MAXIMUM AND COMPARISON PRINCIPLES FOR CONVEX FUNCTIONS ON THE HEISENBERG GROUP

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1. Introduction

The purpose in this paper is to establish pointwise estimates for a class of convex functions on the Heisenberg group. The following integral estimate for classical convex functions in terms of the Monge–Ampère operator $\det D^2u$ was proved by Aleksandrov (see [4, Theorem 1.4.2]):

Theorem (Aleksandrov's maximum principle). *If $\Omega \subset \mathbb{R}^n$ is a bounded open and convex set with diameter Δ and $u \in C(\overline{\Omega})$ is convex with $u = 0$ on $\partial\Omega$, then for all $x_0 \in \Omega$*

$$|u(x_0)|^n \leq c_n \, \Delta^{n-1} \, dist(x_0, \partial\Omega) \, |\partial u(\Omega)|,$$

where c_n is a positive constant depending only on the dimension n, and ∂u is the normal mapping or subdifferential of u, see [4, Definition 1.1.1].

Such estimate is of great importance in the theory of weak solutions for the Monge–Ampère equation [*], and its proof revolves around the geometric features of the normal mapping which yield in addition the useful comparison principle for Monge-Ampère measures [4, Theorem 1.4.6]. A natural question is if similar comparison and maximum principles hold in the setting of Carnot groups. The reasons for this question are that those estimates are the key tools to develop a theory of weak solutions for nondivergence subelliptic operators of the form $a_{ij}X_i X_j$, where a_{ij} is a uniformly elliptic measurable matrix and $X = \{X_1, \dots, X_m\}$ is a system of left invariant vector fields on a Carnot group, and also to understand fully nonlinear equations in this setting. The difficulty for this study is the doubtful existence of a notion of normal mapping in Carnot groups suitable to establish maximum and comparison principles.

In this paper we address this question in the Heisenberg group $\mathbb{H}^1$ and follow a route different from the one described above for standard convex functions, and in particular, we do not use any notion of normal mapping. This approach was recently used by Trudinger and Wang to study Hessian equations [12].

To explain the main results and plan of the paper, we first give some notation and basic definitions. Let $\xi = (x, y, t)$ and $\xi_0 = (x_0, y_0, t_0)$ denote points in $\mathbb{R}^3$. The Heisenberg group $\mathbb{H}^1$ is $\mathbb{R}^3$ endowed with the non-commutative multiplication given by

$$\xi_0 \circ \xi = (x_0 + x, y_0 + y, t_0 + t + 2(xy_0 - yx_0)).$$

The corresponding Lie algebra $\mathcal{G}$ is spanned by the left-invariant vector fields

$$X = \partial_x + 2y\,\partial_t, \quad Y = \partial_y - 2x\,\partial_t, \quad [X, Y] = XY - YX = -4\partial_t,$$

and $\mathcal{G}$ admits a nilpotent stratification of step two, i.e., it decomposes as the vector space direct sum $\mathcal{G} = V_1 \oplus V_2$, where V_1 is the vector space generated by X and Y, and V_2 is the space generated by $[X, Y]$. For this reason, $\mathbb{H}^1$ is the prototype of Carnot groups of step two.

On the Heisenberg group, and more generally in Carnot groups, several notions of convexity have been introduced and compared in [3] (horizontal convexity), and [8] (viscosity convexity). All these definitions are now known to be equivalent even in the general case of Carnot groups, see [2], [7], [9], and [13]. The notion of convex function used in this paper is the following, and throughout the paper convexity is understood in this sense and for continuous functions in the extended sense given in Definition 2.1.

Definition 1.1. *The function $u \in C^2(\Omega)$ is convex in Ω if the symmetric matrix*

$$\mathcal{H}(u) = \begin{bmatrix} X^2u & (XYu + YXu)/2 \\ (XYu + YXu)/2 & Y^2u \end{bmatrix}$$

is positive semidefinite in Ω.

1.1. **Main results and plan of the paper.** Our integral estimates are in terms of the Monge–Ampère type operator

$$\begin{aligned} H(u) &= \det \mathcal{H}(u) + 3/4([X, Y]u)^2 \\ &= \det \mathcal{H}(u) + 12(\partial_t u)^2, \end{aligned} \tag{1.1}$$

[*]If u is smooth, then $|\partial u(\Omega)| = \int_\Omega \det D^2u(x)dx$.

and as we shall see in Section 3, the motivation for this choice in place of the expected $\det \mathcal{H}(u)$ is that $H(u)$ has the null-Lagrangian property from Lemma 3.1.

The first step is to establish by means of integration by parts the following comparison principle for smooth functions.

Theorem 1.2 (Comparison Principle)**.** *Let $u, v \in C^2(\overline{\Omega})$ such that $u + v$ is convex in Ω satisfying $v = u$ on $\partial\Omega$ and $v < u$ in Ω. Then*

$$\int_\Omega H(u)\, d\xi \leq \int_\Omega H(v)\, d\xi.$$

We then extend in Theorem 4.7 this comparison principle to "cones" defined in term of a distance d, locally equivalent to the Carnot-Carathéodory distance generated by X and Y, and defined by

$$d(\xi, \xi_0) = \rho(\xi_0^{-1} \circ \xi) \tag{1.2}$$

where ρ is a smooth gauge in $\mathbb{H}^1$, given by $\rho(\xi) = \left((x^2 + y^2)^2 + t^2\right)^{1/4}$. This together with the geometry in $\mathbb{H}^1$ leads by iteration to our maximum principle of Aleksandrov type on d-balls B_R of radius R.

Theorem 1.3 (Maximum Principle)**.** *Let $u \in C^2(B_R)$ be convex, $u = 0$ on ∂B_R. If $u(\xi_0) = \min_{B_R} u$, then there exists a positive constant c, depending on $d(\xi_0, \partial B_R)$, such that*

$$|u(\xi_0)|^2 \leq c \int_{B_R} H(u)\, d\xi.$$

We explicitly remark that an analogous pointwise estimate does not hold with $\det \mathcal{H}(u)$ instead of $H(u)$, see Proposition 4.5.

We next estimate the oscillation of convex functions.

Theorem 1.4 (Oscillation Estimate)**.** *Let $u \in C^2(\Omega)$ be convex. For any compact domain $\Omega' \Subset \Omega$ there exists a positive constant C depending on Ω' and Ω and independent of u, such that*

$$\int_{\Omega'} H(u)\, d\xi \leq C(\mathrm{osc}_\Omega u)^2. \tag{1.3}$$

Theorem 1.4 permits to extend our definition of Monge–Ampère measure to continuous convex functions and obtain a general comparison principle, Theorem 6.7. Moreover, Theorem 1.4 furnishes an estimate of the L^2 norm of the bracket $[X, Y]u = -4\partial_t u$ of a convex function u. This property is extended by the authors to any dimension in [5] using the second elementary symmetric function of the eigenvalues of the matrix $\mathcal{H}(u)$ to prove the subelliptic version of the Aleksandrov-Busemann-Feller theorem about the twice differentiability a.e. of convex functions in $\mathbb{H}^n$ for any n.

The organization of the paper is the following. Section 2 contains preliminaries about $\mathbb{H}^n$ and equivalent definitions of convexity. In Section 3 we prove the comparison principle Theorem 1.2. Section 4 contains the proof that "cones" agreeing with convex functions u on the boundary are above u inside, and the comparison principle for cones Theorem 4.7. In Section 5 we prove the maximum principle Theorem 1.3. Finally, Section 6 contains

the proof of the oscillation estimates, Theorem 1.4, and the construction of the analogue of Monge–Ampère measures for convex functions in the Heisenberg setting.

Acknowledgements and comments. The results in this paper and the ideas of their proofs have been presented in the following talks: Analysis Seminar, Temple U., October 2002; Fabes–Chiarenza Lectures at Siracusa, December 2002; Pan-American Conference, Santiago de Chile, January 2003; Analysis Seminar, U. of Bologna, March 2003; and Analysis Seminar, U. Texas at Austin, March 2003.
The first author gratefully acknowledges support from the NSF under grant DMS–0070648, and thanks the University of Bologna for its support and kind hospitality during his visits to carry out this project. The second author thanks Temple University for the hospitality during her visit in March-April 2001 when the key Theorem 1.2 was proved.
We like to thank the referee for reading carefully the manuscript and for several useful suggestions to improve the presentation.

2. Preliminaries and convexity

We first recall some basic properties about $\mathbb{H}^1$ that will be used throughout the paper and we refer to the book [11, Chapters XII and XIII] for more details about $\mathbb{H}^n$.

The group $\mathbb{H}^1$ has a family of dilations which are group homomorphisms, given by

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t),$$

for $\lambda > 0$. The distance d defined in (1.2) is homogeneous of degree one with respect to the group of dilations,

$$d(\delta_\lambda \xi, \delta_\lambda \xi_0) = \lambda\, d(\xi, \xi_0),$$

for every $\lambda > 0$ and for every $\xi, \xi_0 \in \mathbb{H}^1$.

Moreover, if we denote by $B_R(\xi_0) = \{\xi \in \mathbb{R}^3 : d(\xi, \xi_0) < R\}$ the ball with center at ξ_0 of radius R with respect to distance d, then by left translation and dilation it is easy to see that the Lebesgue measure of $B_R(\xi_0)$ is

$$|B_R(\xi_0)| = R^Q |B_1(0)|, \tag{2.1}$$

with $Q = \dim(V_1) + 2\dim(V_2) = 4$. Hence, the number Q plays a role of a dimension with respect to the group of dilations, and for this reason it is called the homogeneous dimension of $\mathbb{H}^1$.

2.1. **Convexity.** We extend the Definition 1.1 to continuous functions.

Definition 2.1. *The function u is convex in Ω if there exists a sequence $u_k \in C^2(\Omega)$ of convex functions in Ω in the sense of Definition 1.1 such that $u_k \to u$ uniformly on compact subsets of Ω.*

We shall now prove that our definition of convexity is equivalent to the horizontal convexity in [3]. To this purpose, given $\xi_0 = (x_0, y_0, t_0) \in \mathbb{R}^3$ we introduce the horizontal plane through ξ_0

$$\Pi_{\xi_0} = \{(x, y, t) : t - t_0 - 2(xy_0 - yx_0) = 0\}.$$

That is, Π_{ξ_0} is the plane generated by the vectors $(1, 0, 2y_0)$, $(0, 1, -2x_0)$ and passing through the point ξ_0. Notice that if $h \in \mathbb{H}^1$, then

$$\xi \in \Pi_{\xi_0} \text{ if and only if } h \circ \xi \in \Pi_{h\circ\xi_0}. \tag{2.2}$$

Let $\xi_0 = (x_0, y_0, t_0), \zeta = (x, y, t)$ and

$$g(\zeta) = f(\xi_0 \circ \zeta).$$

We have

$$\partial_x g(0) = Xf(\xi_0), \quad \partial_y g(0) = Yf(\xi_0), \quad \partial_t g(0) = \partial_t f(\xi_0),$$

and

$$\partial_{xx} g(0) = (X^2 f)(\xi_0), \quad \partial_{xy} g(0) = (YXf)(\xi_0) - 2\,\partial_t f(\xi_0), \quad \partial_{xt} g(0) = \partial_{tx} f(\xi_0) + 2\,y_0\,\partial_{tt} f(\xi_0),$$

$$\partial_{yx} g(0) = (XYf)(\xi_0) + 2\,\partial_t f(\xi_0), \quad \partial_{yy} g(0) = (Y^2 f)(\xi_0), \quad \partial_{yt} g(0) = \partial_{ty} f(\xi_0) - 2\,x_0\,\partial_{tt} f(\xi_0),$$

$$\partial_{tx} g(0) = \partial_{tx} f(\xi_0) + 2\,y_0\,\partial_{tt} f(\xi_0), \quad \partial_{ty} g(0) = \partial_{ty} f(\xi_0) - 2\,x_0\,\partial_{tt} f(\xi_0), \quad \partial_{tt} g(0) = \partial_{tt} f(\xi_0).$$

Let

$$A = \begin{bmatrix} (X^2 f)(\xi_0) & (YXf)(\xi_0) - 2\,\partial_t f(\xi_0) & \partial_{tx} f(\xi_0) + 2\,y_0\,\partial_{tt} f(\xi_0) \\ (XYf)(\xi_0) + 2\,\partial_t f(\xi_0) & (Y^2 f)(\xi_0) & \partial_{ty} f(\xi_0) - 2\,x_0\,\partial_{tt} f(\xi_0) \\ \partial_{tx} f(\xi_0) + 2\,y_0\,\partial_{tt} f(\xi_0) & \partial_{ty} f(\xi_0) - 2\,x_0\,\partial_{tt} f(\xi_0) & \partial_{tt} f(\xi_0) \end{bmatrix}.$$

Then the Taylor polynomial of order two of g is

$$\begin{aligned} & f(\xi_0) + (Xf(\xi_0), Yf(y_0), \partial_t f(\xi_0)) \cdot \zeta + \frac{1}{2}\langle A\zeta, \zeta\rangle \\ &= f(\xi_0) + (Xf(\xi_0), Yf(y_0)) \cdot (x, y) + (X^2 f)\,x^2 + (XYf + YXf)\,xy + (Y^2 f)\,y^2 \\ &\quad + t\{f_t(\xi_0) + 2\,f_{tx}\,x + 4y_0\,f_{tt}x + (f_{tx} + f_{ty})y - 4x_0\,f_{tt}y\}. \end{aligned}$$

That is, if $(x, y, t) \in \Pi_0$ then $t = 0$ and so on this plane we have

$$\begin{aligned} g(\zeta) = {} & f(\xi_0) + (Xf(\xi_0), Yf(y_0)) \cdot (x, y) \\ & + (X^2 f)\,x^2 + (XYf + YXf)\,xy + (Y^2 f)\,y^2 + o(x^2 + y^2). \end{aligned}$$

The following proposition yields several equivalent definitions of convexity.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^3$ open and assume that if $\xi_0 \in \Omega$, $\xi \in \Pi_{\xi_0} \cap \Omega$, then $\xi_0 \circ \delta_\lambda(\xi_0^{-1} \circ \xi) \in \Omega$ for $0 < \lambda < 1$. The following are equivalent:*

(1) *$u \in C(\Omega)$ is convex in Ω.*
(2) *(Horizontal convexity) Given $\xi_0 \in \Omega$*

$$u(\xi_0 \circ \delta_\lambda(\xi_0^{-1} \circ \xi)) \le u(\xi_0) + \lambda(u(\xi) - u(\xi_0)), \tag{2.3}$$

for all $\xi \in \Pi_{\xi_0} \cap \Omega$ and $0 \le \lambda \le 1$.

(3) *For each $\xi_0 \in \Omega$ and $\xi = (x, y, t) \in \Pi_{\xi_0} \cap \Omega$ the function u restricted to the segment $[\xi_0, \xi]$ is a convex function of one variable in the standard sense.*

Proof. (1) $\Rightarrow$ (2) It is enough to assume that $u \in C^2(\Omega)$. Then the proof follows by [3, Theorem 5.11].

(2) $\Rightarrow$ (3) Easy follows by [8, Lemma 4.1], by taking into account (2.2).

(3) $\Rightarrow$ (1) First of all let us remark that by [2, Theorem 1.2] a function u satisfying condition (3) is locally Lipschitz continuous with respect to d.

Let $\varphi \in C^\infty(\mathbb{R}^3)$, $\varphi \geq 0$, supp $\varphi \subset B_1(0)$ the metric ball, and $\int_{\mathbb{R}^3} \varphi(h)\, dh = 1$. Let $\varepsilon > 0$ and

$$u_\varepsilon(\xi) = u * \varphi_\varepsilon(\xi) = \varepsilon^{-4} \int_{\mathbb{H}^1} u(h^{-1} \circ \xi)\, \varphi(\delta_{1/\varepsilon} h)\, dh.$$

We have that u_ε is smooth and $u_\varepsilon \to u$ uniformly on compact subsets of Ω. We have from (2.2) and (2.3) that

$$\begin{aligned}
u_\varepsilon\left(\xi_0 \circ \delta_\lambda(\xi_0^{-1} \circ \xi)\right) &= \varepsilon^{-4} \int_{\mathbb{H}^1} u\left(h^{-1} \circ \left(\xi_0 \circ \delta_\lambda(\xi_0^{-1} \circ \xi)\right)\right) \varphi(\delta_{1/\varepsilon} h)\, dh \\
&= \varepsilon^{-4} \int_{\mathbb{H}^1} u\left(\left(h^{-1} \circ \xi_0\right) \circ \delta_\lambda\left((h^{-1} \circ \xi_0)^{-1} \circ (h^{-1} \circ \xi)\right)\right) \varphi(\delta_{1/\varepsilon} h)\, dh \\
&\leq \varepsilon^{-4} \int_{\mathbb{H}^1} \left(u(h^{-1} \circ \xi_0) + \lambda\left(u(h^{-1} \circ \xi) - u(h^{-1} \circ \xi_0)\right)\right) \varphi(\delta_{1/\varepsilon} h)\, dh \\
&= u_\varepsilon(\xi_0) + \lambda\, \left(u_\varepsilon(\xi) - u_\varepsilon(\xi_0)\right).
\end{aligned}$$

□

Remark 2.3. From Proposition 2.2(2) we have that if u is convex in the standard sense, then u is convex. However, the gauge function $\rho(x, y, t) = \left((x^2 + y^2)^2 + t^2\right)^{1/4}$ is convex but is not convex in the standard sense, see Proposition 4.5.

3. Null-Lagrangian property and Comparison Principle

In this section we prove Theorem 1.2. Since this theorem places a central role in the present paper, we briefly sketch the key idea. In the Euclidean case the determinant of the Hessian of a smooth function u has the following null-Lagrangian property

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial(\det D^2 u)}{\partial u_{ij}}\right) = 0, \quad \text{for all } j = 1, \ldots, n,$$

which allows to prove the following monotonicity property of the Hessian measure by simple integrating by parts and without using the normal mapping, see [12, Lemma 2.1].

Theorem (Monotonicity property of the Hessian measure). *Let u and v be smooth standard convex functions in a bounded open set Ω with regular boundary, if $u < v$ in Ω and $u = v$ on $\partial\Omega$ then*

$$\int_\Omega \det D^2 u \leq \int_\Omega \det D^2 v.$$

In our setting the operator $\det \mathcal{H}(u)$ does not have a similar null-Lagrangian property because the vector fields X and Y do not commute. However, from (1.1),

$$H(u) = \det \mathcal{H}(u) + 12(\partial_t u)^2 = X^2 u\, Y^2 u - \left(\frac{XYu + YXu}{2}\right)^2 + 3\left(\frac{XYu - YXu}{2}\right)^2,$$

and if we set $X_1 = X, X_2 = Y$, and $r_{ij} = X_i X_j u$, then

$$H(u) = r_{11}\, r_{22} - \left(\frac{r_{12} + r_{21}}{2}\right)^2 + 3\left(\frac{r_{12} - r_{21}}{2}\right)^2,$$

and we have the following lemma.

Lemma 3.1 (Null-Lagrangian property)**.** *For every smooth function u*

$$\sum_{i=1}^{2} X_i \left(\frac{\partial H(u)}{\partial r_{ij}}\right) = 0, \quad \textit{for } j = 1, 2.$$

Proof.

$$\frac{\partial H(u)}{\partial r_{11}} = r_{22}; \quad \frac{\partial H(u)}{\partial r_{12}} = -\left(\frac{r_{12} + r_{21}}{2}\right) + 3\left(\frac{r_{12} - r_{21}}{2}\right) = r_{12} - 2r_{21};$$

$$\frac{\partial H(u)}{\partial r_{21}} = -\left(\frac{r_{12} + r_{21}}{2}\right) - 3\left(\frac{r_{12} - r_{21}}{2}\right) = r_{21} - 2r_{12}; \quad \frac{\partial H(u)}{\partial r_{22}} = r_{11}.$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{2} X_i \left(\frac{\partial H(u)}{\partial r_{i1}}\right) &= X\left(\frac{\partial H(u)}{\partial r_{11}}\right) + Y\left(\frac{\partial H(u)}{\partial r_{21}}\right) \\
&= X(Y^2 u) + Y(YXu - 2XYu) = X(Y^2 u) - Y(XYu) - Y[X, Y]u \\
&= [X, Y]Yu - Y[X, Y]u = [[X, Y], Y]u \equiv 0, \\
\sum_{i=1}^{2} X_i \left(\frac{\partial H(u)}{\partial r_{i2}}\right) &= X\left(\frac{\partial H(u)}{\partial r_{12}}\right) + Y\left(\frac{\partial H(u)}{\partial r_{22}}\right) \\
&= X(XYu - 2YXu) + Y(X^2 u) = Y(X^2 u) - X(YXu) + X[X, Y]u \\
&= -[X, Y]Xu + X[X, Y]u = [X, [X, Y]]u \equiv 0.
\end{aligned} \tag{3.4}$$

□

Proof of Theorem 1.2. Since convex functions can always be approximated by smooth convex functions as in the proof of Proposition 2.2, it is not restrictive to assume $u, v \in C^\infty(\Omega)$. If $Z = \alpha_1\, \partial_{x_1} + \alpha_2\, \partial_{x_2} + \alpha_3\, \partial_{x_3}$ is a smooth vector field, then

$$\int_\Omega Zu\, dx = \int_{\partial\Omega} \nu_Z\, u\, d\sigma(x) - \int_\Omega ((\alpha_1)_{x_1} + (\alpha_2)_{x_2} + (\alpha_3)_{x_3})\, u\, dx, \tag{3.5}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the outer unit normal to $\partial\Omega$ and $\nu_Z = \alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3$.

Since $v = u$ on $\partial\Omega$, $v < u$ in Ω and both functions are smooth up to the boundary, it follows that the normal to $\partial\Omega$ is $\nu = \dfrac{D(v-u)}{|D(v-u)|}$, and therefore $\nu_X = \dfrac{X(v-u)}{|D(v-u)|}$ and $\nu_Y = \dfrac{Y(v-u)}{|D(v-u)|}$.

Let $0 \le s \le 1$ and $\varphi(s) = H(v + sw)$, with $w = u - v$. Then

$$\begin{aligned}\int_\Omega \{H(u) - H(v)\}\, d\xi &= \int_0^1 \int_\Omega \varphi'(s)\, d\xi ds \\ &= \int_0^1 \int_\Omega \left\{ \sum_{i,j=1}^{2} \frac{\partial H}{\partial r_{ij}}(v + sw)\, X_i X_j w \right\} d\xi ds \\ &= \int_0^1 \int_\Omega \left\{ \sum_{i,j=1}^{2} X_i \left(\frac{\partial H}{\partial r_{ij}}(v + sw)\, X_j w \right) - X_i \left(\frac{\partial H}{\partial r_{ij}}(v + sw) \right) X_j w \right\} d\xi ds \\ &= A - B.\end{aligned}$$

By Lemma 3.1 $B \equiv 0$, and using (3.5) in the first term we get

$$\begin{aligned} A &= \int_0^1 \int_{\partial\Omega} \left\{ \sum_{i,j=1}^{2} \left(\frac{\partial H}{\partial r_{ij}}(v + sw)\, X_j w \cdot \nu_{X_i} \right) \right\} d\sigma(\xi) ds \\ &= - \int_0^1 \int_{\partial\Omega} \left\{ \sum_{i,j=1}^{2} \left(\frac{\partial H}{\partial r_{ij}}(v + sw)\, X_j w \cdot X_i w \right) \right\} \frac{1}{|Dw|}\, d\sigma(\xi) ds \\ &= - \int_0^1 \int_{\partial\Omega} \left\langle \begin{bmatrix} Y^2 & -YX + [X,Y] \\ -XY - [X,Y] & X^2 \end{bmatrix} (v + sw) \begin{pmatrix} Xw \\ Yw \end{pmatrix}, \begin{pmatrix} Xw \\ Yw \end{pmatrix} \right\rangle \frac{d\sigma(\xi)}{|Dw|} ds \\ &\quad \text{(by integrating in } ds \text{ first)} \\ &= -\frac{1}{2} \int_{\partial\Omega} \left\langle \begin{bmatrix} Y^2 & -YX \\ -XY & X^2 \end{bmatrix} (u + v) \begin{pmatrix} Xw \\ Yw \end{pmatrix}, \begin{pmatrix} Xw \\ Yw \end{pmatrix} \right\rangle \frac{1}{|Dw|}\, d\sigma(\xi) \\ &\le 0, \end{aligned}$$

because

$$\left\langle \begin{bmatrix} Y^2 & -YX \\ -XY & X^2 \end{bmatrix} (u + v) \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \right\rangle = \left\langle \mathcal{H}(u + v) \begin{pmatrix} \zeta_2 \\ -\zeta_1 \end{pmatrix}, \begin{pmatrix} \zeta_2 \\ -\zeta_1 \end{pmatrix} \right\rangle \ge 0$$

for every $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$, since $u + v$ is convex. This completes the proof of the theorem. □

4. Weak maximum principle

Let $A = (a_{ij})$ be a 2×2 symmetric matrix such that $A \ge 0$, and trace $A > 0$, $a_{ij} \in C(D)$ where $D \subset \mathbb{R}^3$ is an open set; $X_1 = X, X_2 = Y$, and $L = \sum_{i,j=1}^{2} a_{ij}(\xi) X_i X_j$.

Theorem 4.1. *Let Ω be a bounded open set in $\mathbb{R}^3$, and $w \in C^2(\Omega)$. If $Lw \ge 0$ in Ω and $\limsup_{\xi \to \xi_0} w(\xi) \le 0$ for each $\xi_0 \in \partial\Omega$, then $w \le 0$ in Ω.*

We remark that Theorem 4.1 is true even for viscosity solutions and follows from the subelliptic comparison principle proved in [1].

To prove Theorem 4.1 we need two lemmas.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, and $w \in C(\Omega)$. Then there exists $\xi_0 \in \overline{\Omega}$ such that $\sup_{\Omega\cap B(\xi_0,\rho)} w = \sup_\Omega w$ for every $\rho > 0$, where $B(\xi_0,\rho)$ is the Euclidean ball with radius ρ and center ξ_0.*

Lemma 4.3. *Let Ω be open and bounded. There exists a function $w_0 \in C^2(\Omega)$ such that $w_0 > 0$ and $Lw_0 < 0$ in Ω.*

Proof. Let $\lambda > 0$ and choose $M \in \mathbb{R}$ such that $\sup_{\xi\in\Omega} e^{\lambda x+\lambda y} < M$; $\xi = (x, y, t)$. Let $w_0 = M - e^{\lambda x+\lambda y}$. Then $w_0 > 0$ in Ω, $X_1 w_0 = -\lambda e^{\lambda x}$, $X_1^2 w_0 = -\lambda^2 e^{\lambda x}$, $X_2 w_0 = -\lambda e^{\lambda y}$, $X_2^2 w_0 = -\lambda^2 e^{\lambda y}$, and $X_1X_2 w_0 = X_2X_1 w_0 = 0$. Hence $Lw_0 = -\lambda^2(a_{11}\, e^{\lambda x} + a_{22}\, e^{\lambda y}) < 0$ in Ω. □

Proof of Theorem 4.1. First assume that $Lw > 0$ in Ω. By Lemma 4.2, there exists $\xi_0 \in \overline{\Omega}$ such that $\sup_{\Omega\cap B(\xi_0,\rho)} w = \sup_\Omega w$ for every $\rho > 0$. If $\xi_0 \in \Omega$, then $w(\xi_0) = \sup_\Omega w$ and so $Dw(\xi_0) = 0$ and $D^2w(\xi_0) \le 0$. Hence

$$\begin{aligned}
0 < Lw(\xi_0) &= \operatorname{trace}\left(A\begin{bmatrix} X^2w & XYw \\ YXw & Y^2w\end{bmatrix}\right)(\xi_0)\\
&= \operatorname{trace}\left(A\begin{bmatrix} X^2w & (XYw+YXw)/2 \\ (XYw+YXw)/2 & Y^2w\end{bmatrix}\right)(\xi_0)\\
&= \operatorname{trace}\left(A\begin{bmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x\end{bmatrix} D^2w \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2y & -2x\end{bmatrix}\right)(\xi_0)\\
&= \operatorname{trace}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2y & -2x\end{bmatrix} A \begin{bmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x\end{bmatrix} D^2w\right)(\xi_0)\\
&= \operatorname{trace}(\tilde{A}D^2w)(\xi_0) \le 0,
\end{aligned}$$

since $\tilde{A} \ge 0$ and $D^2w(\xi_0) \le 0$. This is a contradiction. Hence $\xi_0 \in \partial\Omega$ and consequently $w \le 0$ in Ω. If $Lw \ge 0$ in Ω, then for each $\varepsilon > 0$ we set $w_\varepsilon = w - \varepsilon\, w_0$ with w_0 as in Lemma 4.3. We have $Lw_\varepsilon = Lw - \varepsilon Lw_0 > 0$ and $\limsup_{\xi\to\xi_0} w_\varepsilon(\xi) \le \limsup_{\xi\to\xi_0} w(\xi) \le 0$ for each $\xi_0 \in \partial\Omega$. By the previous argument, $w_\varepsilon \le 0$ in Ω for each $\varepsilon > 0$, and so $w \le 0$. □

Let us denote by $\mathcal{H}^*(u)$ the Newton transformation as defined in [10]

$$\mathcal{H}^*(u) = \begin{bmatrix} Y^2u & -(XYu+YXu)/2 \\ -(XYu+YXu)/2 & X^2u\end{bmatrix}.$$

We have

$$\det \mathcal{H}(u) = \frac{1}{2}\operatorname{trace}(\mathcal{H}^*(u)\,\mathcal{H}(u)),$$

and

$$\operatorname{trace}(\mathcal{H}^*(u)\,\mathcal{H}(v)) = \operatorname{trace}(\mathcal{H}^*(v)\,\mathcal{H}(u)). \tag{4.1}$$

From Theorem 4.1 we obtain the following comparison principle.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, $u, v \in C^2(\Omega)$ such that $u + v$ is convex, and* $\operatorname{trace}\{\mathcal{H}(u+v)\} > 0$. *If* $\det \mathcal{H}(u) \geq \det \mathcal{H}(v)$ *in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Proof. We have

$$\begin{aligned}
0 &\leq \det \mathcal{H}(u) - \det \mathcal{H}(v) \\
&= \frac{1}{2}\,(\operatorname{trace}(\mathcal{H}^*(u)\,\mathcal{H}(u)) - \operatorname{trace}(\mathcal{H}^*(v)\,\mathcal{H}(v))) \\
&= \frac{1}{2}\,(\operatorname{trace}(\mathcal{H}^*(u)\,\mathcal{H}(u-v)) + \operatorname{trace}((\mathcal{H}^*(u) - \mathcal{H}^*(v))\,\mathcal{H}(v))) \\
&= \frac{1}{2}\,(\operatorname{trace}(\mathcal{H}^*(u)\,\mathcal{H}(u-v)) + \operatorname{trace}(\mathcal{H}^*(v)\,\mathcal{H}(u-v))) \qquad \text{by (4.1)} \\
&= \frac{1}{2}\operatorname{trace}(\mathcal{H}^*(u+v)\,\mathcal{H}(u-v)) \\
&= \frac{1}{2}\operatorname{trace}(\mathcal{H}^*(u+v)\,\mathcal{H}(w)),
\end{aligned}$$

where $w = u - v \leq 0$ on $\partial\Omega$. Applying Theorem 4.1 to w with $A = \mathcal{H}^*(u+v)$, the proposition follows. □

4.1. **A comparison principle for "cones".** As a consequence of Proposition 4.4 we get that "cones" that agreeing with an convex function u on the boundary of a ball B are above u inside B.

Proposition 4.5. *Let $\Omega = \{\xi \in \mathbb{R}^3 : 0 < d(\xi, \xi_0) < R\}$, and $v(\xi) = m\left(\dfrac{d(\xi,\xi_0)}{R} - 1\right)$. If $m \geq 0$, then v is convex in Ω,* $\det \mathcal{H}(v) = 0$ *in Ω, and* $\det \mathcal{H}(v)$ *is integrable in $\overline{\Omega}$.*

Proof. If $\zeta \in \mathbb{R}^3$ and $g(\xi) = f(\zeta \circ \xi)$, then $Xg(\xi) = (Xf)(\zeta \circ \xi)$ and $Yg(\xi) = (Yf)(\zeta \circ \xi)$. Therefore we can assume that $\xi_0 = 0$. Let $r = (x^2 + y^2)^2 + t^2$ and $h \in C^1((0, +\infty))$. Then $Xr = 4x^3 + 4xy^2 + 4yt$, $Yr = 4yx^2 + 4y^3 - 4xt$, $X^2r = Y^2r = 12(x^2 + y^2)$, $YXr = 4t$, and $XYr = -4t$. If $u(x, y, t) = h(r)$, then $Xu = h'(r)\,Xr$, $Yu = h'(r)\,Yr$, $X^2u = h''(r)\,(Xr)^2 + h'(r)\,X^2r$, $Y^2u = h''(r)\,(Yr)^2 + h'(r)\,Y^2r$, $XYu = h''(r)\,XrYr + h'(r)\,XYr$, $YXu = h''(r)\,YrXr + h'(r)\,YXr$. Thus

$$\det \mathcal{H}(u) = 48\,(x^2 + y^2)^2\,\{4\,r\,h''(r) + 3\,h'(r)\}\,h'(r). \tag{4.2}$$

Therefore $\det \mathcal{H}(u) = 0$ if $h'(r) = 0$ or $4\,r\,h''(r) + 3\,h'(r) = 0$, that is, $h(r) = C$ or $h(r) = r^{1/4}$. If $h(r) = r^{1/4}$, then $X^2h(r) = 3\,r^{-7/4}\,(y(x^2 + y^2) - xt)^2 \geq 0$ and $Y^2h(r) = 3\,r^{-7/4}\,(x(x^2 + y^2) + yt)^2 \geq 0$, and so $r^{1/4}$ is convex in $\mathbb{R}^3 \setminus \{0\}$.

On the other hand, $\det \mathcal{H}(u) \leq C\,r^{-1/2}$ and by (2.1),

$$\int_{r^{1/4} \leq R} \det \mathcal{H}(u)\,d\xi \leq C \int_{r^{1/4} \leq R} r^{-1/2}\,d\xi = C \int_0^R \rho^{Q-1}\,\rho^{-2}\,d\rho = C\,R^2$$

since $Q = 4$. □

Proposition 4.6. *Let $u \in C^2(\Omega)$ be convex, with $\Omega = \{\xi \in \mathbb{R}^3 : 0 < d(\xi, \xi_0) < R\}$, and $u \leq 0$ on $\{\xi \in \mathbb{R}^3 : d(\xi, \xi_0) = R\}$. Then $u \leq v$, where v is defined in Proposition 4.5 with $m = -u(\xi_0)$.*

Proof. Let $\varepsilon > 0$, $\xi_0 = (x_0, y_0, t_0)$, $\xi = (x, y, t)$,

$$u_\varepsilon(\xi) = u(\xi) + \varepsilon\,(x^2 + y^2),$$

and

$$v_\varepsilon(\xi) = -(1 - \sqrt{\varepsilon})\,u(\xi_0)\left(\frac{d(\xi, \xi_0)}{(1 - \sqrt{\varepsilon})\,R} - 1\right).$$

We first claim that $u_\varepsilon(\xi) \leq v_\varepsilon(\xi)$ for all $\xi \in \partial\Omega$ and for all ε sufficiently small. Indeed, if $\xi = \xi_0$, then $u_\varepsilon(\xi_0) \leq v_\varepsilon(\xi_0)$ if and only if $\sqrt{\varepsilon}\,(x_0^2 + y_0^2) \leq -u(\xi_0)$ which holds for all ε sufficiently small. On the other hand, if $d(\xi, \xi_0) = R$, then $v_\varepsilon(\xi) = -\sqrt{\varepsilon}\,u(\xi_0)$ and $u_\varepsilon(\xi) \leq \varepsilon\,(x^2 + y^2) \leq \varepsilon \max_{d(\xi,\xi_0)=R}(x^2 + y^2) = \varepsilon\,M$. Hence $u_\varepsilon(\xi) \leq v_\varepsilon(\xi)$ on $d(\xi, \xi_0) = R$ if $\sqrt{\varepsilon}\,M \leq -u(\xi_0)$ which again holds for all ε sufficiently small.

We also have

$$\det \mathcal{H}(u_\varepsilon) = \det \mathcal{H}(u) + 2\varepsilon \operatorname{trace} \mathcal{H}(u) + 4\,\varepsilon^2 > 0 = \det \mathcal{H}(v_\varepsilon) \tag{4.3}$$

in Ω, and $\operatorname{trace}\{\mathcal{H}(u_\varepsilon + v_\varepsilon)\} = \operatorname{trace} \mathcal{H}(u) + 8\,\varepsilon + \operatorname{trace} \mathcal{H}(v_\varepsilon) > 0$. Therefore from Proposition 4.4 we get $u_\varepsilon \leq v_\varepsilon$ in Ω, and the proposition follows letting $\varepsilon \to 0$.

□

As a consequence of these propositions we get the following extension of Theorem 1.2 needed in the proof of the maximum principle Theorem 1.3.

Theorem 4.7. *Let $\Omega = \{\xi \in \mathbb{R}^3 : 0 < d(\xi, \xi_0) < R\}$, and let $v \in C^2(\overline{B}_R(\xi_0))$ be convex in Ω satisfying $v = 0$ on $\partial B_R(\xi_0)$ and set $u(\xi) = -v(\xi_0)\left(\dfrac{d(\xi, \xi_0)}{R} - 1\right)$. Then*

$$\int_{B_R(\xi_0)} \left\{\det \mathcal{H}(u) + 12\,(\partial_t u)^2\right\} d\xi \leq \int_{B_R(\xi_0)} \left\{\det \mathcal{H}(v) + 12\,(\partial_t v)^2\right\} d\xi.$$

Proof. From Proposition 4.6 we have that $v \leq u$ in $B_R(\xi_0)$. Let $\varepsilon > 0$, we claim that

$$\begin{aligned} &\int_{B_R(\xi_0)\setminus B_\varepsilon(\xi_0)} \left\{\det \mathcal{H}(u) + 12\,(\partial_t u)^2\right\} d\xi \\ &\qquad \leq \int_{B_R(\xi_0)\setminus B_\varepsilon(\xi_0)} \left\{\det \mathcal{H}(v) + 12\,(\partial_t v)^2\right\} d\xi + O(\varepsilon^2), \end{aligned} \tag{4.4}$$

as $\varepsilon \to 0$. We may assume by the invariance of the vector fields that $\xi_0 = 0$. Since the functions u, v are both convex and C^2 except at 0, we proceed as in the proof of Theorem 1.2 applied to the open set $\Omega_\varepsilon = B_R(0) \setminus B_\varepsilon(0)$. The integral A in that theorem has now the

form

$$\begin{aligned} A &= \int_0^1 \int_{\partial B_R(0)} \left\{ \sum_{i,j=1}^{2} \left(\frac{\partial H}{\partial r_{ij}}(v+sw)\, X_j w \cdot \nu_{X_i} \right) \right\} d\sigma(\xi) ds \\ &\quad - \int_0^1 \int_{\partial B_\varepsilon(0)} \left\{ \sum_{i,j=1}^{2} \left(\frac{\partial H}{\partial r_{ij}}(v+sw)\, X_j w \cdot \nu_{X_i} \right) \right\} d\sigma(\xi) ds \\ &= A_1 - A_2. \end{aligned}$$

As in the proof of Theorem 1.2, $A_1 \le 0$. We claim that $A_2 = O(\varepsilon^2)$ as $\epsilon \to 0$, in fact, each summand in A_2 is $O(\varepsilon^2)$. Recall that $w = u - v$. From the computations in the proof of Proposition 4.5, we have that $|X_j d| \le 1$, $|X_i X_j d| \le c\, d^{-1}$ and we see that

$$\begin{aligned} J_{11} &= \int_{\partial B_\varepsilon(0)} \frac{\partial H}{\partial r_{11}}(v+sw)\, X_1 w\, \nu_{X_1}\, d\sigma(\xi) \\ &= -\int_{d(\xi)=\varepsilon} Y^2(v+sw)\, Xw\, \frac{Xd}{|Dd|}\, d\sigma(\xi) \\ &\le C \int_{d(\xi)=\varepsilon} d^{-1}\, \frac{d\sigma(\xi)}{|Dd|} \\ &= C\, \varepsilon^{-1} \int_{d(\xi)=\varepsilon} \frac{d\sigma(\xi)}{|Dd|}, \end{aligned} \tag{4.5}$$

for all $0 < s < 1$. On the other hand, from the coarea formula and by (2.1)

$$\int_0^R \int_{d(\xi)=\tau} \frac{d\sigma(\xi)}{|Dd|}\, d\tau = \int_{d(\xi)\le R} d\xi = C\, R^4.$$

So, by differentiating the previous equality with respect to R, we have $\displaystyle\int_{d(\xi)=R} \frac{d\sigma(\xi)}{|Dd|} = C\, R^3$ and inserting this value in (4.5) we obtain that $J_{11} = O(\varepsilon^2)$. Since all other terms in A_2 can be handled in the same way, we obtain (4.4) and the theorem follows letting $\varepsilon \to 0$. □

As a consequence of Proposition 4.4 we obtain a simple proof of the fact that convex functions are Lipschitz with respect to the distance d. The fact that comparison with cones implies a Lipschitz bound was already noted by Jensen in [6]. A subelliptic version of Jensen's argument is also given in [8].

Proposition 4.8. *Let $\Omega \subset \mathbb{R}^3$ be an open set and $u \in C(\Omega)$ convex in Ω. Then for each ball $\overline{B} \subset \Omega$ there exists a constant C_B such that $|u(x) - u(y)| \le C_B\, d(x,y)$ for all $x, y \in B$.*

Proof. We can assume that $u \in C^2(\Omega)$ and let $B_d(x_0, 2R) \subset \Omega$. Let $y \in B_d(x_0, R)$ and $\phi(x) = u(x) - u(y) + \varepsilon\Big((x_1 - y_1)^2 + (x_2 - y_2)^2\Big)$; $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, with $x \in B_d(y, R)$. We have $\mathcal{H}(\phi + C_\varepsilon\, d(\cdot, y)) > 0$ in $B_d(y, R)$ and $\phi(x) \le C_\varepsilon\, d(x,y)$ for $d(x,y) = R$ where $C_\varepsilon = \dfrac{\operatorname{osc}_{B_d(x_0,2R)} u + \varepsilon \operatorname{diam}(B_d(x_0, 2R))^2}{R}$. We have $\det \mathcal{H}(\phi) \ge \det \mathcal{H}(d(\cdot, y))$ in $B_d(y, R) \setminus \{y\}$ so by the comparison principle Proposition 4.4 we get that $\phi(x) \le C_\varepsilon\, d(x,y)$

for $x \in B_d(y,R)$. Letting $\varepsilon \to 0$ we get $u(x) - u(y) \le C\, d(x,y)$ for $x \in B_d(y,R)$ with $C = \dfrac{\operatorname{osc}_{B_d(x_0,2R)} u}{R}$ and $y \in B_d(x_0,R)$. If $x,y \in B_d(x_0,R/4)$, then $x \in B_d(y,R/2)$ and so $y \in B_d(x,R)$ and by the previous inequality we get $u(y) - u(x) \le C\, d(y,x) = C\, d(x,y)$. Therefore we obtain $|u(x) - u(y)| \le C\, d(x,y)$ for all $x,y \in B_d(x_0,R/4)$. □

5. Maximum Principle of Aleksandrov type

To prove Theorem 1.3, we need some preliminary results.

Proposition 5.1. *Let u be convex in Ω open and bounded. Suppose $u \le 0$ on $\partial\Omega$. Then $u \le 0$ in Ω.*

Proof. Let $\varepsilon > 0$ and $u_\varepsilon(x,y,t) = u(x,y,t) + \varepsilon\,(x^2+y^2)$. We have $\mathcal{H}(u_\varepsilon) = \mathcal{H}(u) + 2\varepsilon\,\mathrm{Id}$, so

$$\det \mathcal{H}(u_\varepsilon) = \det \mathcal{H}(u) + 2\varepsilon \operatorname{trace} \mathcal{H}(u) + 4\,\varepsilon^2.$$

Since $\det \mathcal{H}(\varepsilon(x^2+y^2)) = 4\,\varepsilon^2$, we get $\det \mathcal{H}(u_\varepsilon) \ge \det \mathcal{H}(\varepsilon(x^2+y^2))$ in Ω. Also $u_\varepsilon \le \varepsilon(x^2+y^2)$ on $\partial\Omega$, and $\operatorname{trace}\{\mathcal{H}(u_\varepsilon + \varepsilon(x^2+y^2)\} = \operatorname{trace}\mathcal{H}(u) + 8\,\varepsilon > 0$. The proposition then follows from Proposition 4.4. □

The following lemma will be used repeatedly in the proof of Proposition 5.3.

Lemma 5.2. *Let $\xi_0 \in B_R(0)$ and $\xi \in \Pi_{\xi_0} \cap B_R(0)$. Let $\lambda > 0$ be such that*

$$\xi' = \xi_0 \circ \delta_\lambda(\xi_0^{-1} \circ \xi) \in \Pi_{\xi_0} \cap \partial B_R(0).$$

Suppose u is convex in $B_R(0)$ and $u = 0$ on $\partial B_R(0)$. We have

(1) *If $\xi_0 = (x_0,y_0,t_0)$ and $\xi = (0,0,t_0)$, then $\lambda \ge 2$ and*

$$u(\xi) \le \frac{1}{2}\, u(\xi_0). \tag{5.6}$$

(2) *If $0 < \alpha,\beta < 1$, $\alpha + \beta < 1$, $\rho(\xi_0) \le \alpha\, R$ and $d(\xi_0,\xi) \le \beta\, R$, then $\lambda \ge \dfrac{1-\alpha}{\beta}$ and*

$$u(\xi) \le \frac{1-\alpha-\beta}{1-\alpha}\, u(\xi_0). \tag{5.7}$$

Proof. To prove the first part of (1), if $\eta = (x,y,t) \in \Pi_{\xi_0}$, then we have that

$$\xi_0 \circ \delta_\lambda(\xi_0^{-1} \circ \eta) = (x_0 + \lambda(x - x_0), y_0 + \lambda(y - y_0), t_0 + \lambda(t - t_0)) \in \Pi_{\xi_0},$$

in particular, $\xi' = ((1-\lambda)x_0, (1-\lambda)y_0, t_0)$. Hence

$$\begin{aligned} R^4 = \rho(\xi')^4 &= \left((1-\lambda)^2 x_0^2 + (1-\lambda)^2 y_0^2\right)^2 + t_0^2 \\ &= (1-\lambda)^4 \left(x_0^2 + y_0^2\right)^2 + \rho(\xi_0)^4 - \left(x_0^2 + y_0^2\right)^2 \\ &\le \left((1-\lambda)^4 - 1\right)\left(x_0^2 + y_0^2\right)^2 + R^4, \end{aligned}$$

and so $|1-\lambda| \ge 1$. Since $\lambda > 0$, it follows that $\lambda \ge 2$.

To prove the first part of (2) we write

$$\begin{aligned} R = \rho(\xi') &= \rho((\xi_0^{-1})^{-1} \circ \delta_\lambda(\xi_0^{-1} \circ \xi)) \le \rho(\xi_0^{-1}) + \rho(\delta_\lambda(\xi_0^{-1} \circ \xi)) \\ &= \rho(\xi_0) + \lambda\, \rho(\xi_0^{-1} \circ \xi) = \rho(\xi_0) + \lambda\, d(\xi_0, \xi) \\ &\le \alpha\, R + \lambda \beta\, R, \end{aligned}$$

and so $\lambda \ge \dfrac{1-\alpha}{\beta}$.

To prove (5.6) and (5.7), by definition of ξ' we have that $\xi = \xi_0 \circ \delta_{1/\lambda}(\xi_0^{-1} \circ \xi')$. Since $u(\xi') = 0$ and in any case $\lambda \ge 1$, it follows from (2.3) that $u(\xi) \le \left(1 - \dfrac{1}{\lambda}\right) u(\xi_0)$. Thus (5.6) and (5.7) follow from Proposition 5.1 since $u \le 0$ in $B_R(0)$. □

Proposition 5.3 (Harnack-type inequality for convex functions)**.** *Let u be convex and $u = 0$ on $\partial B_R(0)$. Given $\xi_0 \in B_R(0)$ there exists a positive constant c, depending on $d(\xi_0, \partial B_R(0))$, such that*

$$u(0) \le c\, u(\xi_0).$$

Proof. Let $\xi_0 = (x_0, y_0, t_0)$ and $\xi_1 = \exp(-x_0 X - y_0 Y)(\xi_0) = (0, 0, t_0) \in \Pi_{\xi_0}$. We obviously have that $d(\xi_1, \xi_0) = \sqrt{x_0^2 + y_0^2} \le d(0, \xi_0) < R$. Applying Lemma 5.2(1) with $\xi_0 \rightsquigarrow \xi_0$ and $\xi \rightsquigarrow \xi_1$ we get that

$$u(\xi_1) \le \frac{1}{2} u(\xi_0). \tag{5.8}$$

We shall prove that there exists a constant $C_1 > 0$ depending only of the distance from ξ_1 to $\partial B_R(0)$ such that

$$u(0) \le C_1\, u(\xi_1). \tag{5.9}$$

To prove (5.9) we may assume $\xi_1 \ne 0$, and consider two cases.

Case 1. $d(\xi_1, 0) = |t_0|^{1/2} \le R/2$.

If $t_0 > 0$, define $\sigma = \dfrac{\sqrt{t_0}}{2}$ and put

$$\begin{aligned} \xi_2 &= \exp(\sigma X)\xi_1 = (\sigma, 0, t_0), \\ \xi_3 &= \exp(\sigma Y)\xi_2 = (\sigma, \sigma, t_0 - 2\sigma^2), \\ \xi_4 &= \exp(-\sigma X)\xi_3 = (0, \sigma, t_0 - 2\sigma^2 - 2\sigma^2) = (0, \sigma, t_0 - 4\sigma^2) = (0, \sigma, 0). \end{aligned}$$

By our choice of σ we have

$$\exp(-\sigma Y)\xi_4 = (0, 0, t_0 - 4\sigma^2) = 0.$$

Let us remark that

$$\sigma = \frac{1}{2} d(\xi_1, 0) \le R/4.$$

We have

$$d(\xi_1, \xi_2) = d(\xi_2, \xi_3) = d(\xi_3, \xi_4) = \sigma;$$
$$\rho(\xi_2) = 17^{1/4}\, \sigma; \qquad \rho(\xi_3) = 8^{1/4}\, \sigma; \qquad \rho(\xi_4) = \sigma.$$

Hence $\xi_2, \xi_3, \xi_4 \in B_R$. Applying Lemma 5.2(2) with $\xi_0 \rightsquigarrow \xi_1$, $\xi \rightsquigarrow \xi_2$, $\alpha = 1/2$, and $\beta = 1/4$ we get that

$$u(\xi_2) \leq \frac{1}{2} u(\xi_1).$$

Next, applying Lemma 5.2(2) with $\xi_0 \rightsquigarrow \xi_2$, $\xi \rightsquigarrow \xi_3$, $\alpha = 17^{1/4}/4$, and $\beta = 1/4$, we get that

$$u(\xi_3) \leq \frac{3 - 17^{1/4}}{4 - 17^{1/4}} u(\xi_2) < \frac{3}{8} u(\xi_2).$$

Applying once again Lemma 5.2(2) now with $\xi_0 \rightsquigarrow \xi_3$ and $\xi \rightsquigarrow \xi_4$, $\alpha = 8^{1/4}/4, \beta = 1/4$, we get that

$$u(\xi_4) \leq \frac{3 - 8^{1/4}}{4 - 8^{1/4}} u(\xi_3) < \frac{1}{2} u(\xi_3).$$

Define

$$\xi^{(4)} = \xi_4 \circ \delta_\lambda(\xi_4^{-1}) \in \Pi_{\xi_4}$$

and choose $\lambda > 0$ such that $\xi^{(4)} \in \partial B_R$. Applying Lemma 5.2(2) now with $\xi_0 \rightsquigarrow \xi_4$ and $\xi \rightsquigarrow 0$, $\alpha = 1/4, \beta = 1/4$, we get that

$$u(0) \leq \frac{2}{3} u(\xi_4).$$

This completes the proof of (5.9) for $t_0 > 0$.

If $t_0 < 0$, define $\sigma = \dfrac{\sqrt{-t_0}}{2}$ and put

$$\begin{aligned}
\xi_2 &= \exp(\sigma Y)\xi_1 = (0, \sigma, t_0),\\
\xi_3 &= \exp(\sigma X)\xi_2 = (\sigma, \sigma, t_0 + 2\sigma^2),\\
\xi_4 &= \exp(-\sigma Y)\xi_3 = (\sigma, 0, t_0 + 4\sigma^2).
\end{aligned}$$

By our choice of σ we have

$$\exp(-\sigma X)\xi_4 = (0, 0, t_0 + 4\sigma^2) = 0.$$

Then, arguing as in case $t_0 > 0$, we get (5.9).

Case 2. $R/2 < d(\xi_1, 0) = |t_0|^{1/2} < R$.

Define

$$d := \frac{d(\xi_1, \partial B_R)}{\sqrt{6}} = \frac{\sqrt{R^2 - |t_0|}}{\sqrt{6}}. \tag{5.10}$$

Obviously $d^2 < R^2/8$. It is not restrictive to assume $t_0 > 0$. We first prove that there exists a universal constant $0 < C_2 < 1$ such that

$$u(0, 0, t_0 - 4d^2) \leq C_2\, u(\xi_1). \tag{5.11}$$

Let

$$\begin{aligned}
\xi_1 &= (0,0,t_0)\\
\xi_2 &= \exp(dX)(\xi_1) = (d,0,t_0)\\
\xi_3 &= \exp(dY)(\xi_2) = (d,d,t_0-2d^2)\\
\xi_4 &= \exp(-dX)(\xi_3) = (0,d,t_0-4d^2)\\
\xi_5 &= \exp(-dY)(\xi_4) = (0,0,t_0-4d^2).
\end{aligned}$$

We have $\xi_{i+1} \in \Pi_{\xi_i}$ for $i = 1,2,3,4$. Let

$$\xi_2^{(1)} = \exp(\lambda dX)(\xi_1) = (\lambda d, 0, t_0) = \xi_1 \circ \delta_\lambda(\xi_1^{-1} \circ \xi_2),$$

with $\lambda > 0$ such that $\xi_2^1 \in \Pi_{\xi_1} \cap \partial B_R$. Then

$$R^4 = \rho(\xi_2^{(1)}) = \lambda^4 d^4 + t_0^2 = \lambda^4 d^4 + (R^2 - 6d^2)^2 = (\lambda^4 + 36)d^4 + R^4 - 12d^2R^2,$$

and so

$$12R^2 = (\lambda^4 + 36)d^2 \le (\lambda^4 + 36)R^2/8$$

which yields $\lambda > 2$. Hence,

$$u(\xi_2) \le (1/2)u(\xi_1).$$

We have

$$\begin{aligned}
\rho(\xi_2)^4 &= d^4 + t_0^2 = d^4 + (R^2 - 6d^2)^2\\
&= 37d^4 + R^4 - 12R^2d^2 = d^2(37d^2 - 12R^2) + R^4\\
&\le d^2(37/8 - 12)R^2 + R^4 = \left(\frac{1}{8}\left(\frac{37}{8} - 12\right) + 1\right) R^4 < R^4,
\end{aligned}$$

and

$$d(\xi_2, \xi_3) = d \le \frac{1}{\sqrt{8}} R.$$

If

$$\xi_3^{(2)} = \exp(\lambda dY)(\xi_2) = (d, \lambda d, t_0 - 2\lambda d^2) = \xi_2 \circ \delta_\lambda(\xi_2^{-2} \circ \xi_3)$$

and we pick $\lambda > 0$ such that $\xi_3^{(2)} \in \Pi_{\xi_2} \cap \partial B_R$, then applying Lemma 5.2(2) with $\xi_0 \rightsquigarrow \xi_2$, $\xi \rightsquigarrow \xi_3$, $\alpha = \sqrt[4]{\frac{1}{8}\left(\frac{37}{8} - 12\right) + 1}$, and $\beta = \frac{1}{\sqrt{8}}$, we get that

$$u(\xi_3) \le \frac{\sqrt{8} - \sqrt[4]{5} - 1}{\sqrt{8} - \sqrt[4]{5}}\, u(\xi_2) < \frac{1}{5}\, u(\xi_2).$$

Next,

$$\begin{aligned}
\rho(\xi_3)^4 &= (2d^2)^2 + (t_0 - 2d^2)^2 = 4d^4 + (R^2 - 8d^2)^2 = 68d^4 - 16R^2d^2 + R^4\\
&\le d^2R^2(68/8 - 16) + R^4 \le \left(\frac{1}{8}\left(\frac{68}{8} - 16\right) + 1\right) R^4 < R^4,
\end{aligned}$$

and

$$d(\xi_3, \xi_4) = d \le \frac{1}{\sqrt{8}} R.$$

Let

$$\xi_4^{(3)} = \exp(-\lambda dX)(\xi_3) = ((1-\lambda)d, d, t_0 - 2d^2 - 2\lambda d^2) = \xi_3 \circ \delta_\lambda(\xi_3^{-1} \circ \xi_4),$$

with $\lambda > 0$ such that $\xi_4^{(3)} \in \partial B_R \cap \Pi_{\xi_3}$. Applying Lemma 5.2(2) with $\xi_0 \rightsquigarrow \xi_3$, $\xi \rightsquigarrow \xi_4$, $\alpha = \sqrt[4]{\frac{1}{8}\left(\frac{68}{8} - 16\right) + 1} = \frac{1}{2}$, and $\beta = \frac{1}{\sqrt{8}}$, we get that

$$u(\xi_4) < \frac{1}{4} u(\xi_3).$$

We have

$$\begin{aligned}\rho(\xi_4)^4 &= d^4 + (t_0 - 4d^2)^2 = d^4 + (R^2 - 10d^2)^2 = 101d^4 + R^4 - 20R^2d^2 \\ &= (101d^2 - 20R^2)d^2 + R^4 \le (101/8 - 20)R^2d^2 + R^4 \\ &\le \left(\frac{1}{8}\left(\frac{101}{8} - 20\right) + 1\right) R^4 < R^4,\end{aligned}$$

and

$$d(\xi_4, \xi_5) = d \le \frac{1}{\sqrt{8}} R.$$

Letting

$$\xi_5^{(4)} = \exp(-\lambda dY)(\xi_4) = (0, (1-\lambda)d, t_0 - 4d^2) = \xi_4 \circ \delta_\lambda(\xi_4^{-1} \circ \xi_5)$$

with $\lambda > 0$ such that $\xi_5^{(4)} \in \Pi_{\xi_4} \cap \partial B_R$, and applying Lemma 5.2(2) with $\xi_0 \rightsquigarrow \xi_4$, $\xi \rightsquigarrow \xi_5$, $\alpha = \sqrt[4]{\frac{1}{8}\left(\frac{101}{8} - 20\right) + 1} = \frac{\sqrt[4]{5}}{\sqrt{8}}$, and $\beta = \frac{1}{\sqrt{8}}$, we get that

$$u(\xi_5) \le \frac{1}{5} u(\xi_4).$$

Thus, inequality (5.11) follows.

To complete the proof of (5.9) in Case 2, we iterate the inequality (5.11). Let $d_0 = d$ (defined in (5.10)), $t_1 = t_0 - 4\,d_0^2$, and in general

$$t_{j+1} = t_j - 4\,d_j^2, \qquad \text{and} \qquad d_j^2 = \frac{R^2 - t_j}{6}.$$

We have $d_{j+1}^2 = \dfrac{R^2 - t_{j+1}}{6} = \dfrac{R^2 - t_j + 4\,d_j^2}{6} = \left(1 + \dfrac{2}{3}\right) d_j^2$. Thus,

$$\begin{aligned} t_{N+1} = t_0 - 4\sum_{j=0}^{N} d_j^2 &= t_0 - 4\,d_0^2 \sum_{j=0}^{N} \left(\frac{5}{3}\right)^j \\ &= t_0 - (R^2 - t_0)\left(\left(\frac{5}{3}\right)^{N+1} - 1\right). \end{aligned} \tag{5.12}$$

Pick N such that

$$t_N \le \frac{R^2}{4} < t_{N-1},$$

which amounts

$$N - 1 < \ln\left(\frac{3\,R^2}{4(R^2 - t_0)}\right)^{1/\ln(5/3)} \le N. \tag{5.13}$$

We have $t_N < t_{N-1} < \cdots < t_1 < t_0$ and it is easy to check from (5.12), the choice of N, and (5.10) that $t_N \ge -R^2/4$. Therefore $(0,0,t_j) \in B_R(0) \setminus B_{R/2}(0)$ for $0 \le j \le N-1$ and $(0,0,t_N) \in B_{R/2}(0)$. Iterating (5.11) N times, then yields

$$u(0,0,t_N) \le C_2^N\, u(\xi_1).$$

Since $0 < C_2 < 1$, there is $\gamma > 0$ such that $C_2 = e^{-\gamma}$, and from (5.13) we obtain ($u < 0$)

$$u(0,0,t_N) \le C_2\, \exp\left(-\gamma\, \ln\left(\frac{3\,R^2}{4(R^2 - t_0)}\right)^{1/\ln(5/3)}\right) u(\xi_1).$$

Since $(0,0,t_N) \in B_{R/2}(0)$, we can apply (5.9) in Case 1 to get $u(0) \le C_1\, u(0,0,t_N)$. Consequently,

$$u(0) \le C_1\, C_2 \left(\frac{4(R^2 - t_0)}{3\,R^2}\right)^{\gamma/\ln(5/3)} u(\xi_1),$$

which completes the proof of (5.9) in Case 2.

Finally, combining (5.8) and (5.9) we obtain the proposition. □

Proof of Theorem 1.3. Define

$$u(0) = -m$$

and

$$v(\zeta) = m\left(\frac{d(\zeta,0)}{R} - 1\right).$$

We have $v = u = 0$ on ∂B_R, v is convex in B_R and $v \ge u$ in B_R by Proposition 4.6. From the comparison principle, Theorem 4.7, we then get

$$\int_{B_R} \{\det \mathcal{H}(v) + 12\, v_t^2\}\, d\xi \le \int_{B_R} \{\det \mathcal{H}(u) + 12\, u_t^2\}\, d\xi.$$

Moreover,

$$\begin{aligned}\int_{B_R} \{\det \mathcal{H}(v) + 12\, v_t^2\}\, d\xi &= \left(\frac{m}{R}\right)^2 \int_{B_R} \{\det \mathcal{H}(d(\zeta,0)) + 12\,(\partial_t d(\zeta,0))^2\}\, d\zeta \\ &= 12\left(\frac{m}{R}\right)^2 R^2 \int_{B_1} (\partial_t d(\zeta,0))^2 d\zeta \\ &= c_1 m^2\end{aligned}$$

with

$$c_1 = 12 \int_{B_1} (\partial_t d(\zeta,0))^2 d\zeta > 0.$$

Let

$$u(\xi_0) = \min_{B_R} u = -m_0.$$

By Proposition 5.3 there exists a constant $c > 0$ such that

$$m_0 \leq \frac{1}{c} m.$$

Hence,

$$m_0^2 \leq \frac{1}{c^2} m^2 \leq \frac{1}{c_1 c^2} \int_{B_R} \{\det \mathcal{H}(u) + 12\, u_t^2\}\, d\xi.$$

□

6. Monge-Ampère Measures

6.1. **Oscillation estimate.** In this section we prove that if u is convex, we can control the integral of $\det \mathcal{H}(u) + 12(u_t)^2$ locally in terms of the oscillation of u, Theorem 1.4

Let us start with a lemma on convex functions, which is similar to the Euclidean one for standard convex functions [12, Lemma 2.3].

Lemma 6.1. *If $u_1, u_2 \in C^2(\Omega)$ are convex, and f is convex in $\mathbb{R}^2$ and nondecreasing in each variable, then the composite function $w = f(u_1, u_2)$ is convex.*

Proof. Assume first that $f \in C^2(\mathbb{R}^2)$, and set $X_1 = X, X_2 = Y$. We have

$$X_j w = \sum_{p=1}^{2} \frac{\partial f}{\partial u_p} X_j u_p,$$

$$X_i X_j w = \sum_{p=1}^{2} \left(\frac{\partial f}{\partial u_p} X_i X_j u_p + \sum_{q=1}^{2} \frac{\partial^2 f}{\partial u_q \partial u_p} X_i u_q X_j u_p \right),$$

and for every $h = (h_1, h_2) \in \mathbb{R}^2$

$$\begin{aligned} \langle \mathcal{H}(w)h, h\rangle &= \sum_{i,j=1}^{2} X_i X_j w\, h_i\, h_j \\ &= \sum_{p=1}^{2} \frac{\partial f}{\partial u_p} \langle \mathcal{H}(u_p)h, h\rangle + \sum_{p,q=1}^{2} \frac{\partial^2 f}{\partial u_q \partial u_p} (\sum_{i=1}^{2} X_i u_q h_i)(\sum_{j=1}^{2} X_j u_p h_j) \\ &\geq 0, \end{aligned}$$

since $\mathcal{H}(u_p)$ is non negative definite and $\dfrac{\partial f}{\partial u_p} \geq 0$ for $p = 1, 2$, and the matrix

$$\left(\frac{\partial^2 f}{\partial u_q \partial u_p} \right)_{p,q=1,2}$$

is non negative definite.

If f is only continuous, then given $h > 0$ let

$$f_h(x) = h^{-2} \int_{\mathbb{R}^2} \varphi\left(\frac{x-y}{h}\right) f(y)dy,$$

where $\varphi \in C^\infty$ is nonnegative vanishing outside the unit ball of $\mathbb{R}^2$, and $\int \varphi = 1$. Since f is convex, then f_h is convex and by the previous calculation $w_h = f_h(u_1, u_2)$ is convex. In particular, w_h satisfies Proposition 2.2 and since $w_h \to w$ uniformly on compact sets as $h \to 0$, we get that w is convex. □

Proof of Theorem 1.4. Given $\xi_0 \in \Omega$ let $B_R = B_R(\xi_0)$ be a d–ball of radius R and center at ξ_0 such that $B_R \subset \Omega$. Let $B_{\sigma R}$ be the concentric ball of radius σR, with $0 < \sigma < 1$. Without loss of generality we can assume $\xi_0 = 0$, because the vector fields X and Y are left invariant with respect to the group of translations. Let $M = \max_{B_R} u$, then $u - M \leq 0$ in B_R. Given $\varepsilon > 0$ we shall work with the function $u - M - \varepsilon < -\varepsilon$. In other words, by subtracting a constant, we may assume $u < -\varepsilon$ in B_R, for each given positive constant ε; ε will tend to zero at the end of the proof.

Define

$$m_0 = \inf_{B_R} u,$$

and

$$v(\xi) = \frac{m_0}{(1-\sigma^4)R^4}(R^4 - \|\xi\|^4).$$

Obviously $v = 0$ on ∂B_R and $v = m_0$ on $\partial B_{\sigma R}$. We claim that v is convex in B_R and $v \leq m_0$ in $B_{\sigma R}$. Setting $r = \|\xi\|^4$, $h(r) = \dfrac{m_0}{(1-\sigma^4)R^4}(R^4 - r)$, and following the calculations in the proof of Proposition 4.5 we get

$$\det \mathcal{H}(v) = 144(x^2+y^2)^2 \left(\frac{m_0}{(1-\sigma^4)R^4}\right)^2 \geq 0,$$

and

$$X^2 h = Y^2 h = -12\,(x^2+y^2)\frac{m_0}{(1-\sigma^4)R^4} \geq 0,$$

because m_0 is negative. Hence v is convex in B_R. Since $v - m_0 = 0$ on $\partial B_{\sigma R}$, it follows from Proposition 5.1 that $v \leq m_0$ in $B_{\sigma R}$. In particular, $v \leq u$ in $B_{\sigma R}$.

To continue with the proof we need the following lemma.

Lemma 6.2. *Let $\rho \in C_0^\infty(\mathbb{R}^2)$, radial with support in the Euclidean unit ball, $\int_{\mathbb{R}^2} \rho(x)\,dx = 1$, and let*

$$f_h(x_1, x_2) = h^{-2} \int_{\mathbb{R}^2} \rho((x-y)/h)\, \max\{y_1, y_2\}\, dy_1 dy_2. \tag{6.14}$$

The function f_h satisfies the following conditions:

(1) *If $x_1 > x_2$, then there exists $h_0 > 0$ and a neighborhood V of (x_1, x_2) such that $f_h(y_1, y_2) = y_1$ for all $(y_1, y_2) \in V$ and for all $h \leq h_0$.*
(2) *There exists a positive constant α such that $f_h(x, x) = x + \alpha\, h$ for all $h > 0$ and for all $x \in \mathbb{R}$.*

(3) *For all* $h > 0$, $f_h(\cdot, x_2)$ *is nondecreasing for each* x_2 *and* $f_h(x_1, \cdot)$ *is nondecreasing for each* x_1.

Proof. Proof of (1). If $x_1 > x_2$ then there exists a cube Q centered at (x_1, x_2) such that if $(z_1, z_2) \in Q$ then $z_1 > z_2$. Hence $x_1 - y_1 > x_2 - y_2$ for all $|(y_1, y_2)| < h$ with h sufficiently small. Then

$$\begin{aligned} f_h(x_1, x_2) &= h^{-2} \int_{|y|<h} \rho(y/h)(x_1 - y_1)\, dy_1 dy_2 = x_1 - h^{-2} \int_{|y|<h} \rho(y/h) y_1\, dy_1 dy_2 \\ &= x_1 - h \int_0^1 t^2 \rho(t) \int_{S^1} y_1\, d\sigma(y)\, dt = x_1. \end{aligned}$$

Proof of (2). We have

$$\begin{aligned} f_h(x, x) &= h^{-2} \int_{|y|<h} \rho(y/h) \max\{x - y_1, x - y_2\}\, dy_1 dy_2 \\ &= h^{-2} \int_{|y|<h} \rho(y/h) \left(x + \max\{-y_1, -y_2\}\right)\, dy_1 dy_2 \\ &= x_1 + h^{-2} \int_{|y|<h} \rho(y/h) \max\{-y_1, -y_2\}\, dy_1 dy_2 \\ &= x_1 + h^{-2} \int_{|y|<h} \rho(y/h) \max\{y_1, y_2\}\, dy_1 dy_2 \\ &= x_1 + h \int_{|y|<1} \rho(y) \max\{y_1, y_2\}\, dy_1 dy_2 \\ &= x_1 + h \int_0^1 t^2 \rho(t) \int_{S^1} \max\{y_1, y_2\}\, d\sigma(y)\, dt \\ &= x_1 + h \int_0^1 t^2 \rho(t) \int_{S^1} \frac{|y_1 - y_2| + y_1 + y_2}{2}\, d\sigma(y)\, dt \\ &= x_1 + h \int_0^1 t^2 \rho(t) \int_{S^1} \frac{|y_1 - y_2|}{2}\, d\sigma(y)\, dt = x_1 + \alpha\, h. \end{aligned}$$

The proof of (3) is trivial. □

Continuing with the proof of Theorem 1.4, we define

$$w_h = f_h(u, v).$$

From Lemma 6.1, w_h is convex in B_R. If $y \in B_{\sigma R}$, then $v(y) \le u(y)$. If $v(y) < u(y)$, then $f_h(u, v)(y) = u(y)$ for h sufficiently small; and if $v(y) = u(y)$, then $f_h(u, v)(y) = u(y) + \alpha\, h$. Hence

$$\begin{aligned} \int_{B_{\sigma R}} \{\det \mathcal{H}(u) + 12(\partial_t u)^2\}\, d\xi &= \int_{B_{\sigma R}} \{\det \mathcal{H}(w_h) + 12((w_h)_t)^2\}\, d\xi \\ &\le \int_{B_R} \{\det \mathcal{H}(w_h) + 12((w_h)_t)^2\}\, d\xi. \end{aligned} \tag{6.15}$$

Now notice that $f_h(u,v) \geq v$ in B_R for all h sufficiently small. In addition, $u < 0$ and $v = 0$ on ∂B_R so $f_h(u,v) = 0$ on ∂B_R. Then we can apply Theorem 1.2 to w_h and v to get

$$\begin{aligned}\int_{B_R} \{\det \mathcal{H}(w_h) + 12(\partial_t w_h)^2\}\, d\xi &\leq \int_{B_R} \{\det \mathcal{H}(v) + 12(v_t)^2\}\, d\xi \\ &= 48\left(\frac{m_0}{(1-\sigma)R^4}\right)^2 \int_{B_R} (3(x^2+y^2)^2 + t^2)\, d\xi \\ &= 48\left(\frac{m_0}{(1-\sigma)}\right)^2 \int_{B_1} (3(x^2+y^2)^2 + t^2)\, d\xi.\end{aligned}$$

This inequality combined with (6.15) yields

$$\int_{B_{\sigma R}} \{\det \mathcal{H}(u) + 12(\partial_t u)^2\}\, d\xi \leq C\,(m_0)^2 \leq C\,(\mathrm{osc}_{B_R} u + \varepsilon)^2.$$

The inequality (1.3) then follows letting $\varepsilon \to 0$ and covering Ω' with balls. □

Corollary 6.3. *Let $u \in C^2(\Omega)$ be convex. For any compact domain $\Omega' \Subset \Omega$ there exists a positive constant C, independent of u, such that*

$$\int_{\Omega'} \det \mathcal{H}(u)\, d\xi \leq C(\mathrm{osc}_\Omega u)^2. \tag{6.16}$$

Corollary 6.4. *Let $u \in C^2(\Omega)$ be convex. For any compact domain $\Omega' \Subset \Omega$ there exists a positive constant C, independent of u, such that*

$$\int_{\Omega'} \mathrm{trace}\, \mathcal{H}(u)\, d\xi \leq CR^2 \mathrm{osc}_\Omega u. \tag{6.17}$$

6.2. **Measure generated by a convex function.** We shall prove that the Borel measure defined by $\mu(u) = \int \det \mathcal{H}(u) + 12u_t^2$ when u is smooth can also be defined for general convex functions. We call this measure the measure associated with u, and we shall show that the map $u \in C(\Omega) \to \mu(u)$ is weakly continuous on $C(\Omega)$.

Theorem 6.5. *Given $u \in C(\Omega)$ convex there exists a unique Borel measure $\mu(u)$ such that when $u \in C^2(\Omega)$,*

$$\mu(u)(E) = \int_E \{\det \mathcal{H}(u) + 12u_t^2\}\, d\xi \tag{6.18}$$

for any Borel set $E \subset \Omega$. Moreover, if $u_k \in C(\Omega)$ are convex, and $u_k \to u$ on compact subsets of Ω, then $\mu(u_k)$ converges weakly to $\mu(u)$, that is,

$$\int_\Omega f\, d\mu(u_k) \to \int_\Omega f\, d\mu(u), \tag{6.19}$$

for any $f \in C(\Omega)$ with compact support in Ω.

Proof. Let $u \in C(\Omega)$ be convex, and let $\{u_k\} \subset C^2(\Omega)$ be a sequence of convex functions converging to u uniformly on compacts of Ω. By Theorem 1.4

$$\int_{\Omega'} \{\det \mathcal{H}(u_k) + 12(\partial_t u_k)^2\}\, d\xi$$

are uniformly bounded, for every $\Omega' \Subset \Omega$, and hence a subsequence of $(\det \mathcal{H}(u_k) + 12(\partial_t u_k)^2)$ converges weakly in the sense of measures to a Borel measure $\mu(u)$ on Ω. We now prove that the map $u \in C(\Omega) \to \mu(u) \in M(\Omega)$, the space of finite Borel measures on Ω, is well defined. Accordingly, let $\{v_k\} \subset C^2(\Omega)$ be another sequence of convex functions converging to u uniformly on compacts of Ω. Assume $(\det \mathcal{H}(u_k) + 12(\partial_t u_k)^2)$ and $(\det \mathcal{H}(v_k) + 12(\partial_t v_k)^2)$ converge weakly to Borel measures μ, μ' respectively. Let $B = B_R \Subset \Omega$, and fix $\sigma \in (0,1)$. Let $\eta \in C^2(\overline{\Omega})$ be an convex function such that $\eta = 0$ in $B_{\sigma R}$ and $\eta = 1$ on ∂B_R. In the d–ball $B_R(0)$, the function η can be constructed as follows. If $v(\xi) = \dfrac{1}{1-\sigma^4}\left(\dfrac{\|\xi\|^4}{R^4} - \sigma^4\right)$ and f_h is the function given by (6.14), then define $\eta(\xi) = f_h(v, 0)$ with h sufficiently small. From the uniform convergence of $\{u_k\}$ and $\{v_k\}$ towards u, given $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$-\frac{\varepsilon}{2} \le u_k(x) - v_k(x) \le \frac{\varepsilon}{2}, \quad \text{for all } x \in \overline{B} \text{ and } k \ge k_\varepsilon.$$

Hence

$$u_k + \frac{\varepsilon}{2} \le v_k + \varepsilon\eta$$

on ∂B_R for $k \ge k_\varepsilon$. Define $\Omega_k = \{\xi \in B_R : u_k + \frac{\varepsilon}{2} > v_k + \varepsilon\eta\}$. From Theorem 1.2 we have

$$\begin{aligned}
\int_{\Omega_k} \{\det \mathcal{H}(u_k) + 12(\partial_t u_k)^2\}\, d\xi &\le \int_{\Omega_k} \det \mathcal{H}(v_k + \varepsilon\eta) + 12(\partial_t v_k + \varepsilon\partial_t\eta)^2 \\
&\le \int_{B_R} \det \mathcal{H}(v_k) + 12(\partial_t v_k)^2 + \varepsilon^2 C \\
&\qquad + \varepsilon C \int_{B_R} (\operatorname{trace} \mathcal{H}(v_k) + |\partial_t v_k|) \\
&\le \int_{B_R} \det \mathcal{H}(v_k) + 12(\partial_t v_k)^2 + \varepsilon^2 C \\
&\qquad + \varepsilon C \int_{B_R} \Big(\operatorname{trace} \mathcal{H}(v_k) + |\partial_t v_k|^2 + 1\Big) \qquad (6.20)
\end{aligned}$$

and by Theorem 1.4 and Corollary 6.4 the right hand side is bounded by

$$\int_{B_R} \det \mathcal{H}(v_k) + 12(\partial_t v_k)^2 + \varepsilon C.$$

By definition of Ω_k and since $\eta = 0$ in $B_{\sigma R}$, it follows that $B_{\sigma R} \subset \Omega_k$ and so by (6.20) we get

$$\int_{B_{\sigma R}} \det \mathcal{H}(u_k) + 12(\partial_t u_k)^2 \le \int_{B_R} \det \mathcal{H}(v_k) + 12(\partial_t v_k)^2 + \varepsilon C, \qquad (6.21)$$

and letting $k \to \infty$, we get $\mu(B_{\sigma R}) \le \mu'(B_R) + C\,\varepsilon$. Hence if $\varepsilon \to 0$ and $\sigma \to 1$ we obtain

$$\mu(B) \le \mu'(B).$$

By interchanging $\{u_k\}$ and $\{v_k\}$ we get $\mu = \mu'$.

To prove (6.19), we first claim that it holds when $u_k \in C^2(\Omega)$. Indeed, let u_{k_m} be an arbitrary subsequence of u_k, so $u_{k_m} \to u$ locally uniformly as $m \to \infty$. By definition of

$\mu(u)$, there is a subsequence $u_{k_{m_j}}$ such that $\mu\left(u_{k_{m_j}}\right) \to \mu(u)$ weakly as $j \to \infty$. Therefore, given $f \in C_0(\Omega)$, the sequence $\int_\Omega f\, d\mu(u_k)$ and an arbitrary subsequence $\int_\Omega f\, d\mu(u_{k_m})$, there exists a subsequence $\int_\Omega f\, d\mu(u_{k_{m_j}})$ converging to $\int_\Omega f\, d\mu(u)$ as $j \to \infty$ and (6.19) follows. For the general case, given k there exists $u_j^k \in C^2(\Omega)$ such that $u_j^k \to u_k$ locally uniformly as $j \to \infty$. By definition of $\mu(u_k)$, there exists a subsequence $u_{j_m}^k$ such that $\mu\left(u_{j_m}^k\right) \to \mu(u_k)$ weakly as $m \to \infty$. Let $f \in C_0(\Omega)$, supp $f = K \subset \Omega' \Subset \Omega$. There exists $m_1 < m_2 < \cdots$ such that

$$|u_{j_{m_k}}^k(\xi) - u_k(\xi)| < 1/k, \qquad \text{for all } \xi \in \Omega',$$

and

$$\left|\int_\Omega f\, d\mu\left(u_{j_{m_k}}^k\right) - \int_\Omega f\, d\mu(u_k)\right| < 1/k,$$

for $k = 1, 2, \cdots$. Hence $v_k = u_{j_{m_k}}^k \to u$ uniformly in Ω' as $k \to \infty$, and so from the previous claim

$$\int_\Omega f\, d\mu(v_k) \to \int_\Omega f\, d\mu(u), \quad \text{as } k \to \infty.$$

Therefore,

$$\begin{aligned}\left|\int_\Omega f\, d\mu(u_k) - \int_\Omega f\, d\mu(u)\right| &\le \left|\int_\Omega f\, d\mu(u_k) - \int_\Omega f\, d\mu(v_k)\right| + \left|\int_\Omega f\, d\mu(v_k) - \int_\Omega f\, d\mu(u)\right| \\ &\le \frac{1}{k} + \left|\int_\Omega f\, d\mu(v_k) - \int_\Omega f\, d\mu(u)\right| \to 0, \text{as } k \to \infty,\end{aligned}$$

and the proof of the theorem is complete. □

Corollary 6.6. *If $u, v \in C(\overline{\Omega})$ are convex in Ω, $u = v$ on $\partial\Omega$ and $u \ge v$ in Ω, then $\mu(u)(\Omega) \le \mu(v)(\Omega)$.*

6.3. **Comparison principle for measures.**

Theorem 6.7. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set. If $u, v \in C(\overline{\Omega})$ are convex in Ω, $u \le v$ on $\partial\Omega$ and $\mu(u)(E) \ge \mu(v)(E)$ for each $E \subset \Omega$ Borel set, then $u \le v$ in Ω.*

Proof. Assume $0 \in \Omega$, $\Delta = \text{diam}(\Omega)$, $\varepsilon > 0$, and $u_\varepsilon(x, y, t) = u(x, y, t) + \varepsilon\,(x^2 + y^2 - \Delta^2)$. We have $x^2 + y^2 - \Delta^2 < 0$ for $(x, y, t) \in \overline{\Omega}$, so $u_\varepsilon < u \le v$ in $\partial\Omega$. Suppose there exists $(x_0, y_0, t_0) \in \Omega$ such that $u(x_0, y_0, t_0) > v(x_0, y_0, t_0)$. Hence the set $D = \{(x, y, t) \in \Omega : u_\varepsilon(x, y, t) > v(x, y, t)\}$ is non empty for all ε sufficiently small. In addition, $D \cap \partial\Omega = \emptyset$. So $D \subset \Omega$ and $u_\varepsilon = v$ on ∂D. By Corollary 6.6 we get $\mu(u_\varepsilon)(D) \le \mu(v)(D)$. On the other hand, there exist $u_k \in C^2(\Omega)$ convex in Ω such that $u_k \to u$ uniformly on compact subsets of Ω. Let $u_{k,\varepsilon}(x, y, t) = u_k(x, y, t) + \varepsilon\,(x^2 + y^2 - \Delta^2)$. We have from (4.3) that

$$\begin{aligned}\int_D \{\det \mathcal{H}(u_{k,\varepsilon}) + (u_{k,\varepsilon})_t^2\}\, d\xi &= \int_D \{\det \mathcal{H}(u_k) + 2\varepsilon\, \text{trace}\, \mathcal{H}(u_k) + 4\varepsilon^2 + (u_k)_t^2\}\, d\xi \\ &\ge \mu(u_k)(D) + 4\varepsilon^2\, |D|.\end{aligned}$$

Letting $k \to \infty$ we get from Theorem 6.5 that $\mu(u_\varepsilon)(D) \ge \mu(u)(D) + 4\varepsilon^2\, |D| > \mu(u)(D)$ obtaining a contradiction. □

References

[1] T. Bieske. *On ∞-harmonic functions on the Heisenberg group.* Comm. Partial Differential Equations **27** (2002) no. 3 & 4, 727–761.

[2] Z. M. Balogh and M. Rickly. *Regularity of convex functions on Heisenberg groups.* Preprint, `http://cvgmt.sns.it/papers/balric/convex.pdf`.

[3] D. Danielli, N. Garofalo and D. M. Nhieu. *Notions of convexity in Carnot groups.* Comm. Anal. Geom. 11(2), 2003, 263-341.

[4] C. E. Gutiérrez. *The Monge-Ampère equation.* Birkhäuser, Boston, MA, 2001.

[5] C. E. Gutiérrez and A. Montanari. *On the second order derivatives of convex functions on the Heisenberg group*. Ann. Scuola Norm. Sup. Pisa Cl. Sci., to appear. ArXiv Preprint, `http://arxiv.org/abs/math.AP/0309167`.

[6] R. Jensen. *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient,* Arch. Rational Mech. Anal. **123** (1993), no. 1, 51–74.

[7] P. Juutinen, G. Lu, J. Manfredi and B. Stroffolini. *Convex functions on Carnot groups.* Preprint.

[8] G. Lu, J. Manfredi and B. Stroffolini. *Convex functions on the Heisenberg group.* Calc. Var. Partial Differential Equations 19(1), 2003, 1–22.

[9] V. Magnani. *Lipschitz continuity, Aleksandrov's Theorem and characterization of H-convex functions.* Preprint, `http://cvgmt.sns.it/onthefly.cgi/papers/mag03a/hconvex.pdf`.

[10] R. C. Reilly. *On the Hessian of a function and the curvatures of its graph,* Michigan Math. J. **20**, (1973–74), 373–383.

[11] E. M. Stein. *Harmonic Analysis: Real Variable methods, Orthogonality and Oscillatory Integrals* Vol. 43 of the Princeton Math. Series. Princeton U. Press. Princeton, NJ, 1993.

[12] N. S. Trudinger and Xu-Jia Wang. *Hessian measures I,* Topol. Methods Nonlinear Anal. **10** (1997) 225–239.

[13] C. Y. Wang. *Viscosity convex functions on Carnot groups.* Preprint, `http://arxiv.org/abs/math.AP/0309079`.

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