit normal vector $\nu = \nu(x)$ at the striking point. Since we will be seeking for refracting surfaces $\mathcal{R}$, which, in particular, are convex or concave, the normal vector $\nu(x)$ exists for almost every direction x. Also $t_{\mathcal{R}}(x) = G(x, \nu(x))$, with a function $G(x, x')$ continuous in $\Omega \times \Omega^*$and so $t_{\mathcal{R}}$ is defined for a.e. direction x. We then propose the following model: the refracting surface $\mathcal{R}$ is a solution to our problem if

$$\int_{\mathcal{T}_{\mathcal{R}}(F)} f(x)\, t_{\mathcal{R}}(x)\, dx \geq \int_F g(m)\, dm \tag{5.37}$$

for each Borel subset $F \subset \Omega^*$. Here $\mathcal{T}_{\mathcal{R}}(F)$ is the collection of all directions $x \in \Omega$ that are refracted into a direction in the set F. We prove in [GM13] that if $\mathcal{R}$ is a refractor, then the function $t_{\mathcal{R}}(x)$ is continuous relative to the set $\Omega \setminus S$, where S is the set of directions where $\mathcal{R}ho$ is not differentiable, i.e., $|S| = 0$. Therefore $t_{\mathcal{R}}(x)$ is measurable and so (5.37) is well defined. Since a fraction of the energy is used in internal reflection, to be able to transmit and receive $g(m)$ a little extra energy will be needed at the outset. A refractor $\mathcal{R}$ will be admissible to transmit the amount g if

$$\int_{\Omega} f(x)\, t_{\mathcal{R}}(x)\, dx \geq \int_{\Omega^*} g(m)\, dm. \tag{5.38}$$

Since a priori we only know f, g and not $\mathcal{R}$, we do not know if this is satisfied. In order to make sure this is the case, we proved in Subsection 5.12, that if for example $n_2/n_1 = \kappa < 1$, then $r(x) \leq C_\epsilon < 1$ for all $x \in \Omega$ such that $x \cdot m \geq \kappa + \epsilon$[**] , where $\epsilon > 0$ and with C_ϵ independent of $\mathcal{R}$. So if we assume that the input energy is sufficiently larger than the output energy, then (5.38) holds. More precisely, if

$$\int_{\Omega} f(x)\, dx \geq \frac{1}{1 - C_\epsilon} \int_{\Omega^*} g(m)\, dm,$$

then (5.38) holds.

[**]We recall that the physical constraint for refraction is that $x \cdot m \geq \kappa$ for $\kappa < 1$, see Lemma 2.1.

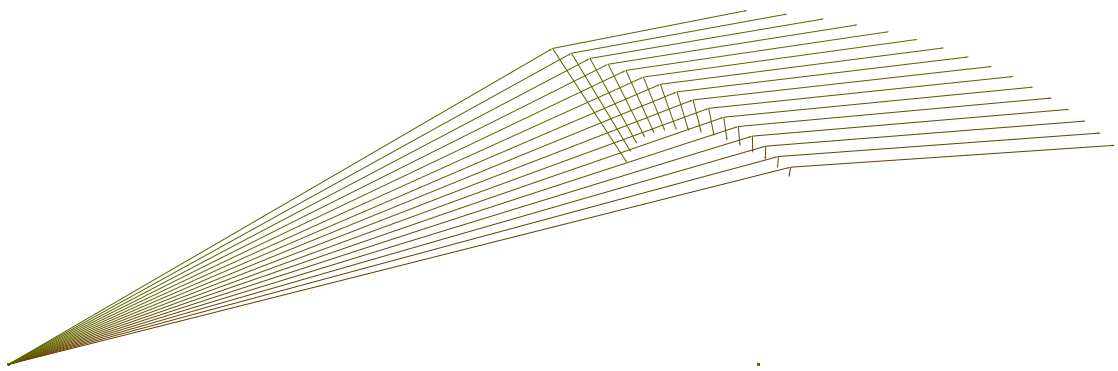


FIGURE 5. Refracted and reflected vectors

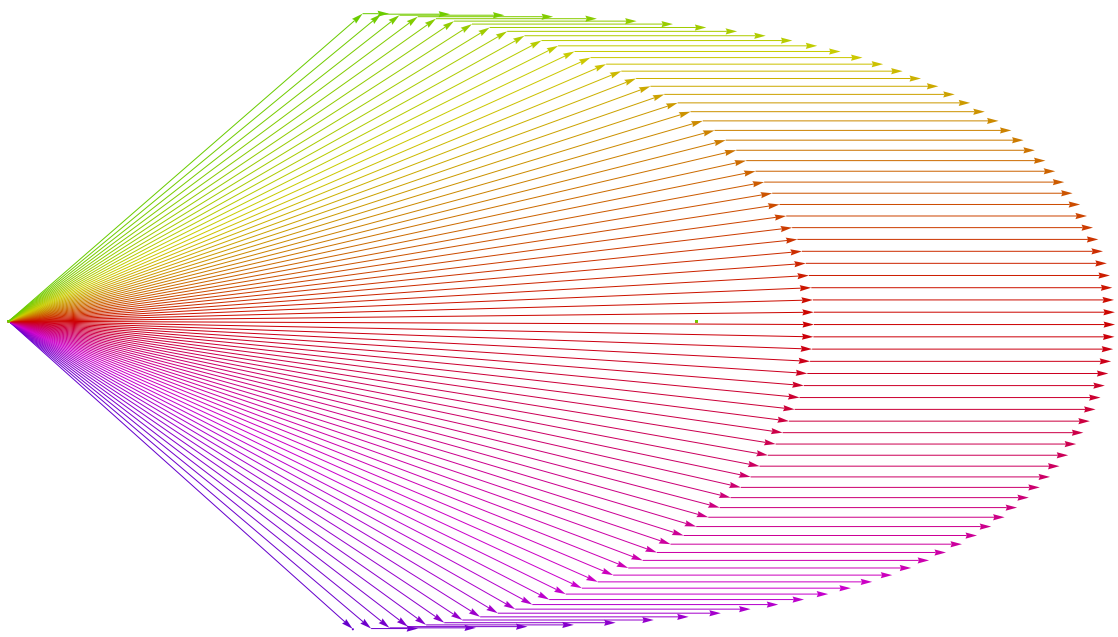


FIGURE 6. Refracted vectors for an ellipse refracting into a fixed direction

Figure 5 represents an arc of ellipse separating glass and air, $\kappa = 2/3$, where the refracted and reflected directions are multiplied by the Fresnel coefficients $t(x)$ and $r(x)$ respectively. Figure 6 represents all the refracted vectors in an ellipse having the uniform refraction property, i.e. all rays are refracted into a fix direction, where the refracted vectors are multiplied by the Fresnel coefficient $t(x)$. Notice that the size of the refracted vectors close to the critical angle, i.e., $x \cdot m = \kappa$ tend to zero.

With this model it is proved in [GM13] existence of solutions, even for Radon measures μ instead of g. The basic geometry of the refractors is described in

[GH09] and depends on κ. Indeed, the surfaces having the uniform refracting property are semi ellipsoids if $\kappa < 1$ and one sheet of hyperboloids of two sheets if $\kappa > 1$, see Subsection 2.3. A difficulty in our case is the presence of the coefficient $t_{\mathcal{R}}(x)$ in (5.37). This prevents us from using the optimal transportation methods used in [GH09]. The route used was explained in Section 4, that is, to solve first the problem when the right hand side is a linear combination of delta functions and then proceed by approximation. To carry out this we need to understand how the Fresnel coefficients evolve when a sequence of refractors converge. When the measure μ in the target is discrete, refractors always overshoot energy in one direction. When μ is any Radon measure, it is proved in [GM13] that refractors transmit more energy in any a priori chosen direction $m_o \in \overline{\Omega^*}$ which lies in the support of μ, and there is one refractor overshooting the least amount of energy in the direction of m_o. That is, for each Borel set $F \subset \Omega^*$ such that $m_o \notin F$ we have $\int_{\mathcal{T}_{\mathcal{R}}(F)} f(x)\, t_{\mathcal{R}}(x)\, dx = \int_F g(m)\, dm$.

To place our results in perspective, we finish enumerating some related results in this area. The refractor problem assuming energy conservation, i.e., $t_{\mathcal{R}}(x) = 1$ in (5.37), was considered for the first time in [GH09] for the far field case, and in [GH08] and [GH13] for the near field case. For reflectors also assuming energy conservation, see [CO08], [Wan96], [CH09], [CGH08] for the far field problem, and [KO97], and [KW10] for the near field.

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