Calc. Var. (2007) 28:275–316
DOI 10.1007/s00526-006-0045-x

Calculus of Variations

On Monge–Ampère type equations arising in optimal transportation problems

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Received: 20 May 2005 / Accepted: 15 May 2006 /
Published online: 18 August 2006

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1 Introduction

The problem of optimal transportation is to find an optimal map that pushes masses from one location to another, where the optimality depends upon the context of the problem. These types of problems appear in several forms and in various areas of mathematics and its applications: economics, probability theory, optimization, meteorology, and computer graphics. We refer to [14] for a detailed and complete description of the probabilistic approach of Kantorovitch to this problem and to the Preface to

The first author was partially supported by NSF grant DMS-0300004.

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Volume I of this work for a large number of examples of applications in econometrics, probability, quality control, etc. In addition, for a recent presentation of the theory and several applications we refer to [18].

The mathematical formulation of the optimal transportation problem considered in this paper originates with Gaspar Monge 1746–1818. Let $f, g \in L^1(\mathbb{R}^n)$ be nonnegative compactly supported with $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$, and let $\mathrm{d}\mu = f\,\mathrm{d}x$, $\mathrm{d}\nu = g\,\mathrm{d}x$. The Borel measurable map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is measure preserving with respect to μ and ν if $\mu(\phi^{-1}(E)) = \nu(E)$ for each Borel set $E \subset \mathbb{R}^n$. Let $\mathcal{S}(\mu, \nu)$ denote the class of all these measure preserving maps, and let $c : \mathbb{R}^n \to \mathbb{R}$ be a convex function,[1] the cost function. The problem is minimizing the cost functional[2]

$$\mathcal{C}(s) = \int_{\mathbb{R}^n} c(x - s(x))\,\mathrm{d}\mu(x) \tag{1}$$

among all $s \in \mathcal{S}(\mu, \nu)$, and the answer is given by the following theorem due to Caffarelli, Gangbo and McCann, see [4,9] and also [17].

Theorem 1.1 *Let $c : \mathbb{R}^n \to \mathbb{R}$ be C^1 and strictly convex, f, g and $\mathcal{C}$ as above. Then*

(a) *There exists $t \in \mathcal{S}(\mu, \nu)$ such that $\mathcal{C}(t) = \inf_{s \in \mathcal{S}(\mu,\nu)} \mathcal{C}(s)$;*
(b) *t is essentially unique, i.e., if the infimum is attained also at $\bar{t}$, then $t(x) = \bar{t}(x)$ for a.e. x in the supp(f);*
(c) *t is essentially one to one, that is, there exists $t^* \in \mathcal{S}(\nu, \mu)$ such that $t^*(t(x)) = x$ for a.e. $x \in$ supp(f), and $t(t^*(y)) = y$ for a.e. $y \in$ supp(g);*
(d) *There exists a c-convex function u such that t is given by the formula*

$$t(x) = x - (Dc)^{-1}(-Du(x)).$$

Monge's original problem is the case $c(x) = |x|$, and the minimizer is not unique, see [7,16] for recent results.

The objective in this paper is to study the following fully nonlinear pde of Monge-Ampère type arising in the problem of optimal transport:

$$g(x - Dc^*(-Du(x)))\det[I + D^2c^*(-Du(x))D^2u(x)] = f(x) \quad \text{in } \Omega, \tag{2}$$

where Ω is a bounded open set, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is positive, $f \in L^1_{\mathrm{loc}}(\Omega)$ is nonnegative, and c^* is the Legendre–Fenchel transform of c, see (3). Solutions to this equation are understood in a weak sense and in a way parallel to the notion of weak solution to the Monge–Ampère equation this time with a notion of subdifferential associated with the cost function c, see Definition 3.1. Our main results are comparison and maximum principles for this equation, and the solvability of the Dirichlet problem in this class of solutions.

We emphasize that unlike the standard subdifferential for convex functions and at the level of generality on the cost functions considered in this paper, the c-subdifferential (see Definition 2.1) is a nonlocal operator on the set of c-convex functions. That is, if the inequality in Definition 2.1 holds in a subdomain $\Omega' \subset \Omega$, then it does not necessarily holds in all of Ω. A similar nonlocality is then inherited by the corresponding Monge–Ampère measures and the resulting pde's. Also in general, the notion of generalized solutions to (2) might not be closed under locally uniform limits. This

[1] The convexity assumption is technical and to be able to use the tools of convex analysis.

[2] A more general cost function can be used: $c = c(x, y)$, but for simplicity we choose $c = c(x - y)$.

closeness property holds under condition (18), see Lemma 3.3 and Remark 3.5, and appears in Sect. 6 in the solution of the Dirichlet problem by approximation.

The paper is organized as follows. In Sect. 2 we study the notions of convexity and subdifferential associated with the cost function c and define the notion of generalized Monge–Ampère measure associated with (2). The notion of weak solution to (2) is given in Sect. 3 where we also prove a stability property, Corollary 3.4. Section 4 contains maximum principles extending to the present setting the Aleksandrov–Bakelman–Pucci estimate for the Monge–Ampère operator. Section 5 contains the proofs of the comparison principles. Sections 4 and 5 have independent interest and are used later to solve the Dirichlet problem. In Sect. 6 we solve the Dirichlet problem for a class of domains strictly convex with respect to the cost function c, see Definition 6.2, first for the homogeneous case, Theorem 6.7, next for the case when the right hand side is a sum of deltas, Theorem 6.11, and finally for general right hand sides, Theorem 6.12 and Corollary 6.13.

2 Generalized Monge–Ampère measures

Let $c : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and Ω be an open set in $\mathbb{R}^n$.

2.1 c-subdifferential and c-convexity[3]

Definition 2.1 *Let $u : \Omega \to \mathbb{R} \cup \{+\infty\}$. The c-subdifferential $\partial_c u(x)$ at $x \in \Omega$ is defined by*

$$\partial_c u(x) = \{p \in \mathbb{R}^n : u(z) \geq u(x) - c(z - p) + c(x - p), \forall z \in \Omega\}.$$

Also for $E \subset \Omega$ we define $\partial_c u(E) = \cup_{x \in E} \partial_c u(x)$.

Definition 2.2 *A function $u : \Omega \to \mathbb{R} \cup \{+\infty\}$, not identically $+\infty$, is c-convex in Ω if there is a set $A \subset \mathbb{R}^n \times \mathbb{R}$ such that*

$$u(x) = \sup_{(y,\lambda) \in A} [-c(x - y) - \lambda] \quad \textit{for all } x \in \Omega.$$

Remark 2.3 The definition of c-convexity is not stable by linear operations. For example, the function $u(x) = 1 - \frac{1}{2}|x|^2$ is $|x|^2/2$-convex, but $2u(x) = 2 - |x|^2$ is not $|x|^2/2$-convex. However it can be proved, see [18], that if u is c-convex then tu is also c-convex for every $t \in [0, 1]$.

We shall consider the following conditions for the cost function c.

$(H1)$ $c : \mathbb{R}^n \to \mathbb{R}$ is a C^1 and strictly convex function.

$(H2)$ $c : \mathbb{R}^n \to \mathbb{R}$ is a strictly convex function and $\lim_{|x| \to +\infty} \dfrac{c(x)}{|x|} = +\infty$.

$(H3)$ $c : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $\lim_{|x| \to +\infty} \dfrac{c(x)}{|x|} = +\infty$.

Notice that the function $c(x) = (1 + |x|^2)^{1/2}$ satisfies $(H1)$ and does not satisfy $(H2)$.

Proposition 2.4 *Suppose c satisfies $(H3)$. If $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and convex, then u is c-convex.*

[3] Introduced in [6,8].

Proof Since every lower semicontinuous convex function is the supremum of affine functions, it is enough to assume that $u(z) = q \cdot z + b$. Since c is continuous and $\lim_{|x|\to+\infty} c(x)/|x| = +\infty$, it follows sliding $-u$ in a parallel fashion that $-u + \lambda$ is a supporting hyperplane to c at some point for some λ. That is, there exist $x_u \in \mathbb{R}^n$ and $\lambda_u \in \mathbb{R}$ such that $c(x_u) = -u(x_u) + \lambda_u$ and $c(z) \geq -u(z) + \lambda_u$ for all $z \in \mathbb{R}^n$. Given $x \in \Omega$, let $y_x = x - x_u$ and $\lambda_x = -u(x) + u(x_u) - \lambda_u$. We have $u(x) = -c(x - y_x) - \lambda_x$ and $u(z) \geq -c(z - y_x) - \lambda_x$ for all $z \in \Omega$ since u is affine. Setting $A = \{(y_x, \lambda_x) : x \in \Omega\}$ we obtain the proposition. □

Remark 2.5 Suppose c satisfies $(H3)$. If u is a c-convex function in Ω that is bounded in a neighborhood of $x_0 \in \Omega$, then $\partial_c u(x_0) \neq \emptyset$. Indeed, without loss of generality we can assume that the set $A \subset \mathbb{R}^n \times \mathbb{R}$ in the definition of u is closed. Then arguing as in Claim 2 of the proof of Lemma 2.10 below, there exists $(y, \lambda) \in A$ such that $u(x_0) = -c(x_0 - y) - \lambda$ and so $u(x) \geq -c(x - y) - \lambda = u(x_0) - c(x - y) + c(x_0 - y)$ for all $x \in \Omega$. Therefore $y \in \partial_c u(x_0)$.

Remark 2.6 Suppose c satisfies $(H3)$. It follows from the convexity of c that if u is c-convex and locally bounded in Ω, then u is locally Lipschitz in Ω. Indeed, let $K \subset \Omega$ be compact and $x_1, x_2 \in K$. From Remark 2.5, we have that $\partial_c u(x_i) \neq \emptyset$ for $i = 1, 2$. Let $y_1 \in \partial_c u(x_1)$. By Lemma 2.14, $|y_1| \leq R$, and since c is locally Lipschitz we have $u(x_2) - u(x_1) \geq -c(x_2 - y_1) + c(x_1 - y_1) \geq -C(K, R)\,|x_2 - x_1|$.

Proposition 2.7 *Let u be a function defined on Ω, and suppose that x_0 is a point of differentiability of u, and $\partial_c u(x_0) \neq \emptyset$. Then we have*

(1) *If $(H1)$ holds, then*

$$\partial_c u(x_0) = \{x_0 - (Dc)^{-1}(-Du(x_0))\}.$$

(2) *If $(H2)$ holds, then*

$$\partial_c u(x_0) = \{x_0 - Dc^*(-Du(x_0))\},$$

where c^* is the Legendre–Fenchel transform[4] *of c defined by*

$$c^*(y) = \sup_{x\in\mathbb{R}^n} [x \cdot y - c(x)]. \tag{3}$$

Proof Suppose first that c satisfies $(H1)$. Let $p \in \partial_c u(x_0)$. Then $u(x) + c(x - p) \geq u(x_0) + c(x_0 - p)$ for all $x \in \Omega$ with equality at $x = x_0$. That is, $u(x) + c(x - p)$ attains a minimum at x_0 and therefore $Dc(x_0 - p) = -Du(x_0)$. Since c is C^1 and strictly convex, $(Dc)^{-1}$ exists on the image of Dc and we have $p = x_0 - (Dc)^{-1}(-Du(x_0))$.

To prove (2) we need the following definition.

Definition 2.8 *The function $u : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is subdifferentiable at $x_0 \in \Omega$ if $u(x_0)$ is finite and there exists $z \in \mathbb{R}^n$ such that*

$$u(x_0 + v) \geq u(x_0) + v \cdot z + o(|v|)$$

as $|v| \to 0$. Let us denote by $M_u(x_0)$ the set of z's satisfying the property above.

[4] See [15, Chapter 11].

Suppose that $(H2)$ holds, and let $p \in \partial_c u(x_0)$ with x_0 a point of differentiability of u. Then

$$u(x_0) - c(x_0 + v - p) + c(x_0 - p) \leq u(x_0 + v) \leq u(x_0) + v \cdot Du(x_0) + o(|v|).$$

Hence $c(x_0 + v - p) \geq c(x_0 - p) + v \cdot (-Du(x_0)) + o(|v|)$ as $|v| \to 0$, and so $-Du(x_0) \in M_c(x_0 - p)$. From [9, Corollary A.2] we get that $x_0 - p = Dc^*(-Du(x_0))$ and the proposition follows. □

Remark 2.9 Suppose c is strictly convex satisfying c and c^* are $C^2(\mathbb{R}^n)$[5] and $u : \Omega \to \mathbb{R}$ has a second derivative $D^2u(x_0)$ at x_0. Then if $\partial_c u(x_0) \neq \emptyset$, we have

$$I + D^2c^*\left(-Du(x_0)\right) D^2u(x_0) \text{ is diagonalisable with nonnegative eigenvalues.} \tag{4}$$

Indeed, let $p \in \partial_c u(x_0)$, then $u(x) + c(x - p) \geq u(x_0) + c(x_0 - p)$ for all $x \in \Omega$. Hence $Du(x_0) + Dc(x_0 - p) = 0$ and by Taylor's theorem $D^2u(x_0) + D^2c(x_0 - p) \geq 0$. So from Proposition 2.7(2) we get

$$D^2u(x_0) + D^2c\left(Dc^*\left(-Du(x_0)\right)\right) \geq 0. \tag{5}$$

On the other hand, from [13, Corollary 23.5.1 and Theorem 26.1] we have that $Dc^*\left(Dc(x)\right) = x$ for every $x \in \mathbb{R}^n$ and $Dc\left(Dc^*(y)\right) = y$ for every y in the image of Dc. Differentiating these equations yields $D^2c^*\left(Dc(x)\right) D^2c(x) = I$, and $D^2c\left(Dc^*(y)\right) D^2c^*(y) = I$, for every $x \in \mathbb{R}^n$ and every y in the image of Dc. From this we derive that for any y in the image of Dc we have $D^2c^*(y)$ is invertible with $[D^2c^*(y)]^{-1} = D^2c\left(Dc^*(y)\right)$, and letting $y = -Du(x_0)$, we obtain from (5) that

$$D^2u(x_0) \geq -[D^2c^*\left(-Du(x_0)\right)]^{-1} \tag{6}$$

Since c^* is convex, the symmetric matrix $D^2c^*\left(-Du(x_0)\right)$ is positive definite as it is invertible. Therefore, (6) implies (4).

Lemma 2.10 *Let $\Omega \subset \mathbb{R}^n$ be an open set and suppose that $(H3)$ holds. If $u_n : \Omega \to \mathbb{R}$ is a sequence of c-convex functions such that $u_n \to u$ locally uniformly in Ω with u locally bounded in Ω, then u is c-convex in Ω.*

Proof By definition of c-convexity we have $u_n(x) = \sup_{(y,\lambda)\in A_n}[-c(x - y) - \lambda]$, $\forall x \in \Omega$, for each n. Since c is continuous on $\mathbb{R}^n$, without loss of generality we can assume that each A_n is a closed subset of $\mathbb{R}^n \times \mathbb{R}$.

Claim 1 If $\{y_k\}$ and $\{\lambda_k\}$ are sequences such that there exists constants A, B, r with

$$A \leq -c(x_0 - y_k) - \lambda_k \tag{7}$$

and

$$-c(x - y_k) - \lambda_k \leq B \tag{8}$$

for all $x \in B(x_0, r) \Subset \Omega$ and for all k, then $\{y_k\}$ is a bounded sequence, and consequently $\{\lambda_k\}$ is also bounded.

Suppose by contradiction that $\{y_k\}$ is unbounded. Passing through a subsequence we can assume that $|y_k| \to +\infty$. Let $v_k := x_0 - y_k$. Since $|v_k| \to +\infty$, we may assume

[5] If c is $C^2(\mathbb{R}^n)$ and $D^2c(x)$ is positive definite for all x, then $c^* \in C^2(\mathbb{R}^n)$, $D^2c^*(x)$ is positive definite for all x, and $(c^*)^* = c$, see [15, Example 11.9, p. 480].

that $|v_k| > 1$ for all k sufficiently large. Setting $\zeta_k := 1 - \dfrac{r}{|v_k|}$, we have $\zeta_k \to 1$. Applying (8) at $x = x_0 + (\zeta_k - 1)v_k$ and using (7) we get

$$\begin{aligned} B &\geq -c(x_0 + (\zeta_k - 1)v_k - y_k)) - \lambda_k = -c(\zeta_k v_k) - \lambda_k \\ &\geq -c(\zeta_k v_k) + c(x_0 - y_k) + A = -c(\zeta_k v_k) + c(v_k) + A. \end{aligned}$$

Hence

$$B - A \geq c(v_k) - c(\zeta_k v_k). \tag{9}$$

Since c is convex, this difference can be bounded using a subgradient $p_k \in \partial c(\zeta_k v_k)$:

$$B - A \geq \langle p_k, v_k - \zeta_k v_k \rangle = \langle p_k, (1 - \zeta_k)v_k \rangle = r\left\langle p_k, \frac{v_k}{|v_k|} \right\rangle. \tag{10}$$

On the other hand, being p_k a subgradient also implies that

$$c(0) \geq c(\zeta_k v_k) + \langle p_k, 0 - \zeta_k v_k \rangle. \tag{11}$$

Since $|v_k| \to +\infty$, we have that $\zeta_k > 0$ and dividing (11) by $\zeta_k |v_k| \to +\infty$ yields

$$\liminf_{k\to\infty} \left\langle p_k, \frac{v_k}{|v_k|} \right\rangle \geq \liminf_{k\to\infty} \frac{c(\zeta_k v_k)}{|\zeta_k v_k|}.$$

The assumption $\lim_{|x|\to+\infty} c(x)/|x| = +\infty$ implies that both these limits diverge, yielding a contradiction with (10). Therefore y_k is bounded and since c is continuous we get from (7) that λ_k is also bounded and Claim 1 is proved.

Claim 2 For each $x \in \Omega$ there exists $N_x \in \mathbb{N}$ and a sequence $(y_n(x), \lambda_n(x)) \in A_n$ such that $u_n(x) = -c(x - y_n(x)) - \lambda_n(x)$ for all $n \geq N_x$.

Let $r_x \in (0,1)$ such that $B(x, r_x) \Subset \Omega$ and u is bounded on $\bar{B}(x, r_x)$. Then since $u_n \to u$ uniformly on $\bar{B}(x, r_x)$, there exist constants $M_x > 0$ and $N_x \in \mathbb{N}$ such that

$$-M_x < u_n(z) < M_x \qquad \forall z \in \bar{B}(x, r_x) \quad \text{and} \quad \forall n \geq N_x.$$

Since u_n is c-convex, for each n we can find a sequence $\{(y_n^k(x), \lambda_n^k(x))\}_{k=1}^{\infty} \subset A_n$ satisfying $u_n(x) = \lim_{k\to\infty}[-c(x - y_n^k(x)) - \lambda_n^k(x)]$, and $u_n(x) - 1 \leq -c(x - y_n^k(x)) - \lambda_n^k(x)$. Hence if $n \geq N_x$, then

$$-M_x - 1 \leq -c(x - y_n^k(x)) - \lambda_n^k(x),$$

and

$$-c(z - y_n^k(x)) - \lambda_n^k(x) \leq u_n(z) \leq M_x \qquad \forall z \in \bar{B}(x, r_x).$$

So, from Claim 1, there exist $(y_n(x), \lambda_n(x)) \in A_n$ and a subsequence $\{(y_n^{k_j}(x), \lambda_n^{k_j}(x))\}_{j=1}^{\infty}$ such that $(y_n^{k_j}(x), \lambda_n^{k_j}(x)) \to (y_n(x), \lambda_n(x))$ as $j \to \infty$. Therefore,

$$u_n(x) = \lim_{j\to\infty} [-c(x - y_n^{k_j}(x)) - \lambda_n^{k_j}(x)] = -c(x - y_n(x)) - \lambda_n(x)$$

and Claim 2 is proved.

Claim 3 Let

$$B_x = \left\{(y,\lambda) \in \mathbb{R}^n \times \mathbb{R} : (y,\lambda) = \lim_{j\to\infty} (y_{n_j}(x), \lambda_{n_j}(x)), \text{ for some subsequence } n_j\right\},$$

and set $A = \cup_{x\in\Omega} B_x$. We claim that

$$u(z) = \sup_{(y,\lambda)\in A} [-c(z-y) - \lambda] \quad \forall z \in \Omega. \tag{12}$$

Let $z \in \Omega$ and choose $r_z \in (0,1)$ as above. Then $\bar{B}(z,r_z) \subset \Omega$ and as before we have $-M(z) < u_n(x) < M(z) \quad \forall x \in \bar{B}(z,r_z), \quad \forall n \geq N_z$. If $(y_n(z), \lambda_n(z))$ is the sequence in Claim 2, we have that for any $n \geq N_z$

$$-M(z) < -c(z - y_n(z)) - \lambda_n(z) \text{ and } -c(x - y_n(z)) - \lambda_n(z) < M(z) \quad \forall x \in \bar{B}(z,r_z).$$

We conclude from Claim 1 that $\{y_n(z)\}_{n=N_z}^{\infty}$ and $\{\lambda_n(z)\}_{n=N_z}^{\infty}$ are bounded. Hence, there exist $(y^*,\lambda^*) \in \mathbb{R}^n \times \mathbb{R}$ and a subsequence $\{(y_{n_k}(z), \lambda_{n_k}(z))\}_{k=1}^{\infty}$ of $\{(y_n(z), \lambda_n(z))\}_{n=N_z}^{\infty}$ such that $(y_{n_k}(z), \lambda_{n_k}(z)) \to (y^*,\lambda^*)$. Therefore, $(y^*,\lambda^*) \in B_z \subset A$ and

$$u(z) = \lim_{k\to\infty} u_{n_k}(z) = \lim_{k\to\infty} [-c(z - y_{n_k}(z)) - \lambda_{n_k}(z)] = -c(z - y^*) - \lambda^*.$$

Thus to prove (12) it is enough to show that $-c(z-y)-\lambda \leq u(z) \quad \forall (y,\lambda) \in A$. Indeed, let $(y,\lambda) \in A$. Then $(y,\lambda) \in B_x$ for some $x \in \Omega$ and hence there exists a subsequence $\{(y_{n_j}(x), \lambda_{n_j}(x))\}_{j=1}^{\infty}$ of $\{(y_n(x), \lambda_n(x))\}_{n=N_x}^{\infty}$ such that $(y_{n_j}(x), \lambda_{n_j}(x)) \to (y,\lambda)$. We have

$$u_{n_j}(z) = -c(z - y_{n_j}(z)) - \lambda_{n_j}(z) \geq -c(z - y_{n_j}(x)) - \lambda_{n_j}(x) \quad \forall j.$$

Letting $j \to \infty$ and since $(y_{n_j}(x), \lambda_{n_j}(x)) \to (y,\lambda)$, we then get $u(z) \geq -c(z-y) - \lambda$. This completes the proof of the lemma. □

2.2 A Monge–Ampère measure associated with the cost function c

In this subsection we define a generalized Monge-Ampère measure, and to do it we need the following lemma, which is a generalization of a classical lemma of Aleksandrov.

Lemma 2.11 *Suppose that either* $(H1)$ *or* $(H2)$ *holds. Let* $X \subset \mathbb{R}^n$ *be a nonempty bounded set and* $u : X \to \mathbb{R} \cup \{+\infty\}$*, not identically* $+\infty$*, be bounded from below on* X*. Then the Lebesgue measure of the set*

$$\tilde{S} = \{p \in \mathbb{R}^n : \ p \in \partial_c u(x_1) \cap \partial_c u(x_2) \ \textit{for some} \ x_1, x_2 \in X, \ x_1 \neq x_2\}$$

is zero.

Proof Define for each $y \in \mathbb{R}^n$,

$$u^*(y) = \sup_{x\in X} [-c(x-y) - u(x)].$$

Since $c \in C(\mathbb{R}^n)$, X is bounded, $u \not\equiv +\infty$ and u is bounded from below on X we get $u^* : \mathbb{R}^n \to \mathbb{R}$. Moreover, as c is locally Lipschitz on $\mathbb{R}^n$, it is clear that u^* is also locally Lipschitz on $\mathbb{R}^n$. Hence, if we let $E = \{x \in \mathbb{R}^n : u^* \text{ is not differentiable at } x\}$, then $|E| = 0$. We shall show that $\tilde{S} \subset E$. Indeed, let $p \in \tilde{S}$ then $p \in \partial_c u(x_1) \cap \partial_c u(x_2)$ for some $x_1, x_2 \in X, \ x_1 \neq x_2$. Hence, $u(z) \geq u(x_1) - c(z-p) + c(x_1 - p)$, and $u(z) \geq u(x_2) - c(z-p) + c(x_2 - p)$, $\forall z \in X$. Thus, $u^*(p) = -c(x_1 - p) - u(x_1)$

and $u^*(p) = -c(x_2 - p) - u(x_2)$. Moreover, by definition of u^* we have $u^*(z) \geq -c(x_1 - z) - u(x_1) \quad \forall z \in \mathbb{R}^n$, and $u^*(z) \geq -c(x_2 - z) - u(x_2) \quad \forall z \in \mathbb{R}^n$. So $u^*(z) \geq u^*(p) - c(x_i - z) + c(x_i - p), \forall z \in \mathbb{R}^n, i = 1, 2$. Hence, we obtain $x_1, x_2 \in \partial_h(u^*, \mathbb{R}^n)(p)$ where we denote $h(x) = c(-x)$ for every $x \in \mathbb{R}^n$. Note that h satisfies the same assumptions as c. Then by Proposition 2.7 we must have $p \in E$ since $x_1 \neq x_2$. The proof is complete. □

Corollary 2.12 *Let Ω be an open set in $\mathbb{R}^n$ and suppose that either $(H1)$ or $(H2)$ holds. Let $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ be such that on any bounded open set $U \Subset \Omega$, u is not identical to $+\infty$ and bounded from below. Then the Lebesgue measure of the set*

$$S = \{p \in \mathbb{R}^n : \ \textit{there exist } \ x, y \in \Omega, x \neq y \ \textit{ and } \ p \in \partial_c u(x) \cap \partial_c u(y)\}$$

is zero.

Proof We can write $\Omega = \cup_k \Omega_k$ where $\Omega_k \subset \Omega_{k+1}$ are open and $\bar{\Omega}_k \subset \Omega$ are compact. If $p \in S$ then there exist $x, y \in \Omega, x \neq y$ with $u(z) \geq u(x) - c(z - p) + c(x - p) \ \forall z \in \Omega$, and $u(z) \geq u(y) - c(z - p) + c(y - p) \ \forall z \in \Omega$. Since Ω_k increases, $x, y \in \Omega_m$ for some m. That is, if

$$S_m = \{p \in \mathbb{R}^n : \ \text{there exist } \ x, y \in \Omega_m, x \neq y \ \text{ and } \ p \in \partial_c(u, \Omega_m)(x) \cap \partial_c(u, \Omega_m)(y)\},$$

then we have $p \in S_m$, i.e., $S \subset \cup_{m=1}^{\infty} S_m$. But by the assumptions and Lemma 2.11 we get $|S_m| = 0$ for all m. Hence the proof is complete. □

Remark 2.13 Suppose c satisfies $(H3)$. Let Ω be a bounded open set in $\mathbb{R}^n$ and $u \in C(\Omega)$ be c-convex. Then

$$p \in \partial_c u(x) \text{ if and only if } x \in \partial_h(u^*, \mathbb{R}^n)(p),$$

where $h(x) = c(-x)$ and $u^*(y) = \sup_{x \in \Omega} [-c(x - y) - u(x)]$.

Proof It follows by the argument in Lemma 2.11 that if $p \in \partial_c u(x)$ then $x \in \partial_h(u^*, \mathbb{R}^n)(p)$. Now if $x \in \partial_h(u^*, \mathbb{R}^n)(p)$, then $u^*(y) \geq u^*(p) - c(x - y) + c(x - p)$ for all $y \in \mathbb{R}^n$. This gives by the definition of u^* that for each $y \in \mathbb{R}^n$, $u^*(y) \geq -c(z - p) - u(z) - c(x - y) + c(x - p)$ for all $z \in \Omega$. Picking $y \in \partial_c u(x)$ which is nonempty by Remark 2.5, then as $u^*(y) = -c(x - y) - u(x)$ we obtain $u(z) \geq u(x) - c(z - p) + c(x - p)$ for all $z \in \Omega$. That is, $p \in \partial_c u(x)$ as desired. □

Lemma 2.14 *Suppose c satisfies $(H3)$. Let $u : \Omega \to \mathbb{R}$ be a locally bounded function in Ω. If $K \subset \Omega$ is compact, then there exists $R > 0$, depending only on K and the L^∞-norm of u over a small neighborhood of K, such that*

$$\partial_c u(K) \subset B(0, R).$$

Proof Indeed, assume that this is not true. Then for each $n \in \mathbb{N}$, there exists $x_n \in K$ and $p_n \in \partial_c u(x_n)$ with $|p_n| > n$. Hence $u(x) \geq u(x_n) - c(x - p_n) + c(x_n - p_n) \forall x \in \Omega$, and since u is locally bounded, there exists $M > 0$ such that

$$-c(x - p_n) + c(x_n - p_n) \leq M \qquad \forall x \in K_\delta \ \text{ and } \forall n \in \mathbb{N}, \tag{13}$$

where $\delta = (1/2) \min\{\text{dist}(x, \partial\Omega), 1\}$ and $K_\delta = \{x \in \Omega : \text{dist}(x, K) \leq \delta\}$. Let $v_n = x_n - p_n$. Since $|v_n| \to +\infty$, we may assume $|v_n| > 1 \quad \forall n$. Setting $\zeta_n = 1 - \delta/|v_n|$ and evaluating (13) at $x = x_n + (\zeta_n - 1)v_n \in K_\delta$ yields $M \geq c(v_n) - c(\zeta_n v_n)$. Applying the argument used after the inequality (9) yields a contradiction. This proves the lemma. □

Lemma 2.15 *Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in C(\Omega)$, and $\mathcal{B} = \{E \subset \Omega : \partial_c u(E)$ is Lebesgue measurable$\}$. We have*

(i) *If $K \subset \Omega$ is compact, then $\partial_c u(K)$ is closed. Moreover, if $(H3)$ holds then $\partial_c u(K)$ is compact.*
(ii) *$\mathcal{B}$ contains all closed subsets and all open subsets of Ω.*
(iii) *If either $(H1)$ or $(H2)$ holds, then $\mathcal{B}$ is a σ-algebra on Ω containing all Borel subsets of Ω. Moreover,*

$$|\partial_c u(\Omega \setminus E)| = |\partial_c u(\Omega) \setminus \partial_c u(E)| \qquad \forall E \in \mathcal{B}.$$

Proof (i) Let K be a compact subset of Ω, $\{p_n\}_{n=1}^{\infty} \subset \partial_c u(K)$, and suppose $p_n \to p$. We shall show that $p \in \partial_c u(K)$. For each n, since $p_n \in \partial_c u(K)$ we have $p_n \in \partial_c u(x_n)$ for some $x_n \in K$. But K is compact, so there exist $x \in K$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$. We have $u(z) \geq u(x_{n_k}) - c(z - p_{n_k}) + c(x_{n_k} - p_{n_k})$, $\forall z \in \Omega$. Passing to limit we obtain $p \in \partial_c u(x) \subset \partial_c u(K)$. Hence, $\partial_c u(K)$ is closed. The second statement then follows from Lemma 2.14.

(ii) Let E be a closed subset of Ω. Then we can write $E = \cup_{n=1}^{\infty} K_n$ where K_n are compact. Therefore, $\partial_c u(E) = \partial_c u(\cup_{n=1}^{\infty} K_n) = \cup_{n=1}^{\infty} \partial_c u(K_n)$. By (i), each $\partial_c u(K_n)$ is Lebesgue measurable. So $\partial_c u(E)$ is measurable, i.e., $E \in \mathcal{B}$. The proof is identical if E is open.

(iii) Suppose $\{E_i\}_{i=1}^{\infty} \subset \mathcal{B}$. Since $\partial_c u(\cup_{i=1}^{\infty} E_i) = \cup_{i=1}^{\infty} \partial_c u(E_i)$ we then get $\partial_c u(\cup_{i=1}^{\infty} E_i)$ is Lebesgue measurable. So $\cup_{i=1}^{\infty} E_i \in \mathcal{B}$. We also have $\Omega \in \mathcal{B}$ by (ii). Now suppose $E \in \mathcal{B}$, we shall show that $\Omega \setminus E \in \mathcal{B}$. Indeed, we have

$$\partial_c u(\Omega \setminus E) = [\partial_c u(\Omega) \setminus \partial_c u(E)] \cup [\partial_c u(\Omega \setminus E) \cap \partial_c u(E)].$$

By Corollary 2.12, the second set on the right hand side has measure zero. So $\partial_c u(\Omega \setminus E)$ is Lebesgue measurable and $|\partial_c u(\Omega \setminus E)| = |\partial_c u(\Omega) \setminus \partial_c u(E)|$. Also since $\mathcal{B}$ is a σ-algebra and by (ii) $\mathcal{B}$ contains all closed subsets of Ω we get $\mathcal{B}$ contains all Borel subsets of Ω. □

From Lemma 2.15 we then define

Definition 2.16 *Let g be a locally integrable function which is positive a.e. on $\mathbb{R}^n$. Suppose that c satisfies either (H1) or (H2), and Ω is an open set in $\mathbb{R}^n$. Then for each given function $u \in C(\Omega)$, the generalized Monge-Ampère measure of u associated with the cost function c and the weight g is the Borel measure defined by*

$$\omega_c(g, u)(E) = \int_{\partial_c u(E)} g(y)\, \mathrm{d}y$$

for every Borel set $E \subset \Omega$. When $g \equiv 1$, we simply write the measure as $\omega_c(u)$.

Remark 2.17 If c satisfies $(H2)$ and $u \in C(\Omega)$, then we know from Lemma 2.15(i) that $\partial_c u(K)$ is compact for every compact set $K \subset \Omega$. Therefore the measure $\omega_c(g, u)$ is finite on compact subsets of Ω, and so $\omega_c(g, u)$ is a regular measure. Namely, it has the following regularity properties

$$\omega_c(g, u)(E) = \inf\{\omega_c(g, u)(U) : \ E \subset U \subset \Omega, \ \ U \text{ open}\}$$

for all Borel sets $E \subset \Omega$, and

$$\omega_c(g, u)(U) = \sup\{\omega_c(g, u)(K) : \ K \subset U, \ \ K \text{ compact}\}$$

for all open sets $U \subset \Omega$.

Definition 2.18 *Let $u \in C(\Omega)$ and $x_0 \in \Omega$. Then u is called strictly c-convex at x_0 if $\partial_c u(x_0) \neq \emptyset$ and for any $p \in \partial_c u(x_0)$ we have $u(x) > u(x_0) - c(x-p) + c(x_0 - p)$ for all $x \in \Omega \setminus \{x_0\}$.*

For $u \in C(\Omega)$, define $\Gamma_u = \{x \in \Omega : \partial_c u(x) \neq \emptyset\}$. Then Γ_u is a relatively closed set in Ω since $\Gamma_u = \{x \in \Omega : u_*(x) = u(x)\}$, where u_* is the continuous function defined in (23). We then have the following result noticing that $\Gamma_u = \Omega$ iff u is c-convex in Ω.

Proposition 2.19 *Suppose that c satisfies $(H2)$ and $c^* \in C^2(\mathbb{R}^n)$. We have*

(1) *If $u \in C^2(\Omega)$, then*

$$\omega_c(g,u)(E) = \int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)|\,\mathrm{d}x$$

for all Borel sets $E \subset \Omega$.

(2) *If in addition $c \in C^2(\mathbb{R}^n)$ then for any $u \in C^2(\Omega)$,*

$$\omega_c(g,u)(E) = \int_{E\cap\Gamma_u} g(x - Dc^*(-Du))\det(I + D^2c^*(-Du)D^2u)\,\mathrm{d}x$$

for all Borel sets $E \subset \Omega$.

Proof (1) Define $s(x) = x - Dc^*(-Du(x))$ for every x in Ω. Since $c^* \in C^2(\mathbb{R}^n)$ and $u \in C^2(\Omega)$, it follows that $s : \Omega \to \mathbb{R}^n$ is a C^1 mapping, and by Proposition 2.7 we have $\partial_c u(x) = \{s(x)\}$ for every x in Ω satisfying $\partial_c u(x) \neq \emptyset$. Let $\tilde{A} = \{x \in \Omega : u \text{ is not strictly } c\text{-convex at } x\}$ and let S be defined as in Corollary 2.12. Then $|S| = 0$ and we claim that $\tilde{A} = (\Omega - \Gamma_u) \cup (\Gamma_u \cap s^{-1}(S))$. Indeed, if $x \in \tilde{A}$ and $x \in \Gamma_u$ then $p = s(x) \in \partial_c u(x)$. Since $\partial_c u(x)$ can not contain more than one element, there exists $y \in \Omega$, $y \neq x$ such that $u(y) = u(x) - c(y-p) + c(x-p)$. Hence, $u(z) \geq u(x) - c(z-p) + c(x-p) = u(y) - c(z-p) + c(y-p)$ for every $z \in \Omega$. So $p \in \partial_c u(x) \cap \partial_c u(y)$, i.e., $x \in s^{-1}(S)$. This implies the claim as the reverse relation is obvious. We now let $S' \subset \mathbb{R}^n$ be a Borel set such that $S \subset S'$ and $|S'| = 0$. Put $A = (\Omega - \Gamma_u) \cup (\Gamma_u \cap s^{-1}(S'))$, then clearly A is a measurable set, $\tilde{A} \subset A$, and as $\partial_c u(A) = \partial_c u(\Gamma_u \cap s^{-1}(S')) = s(\Gamma_u \cap s^{-1}(S')) \subset S'$ we have $|\partial_c u(A)| = 0$. We now proceed the proof as follows. From the definition of $\tilde{A}$ it is easy to see that s is one-to-one on $\Omega \setminus \tilde{A}$, and hence one-to-one on $\Omega \setminus A$. Therefore for any Borel subset E of Ω, by using the usual change variables formula we obtain

$$\begin{aligned}
&\int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)|\,\mathrm{d}x \\
&\quad = \int_{E\cap\Gamma_u} g(s(x))|\det Ds(x)|\,\mathrm{d}x \geq \int_{E\setminus A} g(s(x))|\det Ds(x)|\,\mathrm{d}x \\
&\quad = \int_{s(E\setminus A)} g(y)\,\mathrm{d}y = \int_{\partial_c u(E\setminus A)} g(y)\,\mathrm{d}y \overset{(*)}{=} \int_{\partial_c u(E)} g(y)\,\mathrm{d}y = \omega_c(g,u)(E).
\end{aligned}$$

Note that the equality $(*)$ holds since $|\partial_c u(E) - \partial_c u(E \setminus A)| = |\partial_c u(E - (E \setminus A))| = |\partial_c u(E \cap A)| = 0$. Thus we have proved that

$$\int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)|\,\mathrm{d}x \geq \omega_c(g,u)(E)$$

for every Borel set $E \subset \Omega$. Hence (1) will be proved if we show the reverse inequality. To do that, let $B = \{x \in \Omega : \det Ds(x) = 0\}$ and let E be a Borel set in Ω. Then for any open set U with $E \subset U \subset \Omega$, we can write the open set $U \backslash B$ as $U \backslash B = \cup_{i=1}^{\infty} C_i$ where $\{C_i\}_{i=1}^{\infty}$ are cubes with disjoint interior and sides parallel to the coordinate axes. We can choose C_i small enough so that $s : C_i \to s(C_i)$ is a diffeomorphism. We therefore have

$$\begin{aligned}
&\int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)|\,\mathrm{d}x \\
&\quad\le \int_{U\cap\Gamma_u} g(s(x))|\det Ds(x)|\,\mathrm{d}x = \int_{(U\backslash B)\cap\Gamma_u} g(s(x))|\det Ds(x)|\,\mathrm{d}x \\
&\quad= \int_{(\cup_{i=1}^{\infty} C_i)\cap\Gamma_u} g(s(x))|\det Ds(x)|\,\mathrm{d}x = \int_{(\cup_{i=1}^{\infty} \mathring{C}_i)\cap\Gamma_u} g(s(x))|\det Ds(x)|\,\mathrm{d}x \\
&\quad= \sum_{i=1}^{\infty} \int_{\mathring{C}_i\cap\Gamma_u} g(s(x))|\det Ds(x)|\,\mathrm{d}x = \sum_{i=1}^{\infty} \int_{s(\mathring{C}_i\cap\Gamma_u)} g(y)\,\mathrm{d}y \\
&\quad= \sum_{i=1}^{\infty} \omega_c(g,u)(\mathring{C}_i \cap \Gamma_u) = \omega_c(g,u)((\cup_{i=1}^{\infty} \mathring{C}_i) \cap \Gamma_u) \le \omega_c(g,u)(U).
\end{aligned}$$

Hence since the measure $\omega_c(g,u)$ is regular, we deduce that

$$\int_{E\cap\Gamma_u} g(x - Dc^*(-Du))|\det(I + D^2c^*(-Du)D^2u)|\,\mathrm{d}x \le \omega_c(g,u)(E).$$

This combined with the previous inequality yield the desired result for (1).

(2) This is a consequence of (1) and Remark 2.9. □

3 Aleksandrov solutions

The equation (2) is highly fully nonlinear and when the cost function c is nice enough, it is degenerate elliptic on the set of c-convex functions. Motivated by Proposition 2.19 and by using the previous results we shall define a notion of weak solutions for (2) and study the stability property of the solutions.

Definition 3.1 *We say that a c-convex function $u \in C(\Omega)$ is a generalized solution of* (2) *in the sense of Aleksandrov, or simply Aleksandrov solution, if*

$$\omega_c(g,u)(E) = \int_E f(x)\,\mathrm{d}x$$

for any Borel set $E \subset \Omega$.

Proposition 3.2 *Suppose c satisfies* (*H*2), *and that $c, c^* \in C^2(\mathbb{R}^n)$. Let $u \in C(\Omega)$ be a c-convex function. Then u is an Aleksandrov solution of* (2) *iff $\omega_c(g,u)$ is absolutely continuous w.r.t. the Lebesgue measure and* (2) *is satisfied pointwise a.e. on Ω.*

Proof Observing first that by [9, Corollary C.5], u is locally semi-convex[6] and hence twice differentiable a.e. on Ω in the sense of Aleksandrov. Also, by using Remark 2.9 and an argument similar to [11, Proposition A.2] we have that whenever u has an Aleksandrov second derivative $D^2u(x_0)$ at $x_0 \in \Omega$ then

$$\lim_{r\to 0^+} \frac{|\partial_c u(B_r(x_0))|}{|B_r(x_0)|} = \det[I + D^2c^*(-Du(x_0))D^2u(x_0)], \tag{14}$$

and if in addition $I + D^2c^*(-Du(x_0))D^2u(x_0)$ is invertible, then $\partial_c u(B_r(x_0))$ shrinks nicely to $x_0 - Dc^*(-Du(x_0))$.

Suppose $\omega_c(g,u)$ is absolutely continuous w.r.t. the Lebesgue measure on the σ-algebra of Borel sets in Ω and with density $F(x)$. The proposition will be proved if we show that

$$F(x) = g(x - Dc^*(-Du(x)))\det[I + D^2c^*(-Du(x))D^2u(x)] \text{ a.e. } \quad x \text{ in } \Omega. \tag{15}$$

Since $g > 0$ a.e. on $\mathbb{R}^n$ we get $\omega_c(u)$ is also absolutely continuous w.r.t. the Lebesgue measure. Combining this with (14) we see that $\det[I + D^2c^*(-Du(x))D^2u(x)]$ is the density of $\omega_c(u)$. Now let M be the set of points $x \in \Omega$ satisfying u has Aleksandrov second derivative at x and $\det[I + D^2c^*(-Du(x))D^2u(x)] > 0$, and let H be a Borel set in $\mathbb{R}^n$ with Lebesgue measure zero such that every point in $\mathbb{R}^n \setminus H$ is a Lebesgue point of g. Define $E = \{x \in M : x - Dc^*(-Du(x)) \in H\}$. Then it is clear from Remark 2.13 that $E = \partial_h(u^*, \mathbb{R}^n)(H) \cap M$ and hence E is Lebesgue measurable by Lemma 2.15 as $u^* \in C(\mathbb{R}^n)$. We claim that $|E| = 0$. Indeed, let $K \subset E$ be a compact set then we have $\partial_c u(K) \subset \partial_c u(E) \subset H$. Hence, $\int_K \det[I + D^2c^*(-Du(x))D^2u(x)]\, dx = \omega_c(u)(K) = 0$. This implies that $|K| = 0$ as the integrand is positive on K. Since E is Lebesgue measurable, the claim follows because $|E| = \sup\{|K| : K \subset E,\ \text{K is compact}\}$.

For each $x \in M - E$ since $I + D^2c^*(-Du(x))D^2u(x)$ is positive definite we get $\partial_c u(B_r(x))$ shrinks nicely to $x - Dc^*(-Du(x))$, a Lebesgue point of g. Consequently,

$$\begin{aligned}\lim_{r\to 0^+} \frac{\omega_c(g,u)(B_r(x))}{|B_r(x)|} &= \lim_{r\to 0^+} \frac{|\partial_c u(B_r(x))|}{|B_r(x)|} \frac{1}{|\partial_c u(B_r(x))|} \int\limits_{\partial_c u(B_r(x))} g(y)\, \mathrm{d}y \\ &= g(x - Dc^*(-Du(x)))\det[I + D^2c^*(-Du(x))D^2u(x)] \quad \forall x \in M - E.\end{aligned}$$

Thus we obtain

$$F(x) = g(x - Dc^*(-Du(x)))\det[I + D^2c^*(-Du(x))D^2u(x)] \quad \text{a.e. on } M. \tag{16}$$

On the other hand, by letting B be a Borel set in Ω such that $M \subset B$ and $|M| = |B|$ we have $|\partial_c u(\Omega - B)| = \int_{\Omega - B} \det[I + D^2c^*(-Du(x))D^2u(x)]\, \mathrm{d}x = 0$ since $\det[I + D^2c^*(-Du(x))D^2u(x)]$ is zero a.e. on $\Omega - M$. Therefore,

$$\int\limits_{\Omega - B} F(x)\, \mathrm{d}x = \omega_c(g,u)(\Omega - B) = \int\limits_{\partial_c u(\Omega - B)} g(y)\, \mathrm{d}y = 0,$$

which gives $F(x) = 0$ a.e. on $\Omega - B$. This implies that

$$F(x) = g(x - Dc^*(-Du(x)))\det[I + D^2c^*(-Du(x))D^2u(x)] \quad \text{a.e. on } \Omega - M. \tag{17}$$

From (16) and (17) we get (15) and the proof is complete. □

[6] This means that given $x \in \Omega$ there exist a ball $B_r(x)$ and a nonnegative constant λ such that $u(x) + \lambda|x|^2$ is convex on $B_r(x)$ in the standard sense, see [9, p. 134].

In order to solve the Dirichlet problem for the equation (2), the following lemma is needed in Sect. 6. Condition (18) is used to pass from local inequalities to global ones, see the argument after (20).

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u_k \in C(\Omega)$ be a sequence such that $u_k \to u$ uniformly on compact subsets of Ω.*

(i) *If $K \subset \Omega$ is compact, then*

$$\limsup_{k\to\infty} \partial_c u_k(K) \subset \partial_c u(K),$$

and by Fatou

$$\limsup_{k\to\infty} \omega_c(g, u_k)(K) \le \omega_c(g, u)(K).$$

(ii) *Assume further that (H2) holds, u_k are c-convex on Ω and for every subsequence $\{k_j\}$ and $\{z_{k_j}\} \subset \Omega$ with $z_{k_j} \to z_0 \in \partial\Omega$, we have*

$$\liminf_{j\to\infty} u(z_{k_j}) \le \limsup_{j\to\infty} u_{k_j}(z_{k_j}). \tag{18}$$

If K is compact and U is open such that $K \subset U \subset \Omega$, then we get

$$\partial_c u(K) \subset \liminf_{k\to\infty} \partial_c u_k(U)$$

where the inclusion holds for almost every point of the set on the left hand side, and by Fatou

$$\omega_c(g, u)(K) \le \liminf_{k\to\infty} \omega_c(g, u_k)(U).$$

Proof (i) Let $p \in \limsup_{k\to\infty} \partial_c u_k(K)$. Then for each n, there exist k_n and $x_{k_n} \in K$ such that $p \in \partial_c u_{k_n}(x_{k_n})$. By selecting a subsequence $\{x_j\}$ of $\{x_{k_n}\}$ we may assume $x_j \to x_0 \in K$. On the other hand, $u_j(x) \ge u_j(x_j) - c(x-p) + c(x_j - p)$, $\forall x \in \Omega$, and by letting $j \to \infty$, the uniform convergence of u_j on compacts yields

$$u(x) \ge u(x_0) - c(x-p) + c(x_0 - p) \qquad \forall x \in \Omega$$

that is, $p \in \partial_c u(x_0)$.

(ii) Without loss of generality we can assume that $\overline{U} \subset \Omega$.
Let $A = \{(x,p) | x \in K \text{ and } p \in \partial_c u(x)\}$ and for every $z \in \mathbb{R}^n$ we define the auxiliary function $v(z) = \sup_{(x,p)\in A} f_{x,p}(z)$, where for each $(x,p) \in A$ we denote $f_{x,p}(z) = -c(z-p) + c(x-p) + u(x)$ for all $z \in \mathbb{R}^n$. We first observe that since $\partial_c u(K)$ is bounded by Lemma 2.14 and c is locally Lipschitz it is easy to see that v is locally Lipschitz on $\mathbb{R}^n$. If $(x,p) \in A$, then $u(z) \ge f_{x,p}(z)$ for $z \in \Omega$ and so $v \le u$ on Ω. Since by Lemma 2.10 u is c-convex, then by taking $p \in \partial_c u(z)$ we have $v(z) \ge f_{z,p}(z) = u(z)$ for $z \in K$ and so $v = u$ in K. Moreover, $\partial_c(v, \mathbb{R}^n)(x) = \partial_c u(x)$ for every x in K. Now let $S = \{p \in \mathbb{R}^n : p \in \partial_c(v, \mathbb{R}^n)(x_1) \cap \partial_c(v, \mathbb{R}^n)(x_2)$ for some $x_1, x_2 \in \mathbb{R}^n, x_1 \ne x_2\}$. By Corollary 2.12, $|S| = 0$. Therefore (*ii*) will be proved if we show that $\partial_c(v, \mathbb{R}^n)(K) \setminus S \subset \liminf_{k\to\infty} \partial_c u_k(U)$. Let $p \in \partial_c(v, \mathbb{R}^n)(K) \setminus S$, then there exists $x_0 \in K$ such that $p \in \partial_c(v, \mathbb{R}^n)(x_0)$ and $p \notin \partial_c(v, \mathbb{R}^n)(x)$ for every x in $\mathbb{R}^n \setminus \{x_0\}$. Hence we have

$$v(x) > v(x_0) - c(x-p) + c(x_0 - p) \qquad \forall x \in \mathbb{R}^n \setminus \{x_0\}. \tag{19}$$

Now let $\delta_k := \min_{x\in\overline{U}}\{u_k(x) - u_k(x_0) + c(x-p) - c(x_0-p)\}$. Then this minimum is attained at some $x_k \in \overline{U}$. So $\delta_k = u_k(x_k) - u_k(x_0) + c(x_k - p) - c(x_0 - p)$ and $u_k(x) \geq u_k(x_0) - c(x-p) + c(x_0-p) + \delta_k \quad \forall x \in \overline{U}$. Thus we obtain

$$u_k(x) \geq u_k(x_k) - c(x-p) + c(x_k - p) \qquad \forall x \in \overline{U}. \tag{20}$$

We first claim that $x_k \to x_0$. Indeed, let $\{x_{k_j}\}$ be any convergent subsequence of $\{x_k\}$, say to $\bar{x} \in \overline{U}$. If $\bar{x} \neq x_0$ then since $u_k \to u$ uniformly on $\overline{U}$, passing to the limit in (20) and using (19) we get

$$\begin{aligned} u(x) &\geq u(\bar{x}) - c(x-p) + c(\bar{x}-p) \\ &\geq v(\bar{x}) - c(x-p) + c(\bar{x}-p) \\ &> v(x_0) - c(\bar{x}-p) + c(x_0-p) - c(x-p) + c(\bar{x}-p) \\ &= u(x_0) - c(x-p) + c(x_0-p) \quad \forall x \in \overline{U}, \end{aligned}$$

in particular, $u(x_0) > u(x_0)$, a contradiction. So we must have $x_{k_j} \to x_0$ and hence we obtain $x_k \to x_0 \in U$.

We next claim that there exists k_0 such that for all $k \geq k_0$ we have

$$u_k(x) \geq u_k(x_k) - c(x-p) + c(x_k - p) \quad \forall x \in \Omega,$$

in other words, the inequality (20) holds true in Ω. Otherwise, we can find a subsequence $\{k_j\}$ and $\{z_{k_j}\} \subset \Omega \setminus \overline{U}$ such that

$$u_{k_j}(z_{k_j}) < u_{k_j}(x_{k_j}) - c(z_{k_j} - p) + c(x_{k_j} - p) \quad \forall j. \tag{21}$$

Since Ω is bounded, passing through a subsequence, we can assume that $z_{k_j} \to z_0 \in \overline{\Omega} \setminus U$. If $z_0 \in \Omega \setminus U$, then by letting $j \to \infty$ in (21) and using the assumption that $u_{k_j} \to u$ uniformly on compact subsets of Ω, we deduce that

$$v(z_0) \leq u(x_0) - c(z_0 - p) + c(x_0 - p) = v(x_0) - c(z_0 - p) + c(x_0 - p). \tag{22}$$

On the other hand, if $z_0 \in \partial\Omega$, then from (21) we obtain

$$\limsup_{j\to\infty} u_{k_j}(z_{k_j}) \leq u(x_0) - c(z_0 - p) + c(x_0 - p) = v(x_0) - c(z_0 - p) + c(x_0 - p).$$

But (18) and since $u \geq v$ on Ω yield

$$\limsup_{j\to\infty} u_{k_j}(z_{k_j}) \geq \liminf_{j\to\infty} u(z_{k_j}) \geq \liminf_{j\to\infty} v(z_{k_j}) = v(z_0),$$

and therefore (22) also holds. This gives a contradiction with (19) since $z_0 \neq x_0$. So the claim is proved. But then we get $p \in \partial_c u_k(x_k)$ for all $k \geq k_0$ and hence $p \in \liminf_{k\to\infty} \partial_c u_k(U)$ as $x_k \to x_0 \in U$. This completes the proof. □

As an immediate consequence we have the following stability property, which will be useful in various contexts later.

Corollary 3.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and suppose that (H2) holds. If $\{u_k\} \subset C(\Omega)$ is a sequence of c-convex functions converging locally uniformly in Ω to a function u and condition* (18) *holds, then $\omega_c(g, u_k)$ tend to $\omega_c(g, u)$ weakly, i.e.,*

$$\int_\Omega f(x)\, \mathrm{d}\omega_c(g, u_k) \to \int_\Omega f(x)\, \mathrm{d}\omega_c(g, u)$$

for any f in $C_0(\Omega)$.

Remark 3.5 We note that by following the proof of Lemma 3.3 we see that if either u_k are in $C^1(\Omega)$ or u_k are convex on a convex set Ω, then the above results still hold without condition (18). The reason is that in these cases we have that (20) holds for every x in an open neighborhood of x_k iff it holds for every x in Ω. However, this is no longer true if u_k are merely c-convex. We also remark that (18) is satisfied if either $u_k \to u$ locally uniformly and $u_k \geq u$ on Ω or $u_k \to u$ uniformly on Ω.

4 Maximum principles

Let Ω be a bounded open set in $\mathbb{R}^n$ and $u \in C(\Omega)$. Consider the classes of functions

$$F(u) := \{v : \ v \text{ is } c\text{-convex in } \Omega \ \text{ and } \ v(x) \leq u(x) \ \ \forall x \in \Omega\},$$

$$G(u) := \{w : \ w \text{ is } c\text{-concave in } \Omega \ \text{ and } \ w(x) \geq u(x) \ \ \forall x \in \Omega\},$$

where w is called c-concave if $-w$ is c-convex. Let

$$u_*(x) := \sup_{v \in F(u)} v(x) \quad \text{and} \quad u^*(x) := \inf_{w \in G(u)} w(x). \tag{23}$$

Then u_* is c-convex and u^* is c-concave on Ω. Moreover, if c satisfies condition $(H3)$, then it follows from Remark 2.6 that u_* and u^* are in $C(\Omega)$. We call these functions the c-convex and c-concave envelopes of u in Ω respectively, and we have the inequalities

$$u_*(x) \leq u(x) \leq u^*(x) \quad \forall x \in \Omega.$$

We also have that $F(-u) = -G(u)$, and hence

$$-(u^*)(x) = -\inf_{w \in G(u)} w(x) = \sup_{v \in -G(u)} v(x) = \sup_{v \in F(-u)} v(x) = (-u)_*(x). \tag{24}$$

Consider the set of contact points

$$C_*(u) := \{x \in \Omega : \ u_*(x) = u(x)\}; C^*(u) := \{x \in \Omega : \ u^*(x) = u(x)\}$$

which are relative closed in Ω if c satisfies $(H3)$. Then by (24) we get

$$C_*(u) = C^*(-u). \tag{25}$$

From the definitions it is clear that

$$\partial_c(u_*)(C_*(u)) = \partial_c u(C_*(u)). \tag{26}$$

Also it is easy to check if $x_0 \notin C_*(u)$, then $\partial_c u(x_0) = \emptyset$. Hence, $\partial_c u(\Omega) = \partial_c u(C_*(u)) \cup \partial_c u(\Omega \setminus C_*(u)) = \partial_c u(C_*(u))$. Therefore by combining with (26) we obtain

$$\partial_c u(\Omega) = \partial_c u(C_*(u)) = \partial_c(u_*)(C_*(u)). \tag{27}$$

Let

$$\partial^c u(x_0) = \{p \in \mathbb{R}^n : \ u(x) \leq u(x_0) + c(x - p) - c(x_0 - p) \ \ \forall x \in \Omega\}$$

be the c-superdifferential of u at x_0. Notice then that $\partial^c(-u)(x_0) = \partial_c u(x_0)$.

Lemma 4.1 *Let $c : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose $u \in C(\bar{\Omega})$ is such that $u \leq 0$ on $\partial\Omega$. Then for any $x_0 \in \Omega$ with $u(x_0) > 0$, we have*

$$\Omega(x_0, u(x_0)) \subset \partial^c(u^*)(C^*(u)),$$

where $\Omega(x,t) = \{y \in \mathbb{R}^n : c(z-y) - c(x-y) + t > 0 \quad \forall z \in \bar{\Omega}\}$.

Proof Let $y \in \Omega(x_0, u(x_0))$, then

$$c(z-y) - c(x_0-y) + u(x_0) > 0 \qquad \forall z \in \bar{\Omega}. \tag{28}$$

Let $\lambda_0 := \inf\{\lambda : \ \lambda + c(z-y) - c(x_0-y) \geq u(z) \quad \forall z \in \bar{\Omega}\}$. By continuity we have

$$\lambda_0 + c(z-y) - c(x_0-y) \geq u(z) \qquad \forall z \in \bar{\Omega}. \tag{29}$$

Consider the minimum $\min_{z\in\bar{\Omega}} [\lambda_0 + c(z-y) - c(x_0-y) - u(z)]$ which is nonnegative by (29). This minimum is attained at some point $\bar{z} \in \bar{\Omega}$, and we have

$$\lambda_0 + c(\bar{z}-y) - c(x_0-y) - u(\bar{z}) = 0, \tag{30}$$

because on the contrary $\lambda_0 + c(z-y) - c(x_0-y) - u(z) \geq \epsilon > 0, \forall z \in \bar{\Omega}$, and λ_0 would not be the minimum. We now claim that $\bar{z} \in \Omega$. Indeed, since $u \leq 0$ on $\partial\Omega$, the claim will be proved if we show that $u(\bar{z}) > 0$. By taking $z = x_0$ in (29) we get $\lambda_0 \geq u(x_0)$, and consequently by (28)$c(z-y) - c(x_0-y) + \lambda_0 > 0, \forall z \in \bar{\Omega}$. Combining with (30) yields $u(\bar{z}) = c(\bar{z}-y) - c(x_0-y) + \lambda_0 > 0$. Thus we must have $\bar{z} \in \Omega$. Therefore we have proved that if $y \in \Omega(x_0, u(x_0))$, then there exists $\bar{z} \in \Omega$ such that $u(\bar{z}) = c(\bar{z}-y) - c(x_0-y) + \lambda_0$, and since the above minimum is zero we also have

$$u(z) \leq \lambda_0 + c(z-y) - c(x_0-y) \quad \forall z \in \bar{\Omega}.$$

Therefore by definition of u^* we obtain

$$u(z) \leq u^*(z) \leq \lambda_0 + c(z-y) - c(x_0-y) \quad \forall z \in \Omega.$$

In particular,

$$u(\bar{z}) \leq u^*(\bar{z}) \leq \lambda_0 + c(\bar{z}-y) - c(x_0-y) = u(\bar{z}).$$

So $u^*(\bar{z}) = \lambda_0 + c(\bar{z}-y) - c(x_0-y) = u(\bar{z})$ and hence $\bar{z} \in C^*(u)$. Moreover by combining with the above inequality we get

$$u^*(z) \leq \lambda_0 + c(z-y) - c(x_0-y) = u^*(\bar{z}) + c(z-y) - c(\bar{z}-y) \quad \forall z \in \Omega.$$

So $y \in \partial^c(u^*)(\bar{z}) \subset \partial^c(u^*)(C^*(u))$ and this completes the proof. □

We notice that from Lemma 4.1, (24), (25) and (27) we have

$$\begin{aligned}\Omega(x_0, u(x_0)) &\subset \partial^c(u^*)(C^*(u)) = \partial_c(-(u^*))(C^*(u)) \\ &= \partial_c((-u)_*)(C^*(u)) = \partial_c((-u)_*)(C_*(-u)) = \partial_c(-u)(C_*(-u)).\end{aligned} \tag{31}$$

Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a continuous cost function satisfying $c(0) = \min_{x\in\mathbb{R}^n} c(x)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For each x in $\overline{\Omega}$ and each $t \geq 0$, let $\Omega(x,t)$ be as in Lemma 4.1, i.e.,

$$\Omega(x,t) = \{y \in \mathbb{R}^n : -t - c(z-y) + c(x-y) < 0, \forall z \in \bar{\Omega}\},$$

and define

$$\tilde{\Omega}(x,t) = \{y \in \mathbb{R}^n : -t - c(z-y) + c(x-y) \le 0, \forall z \in \bar{\Omega}\}.$$

First observe that $\Omega(x,0) = \emptyset$, $x \in \Omega(x,t)$ for any $t > 0$ and $\tilde{\Omega}(x,t)$ is closed. Since c is uniformly continuous on any bounded set of $\mathbb{R}^n$, it is easy to see that $\Omega(x,t)$ is an open set for every x in $\overline{\Omega}$ and every $t \ge 0$. Also if $0 \le t_1 < t_2$ then $\overline{\Omega(x,t_1)} \subset \tilde{\Omega}(x,t_1) \subset \Omega(x,t_2)$, and $\cup_{t\ge 0}\Omega(x,t) = \mathbb{R}^n$. Particularly, we have $\overline{\Omega(x,t_1) \cap B(x,t_1)} \subset \Omega(x,t_2) \cap B(x,t_2)$ where the first set is compact and the later is a nonempty open set. Therefore, for any $x \in \overline{\Omega}$ and any $0 \le t_1 < t_2$, we have that

$$|\overline{\Omega(x,t_1) \cap B(x,t_1)}| < |\Omega(x,t_2) \cap B(x,t_2)|. \tag{32}$$

The following lemma will be needed later.

Lemma 4.2 *Suppose c satisfies either $(H1)$ or $(H2)$, and $c(0) = \min_{x\in\mathbb{R}^n} c(x)$. Then $|\tilde{\Omega}(x,t) - \Omega(x,t)| = 0$ for all $x \in \overline{\Omega}$ and all $t > 0$.*

Proof Let $x_0 \in \overline{\Omega}$ and $t_0 > 0$. Define

$$\mathcal{F} = \{c\text{-convex function } v \in C(\overline{\Omega}) : v \le 0 \text{ on } \overline{\Omega} \text{ and } v(x_0) \le -t_0\}$$

and

$$w(x) = \sup_{v\in\mathcal{F}} v(x) \quad \forall x \in \overline{\Omega}.$$

If we let $v(x) = -t_0 - c(x - x_0) + c(0)$ then it is clear that $v \in \mathcal{F}$. So $\mathcal{F} \ne \emptyset$ and moreover we have w is bounded from below on $\overline{\Omega}$, $w \le 0$ on $\overline{\Omega}$ and $w(x_0) = -t_0$. By using $w(x_0) = -t_0$ it is easy to see that $\tilde{\Omega}(x_0,t_0) = \partial_c(w,\overline{\Omega})(x_0)$.[7] Now let

$$\tilde{S} = \{p \in \mathbb{R}^n : \ p \in \partial_c(w,\overline{\Omega})(x_1) \cap \partial_c(w,\overline{\Omega})(x_2) \ \text{ for some } \ x_1, x_2 \in \overline{\Omega}, \ x_1 \ne x_2\}.$$

Then $|\tilde{S}| = 0$ by Lemma 2.11. We shall complete the proof by showing that $\tilde{\Omega}(x_0,t_0) - \Omega(x_0,t_0) \subset \tilde{S}$. Indeed, if $y \in \tilde{\Omega}(x_0,t_0)$ is such that $y \notin \tilde{S}$ then as $y \in \partial_c(w,\overline{\Omega})(x_0)$ we get $-t_0 - c(z-y) + c(x_0-y) < w(z)$ for all z in $\overline{\Omega} \setminus \{x_0\}$. Particularly, since $w \le 0$ on $\overline{\Omega}$ and $-t_0 < 0$ we obtain $-t_0 - c(z-y) + c(x_0-y) < 0$ for all z in $\overline{\Omega}$. That is, $y \in \Omega(x_0,t_0)$ and hence $\tilde{\Omega}(x_0,t_0) - \Omega(x_0,t_0) \subset \tilde{S}$ as desired. □

Now suppose $g \in L^1_{loc}(\mathbb{R}^n)$ is positive a.e. on $\mathbb{R}^n$. Let $B(g) = \int_{\mathbb{R}^n} g(y)\,\mathrm{d}y$ and for each $t \ge 0$, define

$$h(t) = \inf_{x\in\overline{\Omega}} \int\limits_{\Omega(x,t)\cap B(x,t)} g(y)\,\mathrm{d}y. \tag{33}$$

Then clearly $h : [0,+\infty) \to [0,B(g))$ with $h(0) = 0$. We remark that by using the Dominated Convergence Theorem and Lemma 4.2 it can be shown easily that the function $f(x) := \int_{\Omega(x,t)\cap B(x,t)} g(y)\,\mathrm{d}y$ is continuous on $\overline{\Omega}$ (a similar argument will be employed in the proof of Lemma 4.3 below). Therefore, in the definition of h the infimum is achieved, i.e.,

$$h(t) = \min_{x\in\overline{\Omega}} \int\limits_{\Omega(x,t)\cap B(x,t)} g(y)\,\mathrm{d}y.$$

By this fact and (32), we also have h is strictly increasing.

[7] Notice that if c satisfies $(H3)$, then from Lemma 2.14 the set $\tilde{\Omega}(x,t)$ is bounded whenever $x \in \Omega$.

Lemma 4.3 *Suppose c satisfies either $(H1)$ or $(H2)$, and $c(0) = \min_{\mathbb{R}^n} c(x)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then the map $h : [0, +\infty) \to [0, B(g))$ is continuous, strictly increasing, and onto with $h(0) = 0$. Consequently, it is invertible and $h^{-1} : [0, B(g)) \to [0, +\infty)$ is also continuous, strictly increasing with $h^{-1}(0) = 0$.*

Proof It remains to prove h is continuous and onto. Firstly, let $t_0 \in [0, +\infty)$ and we want to show h is continuous at t_0. For this it suffices to prove that for any sequence $\{t_n\} \subset (0, +\infty)$ with $t_n \to t_0$, there exists a subsequence $\{n_j\}$ such that $h(t_{n_j}) \to h(t_0)$. If $\{t_n\}$ is such a sequence, then by the remark before this lemma there exists $\{x_n\} \subset \overline{\Omega}$ satisfying

$$h(t_n) = \int\limits_{\Omega(x_n,t_n)\cap B(x_n,t_n)} g(y)\,\mathrm{d}y \quad \forall n.$$

As Ω is bounded we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x_0 \in \overline{\Omega}$ so that $x_{n_j} \to x_0$ as $j \to \infty$. We consider the following two cases.

Case 1 $t_0 = 0$. We have

$$h(t_{n_j}) = \int\limits_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})} g(y)\,\mathrm{d}y = \int\limits_{B(x_0,1)} g(y)\chi_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})}(y)\,\mathrm{d}y \longrightarrow 0 = h(0)$$

by the Dominated Convergence Theorem since $\chi_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})}(y) \longrightarrow 0$ for all $y \neq x_0$.

Case 2 $t_0 > 0$. We have

$$\begin{aligned} h(t_{n_j}) &= \int\limits_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})} g(y)\,\mathrm{d}y = \int\limits_{B(x_0,2t_0)} g(y)\chi_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})}(y)\,\mathrm{d}y \\ &\longrightarrow \int\limits_{B(x_0,2t_0)} g(y)\chi_{\Omega(x_0,t_0)\cap B(x_0,t_0)}(y)\,\mathrm{d}y = \int\limits_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\,\mathrm{d}y \end{aligned}$$

by the Dominated Convergence Theorem since $\chi_{\Omega(x_{n_j},t_{n_j})\cap B(x_{n_j},t_{n_j})}(y) \longrightarrow \chi_{\Omega(x_0,t_0)\cap B(x_0,t_0)}(y)$ for all $y \notin E := [(\tilde{\Omega}(x_0,t_0) - \Omega(x_0,t_0)) \cap \overline{B}(x_0,t_0)] \cup [\tilde{\Omega}(x_0,t_0) \cap \partial B(x_0,t_0)]$, which has Lebesgue measure zero by Lemma 4.2. This can be easily verified by noticing the fact that $\mathbb{R}^n = [\Omega(x_0,t_0) \cap B] \cup \tilde{\Omega}(x_0,t_0)^c \cup \overline{B}(x_0,t_0)^c \cup E$. Thus we have shown that

$$h(t_{n_j}) \longrightarrow \int\limits_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\,\mathrm{d}y.$$

Now we claim that $\int_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\,\mathrm{d}y = h(t_0)$. Indeed, for each $x \in \overline{\Omega}$ we have

$$\begin{aligned} \int\limits_{\Omega(x,t_0)\cap B(x,t_0)} g(y)\,\mathrm{d}y &= \lim_{j\to\infty} \int\limits_{\Omega(x,t_{n_j})\cap B(x,t_{n_j})} g(y)\,\mathrm{d}y \\ &\geq \liminf_{j\to\infty} h(t_{n_j}) = \int\limits_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\,\mathrm{d}y \end{aligned}$$

where we have again used Lemma 4.2 in the first equality. Therefore, by taking the infimum on the left hand side we obtain $h(t_0) \geq \int_{\Omega(x_0,t_0)\cap B(x_0,t_0)} g(y)\,\mathrm{d}y$. So the claim is proved since the reverse inequality is obvious. Thus we get $h(t_{n_j}) \to h(t_0)$ as desired. This implies that h is continuous at t_0.

Secondly, we want to show that h is onto. We know that $h(0) = 0$. Now if we let $a \in (0, B(g))$ then since

$$\int\limits_{\Omega(0,t)\cap B(0,t)} g(y)\,\mathrm{d}y \longrightarrow B(g) \quad \text{as } t \to +\infty$$

we can find a $t_0 > 0$ such that $a < \int_{\Omega(0,t_0)\cap B(0,t_0)} g(y)\,\mathrm{d}y < B(g)$. For any $x \in \overline{\Omega}$ and any $y \in \Omega(0,t_0) \cap B(0,t_0)$ we have

$$\begin{aligned} -c(z-y) + c(x-y) &= -c(z-y) + c(-y) + [c(x-y) - c(-y)] \\ &< t_0 + \sup_{w_1\in\Omega_{t_0};w_2\in B(0,t_0)} |c(w_1) - c(w_2)| =: t_1 < +\infty \quad \forall z \in \overline{\Omega}, \end{aligned}$$

where $\Omega_{t_0} = \{y \in \mathbb{R}^n : \mathrm{dist}(y, \overline{\Omega}) < t_0\}$. Consequently, $\Omega(0,t_0) \cap B(0,t_0) \subset \Omega(x,t_1) \cap B(0,t_0)$ for all x in $\overline{\Omega}$. Hence, by picking t_1 sufficiently large if necessary we can assume that $\Omega(0,t_0) \cap B(0,t_0) \subset \Omega(x,t_1) \cap B(x,t_1)$ for all x in $\overline{\Omega}$. This implies that $\int_{\Omega(0,t_0)\cap B(0,t_0)} g(y)\,\mathrm{d}y \leq h(t_1)$. Therefore, we obtain

$$h(0) = 0 < a < \int\limits_{\Omega(0,t_0)\cap B(0,t_0)} g(y)\,\mathrm{d}y \leq h(t_1) < B(g).$$

But then since h is continuous on $[0, +\infty)$, there must exist a $t_2 \in (0, +\infty)$ so that $h(t_2) = a$, which means that h is onto. □

From now on for convenience we will consider h^{-1} as a one-to-one function from $[0, B(g)]$ onto $[0, +\infty]$ with $h^{-1}(B(g)) = +\infty$. Therefore, $h^{-1}(a) < +\infty$ only if $0 \leq a < B(g)$. By combining the previous results we obtain the following maximum principle which holds for any continuous function on $\bar{\Omega}$.

Theorem 4.4 *Suppose c satisfies either $(H1)$ or $(H2)$, and $c(0) = \min_{\mathbb{R}^n} c(x)$. Let $g \in L^1_{loc}(\mathbb{R}^n)$ be positive a.e. and Ω be a bounded open set in $\mathbb{R}^n$. If $u \in C(\overline{\Omega})$ then*

$$\max_{\overline{\Omega}} u(x) \leq \max_{\partial\Omega} u(x) + h^{-1}(\omega_c(g, -u)(\Omega)).$$

Proof Let $M = \max_{\partial\Omega} u(x)$ and let $x_0 \in \Omega$ be such that $u(x_0) > M$. By Lemma 4.1 and (31) we have $\Omega(x_0, u(x_0) - M) \subset \partial_c(-u + M)(C_*(-u + M)) = \partial_c(-u)(\Omega)$. This gives

$$h(u(x_0) - M) \leq \int\limits_{\Omega(x_0,u(x_0)-M)} g(y)\,\mathrm{d}y \leq \int\limits_{\partial_c(-u)(\Omega)} g(y)\,\mathrm{d}y = \omega_c(g, -u)(\Omega).$$

Hence by taking the inverse we obtain $u(x_0) \leq M + h^{-1}(\omega_c(g, -u)(\Omega))$ and the proof is complete. □

We end this section noticing that if the cost function c is convex, C^1 and satisfies that there exist positive constants A, α such that $|Dc(x)| \leq A|x|^\alpha$ for all x in $\mathbb{R}^n$ then Theorem 4.4 also holds with the function h defined in a simpler way, namely

$h(t) = \inf_{x\in\bar{\Omega}} \int_{B(x,t)} g(y)\,\mathrm{d}y = \min_{x\in\bar{\Omega}} \int_{B(x,t)} g(y)\,\mathrm{d}y$. The advantage of this definition is that it is independent of c and in many cases when the function g is simple enough we can calculate h and h^{-1} exactly. For example, when $g \equiv 1$ we have the following result which is an extension of the well known Aleksandrov–Bakelman–Pucci maximum principle.

Theorem 4.5 *Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a C^1 and convex function satisfying there exist positive constants A, α such that $|Dc(x)| \le A|x|^\alpha$ for all x in $\mathbb{R}^n$. Let Ω be a bounded open set in $\mathbb{R}^n$. We have*

(a) If $u \in C(\overline{\Omega})$ then

$$\max_{\overline{\Omega}} u(x) \le \max_{\partial\Omega} u(x) + A\,\omega_n^{-\frac{\alpha}{n}} \operatorname{diam}(\Omega)\,|\partial_c(-u)(C_*(-u))|^{\frac{\alpha}{n}}.$$

(b) If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and in addition c satisfies $(H2)$ with $c^* \in C^2(\mathbb{R}^n)$, then

$$\max_{\overline{\Omega}} u(x) \le \max_{\partial\Omega} u(x) + A\,\omega_n^{-\frac{\alpha}{n}} \operatorname{diam}(\Omega) \left(\int_{C_*(-u)} |\det(I - D^2c^*(Du)D^2u)|dx \right)^{\frac{\alpha}{n}}.$$

Proof (a) Let $M = \max_{\partial\Omega} u(x)$ and $x_0 \in \Omega$ be such that $u(x_0) > M$. For any $z \in \bar{\Omega}$ and $y \in \mathbb{R}^n$, we have from the convexity of c and the assumptions that

$$\begin{aligned} c(z-y) - c(x_0-y) + u(x_0) - M &\ge Dc(x_0-y)\cdot(z-x_0) + u(x_0) - M \\ &\ge -A|x_0-y|^\alpha|z-x_0| + u(x_0) - M \\ &\ge -A\,\operatorname{diam}(\Omega)|x_0-y|^\alpha + u(x_0) - M. \end{aligned}$$

So if $y \in B(x_0,R)$ where $R = \left(\dfrac{u(x_0)-M}{A\,\operatorname{diam}(\Omega)}\right)^{\frac{1}{\alpha}}$, then we get $c(z-y) - c(x_0 - y) + u(x_0) - M > 0$, $\forall z \in \bar{\Omega}$. That is, $B(x_0,R) \subset \Omega(x_0, u(x_0) - M)$. Therefore, by Lemma 4.1 and (31) we obtain

$$\omega_n R^n = |B(x_0,R)| \le |\Omega(x_0, u(x_0) - M)| \le |\partial_c(-u)(C_*(-u))|$$

or $u(x_0) - M \le A\,w_n^{-\frac{\alpha}{n}} \operatorname{diam}(\Omega)|\partial_c(-u)(C_*(-u))|^{\frac{\alpha}{n}}$. This completes the proof of part (a).

(b) This follows from (a) and the first part of Proposition 2.19.

5 Comparison principles

We begin with the following basic lemma.

Lemma 5.1 *Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u, v \in C(\bar{\Omega})$. If $u = v$ on $\partial\Omega$ and $v \ge u$ in Ω, then*

$$\partial_c v(\Omega) \subset \partial_c u(\Omega).$$

Proof The proof is the same as in [10, Lemma 1.4.1] for the standard subdifferential but we include it here for convenience.

Let $p \in \partial_c v(\Omega)$. There exists $x_0 \in \Omega$ such that $v(x) \geq v(x_0) - c(x-p) + c(x_0-p)$ $\forall x \in \Omega$. Define

$$a = \sup_{x \in \bar{\Omega}} \{v(x_0) - c(x-p) + c(x_0-p) - u(x)\}.$$

Since $v(x_0) \geq u(x_0)$ we have $a \geq 0$. Also, there exists $x_1 \in \bar{\Omega}$ such that $a = v(x_0) - c(x_1-p) + c(x_0-p) - u(x_1)$ and so $u(x) \geq u(x_1) - c(x-p) + c(x_1-p), \forall x \in \Omega$. Moreover, $v(x_1) \geq v(x_0) - c(x_1-p) + c(x_0-p) = u(x_1) + a$. Hence, if $a > 0$, then $x_1 \in \Omega$ and we get $p \in \partial_c u(x_1) \subset \partial_c u(\Omega)$. If $a = 0$, then $u(x) \geq u(x_0) - c(x-p) + c(x_0-p)$, $\forall x \in \Omega$, and we obtain $p \in \partial_c u(x_0) \subset \partial_c u(\Omega)$. This completes the proof. □

We next have the following result which gives a stronger conclusion than Lemma 5.1.

Lemma 5.2 *Suppose $c : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz continuous function. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u, v \in C(\Omega)$. Suppose that the set $G := \{x \in \Omega : v(x) > u(x)\}$ satisfies $\overline{G} \subset \Omega$. Then $\partial_c v(G) \subset \mathrm{Int}(\partial_c u(G))$.*

Proof If $p \in \partial_c v(G)$, then there exists $x_0 \in G$ such that $v(x) \geq v(x_0) - c(x-p) + c(x_0-p)$ for all $x \in \Omega$. Let $\epsilon = v(x_0) - u(x_0) > 0$ and consider the hypersurface of the form $v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2}$, where q will be chosen in a moment. Fix a ball B sufficiently large such that $x - z \in B$ for all $(x, z) \in \Omega \times B(p, 1)$. Choose $M_\epsilon = \dfrac{\epsilon}{4\|c\|_{Lip(B)} + \epsilon}$. Then for any $q \in B(p, M_\epsilon)$ we have

$$\begin{aligned} v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} &= v(x_0) - c(x-p) + c(x_0-p) \\ &\quad + [c(x-p) - c(x-q)] \\ &\quad + [c(x_0-q) - c(x_0-p)] - \frac{\epsilon}{2} \\ &\leq v(x) + 2\|c\|_{Lip(B)}|p-q| - \frac{\epsilon}{2} \leq v(x) \quad \forall x \in \Omega. \end{aligned} \tag{34}$$

We shall show that $B(p, M_\epsilon) \subset \partial_c u(G)$. Indeed, for any $q \in B(p, M_\epsilon)$ let

$$a = \sup_{x \in \bar{G}} \left\{v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} - u(x)\right\}.$$

Since $v(x_0) - u(x_0) = \epsilon$, we get $a > 0$. Also observe that if x is in $\Omega \setminus \bar{G}$, then $v(x) \leq u(x)$ and so by combining with (34) we get

$$\begin{aligned} v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} - u(x) &\leq v(x_0) - c(x-q) \\ &\quad + c(x_0-q) - \frac{\epsilon}{2} - v(x) \leq 0 < a. \end{aligned}$$

So in fact $a = \sup_{x \in \Omega} \{v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} - u(x)\}$. Now by the definition of a there exists $x_1 \in \bar{G}$ such that $a = v(x_0) - c(x_1-q) + c(x_0-q) - \dfrac{\epsilon}{2} - u(x_1)$ and hence

$$v(x_0) - c(x_1-q) + c(x_0-q) - \frac{\epsilon}{2} - u(x_1) \geq v(x_0) - c(x-q) + c(x_0-q) - \frac{\epsilon}{2} - u(x)$$

for all $x \in \Omega$, or equivalently,

$$u(x) \geq u(x_1) - c(x-q) + c(x_1 - q) \quad \forall x \in \Omega. \tag{35}$$

On the other hand, applying (34) at $x = x_1$ yields

$$v(x_1) \geq v(x_0) - c(x_1 - q) + c(x_0 - q) - \frac{\epsilon}{2} = u(x_1) + a > u(x_1).$$

Therefore, $x_1 \in G$ and hence from (35) we get $q \in \partial_c u(x_1) \subset \partial_c u(G)$, i.e., $B(p, M_\epsilon) \subset \partial_c u(G)$. This completes the proof of the lemma. □

We recall a lemma from [9] adapted to the case of the c-subdifferential.

Lemma 5.3 (Lemma 4.3 from [9]) *Suppose c satisfies $(H3)$. Let $\Omega \subset \mathbb{R}^n$ be an open set and $u, v \in C(\Omega)$. Assume that $G = \{x \in \Omega : v(x) > u(x)\}$ is bounded, and $X = \{x \in \Omega : \partial_c u(x) \cap \partial_c v(G) \neq \emptyset\}$ is nonempty. If $p \in \Omega$ with $u(p) = v(p)$ and $\partial_c u(p) \cap \partial_c v(p) = \emptyset$, then dist$(p, X) > 0$.*

Proof Suppose $\text{dist}(p, X) = 0$. Then there exist $x_n \in X$ such that $x_n \to p$. Hence there exist $z_n \in G$ and y_n such that $y_n \in \partial_c u(x_n) \cap \partial_c v(z_n)$. Since $p \in \Omega$, it follows from Lemma 2.14 that $\partial_c u(\cup_n \{x_n\})$ is bounded. So passing through a subsequence, we may assume that $z_n \to z_0$ and $y_n \to y_0$. Since $y_n \in \partial_c u(x_n)$, we have $u(z) \geq u(x_n) - c(z - y_n) + c(x_n - y_n)$ for all $z \in \Omega$. Letting $n \to \infty$ yields $y_0 \in \partial_c u(p)$, and from the hypotheses $y_0 \notin \partial_c v(p)$. So there exists $t \in \Omega$ such that

$$v(t) < v(p) - c(t - y_0) + c(p - y_0). \tag{36}$$

On the other hand, since $y_n \in \partial_c v(z_n)$, we have

$$\begin{aligned} v(t) &\geq v(z_n) - c(t - y_n) + c(z_n - y_n) \\ &\geq u(z_n) - c(t - y_n) + c(z_n - y_n), \text{ since } z_n \in G \\ &\geq u(p) - c(z_n - y_0) + c(p - y_0) - c(t - y_n) + c(z_n - y_n), \text{ since } y_0 \in \partial_c u(p). \end{aligned}$$

Letting $n \to \infty$ we get $v(t) \geq u(p) - c(t - y_0) + c(p - y_0) = v(p) - c(t - y_0) + c(p - y_0)$ contradicting (36). □

We now consider conditions which force $\omega_c(g, u)(G) > \omega_c(g, v)(G)$. This will be used to prove the comparison principle Theorem 5.5.

Lemma 5.4 *Suppose c satisfies $(H2)$ and g is positive a.e. and locally integrable in $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $G = \{x \in \Omega : v(x) > u(x)\}$ where $u, v \in C(\Omega)$. Suppose that $G \neq \emptyset$, $\overline{G} \subset \Omega$ and $\partial_c u(x_0) \cap \partial_c v(x_0) = \emptyset$ for some $x_0 \in \partial G$. Assume further that $x_0 \in Int(spt(\omega_c(g, u)))$. Then we have*

$$\omega_c(g, u)(G) > \omega_c(g, v)(G).$$

Proof Let $X = \{x \in \Omega : \partial_c u(x) \cap \partial_c v(G) \neq \emptyset\}$. Then if $X \neq \emptyset$ we have from Lemma 5.3 that $\text{dist}(x_0, X) > 0$. Therefore, there exists $r > 0$ such that $B(x_0, r) \subset \Omega$ and $\partial_c u(B(x_0, r)) \cap \partial_c v(G) = \emptyset$, in particular, $\partial_c u(G \cap B(x_0, r)) \cap \partial_c v(G) = \emptyset$. This obviously holds if $X = \emptyset$. From Lemma 5.2, $\partial_c v(G) \subset \partial_c u(G)$. Thus we must have $\partial_c v(G) \subset \partial_c u(G \backslash B(x_0, r))$. Since $x_0 \in \text{Int}(\text{spt}(\omega_c(g, u)))$, there exists $r > 0$ small enough such that

$B(x_0, r) \subset \mathrm{spt}(\omega_c(g,u))$. As $x_0 \in \partial G$, we then get that $\emptyset \neq G \cap B(x_0, r) \subset \mathrm{spt}(\omega_c(g,u))$. We therefore obtain

$$\begin{aligned}\omega_c(g,u)(G) &= \omega_c(g,u)(G \setminus B(x_0,r)) + \omega_c(g,u)(G \cap B(x_0,r)) \\ &\geq \omega_c(g,v)(G) + \omega_c(g,u)(G \cap B(x_0,r)) > \omega_c(g,v)(G).\end{aligned}$$

This completes the proof. □

By using Lemma 5.4 we are able to prove the following comparison principle, which in particular gives the uniqueness of solutions for the Dirichlet problems considered in the next section. In the following theorem we denote $S := \mathrm{spt}(\omega_c(u)) \setminus \overline{\mathrm{Int}(\mathrm{spt}(\omega_c(u)))}$.

Theorem 5.5 *Suppose c satisfies (H2). Let g be positive a.e. and locally integrable in $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u, v \in C(\bar{\Omega})$ be c-convex in Ω, and*

$$\omega_c(g,u)(E) \leq \omega_c(g,v)(E)] \quad \textit{for all Borel sets } E \subset \Omega. \tag{37}$$

Assume that

$$\begin{gathered}\textit{for every open set } D \Subset \Omega \textit{ with } |\partial_c v(D \setminus spt(\omega_c(u)))| = 0, \textit{ there exists} \\ \textit{a closed set } F \subset \partial_c v(S \cap D) \textit{ such that } |\partial_c v(S \cap D) \setminus F| = 0.\end{gathered} \tag{38}$$

Then we have

$$\min_{\bar{\Omega}} \{u(x) - v(x)\} = \min_{\partial\Omega} \{u(x) - v(x)\}.$$

Proof For simplicity we shall present the proof when $g \equiv 1$, i.e., when $\omega_c(g,u)$ and $\omega_c(g,v)$ are replaced by $\omega_c(u)$ and $\omega_c(v)$ respectively. However, it can be readily checked that the same argument works for general g. To this end, we only need to note that from the assumptions on g we have $\mathrm{spt}(\omega_c(g,u)) = \mathrm{spt}(\omega_c(u))$ and if $E, F \subset \mathbb{R}^n$ are two measurable sets with $E \subset F$ and $\int_E g < \infty$, then $\int_E g = \int_F g$ if and only if $|E| = |F|$.

By adding a constant to v if necessary, we can assume without loss of generality that $\min_{\partial\Omega} \{u(x) - v(x)\} = 0$. We shall prove that $u(x) \geq v(x) \ \ \forall x \in \bar{\Omega}$. Indeed, suppose not, then there exists $\bar{x} \in \Omega$ such that $v(\bar{x}) - u(\bar{x}) = \max_{\bar{\Omega}} [v(x) - u(x)] > 0$. Let $\bar{\delta} = [v(\bar{x}) - u(\bar{x})] > 0$. For every $0 < \delta < \bar{\delta}$, define $w_\delta(x) := v(x) - \delta$ in $\overline{\Omega}$ and

$$D_\delta := \{x \in \overline{\Omega} : w_\delta(x) > u(x)\} = \{x \in \Omega : w_\delta(x) > u(x)\}.^8$$

We have $\bar{x} \in D_\delta$, and $w_\delta(x) = v(x) - \delta \leq u(x) - \delta < u(x)$ for $x \in \partial\Omega$. Hence $\overline{D_\delta} \subset \Omega$ and $\partial D_\delta = \{x \in \Omega : w_\delta(x) = u(x)\}$. Applying Lemma 5.2 we obtain

$$\partial_c v(D_\delta) = \partial_c w_\delta(D_\delta) \subset \mathrm{Int}(\partial_c u(D_\delta)). \tag{39}$$

Since v is c-convex, it follows from Remark 2.5 that $\partial_c v(D_\delta) \neq \emptyset$.

Therefore $\omega_c(u)(D_\delta) > 0$ and consequently

$$\mathrm{spt}(\omega_c(u)) \cap D_\delta \neq \emptyset, \qquad \text{for } 0 < \delta < \bar{\delta}. \tag{40}$$

Denote $V = \mathrm{Int}(\mathrm{spt}(\omega_c(u)))$ and fix a $\delta \in (0, \bar{\delta})$. We then consider the following cases.

[8] If $\delta_1 < \delta_2$, then $\overline{D_{\delta_2}} \subset \mathrm{Int}(D_{\delta_1})$.

Case 1 $\overline{V} \cap D_\delta = \emptyset$. Since $\mathrm{spt}(\omega_c(u)) = S \cup \bar{V}$, from (40) we get that $S \cap D_\delta = \mathrm{spt}(\omega_c(u)) \cap D_\delta \neq \emptyset$, and so

$$\begin{aligned}|\partial_c v(D_\delta)| \leq |\partial_c u(D_\delta)| &= \omega_c(u)(\mathrm{spt}(\omega_c(u)) \cap D_\delta) \\ &= \omega_c(u)(S \cap D_\delta) \leq |\partial_c v(S \cap D_\delta)| \leq |\partial_c v(D_\delta)|.\end{aligned}$$

Therefore $|\partial_c v(D_\delta \setminus \mathrm{spt}(\omega_c(u))| = 0$. Thus, from (38) there exists F_δ closed, $F_\delta \subset \partial_c v(S \cap D_\delta)$ with $|\partial_c v(S \cap D_\delta) \setminus F_\delta| = 0$. So

$$|\partial_c v(D_\delta)| = |\partial_c v(S \cap D_\delta)| = |F_\delta|. \tag{41}$$

In addition, $F_\delta \subset \partial_c v(\overline{D_\delta})$, so it follows from Lemma 2.14 that F_δ is a compact set. Moreover, $F_\delta \subset \partial_c v(D_\delta) \subset \mathrm{Int}(\partial_c u(D_\delta))$. Therefore, we obtain

$$|F_\delta| < |\partial_c u(D_\delta)|. \tag{42}$$

From (41) and (42) we deduce that $|\partial_c v(D_\delta)| < |\partial_c u(D_\delta)|$ which contradicts (37).

Case 2 $\overline{V} \cap D_\delta \neq \emptyset$.

Since D_δ is open, $V \cap D_\delta \neq \emptyset$. We then decompose the nonempty open set V into the union of its disjoint connected open components $V_1 \cup V_2 \cup \cdots \cup V_k \cup \cdots$ Note that the number of connected components of V is at most countable.

Case 2 A there exists a connected component V_j of V such that $\overline{V_j} \cap D_\delta \neq \emptyset$ and $\overline{V_j} \cap D_\delta^c \neq \emptyset$.

This implies that $V_j \cap D_\delta \neq \emptyset$ and $\overline{V_j} \cap D_\delta^c \neq \emptyset$. But as V_j is connected, then we claim that there exists a connected open component, say O, of the nonempty open set $V_j \cap D_\delta$ such that $\overline{O} \cap \partial D_\delta \neq \emptyset$. To prove this claim we may assume $V_j \cap D_\delta^c \neq \emptyset$, since otherwise $V_j \subset D_\delta$ and so $O = V_j$ does the job. Let $x_1 \in V_j \cap D_\delta$ and $x_2 \in V_j \cap D_\delta^c$. Since V_j is connected, there is a curve $\gamma \subset V_j$ connecting x_1 and x_2. Since $x_1 \in D_\delta$ and $x_2 \in D_\delta^c$, if we go along γ from x_1 to x_2, then γ will hit ∂D_δ for the first time at some point z. Now let γ_1 be the sub curve of γ connecting x_1 and z. Clearly $\gamma_1 \setminus \{z\} \subset V_j \cap D_\delta$ and hence $\gamma_1 \setminus \{z\}$ lies in some connected component O of $V_j \cap D_\delta$ and so $z \in \bar{O}$. The claim is then proved.

Next let

$$\tilde{O} = \{x \in O \mid u \text{ and } v \text{ are differentiable at } x\}.$$

Since u and v are differentiable a.e. on Ω we get $\tilde{O} = O$ a.e. Now if $\partial_c w_\delta(x) \cap \partial_c u(x) \neq \emptyset$ for all $x \in \tilde{O}$, then from Proposition 2.7(2) we have that $x - Dc^*(-Dw_\delta(x)) = x - Dc^*(-Du(x)) \quad \forall x \in \tilde{O}$. Therefore we obtain $Du(x) = Dw_\delta(x)$ for a.e. $x \in O$. It follows that $u - w_\delta$ is constant on O, and hence constant on $\overline{O}$ by continuity. By using the fact that $u = w_\delta$ on ∂D_δ and $\overline{O} \cap \partial D_\delta \neq \emptyset$ we then get $u = w_\delta$ on O. Since O is nonempty and $O \subset D_\delta$, this contradicts the definition of D_δ. Thus we must have $\partial_c w_\delta(x_0) \cap \partial_c u(x_0) = \emptyset$ for some $x_0 \in \tilde{O}$. We first claim that $0 < w_\delta(x_0) - u(x_0) < \max_{\overline{\Omega}}[w_\delta(x) - u(x)] = \max_{\overline{D_\delta}}[w_\delta(x) - u(x)]$. Indeed, we only need to show the second inequality. Suppose by contradiction that it is false, then by letting $p \in \partial_c w_\delta(x_0)$ we have for every $x \in \Omega$

$$\begin{aligned}u(x_0) - c(x-p) + c(x_0 - p) &= w_\delta(x_0) - c(x-p) + c(x_0-p) - [w_\delta(x_0) - u(x_0)] \\ &\leq w_\delta(x) - \max_{\overline{\Omega}}[w_\delta(x') - u(x')] \leq w_\delta(x) - [w_\delta(x) - u(x)] = u(x),\end{aligned}$$

so $p \in \partial_c u(x_0)$, a contradiction. This proves the claim. Now define $w_\delta^*(x) = w_\delta(x) - [w_\delta(x_0) - u(x_0)]$ in $\overline{\Omega}$ and let $G := \{x \in \Omega : w_\delta^*(x) > u(x)\} \subset D_\delta$. Then we get G is nonempty by the claim. Moreover it is clear that G is open, $\overline{G} \subset \Omega$ and $\partial G = \{x \in \Omega : w_\delta^*(x) = u(x)\}$. We also have $w_\delta^*(x_0) = w_\delta(x_0) - [w_\delta(x_0) - u(x_0)] = u(x_0)$. Therefore, $x_0 \in \partial G$ and since $x_0 \in O \subset \mathrm{Int}(\mathrm{spt}(\omega_c(u)))$, applying Lemma 5.4 we obtain that $|\partial_c u(G)| > |\partial_c v(G)|$, a contradiction.

Case 2 B otherwise we have that each $\overline{V_i}$ is either contained in D_δ or contained in D_δ^c. Define two open sets V_δ^* and V_δ^{**} as follows

$$V_\delta^* := \cup_{\overline{V_i} \subset D_\delta} V_i \subset D_\delta \quad \text{and} \quad V_\delta^{**} := \cup_{\overline{V_i} \subset D_\delta^c} V_i \subset D_\delta^c.$$

Then we have $V = V_\delta^* \cup V_\delta^{**}$, $V_\delta^* \cap V_\delta^{**} = \emptyset$, $\overline{V} = \overline{V_\delta^*} \cup \overline{V_\delta^{**}}$. Also $V_\delta^* \neq \emptyset$, $\overline{V_\delta^*} \subset \overline{D_\delta}$ and $\overline{V_\delta^{**}} \subset D_\delta^c$. Let $\tilde{V}_\delta^* = \{x \in V_\delta^* \mid u \text{ and } v \text{ are differentiable at } x\}$. If we can find $x_0 \in \tilde{V}_\delta^*$ such that $\partial_c w_\delta(x_0) \cap \partial_c u(x_0) = \emptyset$, then by arguing as in Case 2 A we obtain a contradiction. Therefore, we can assume that $\partial_c w_\delta(x) \cap \partial_c u(x) \neq \emptyset$ for all $x \in \tilde{V}_\delta^*$. But again as in Case 2 A these yield that $v - u$ is constant on each connected open component of V_δ^*. We claim that $\overline{V_\delta^*} \subset D_\delta$. Indeed, since otherwise there exist $\hat{x} \in \partial D_\delta$ and a sequence $\{x_n\} \subset V_\delta^*$ such that $x_n \to \hat{x}$. Therefore, $v(x_n) - u(x_n) \to v(\hat{x}) - u(\hat{x}) = \delta$. Since $0 < \delta < \bar{\delta}$ and $v(x) - u(x) > \delta$ for all $x \in D_\delta$, we can pick a n_0 large enough such that $\delta < v(x_{n_0}) - u(x_{n_0}) < \bar{\delta}$. Define $\hat{\delta} := v(x_{n_0}) - u(x_{n_0})$, then $\hat{\delta} \in (0, \bar{\delta})$. As $x_{n_0} \in V_\delta^*$, x_{n_0} must belong to some connected open component V_j of V_δ^*. But since $v - u$ is constant on V_j, we then deduce that $v(x) - u(x) = \hat{\delta}$ for all $x \in V_j$. However, this implies that $V_j \subset \partial D_{\hat{\delta}}$, which is impossible because V_j is a nonempty open set. This yields a contradiction and the claim is proved. Now as $\overline{V_\delta^{**}} \subset D_\delta^c$ and by the claim, we have

$$\mathrm{spt}(\omega_c(u)) \cap D_\delta = (\overline{V_\delta^*} \cup \overline{V_\delta^{**}} \cup S) \cap D_\delta = \overline{V_\delta^*} \cup (S \cap D_\delta).$$

So we obtain

$$\begin{aligned} |\partial_c v(D_\delta)| &\leq |\partial_c u(D_\delta)| = \omega_c(u)(\mathrm{spt}(\omega_c(u)) \cap D_\delta) = \omega_c(u)(\overline{V_\delta^*} \cup (S \cap D_\delta)) \\ &\leq |\partial_c v(\overline{V_\delta^*} \cup (S \cap D_\delta))| = |\partial_c v(\overline{V_\delta^*}) \cup \partial_c v(S \cap D_\delta)| \leq |\partial_c v(D_\delta)|, \end{aligned}$$

and $\left|\partial_c v\left(D_\delta \setminus \mathrm{spt}(\omega_c(u))\right)\right| = 0$. Thus from (38) there exists a closed set $F_\delta \subset \partial_c v(S \cap D_\delta)$ with $|\partial_c v(S \cap D_\delta) \setminus F_\delta| = 0$ and therefore

$$|\partial_c v(D_\delta)| = |\partial_c v(\overline{V_\delta^*}) \cup \partial_c v(S \cap D_\delta)| = |\partial_c v(\overline{V_\delta^*}) \cup F_\delta|. \tag{43}$$

Since F_δ is closed, $F_\delta \subset \partial_c v(S \cap D_\delta) \subset \partial_c v(\overline{D_\delta})$, and the last set is bounded by Lemma 2.14, we have that F_δ is compact. Also $\partial_c v(\overline{V_\delta^*})$ is compact. Moreover, from (39), we have $\partial_c v(\overline{V_\delta^*}) \cup F_\delta \subset \partial_c v(D_\delta) \subset \mathrm{Int}(\partial_c u(D_\delta))$. Therefore, we get

$$|\partial_c v(\overline{V_\delta^*}) \cup F_\delta| < |\partial_c u(D_\delta)|. \tag{44}$$

From (43) and (44) we deduce $|\partial_c v(D_\delta)| < |\partial_c u(D_\delta)|$ obtaining a contradiction. The proof is completed.

Remark 5.6 (On condition (38)) We remark that condition (38) holds if

$$S \text{ has no limit points in the interior of } \Omega. \tag{45}$$

Indeed, it is easy to see that (45) holds if and only if for each $D \Subset \Omega$ open, the set $S \cap D$ is closed. Thus, if $\mathrm{spt}(\omega_c(u)) = \bar{V}$ with V open subset of Ω or $\mathrm{spt}(\omega_c(u))$ is convex with nonempty interior or $\mathrm{spt}(\omega_c(u))$ is a finite set, then (38) holds.

6 Dirichlet problem

6.1 Homogeneous Dirichlet problem

Definition 6.1 *A bounded set $\Omega \subset \mathbb{R}^n$ is called strictly convex if for any $z \in \partial\Omega$, there exists a supporting hyperplane H of $\overline{\Omega}$ such that $H \cap \overline{\Omega} = \{z\}$.*

Definition 6.2 *Let $c : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. A bounded set $\Omega \subset \mathbb{R}^n$ is called strictly c-convex if for any $z \in \partial\Omega$, any $\delta > 0$ and any $a > 0$, there exist $y, y^* \in \mathbb{R}^n$ such that*

$$c(x-y)-c(z-y) \geq 0 \quad \forall x \in \partial\Omega, \quad \textit{and} \quad c(x-y)-c(z-y) \geq a \quad \forall x \in \bar{\Omega} \setminus B(z,\delta) \tag{46}$$

and

$$c(z-y^*)-c(x-y^*) \geq 0 \forall x \in \partial\Omega, \quad \textit{and} \quad c(z-y^*)-c(x-y^*) \geq a \ \forall x \in \partial\Omega \setminus B(z,\delta). \tag{47}$$

To illustrate this definition we give several remarks.

Remark 6.3 If $c(x) = \phi(|x|)$ with $\phi : [0,\infty) \to \mathbb{R}$ continuous and strictly increasing, then (46) implies that Ω satisfies the exterior sphere condition, that is, for every $z \in \partial\Omega$, there exists an open ball B satisfying $\bar{B} \cap \bar{\Omega} = \{z\}$.

Proof Let $z \in \partial\Omega$. We claim that we can find a $y \in \mathbb{R}^n \setminus \bar{\Omega}$ such that

$$|x-y| \geq |z-y|, \quad \text{for all } x \in \bar{\Omega}. \tag{48}$$

If we assume the claim for a moment, then we see that $B(y,|z-y|) \cap \bar{\Omega} = \emptyset$ and $z \in \overline{B}(y,|z-y|) \cap \bar{\Omega}$. Therefore, if we let $\bar{y}$ be the midpoint of the segment $\overline{yz}$ and B be the open ball centered at $\bar{y}$ and with radius $|z-y|/2$, then it is clear that $\bar{B} \cap \bar{\Omega} = \{z\}$ as desired.

It remains to prove the claim. Let a be such that $a > \max_{x,\tilde{y}\in\bar{\Omega}} [c(x-\tilde{y}) - c(z-\tilde{y})]$ and let $\delta = \mathrm{diam}(\Omega)/2$. Then by (46) there exists $y \in \mathbb{R}^n$ such that

$$c(x-y) - c(z-y) \geq 0, \quad \text{for all } x \in \partial\Omega, \tag{49}$$

and

$$c(x-y) - c(z-y) \geq a, \quad \text{for all } x \in \bar{\Omega} \setminus B\left(z, \frac{\mathrm{diam}(\Omega)}{2}\right). \tag{50}$$

From (50) and the choice of a, we must have $y \notin \bar{\Omega}$. Then if $x \in \Omega$, let $\bar{x}$ be a point on the segment $\overline{xy}$ and on $\partial\Omega$. From the form of c and (49) we have, $c(x-y) \geq c(\bar{x}-y) \geq c(z-y)$ and we are done. □

Remark 6.4 If $c(x) = \phi(|x|)$ is convex with $\phi : [0,\infty) \to \mathbb{R}$ continuous and strictly increasing, and the open set Ω verifies the first inequality in (47), then Ω satisfies the enclosing sphere condition, that is, for each $z \in \partial\Omega$ there exists a ball $B_R \supset \Omega$ with $z \in \partial B_R$.

Proof We first see that the first inequality in (47) holds in Ω. Because if $x \in \Omega$, then there exist two points $x_1, x_2 \in \partial\Omega$ such that $x = tx_1 + (1-t)x_2$ for some $0 < t < 1$. Hence

$$\begin{aligned} c(x - y^*) &= c(t(x_1 - y^*) + (1-t)(x_2 - y^*)) \\ &\le tc(x_1 - y^*) + (1-t)c(x_2 - y^*) \\ &\le tc(z - y^*) + (1-t)c(z - y^*) = c(z - y^*). \end{aligned}$$

Therefore

$$\begin{aligned} \Omega \subset \{x : \phi(|x - y^*|) &\le \phi(|z - y^*|)\} \\ &= \{x : |x - y^*| \le |z - y^*|\} \\ &= B_{|z-y^*|}(y^*), \end{aligned}$$

that is, Ω satisfies the enclosing sphere condition and in particular, Ω is strictly convex. □

Remark 6.5 If $c : \mathbb{R}^n \to \mathbb{R}$ is convex and $\lim_{|x|\to\infty} \dfrac{c(x)}{|x|} = +\infty$, and Ω is strictly convex, then (46) holds.

Proof Let $z \in \partial\Omega$, $\delta > 0$, $a > 0$, and $P(x) = 0$ be the equation of the supporting hyperplane to Ω at z. We can assume $\bar{\Omega} \subset \{x : P(x) \ge 0\}$. Since Ω is strictly convex, there exists $\eta > 0$ such that $\{x \in \bar{\Omega} : P(x) \le \eta\} \subset B(z, \delta)$. That is, $P(x) \ge \eta$ for all $x \in \bar{\Omega} \setminus B(x, \delta)$. We can write $P(x) = A \cdot (x - z)$ with $A \in \mathbb{R}^n$. Since $\partial c(\mathbb{R}^n) = \mathbb{R}^n$ (∂c means the standard subdifferential of c), we get that $\frac{a}{\eta}A \in \partial c(w)$ for some $w \in \mathbb{R}^n$. If $y = z - w$, then $\frac{a}{\eta}A \in \partial c(z - y)$ and hence

$$c(x - y) - c(z - y) \ge \frac{a}{\eta}A \cdot (x - z) = \frac{a}{\eta}P(x) \ge 0$$

for all $x \in \bar{\Omega}$, and

$$c(x - y) - c(z - y) \ge \frac{a}{\eta}A \cdot (x - z) = \frac{a}{\eta}P(x) \ge \frac{a}{\eta}\eta = a$$

for all $x \in \bar{\Omega} \setminus B(z, \delta)$. □

Remark 6.6 Let $c : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $c(x) = \phi(|x|)$ for some nondecreasing function $\phi : [0, \infty) \to \mathbb{R}$ satisfying $\phi \in C^1(m, \infty)$ for some $m \ge 0$ and $\lim_{t\to+\infty} \phi'(t) = +\infty$. If $\Omega \subset \mathbb{R}^n$ is a bounded open set satisfying the enclosing sphere condition, then Ω is strictly c-convex.

Proof In view of Remark 6.5, we only need to verify (47). We may assume $m = 0$ because we will see below that we can pick y^* as far as we want from $\overline{\Omega}$. Let $z \in \partial\Omega$, $\delta > 0$, $a > 0$, and $B_R(z_0)$ be an enclosing ball for z. If H is the supporting hyperplane to the ball $B_R(z_0)$ at z, then $H \cap \overline{\Omega} = \{z\}$. Let H^- denote the halfspace containing Ω and let H^+ denote the complementary halfspace. Consider the line passing through z and orthogonal to H. Let L and L' be the rays starting from z and lying in H^+ and H^- respectively. We have that $z_0 \in L'$.

Let $L'_{z_0} = \{y \in L' : |y - z| \ge R\}$. If $y \in L'_{z_0}$, then $\Omega \subset B(y, |y - z|)$ and hence $c(z - y) - c(x - y) \ge 0$ for all $x \in \partial\Omega$. Therefore, (47) will be proved if we show that

there exists $y^* \in L'_{z_0}$ such that $c(z-y^*) - c(x-y^*) \geq a$ for all $x \in \partial\Omega \setminus B(z,\delta)$. From the strict convexity of Ω, there exists $\beta > 0$ such that

$$\frac{x-y}{|x-y|} \cdot \frac{z-x}{|z-x|} \geq \beta, \quad \forall x \in \partial\Omega \setminus B(z,\delta), \quad \text{and} \quad \forall y \in L'_{z_0} \text{ with } |y-z| \text{ large.}$$

Again from the convexity of c we then get

$$\begin{aligned} c(z-y) - c(x-y) \geq Dc(x-y)\cdot(z-x) &= \phi'(|x-y|)\,|z-x|\,\frac{x-y}{|x-y|}\cdot\frac{z-x}{|z-x|} \\ &\geq \phi'(|x-y|)\,\beta\,\delta, \end{aligned}$$

for all $x \in \partial\Omega \setminus B(z,\delta)$ and all $y \in L'_{z_0}$ with $|y-z|$ large. Since $\lim_{t\to+\infty}\phi'(t) = +\infty$, (47) follows picking $y = y^* \in L'_{z_0}$ sufficiently far from z.

This completes the proof. □

Under the assumption that the domain is strictly c-convex we solve the homogeneous Dirichlet problem as follows.

Theorem 6.7 *Suppose that c satisfies condition $(H1)$ and $\lim_{|x|\to+\infty}\frac{c(x)}{|x|} = +\infty$. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set and $\psi : \partial\Omega \to \mathbb{R}$ be a continuous function. Then there exists a unique c-convex function $u \in C(\bar{\Omega})$ Aleksandrov generalized solution to the problem*

$$\begin{aligned} \det[I + D^2c^*(-Du(x))D^2u(x)] &= 0 \quad \text{in } \Omega, u \\ &= \psi \quad \text{on } \partial\Omega. \end{aligned}$$

Proof Define

$$\mathcal{F} := \{f(x) = -c(x-y) - \lambda : \ y \in \mathbb{R}^n, \lambda \in \mathbb{R} \text{ and } f(x) \leq \psi(x) \text{ on } \partial\Omega\}.$$

Then since ψ is continuous on $\partial\Omega$ we have that $\mathcal{F}$ is nonempty. Let

$$u(x) = \sup\{f(x) : f \in \mathcal{F}\}. \tag{51}$$

Claim 1 $u = \psi$ on $\partial\Omega$.

It is clear from the definition of u that $u \leq \psi$ on $\partial\Omega$. Now let $z \in \partial\Omega$, and $\epsilon > 0$. Then we can find $\delta > 0$ such that $|\psi(x) - \psi(z)| < \epsilon$ for all $x \in B(z,\delta) \cap \partial\Omega$. Choose $a = \psi(z) - \epsilon - m$ where $m = \min\{\psi(x) | x \in \partial\Omega \setminus B(z,\delta)\}$. Since Ω is c-strictly convex, there exists $y \in \mathbb{R}^n$ such that $c(x-y) - c(z-y) \geq 0 \forall x \in \partial\Omega$ and $c(x-y) - c(z-y) \geq a \forall x \in \partial\Omega \setminus B(z,\delta)$.

Let $f(x) := -[c(x-y) - c(z-y)] + \psi(z) - \epsilon$. We claim that $f \leq \psi$ on $\partial\Omega$. Indeed, if $x \in B(z,\delta) \cap \partial\Omega$, then $f(x) = -[c(x-y) - c(z-y)] + \psi(z) - \epsilon \leq \psi(z) - \epsilon \leq \psi(x)$. On the other hand, if $x \in \partial\Omega \setminus B(z,\delta)$, then we have $c(x-y) - c(z-y) \geq a$ and hence

$$f(x) = -[c(x-y) - c(z-y)] + \psi(z) - \epsilon \leq -[\psi(z) - \epsilon - m] + \psi(z) - \epsilon = m \leq \psi(x).$$

Therefore $f \in \mathcal{F}$. Thus, $u(z) \geq f(z) = \psi(z) - \epsilon$ for all $\epsilon > 0$. Hence, $u(z) \geq \psi(z)$ and this proves Claim 1.

Claim 2 u is c-convex and $u \in C(\Omega)$.

From the definition it is clear that u is uniformly bounded from below on $\bar{\Omega}$. Now let $g(x) := -c(x) + \max_{\partial\Omega}[\psi + c]$. It is clear that $g \geq \psi$ on $\partial\Omega$, g is c-convex and as $c \in C^1(\mathbb{R}^n)$ we have $\partial_c g(\Omega) = \{0\}$ and so $|\partial_c g(\Omega)| = 0$. Hence for each $f(x) = -c(x-y) - \lambda \in \mathcal{F}$, it follows from the comparison principle Theorem 5.5 that $f \leq g$ in $\overline{\Omega}$ and therefore u is uniformly bounded from above on $\bar{\Omega}$. Thus, we get u is uniformly bounded on $\bar{\Omega}$. Particularly, this implies that u is c-convex and moreover from Remark 2.6 we obtain $u \in C(\Omega)$.

Claim 3 u is continuous up to the boundary.

Let $z \in \partial\Omega$ and $\{x_n\} \subset \Omega$ be a sequence such that $x_n \to z$. For any $\epsilon > 0$, we can find $\delta > 0$ such that $|\psi(x) - \psi(z)| < \epsilon$ for all $x \in B(z,\delta) \cap \partial\Omega$. Choose $b = M - \psi(z) - \epsilon$ where $M = \max\{\psi(x) | x \in \partial\Omega \setminus B(z,\delta)\}$. Since Ω is c-strictly convex, there exists $y^* \in \mathbb{R}^n$ such that $c(z-y^*) - c(x-y^*) \geq 0 \quad \forall x \in \partial\Omega$ and $c(z-y^*) - c(x-y^*) \geq b \quad \forall x \in \partial\Omega \setminus B(z,\delta)$.

Let $h(x) := c(z-y^*) - c(x-y^*) + \psi(z) + \epsilon$. For $x \in B(z,\delta) \cap \partial\Omega$ we have $h(x) \geq \psi(z) + \epsilon > \psi(x)$, and for $x \in \partial\Omega \setminus B(z,\delta)$ we have $h(x) \geq b + \psi(z) + \epsilon \geq \psi(x)$. Therefore, we get $h(x) \geq \psi(x)$ on $\partial\Omega$. Moreover, h is c-convex, $\partial_c h(\Omega) = \{y^*\}$, so $|\partial_c h(\Omega)| = 0$. Thus for any $f(x) = -c(x-y) - \lambda \in \mathcal{F}$, as in Claim 2 we obtain $f \leq h$ in $\overline{\Omega}$. Hence $u(x) \leq h(x) = c(z-y^*) - c(x-y^*) + \psi(z) + \epsilon$ in $\overline{\Omega}$. This yields $\limsup u(x_n) \leq \psi(z) + \epsilon$ and hence $\limsup u(x_n) \leq \psi(z)$ since $\epsilon > 0$ was chosen arbitrary. On the other hand, for any $\epsilon > 0$, by constructing a function $f \in \mathcal{F}$ as in Claim 1 we get $u(x_n) \geq f(x_n) = -[c(x_n - y) - c(z-y)] + \psi(z) - \epsilon$. So $\liminf u(x_n) \geq \psi(z) - \epsilon$ for any $\epsilon > 0$ and hence $\liminf u(x_n) \geq \psi(z)$. Thus $\lim_{n\to+\infty} u(x_n) = \psi(z) = u(z)$ and we obtain $u \in C(\bar{\Omega})$.

Claim 4 $|\partial_c u(\Omega)| = 0$.

Let $p \in \partial_c u(\Omega)$. Then there exists $x_0 \in \Omega$ such that

$$u(x) \geq u(x_0) - c(x-p) + c(x_0 - p) \quad \forall x \in \Omega.$$

Therefore if we let $f(x) := u(x_0) - c(x-p) + c(x_0-p)$, then since $u(x) = \psi(x)$ on $\partial\Omega$ we get $f(x) \leq \psi(x)$ on $\partial\Omega$. We now claim that in fact there is $\zeta \in \partial\Omega$ satisfying $f(\zeta) = \psi(\zeta)$. Indeed, since otherwise there exists $\epsilon > 0$ such that $f + \epsilon \leq \psi$ on $\partial\Omega$. Then the function $f(x) + \epsilon \in \mathcal{F}$ and hence $u(x) \geq f(x) + \epsilon$ for all $x \in \Omega$. In particular, $u(x_0) \geq f(x_0) + \epsilon = u(x_0) + \epsilon$. This is a contradiction. So $f(\zeta) = \psi(\zeta) = u(\zeta)$ for some $\zeta \in \partial\Omega$. But then we get

$$\begin{aligned} u(x) &\geq u(x_0) - c(x-p) + c(x_0-p) \\ &= u(x_0) - c(\zeta - p) + c(x_0 - p) - c(x-p) + c(\zeta - p) \\ &= f(\zeta) - c(x-p) + c(\zeta - p) = u(\zeta) - c(x-p) + c(\zeta - p) \quad \forall x \in \Omega. \end{aligned}$$

So $p \in \partial_c(u, \overline{\Omega})(x_0) \cap \partial_c(u, \overline{\Omega})(\zeta)$, i.e., $p \in \tilde{S}$ where $\tilde{S}$ is defined as in Lemma 2.11. That is $\partial_c u(\Omega) \subset \tilde{S}$ and the claim follows from Lemma 2.11.

Thus we have shown the existence of a generalized c-convex solution. The uniqueness follows from the comparison principle Theorem 5.5 and this completes the proof.

□

Remark 6.8 In case $c(x) = \frac{1}{p}|x|^p$, with $1 < p < \infty$, then the conclusions of Theorem 6.7 and Lemma 6.10 hold when Ω satisfies only condition (46), in particular, this is

satisfied when Ω is strictly convex by Remark 6.5. And consequently, for power cost functions Theorems 6.11 and 6.12 and Corollary 6.13 below are also true when the domain Ω satisfies only condition (46).

Proof We notice that condition (47) is only used in Claim 3 to prove that for any $z \in \partial\Omega$ and any $\epsilon > 0$ we have $\limsup_{x \to z, x \in \Omega} u(x) \leq \psi(z) + \epsilon$. Therefore the remark will be proved if we establish this using only condition (46). In fact, from Remark 6.3, Ω satisfies the exterior sphere condition and thus Ω is q-regular with q the conjugate of p. From [3, Theorem 4.7] there exists $w \in W^{1,q}(\Omega) \cap C(\bar{\Omega})$ weak solution to the q-Laplacian

$$-\text{div}\left(|Dw(x)|^{q-2}Dw(x)\right) + n = 0, \quad \text{in } \Omega \quad \text{and} \quad w = -\psi \text{ on } \partial\Omega.$$

Notice that $\text{div}\left(|Dw(x)|^{q-2}Dw(x)\right) = \text{div}\left(Dc^*\left(Dw(x)\right)\right)$. For each $f(x) = -c(x - y) - \lambda \in \mathcal{F}$, we have $-f(x) \geq -\psi(x)$ on $\partial\Omega$, and $-\text{div}\left(Dc^*\left(-Df(x)\right)\right) + n = 0$. Hence by the comparison principle [3, Theorem 3.1] for the q-Laplacian we get that $-f \geq w$ in $\bar{\Omega}$, and therefore $u(x) = \sup_{f \in \mathcal{F}} f(x) \leq -w(x)$ for all $x \in \bar{\Omega}$. □

Remark 6.9 When ψ is a constant function, the proof above shows that Theorem 6.7 holds when Ω satisfies a condition weaker than c-strictly convex, namely: for any $z \in \partial\Omega$, there exist $y, y^* \in \mathbb{R}^n$ such that $c(x-y) - c(z-y) \geq 0$ and $c(z-y^*) - c(x-y^*) \geq 0$ for all x on $\partial\Omega$.

6.2 Nonhomogeneous Dirichlet problem

Throughout this section we assume that $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and is positive a.e. We begin with the following lemma.

Lemma 6.10 *Suppose that c satisfies condition $(H1)$, and $\lim_{|x| \to +\infty} \dfrac{c(x)}{|x|} = +\infty$. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, and $\psi : \partial\Omega \to \mathbb{R}$ be a continuous function. Then for any set $E \Subset \Omega$ and for any number $\alpha \in \mathbb{R}$, there exists a c-convex function $u \in C(\overline{\Omega})$ satisfying $u = \psi$ on $\partial\Omega$ and $u \leq \alpha$ on E.*

Proof Define

$$\mathcal{F} = \{f(x) = -c(x - y) - \lambda : \ y \in \mathbb{R}^n, \lambda \in \mathbb{R}, \ f \leq \psi \text{ on } \partial\Omega \text{ and } f \leq \alpha \text{ on } E\}.$$

Since ψ is continuous on $\partial\Omega$ we have that $\mathcal{F}$ is nonempty. Let

$$u(x) = \sup_{f \in \mathcal{F}} f(x) \text{ for all } x \in \overline{\Omega}.$$

The proof now follows as in Theorem 6.7 with some obvious modifications in Claim 1. □

Theorem 6.11 *Suppose that c satisfies condition $(H1)$, $\lim_{|x| \to \infty} \dfrac{c(x)}{|x|} = +\infty$, and $c(0) = \min_{x \in \mathbb{R}^n} c(x)$. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, $\psi \in C(\partial\Omega)$, distinct points $x_1, \ldots, x_N \in \Omega$, and $a_1, \ldots, a_N$ positive numbers. If*

$$a_1 + \cdots + a_N < \int_{\mathbb{R}^n} g(y)\, \mathrm{d}y, \tag{52}$$

then there exists a unique function $u \in C(\overline{\Omega})$, c-convex solution to the problem

$$\begin{aligned} \omega_c(g,u) &= \textstyle\sum_{i=1}^N a_i\,\delta_{x_i} \quad \text{in } \Omega, \\ u &= \psi \quad \text{on } \partial\Omega. \end{aligned} \tag{53}$$

Proof Let

$$\begin{aligned} \mathcal{H} = \big\{ & v \in C(\overline{\Omega}) : v \text{ is } c\text{-convex in } \Omega,\ v|_{\partial\Omega} = \psi, \omega_c(g,v)(\Omega) \\ & = \sum_{i=1}^N \omega_c(g,v)(\{x_i\}), \quad \text{and} \quad \int_{\partial_c v(x_i)} g(y)\,\mathrm{d}y \le a_i \text{ for } i = 1,\dots,N \big\}. \end{aligned}$$

From Theorem 6.7, let W be the solution to $\omega_c(g,W) = 0$ and $W = \psi$ on $\partial\Omega$. We have $W \in \mathcal{H}$, and it follows from definition (51) that

$$v \le W, \quad \text{for each } v \in \mathcal{H}. \tag{54}$$

For each $v \in \mathcal{H}$ define

$$V[v] = \int_\Omega [W(x) - v(x)]\,\mathrm{d}x \ge 0,$$

and let

$$\beta = \sup_{v \in \mathcal{H}} V[v].$$

We shall prove that there exists $u \in \mathcal{H}$ such that $\beta = V[u]$ and u is the desired solution to the problem (53).

Claim 1 There exists $L \in \mathbb{R}$ such that

$$L \le v \text{ in } \Omega \text{ and for all } v \in \mathcal{H}. \tag{55}$$

Proof of claim 1 Applying Theorem 4.4 to $-v$ we get

$$\max_\Omega(-v) \le \max_{\partial\Omega}(-v) + h^{-1}(\omega_c(g,v)(\Omega)),$$

where the function h is given in (33). Since $\omega_c(g,v)(\Omega) \le a_1 + \cdots + a_N < B(g)$, and h^{-1} is increasing, we obtain that

$$\min_\Omega v \ge L := \min_{\partial\Omega} \psi - h^{-1}(a_1 + \cdots + a_N) > -\infty.$$

Claim 2 There exists a c-convex function $w \in C(\bar{\Omega})$ with $w = \psi$ on $\partial\Omega$ and

$$w(x) \le v(x), \quad \text{in } \bar{\Omega} \text{ and for all } v \in \mathcal{H}.$$

Proof of Claim 2 Using Lemma 6.10 we can construct a c-convex function $w \in C(\bar{\Omega})$ such that $w = \psi$ on $\partial\Omega$, and $w \le L$ on $\{x_1,\dots,x_N\}$. Next let $v \in \mathcal{H}$ and define

$$\mathcal{G}_v = \{h \in C(\overline{\Omega}) : h \text{ is } c\text{-convex in } \Omega,\ h|_{\partial\Omega} \le \psi,\ h(x_i) \le v(x_i) \text{ for } i = 1,\dots,N\},$$

and

$$\tilde{v}(x) = \sup_{h \in \mathcal{G}_v} h(x), \quad x \in \bar{\Omega}.$$

Notice that $v \in \mathcal{G}_v$, and so $v \le \tilde{v}$. Also $v(x_i) = \tilde{v}(x_i)$ for $1 \le i \le N$. We claim that $v = \tilde{v}$. We have $\tilde{v} = v$ on $\partial\Omega$. So if we prove that $\omega_c(g, v) \le \omega_c(g, \tilde{v})$, the claim will follow from the comparison principle Theorem 5.5. Since the measure $\omega_c(g, v)$ is concentrated on $\{x_1, \ldots, x_N\}$, it is then enough to show that

$$\partial_c v(x_i) \subset \partial_c \tilde{v}(x_i), \quad i = 1, \ldots, N.$$

If $p \in \partial_c v(x_i)$, then $v(x) \ge v(x_i) - c(x-p) + c(x_i - p) =: h(x)$ for all $x \in \bar{\Omega}$. But $h \in \mathcal{G}_v$ so $h \le \tilde{v}$ in $\bar{\Omega}$. Since $h(x_i) = v(x_i) = \tilde{v}(x_i)$, we then get $p \in \partial_c \tilde{v}(x_i)$ as desired. We now notice that from Claim 1, we have that $w(x_i) \le L \le v(x_i)$ for $1 \le i \le N$ and so $w \in \mathcal{G}_v$, and consequently $w \le v$. This completes the proof of Claim 2.

Recall that $\beta = \sup_{v \in \mathcal{H}} V[v]$, and from Claim 2, $\beta \le V[w] < \infty$. Then there exists a sequence $\{u_n\} \subset \mathcal{H}$ such that $V[u_n] \uparrow \beta$ as $n \to \infty$. From (54) and Claim 2 we have that

$$w(x) \le u_n(x) \le W(x), \quad \forall x \in \bar{\Omega}. \tag{56}$$

Claim 2A There is a subsequence $\{u_{n_k}\}$ and $u \in C(\bar{\Omega})$ with $u = \psi$ on $\partial\Omega$ such that $u_{n_k} \to u$ locally uniformly in Ω as $k \to \infty$. We denote this subsequence u_k.

From (56), $\{u_n\}$ is uniformly bounded in $\bar{\Omega}$. Since u_n is c-convex in Ω, from Remark 2.3 we know that given $K \subset \Omega$ compact, u_n is Lipschitz in K, say with constant $C(K, n)$. We claim that $C(K, n)$ is uniformly bounded in n. Indeed, from the hypotheses on c we have that c satisfies condition $(H3)$ and therefore by Lemma 2.14 there exists $R > 0$ such that $\partial_c u_n(K) \subset B(0, R)$ for all $n = 1, 2, \ldots$. Choose the ball B which is large enough such that $z - p \in B$ for all $z \in K$ and $p \in B(0, R)$. Then for any $x, y \in K$, by choosing $p \in \partial_c u_n(y)$ and since c is convex on $\mathbb{R}^n$ we have

$$u_n(x) - u_n(y) \ge -c(x-p) + c(y-p) \ge -\|c\|_{\mathrm{Lip}(B)}|x-y|.$$

Similarly, we also have $u_n(y) - u_n(x) \ge -\|c\|_{\mathrm{Lip}(B)}|x-y|$. Thus $|u_n(x) - u_n(y)| \le \|c\|_{\mathrm{Lip}(B)}|x-y|$ for all $x, y \in K$, that is, $C(K, n) \le \|c\|_{\mathrm{Lip}(B)}$ for all n. This proves the claim. Therefore $\{u_n\}$ are equicontinuous on K and uniformly bounded in Ω. From (56) and since $w = W = \psi$ on $\partial\Omega$, by Arzela-Ascoli's lemma there exists a subsequence $\{u_{n_k}\}$ converging uniformly on compact subsets of Ω to a function $u \in C(\overline{\Omega})$ satisfying $u = \psi$ on $\partial\Omega$. Also u is c-convex by Lemma 2.10. This completes the proof of Claim 2A.

Claim 3 $u \in \mathcal{H}$, and $V[u] = \sup_{v \in \mathcal{H}} V[v]$.

It is enough to show $u \in \mathcal{H}$. We first see that $w_c(g, u_k) \to w_c(g, u)$ weakly in Ω. To prove this we use Lemma 3.3 and so we only have to check that condition (18) holds. Indeed, if $z_{k_j} \to z_0 \in \partial\Omega$, then from (56) we have

$$\begin{aligned} \limsup u_{k_j}(z_{k_j}) &\ge \limsup w(z_{k_j}) = w(z_0) = \psi(z_0) \\ &= \lim W(z_{k_j}) = \liminf W(z_{k_j}) \ge \liminf u(z_{k_j}). \end{aligned}$$

Second, since the measures $w_c(g, u_k)$ are concentrated on $\{x_1, \ldots, x_N\}$, it follows from Lemma 3.3 that

$$\int_{\partial_c u_k(x_i)} g(y)\, \mathrm{d}y = w_c(g, u_k)(\{x_i\}) \to w_c(g, u)(\{x_i\}) = \int_{\partial_c u(x_i)} g(y)\, \mathrm{d}y,$$

as $k \to \infty$ for $1 \le i \le N$. This implies that $\int_{\partial_c u(x_i)} g(y)\,\mathrm{d}y \le a_i$ for $1 \le i \le N$. Also the measure $\omega_c(g,u)$ is concentrated on $\{x_1,\dots,x_N\}$ since it is the limit of measures concentrated on that set. Thus, we get Claim 3.

Claim 4 The function u solves the nonhomogeneous Dirichlet problem (53).

It is enough to show that $\int_{\partial_c u(x_i)} g(y)\,\mathrm{d}y = a_i$ for $1 \le i \le N$. Suppose by contradiction this is not true. Then there exists $1 \le i_0 \le N$ such that $\int_{\partial_c u(x_{i_0})} g(y)\,\mathrm{d}y < a_{i_0}$. By relabelling the indices we may assume that there exists a number $1 \le \ell \le N$ such that

$$\int\limits_{\partial_c u(x_i)} g(y)\,\mathrm{d}y < a_i, \quad \text{for } 1 \le i \le \ell \quad \text{and} \quad \int\limits_{\partial_c u(x_i)} g(y)\,\mathrm{d}y = a_i, \quad \text{for } \ell+1 \le i \le N.$$

Given $n \in \mathbb{N}$ define

$$G_n = \{v \in C(\overline{\Omega}) : v \text{ is } c\text{-convex in } \Omega,\ v|_{\partial\Omega} \le \psi,\ v(x_k) \le u(x_k) - \frac{1}{n} \text{ for } 1 \le k \le \ell \\ \text{and } v(x_m) \le u(x_m) \text{ for } \ell+1 \le m \le N\}.$$

Since $u - \dfrac{1}{n} \in G_n$, we have $G_n \ne \emptyset$. Define $w_n(x) = \sup_{v \in G_n} v(x)$ for $x \in \bar{\Omega}$. We have

$$w_n(x_k) = u(x_k) - \frac{1}{n}, \text{ for } 1 \le k \le \ell \text{ and } w_n(x_m) \le u(x_m), \text{ for } \ell+1 \le m \le N.$$

We now claim that

$$\exists n_0 \text{ such that } \forall n \ge n_0,\ w_n(x_m) = u(x_m) \quad \text{for } \ell+1 \le m \le N. \tag{57}$$

Indeed, by definition $\int_{\partial_c u(x_m)} g(y)\,\mathrm{d}y = a_m > 0$ for $\ell+1 \le m \le N$, and so $|\partial_c u(x_m)| > 0$ for $\ell+1 \le m \le N$. Hence there exists $p_m \in \partial_c u(x_m)$ such that $f_m(x_k) < u(x_k)$ for $1 \le k \le \ell$, where $f_m(x) = u(x_m) - c(x-p_m) + c(x_m - p_m)$. Because if on the contrary for each $p \in \partial_c u(x_m)$ there exists x_k for some $1 \le k \le \ell$ and with $f_m(x_k) = u(x_k)$, then $p \in \partial_c u(x_k)$ and so

$$\partial_c u(x_m) \subset \{p \in \mathbb{R}^n : p \in \partial_c u(x) \cap \partial_c u(y) \text{ for some } x, y \in \Omega,\ x \ne y\}.$$

Hence by Corollary 2.12, it follows that $|\partial_c u(x_m)| = 0$, a contradiction. Therefore there exists n_0 such that

$$f_m(x_k) \le u(x_k) - \frac{1}{n_0}$$

for all $\ell+1 \le m \le N$ and $1 \le k \le \ell$, and so the last inequality holds for all $n \ge n_0$. Hence $f_m \in G_n$ for $\ell+1 \le m \le N$ and for all $n \ge n_0$, and consequently, $w_n(x_m) \ge f_m(x_m) = u(x_m)$ and (57) is proved.

Claim 4A $w_n \in \mathcal{H}$ for sufficiently large n.

Notice that constructing a sufficiently negative c-convex function $w(x)$ whose values are ψ on $\partial\Omega$ as in Claim 2, we have that $w \in G_n$ and so $w_n \in C(\bar{\Omega})$ with $w_n = \psi$ on $\partial\Omega$.

We first show that the measures $\omega_c(g, w_n)$ are concentrated on $\{x_1,\dots x_N\}$. Let $B \subset \Omega$ be an open ball such that $B \cap \{x_1,\dots x_N\} = \emptyset$. We claim that $|\partial_c w_n(B)| = 0$. Let $p \in \partial_c w_n(B)$. Then there exists $z \in B$ such that $w_n(x) \ge w_n(z) - c(x-p) + c(z-p) =:$

$f(x)$ for all $x \in \bar{\Omega}$. We claim that there exists $y \in \bar{\Omega} \setminus B$ such that $f(y) = w_n(y)$. Suppose by contradiction this is not true, so $f(x) < w_n(x)$ for all $x \in \bar{\Omega} \setminus B$. Then $f(x) + \epsilon < w_n(x)$ for all $x \in \bar{\Omega} \setminus B$ for some $\epsilon > 0$ sufficiently small. Notice that $f(x) + \epsilon$ is c-convex in Ω and therefore $f + \epsilon \in G_n$. Hence $f(z) + \epsilon \leq w_n(z) = f(z)$, a contradiction. It then follows from the claim that $p \in \partial_c(w_n, \bar{\Omega})(y)$ and then

$$\partial_c w_n(B) \subset \{p \in \mathbb{R}^n : p \in \partial_c(w_n, \bar{\Omega})(x) \cap \partial_c(w_n, \bar{\Omega})(y) \text{ for some } x, y \in \bar{\Omega},\ x \neq y\}.$$

From Lemma 2.11 we then conclude that $|\partial_c w_n(B)| = 0$.

We have

$$u(x) - \frac{1}{n} \leq w_n(x) \leq u(x), \qquad \text{for } x \in \bar{\Omega} \tag{58}$$

where the first inequality holds because $u - \frac{1}{n} \in G_n$ and the second holds because $G_n \subset \mathcal{G}_u$. Consequently, $-\frac{1}{n} \leq w_n(x) - u(x) \leq 0$ in $\bar{\Omega}$ so $w_n \to u$ uniformly in $\bar{\Omega}$ and therefore $w_c(g, w_n) \to w_c(g, u)$ weakly, and since $w_c(g, w_n)$ are concentrated on $\{x_1, \ldots, x_N\}$ we get from Lemma 3.3 that

$$\int_{\partial_c w_n(x_k)} g(y)\, \mathrm{d}y \to \int_{\partial_c u(x_k)} g(y)\, \mathrm{d}y < a_k$$

for $1 \leq k \leq \ell$, and so there exists $n_1 \geq n_0$ such that for all $n \geq n_1$ we get $\int_{\partial_c w_n(x_k)} g(y)\, \mathrm{d}y < a_k$ for $1 \leq k \leq \ell$. Now let $n \geq n_1$, we shall show that $w_n \in \mathcal{H}$. Indeed, from (57) and (58) we have that $\partial_c w_n(x_m) \subset \partial_c u(x_m)$, for $\ell + 1 \leq m \leq N$, and so

$$\int_{\partial_c w_n(x_m)} g(y)\, \mathrm{d}y \leq \int_{\partial_c u(x_m)} g(y)\, \mathrm{d}y = a_m, \quad \text{for } \ell + 1 \leq m \leq N.$$

Thus, Claim 4A is proved.

Finally, from (58) and since $u(x_k) - w_{n_1}(x_k) = \frac{1}{n_1}$ for $1 \leq k \leq \ell$ we then get that $V[w_{n_1}] = \int_\Omega [W(x) - w_{n_1}(x)]\, \mathrm{d}x = V[u] + \int_\Omega [u(x) - w_{n_1}(x)]\, \mathrm{d}x > V[u]$, a contradiction by Claim 4A. This completes the proof of Claim 4 and hence the theorem is proved since the uniqueness follows from Theorem 5.5. □

6.3 Nonhomogeneous Dirichlet problem with general right hand side

Theorem 6.12 *Suppose that c satisfies condition (H1), $\lim_{|x| \to +\infty} \frac{c(x)}{|x|} = +\infty$, $c(0) = \min_{x \in \mathbb{R}^n} c(x)$, and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is positive a.e. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, and $\psi \in C(\partial\Omega)$. Suppose that μ is a Borel measure in Ω satisfying $spt(\mu) \subset \Omega$ and*

$$\mu(\Omega) < \int_{\mathbb{R}^n} g(y)\, \mathrm{d}y. \tag{59}$$

Then there exists $u \in C(\overline{\Omega})$ that is a c-convex weak solution to the problem $\omega_c(g, u) = \mu$ in Ω and $u = \psi$ on $\partial\Omega$. Moreover, the solution is unique if in addition μ satisfies

$$\text{the set } S := spt(\mu) \setminus \overline{\mathrm{Int}(spt(\mu))} \text{ has no limit points in the interior of } \Omega. \tag{60}$$

Proof First fix a subdomain Ω' of Ω such that $\mathrm{spt}(\mu) \subset \Omega' \Subset \Omega$. By the assumptions we can select a sequence of measures $\{\mu_j\}$ satisfying $\mu_j \rightharpoonup \mu$ weakly in Ω, each μ_j is a finite combination of delta masses with $\mathrm{spt}(\mu_j) \subset \Omega'$ and $\{\mu_j(\Omega)\}$ is uniformly bounded by a positive constant A which is strictly less than $\int_{\mathbb{R}^n} g(y)\,\mathrm{d}y$. Hence for each j, by Theorem 6.11 there exists a unique c-convex weak solution $u_j \in C(\overline{\Omega})$ to the problem $\omega_c(g, u_j) = \mu_j$ in Ω and $u_j = \psi$ on $\partial\Omega$. Then by using the maximum principle Theorem 4.4, we have

$$\begin{aligned} -\min_{\Omega} u_j &= \max_{\Omega}(-u_j) \le \max_{\partial\Omega}(-u_j) + h^{-1}(\omega_c(g, u_j)(\Omega)) \\ &= -\min_{\partial\Omega}\psi + h^{-1}(\mu_j(\Omega)) \le -\min_{\partial\Omega}\psi + h^{-1}(A) =: C < +\infty. \end{aligned}$$

Moreover, by Lemma 6.10 we can find a c-convex function $w \in C(\overline{\Omega})$ satisfying $w = \psi$ on $\partial\Omega$ and $w \le -C$ on Ω'. From Theorem 6.7, let $v \in C(\overline{\Omega})$ be the unique c-convex weak solution to the problem $\omega_c(v) = 0$ in Ω and $v = \psi$ on $\partial\Omega$. Then we have

$$w(x) \le u_j(x) \le v(x) \qquad \forall x \in \overline{\Omega} \tag{61}$$

for every j, where the first inequality is proved using the argument in the proof of Claim 2, Theorem 6.11, and the second follows from Theorem 5.5. It can be showed easily that $\{u_j\}$ has a subsequence converging locally uniformly to some function u in Ω. Then by (61) we have $w \le u \le v$. Therefore $u \in C(\overline{\Omega})$ with $u = \psi$ on $\partial\Omega$. Hence by Lemma 2.10 and Corollary 3.4 we obtain that u is a weak solution to the Dirichlet problem. The uniqueness follows from the comparison principle Theorem 5.5. □

In the last theorem, the assumption that the measure μ has compact support in Ω can be substituted by the existence of a c-convex subsolution to the equation with the given boundary condition. Indeed, we have the following.

Corollary 6.13 *Suppose that c satisfies condition (H1), $\lim_{|x|\to+\infty} \dfrac{c(x)}{|x|} = +\infty$, $c(0) = \min_{x\in\mathbb{R}^n} c(x)$, and $g \in L^1_{loc}(\mathbb{R}^n)$ is positive a.e. Let $\Omega \subset \mathbb{R}^n$ be a strictly c-convex open set, and $\psi \in C(\partial\Omega)$. Suppose that μ is a finite Borel measure in Ω satisfying* (60) *and there exists a c-convex function $w \in C(\overline{\Omega})$ such that $\omega_c(g, w) \ge \mu$ in Ω and $w = \psi$ on $\partial\Omega$. Then there exists a unique $u \in C(\overline{\Omega})$ that is a c-convex weak solution to the problem $\omega_c(g, u) = \mu$ in Ω and $u = \psi$ on $\partial\Omega$.*

Proof Let $S = \mathrm{spt}(\mu) \setminus \overline{\mathrm{Int}(\mathrm{spt}(\mu))}$ and for each positive integer j denote $\Omega_j = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 1/j\}$. Define

$$K_j = (\Omega_j \cap S) \cup \overline{\Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))}$$

which is a subset of $\mathrm{spt}(\mu)$, and $\mu_j(E) = \dfrac{j}{j+1}\mu(E \cap K_j)$ for every Borel set $E \subset \Omega$. Then since μ satisfies condition (60) we have K_j is a closed set and hence $\mathrm{spt}(\mu_j) = \mathrm{spt}(\mu|_{K_j}) = K_j$. Moreover, as S does not contain any interior point and the sets $\Omega_j \cap S$ and $\overline{\Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))}$ are disjoint, we get $\mathrm{Int}(\mathrm{spt}(\mu_j)) = \mathrm{Int}(K_j) = \mathrm{Int}(\overline{\Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))}) = \Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))$. Therefore, if for each j we let $S_j = \mathrm{spt}(\mu_j) \setminus \overline{\mathrm{Int}(\mathrm{spt}(\mu_j))}$, then we obtain $S_j = \Omega_j \cap S$. This implies that condition (60) is satisfied with S replaced by S_j. We note also that $\mu_j \rightharpoonup \mu$ weakly in Ω since for each compact set $K \subset \Omega$ we have $\mu_j(K) \to \mu(K)$ because

$$\lim_{j\to\infty} \mu(K \cap K_j) = \lim_{j\to\infty} \mu((K \cap \Omega_j \cap S) \cup (K \cap \overline{\Omega_j \cap \mathrm{Int}(\mathrm{spt}(\mu))}))$$
$$= \mu((K \cap S) \cup (K \cap \overline{\mathrm{Int}(\mathrm{spt}(\mu))})) = \mu(K \cap \mathrm{spt}(\mu)) = \mu(K).$$

Now since $\mu_j(\Omega) \le \dfrac{j}{j+1}\mu(\Omega)$ and $\mu(\Omega) \le \omega_c(g,w)(\Omega) \le \int_{\mathbb{R}^n} g$, we get $\mu_j(\Omega) < \int_{\mathbb{R}^n} g$ as μ is a finite measure. Thus, by Theorem 6.12 for each j there exists $u_j \in C(\overline{\Omega})$ that is a c-convex weak solution to the problem $\omega_c(g,u_j) = \mu_j$ in Ω and $u_j = \psi$ on $\partial\Omega$. Hence as $\omega_c(g,u_j) = \mu_j \le \mu \le \omega_c(g,w)$ and the measures μ_j satisfy condition (60), by applying the comparison principle Theorem 5.5 we obtain

$$w(x) \le u_j(x) \le v(x) \qquad \forall x \in \overline{\Omega} \tag{62}$$

for every j, where $v \in C(\bar{\Omega})$ is the c-convex weak solution to the homogeneous Dirichlet problem. Then by following the argument in the proof of Theorem 6.12 we get the desired result. □

In general to find a c-convex subsolution is a nontrivial task. However, in the following by using Bakelman's result on the existence of a convex weak solution to the R-curvature problem we shall show that c-convex subsolutions do exist in a number of important cases. In order to do that we need to recall some concepts introduced by Bakelman. Suppose $R \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is positive a.e. on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set. For each convex function u on Ω, define the measure

$$\omega(R,u,E) = \int_{\partial u(E)} R(y)\mathrm{d}y \qquad \text{for all Borel sets } E \subset \Omega,$$

where ∂u denotes the standard subdifferential. We then have the following relation which in particular says that when g is bounded from below by a positive constant c_0 then a convex subsolution to the R-curvature problem with $R(y) = c_0 \det D^2c^*(-y)$ is indeed a c-convex subsolution to our problem.

Lemma 6.14 *Suppose c satisfies condition (H2) and $c^* \in C^2(\mathbb{R}^n \setminus \{z_0\})$ for some z_0 in $\mathbb{R}^n$, and $g(y) \ge c_0 > 0$ for a.e. y. Let $R(y) = c_0 \det D^2c^*(-y)$ a.e. on $\mathbb{R}^n$ and assume that R is positive a.e., and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then for any convex function $u \in C(\Omega)$, we have*

$$\omega_c(g,u)(E) \ge c_0|E| + \omega(R,u,E) \qquad \textit{for all Borel sets } E \subset \Omega.$$

Proof Noticing that since c^* is convex, we have $R \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (see for example [11, Corollary 4.3]). Also, as $\omega_c(g,u) \ge c_0\omega_c(u)$, it is enough to show the above estimate with the left hand side is $\omega_c(u)$ and with $c_0 = 1$. Assume first that u is a convex function satisfying $u \in C^2(\Omega)$ and $D^2u(x) > 0$ in Ω. Let $s(x) = x - Dc^*(-Du(x))$ for x in Ω, and define $K_u = \{x \in \Omega : Du(x) = -z_0\}$. Then K_u is relatively closed in Ω and $s \in C^1(\Omega \setminus K_u)$. We observe that since $D^2c^*(y)$ is symmetric nonnegative definite for every y in $\mathbb{R}^n - \{z_0\}$ and $D^2u(x)$ is symmetric positive definite for all x in Ω, we have $D^2c^*(y)D^2u(x)$ is diagonalizable with nonnegative eigenvalues for every such x and y. But then by diagonalizing, it is easy to see that $\det(I + D^2c^*(-Du(x))D^2u(x)) \ge 1 + \det(D^2c^*(-Du(x))D^2u(x))$ for all x in $\Omega \setminus K_u$. Particularly we have $\det Ds(x) > 0$ on $\Omega \setminus K_u$.

Now let E be an arbitrary Borel set in Ω. We claim that

$$\omega_c(u)(E \setminus K_u) \geq \int\limits_{E\setminus K_u} |\det(I + D^2c^*(-Du)D^2u)| \, \mathrm{d}x. \tag{63}$$

Indeed, let U be any open set with $E \setminus K_u \subset U \subset \Omega$. We can write the open set $U \setminus K_u$ as $U \setminus K_u = \cup_{i=1}^{\infty} C_i$ where $\{C_i\}_{i=1}^{\infty}$ are cubes with disjoint interior and sides parallel to the coordinate axes. As $\det Ds(x) > 0$ on $U\setminus K_u$, we can choose C_i small enough so that $s : C_i \to s(C_i)$ is a diffeomorphism. We therefore have

$$\begin{aligned}
\int\limits_{E\setminus K_u} |\det(I + D^2c^*(-Du)D^2u)| \, \mathrm{d}x &\leq \int\limits_{U\setminus K_u} |\det Ds(x)| \, \mathrm{d}x \\
&= \int\limits_{\cup_{i=1}^{\infty} C_i} |\det Ds(x)| \, \mathrm{d}x = \int\limits_{\cup_{i=1}^{\infty} \mathring{C}_i} |\det Ds(x)| \, \mathrm{d}x \\
&= \sum_{i=1}^{\infty} \int\limits_{\mathring{C}_i} |\det Ds(x)| \, \mathrm{d}x = \sum_{i=1}^{\infty} \int\limits_{s(\mathring{C}_i)} \mathrm{d}y \\
&= \sum_{i=1}^{\infty} |\partial_c u(\mathring{C}_i)| \text{ by Proposition 2.7 (2)} \\
&= \omega_c(u)(\cup_{i=1}^{\infty} \mathring{C}_i) \leq \omega_c(u)(U \setminus K_u) \leq \omega_c(u)(U).
\end{aligned}$$

Since c satisfies condition $(H2)$, the measure $\omega_c(u)$ is regular. Hence, we deduce from the last inequality that (63) holds. Next, as $\partial u(E\cap K_u) = \{Du(x) : x \in E \cap K_u\} \subset \{-z_0\}$ we have $|\partial u(E \cap K_u)| = 0$. Hence,

$$\omega(R, u, E \cap K_u) = 0. \tag{64}$$

We also note that since u is convex and $u \in C^2(\Omega)$ we get from Proposition 2.7 (2) that

$$\partial_c u(E \cap K_u) = \{x - Dc^*(-Du(x)) : x \in E \cap K_u\} = \{x - Dc^*(z_0) : x \in E \cap K_u\}.$$

Thus,

$$\omega_c(u)(E \cap K_u) = |E \cap K_u|. \tag{65}$$

Combining (63), (64), (65) and by the above observation we obtain

$$\begin{aligned}
\omega_c(u)(E) &= \omega_c(u)(E \cap K_u) + \omega_c(u)(E \setminus K_u) \\
&\geq |E \cap K_u| + \int\limits_{E\setminus K_u} |\det(I + D^2c^*(-Du)D^2u)| \mathrm{d}x \\
&\geq |E \cap K_u| + \int\limits_{E\setminus K_u} [1 + \det D^2c^*(-Du) \ \det D^2u] \, \mathrm{d}x \\
&= |E| + \int\limits_{E\setminus K_u} R(Du(x)) \ \det D^2u(x) \, \mathrm{d}x = |E| + \omega(R, u, E \setminus K_u) \\
&= |E| + \omega(R, u, E).
\end{aligned}$$

So the lemma holds for any convex function $u \in C^2(\Omega)$ satisfying $D^2u > 0$ in Ω. For the general case, choose a sequence of convex functions $\{v_m\} \subset C^2(\Omega)$ such that $v_m \to u$ locally uniformly in Ω. Define $u_m(x) = v_m(x) + \frac{1}{m}|x|^2$. Then $D^2u_m > 0$ in Ω for every m and $\{u_m\}$ still share the above properties of $\{v_m\}$. Hence, by applying the previous result we get $\omega_c(u_m) \geq |\cdot| + \omega(R, u_m, .)$ in Ω for all m. But as $\{u_m\}$ are convex and $u_m \to u$ locally uniformly we have $\omega_c(u_m)$, $\omega(R, u_m, .)$ converge weakly to $\omega_c(u)$ and $\omega(R, u, .)$ respectively (see Remark 3.5). Therefore, by passing to the limit we get the desired result. □

We will need the following proposition which is a simple extension of Aleksandrov maximum principle (see [10, Theorem 1.4.2]). Note that here we only need $u \geq 0$ on $\partial\Omega$ instead of $u = 0$ on $\partial\Omega$ as in Aleksandrov's result.

Proposition 6.15 *Suppose $R(y) \geq c_1|y|^{-2k}$ for a.e. y in $\mathbb{R}^n$ with $k < \frac{1}{2}$. Let Ω be a bounded convex open set in $\mathbb{R}^n$, and $u \in C(\overline{\Omega})$ a convex function with $u \geq 0$ on $\partial\Omega$. We have the following:*

(i) *If $k \leq 0$, then*

$$u(x) \geq -\left[\frac{1}{c_1 c(n,k)} \operatorname{diam}(\Omega)^{n-2k-1} \operatorname{dist}(x, \partial\Omega)\, \omega(R, u, \Omega)\right]^{1/(n-2k)} \qquad \forall x \in \Omega.$$

(ii) *If $0 \leq k < \frac{1}{2}$, then*

$$u(x) \geq -\left[\frac{1}{c_1 c(n)} \operatorname{diam}(\Omega)^{n-1} \operatorname{dist}(x, \partial\Omega)^{1-2k}\, \omega(R, u, \Omega)\right]^{1/(n-2k)} \qquad \forall x \in \Omega.$$

Proof Let $x_0 \in \Omega$ be such that $u(x_0) < 0$ and let $\mathcal{F} = \{v \in C(\bar{\Omega}) : v \text{ is convex}, v \leq u \text{ on } \partial\Omega, \text{and } v(x_0) \leq u(x_0)\}$. Then $\mathcal{F} \neq \emptyset$ as $u \in \mathcal{F}$. Define

$$w(x) = \sup_{v \in \mathcal{F}} v(x) \qquad \text{for } x \in \bar{\Omega}.$$

Since Ω is convex, there exists $h \in C(\bar{\Omega})$ which is harmonic in Ω and $h = u$ on $\partial\Omega$. Then by writing w as a supremum of affine functions it is easy to see that $u(x) \leq w(x) \leq h(x)$ in $\bar{\Omega}$, and hence the convex function w is in $C(\bar{\Omega})$ with $w = u$ on $\partial\Omega$ and $w(x_0) = u(x_0)$. This implies that $\partial w(\Omega) \subset \partial u(\Omega)$ from [10, Lemma 1.4.1]. On the other hand, by the definitions of subdifferential and the function w we have

$$\begin{aligned} \partial w(x_0) \quad &= \{p \in \mathbb{R}^n : w(x_0) + p \cdot (x - x_0) \leq u(x) \text{ on } \partial\Omega\} \\ &\supset \{p \in \mathbb{R}^n : u(x_0) + p \cdot (x - x_0) \leq 0 \text{ on } \partial\Omega\} = \partial v(x_0), \end{aligned}$$

where v is the convex function whose graph is the upside down cone with vertex $(x_0, u(x_0))$ and base Ω, with $v = 0$ on $\partial\Omega$. Therefore, $\partial v(x_0) \subset \partial u(\Omega)$. Moreover, by the proof in [10, Theorem 1.4.2] there exists $p_0 \in \mathbb{R}^n$ with $|p_0| = \dfrac{-u(x_0)}{\operatorname{dist}(x_0, \partial\Omega)}$ such that the convex hull K of the n-dimensional ball $B\left(0, \frac{-u(x_0)}{\operatorname{diam}(\Omega)}\right)$ and p_0 is contained in $\partial v(x_0)$. Consequently,

$$\omega(R, u, \Omega) = \int_{\partial u(\Omega)} R(y)\, \mathrm{d}y \geq c_1 \int_K |y|^{-2k}\, \mathrm{d}y \geq c_1 \int_{H(t,\alpha,\beta)} |y|^{-2k}\, \mathrm{d}y, \tag{66}$$

where $t = -u(x_0) > 0$, $\alpha = \operatorname{dist}(x_0, \partial\Omega)$, $\beta = \operatorname{diam}(\Omega)$, and $H(t, \alpha, \beta)$ denotes the convex cone in $\mathbb{R}^n$ with vertex at $(0, ..., 0, t/\alpha)$ and with base the ball $B(0, t/\beta) \subset \mathbb{R}^{n-1}$, the orthogonal hyperplane to the x_n-axis at the origin. Noticing that in the last inequality above we have used the fact that the function $|y|^{-2k}$ is radial. If $0 \le k < \frac{1}{2}$, then we have

$$\int\limits_{H(t,\alpha,\beta)} |y|^{-2k}\,\mathrm{d}y \ge \left(\frac{t}{\alpha}\right)^{-2k} |H(t,\alpha,\beta)| = \omega_{n-2}\left(\frac{t}{\alpha}\right)^{-2k} \int\limits_0^{t/\alpha}\left[\int\limits_0^{t-\alpha z/\beta} h^{n-2}\,\mathrm{d}h\right]\mathrm{d}z$$
$$= \frac{\omega_{n-2}}{n-1}\left(\frac{t}{\alpha}\right)^{-2k}\int\limits_0^{t/\alpha}\left(\frac{t-\alpha z}{\beta}\right)^{n-1}\mathrm{d}z = \frac{\omega_{n-2}}{n(n-1)}\frac{t^{n-2k}}{\alpha^{1-2k}\beta^{n-1}}. \tag{67}$$

If on the other hand $k \le 0$, then

$$\int\limits_{H(t,\alpha,\beta)} |y|^{-2k}\,\mathrm{d}y = \omega_{n-2}\int\limits_0^{t/\alpha}\left[\int\limits_0^{t-\alpha z/\beta}(h^2+z^2)^{-k}h^{n-2}\,\mathrm{d}h\right]\mathrm{d}z$$
$$\ge \omega_{n-2}\int\limits_0^{t/\alpha}\left[\int\limits_{t-\alpha z/2\beta}^{t-\alpha z/\beta}(h^2+z^2)^{-k}h^{n-3}h\,\mathrm{d}h\right]\mathrm{d}z$$
$$\ge \omega_{n-2}\min\left\{2^{n-3},1\right\}\int\limits_0^{t/\alpha}\left(\frac{t-\alpha z}{2\beta}\right)^{n-3}\left[\int\limits_{t-\alpha z/2\beta}^{t-\alpha z/\beta}(h^2+z^2)^{-k}h\,\mathrm{d}h\right]\mathrm{d}z$$
$$= \frac{c(n)}{2}\int\limits_0^{t/\alpha}\left(\frac{t-\alpha z}{2\beta}\right)^{n-3}\left[\int\limits_{z^2+(\frac{t-\alpha z}{2\beta})^2}^{z^2+(\frac{t-\alpha z}{\beta})^2} s^{-k}\,\mathrm{d}s\right]\mathrm{d}z$$
$$= \frac{c(n)}{2(1-k)}\int\limits_0^{t/\alpha}\left(\frac{t-\alpha z}{2\beta}\right)^{n-3}\left\{\left[z^2+\left(\frac{t-\alpha z}{\beta}\right)^2\right]^{1-k} - \left[z^2+\left(\frac{t-\alpha z}{2\beta}\right)^2\right]^{1-k}\right\}\mathrm{d}z$$
$$\ge \frac{c(n)3^{1-k}}{2(1-k)}\int\limits_0^{t/\alpha}\left(\frac{t-\alpha z}{2\beta}\right)^{n-2k-1}$$
$$\mathrm{d}z = \frac{c(n)\,3^{1-k}}{2^{n-2k}(1-k)(n-2k)}\frac{t^{n-2k}}{\alpha\,\beta^{n-2k-1}}. \tag{68}$$

From (66), (67), (68) and the definitions of t, α, and β we derive the desired results. □

We next recall the notion of local parabolic support due to Bakelman which describes more precisely the geometry of domains which are between being strictly convex and satisfying the enclosing sphere condition. For a bounded open convex set $\Omega \subset \mathbb{R}^n$,

let $z \in \partial\Omega$. Then there exist a supporting hyperplane α to $\bar{\Omega}$ at z and an open ball $B_R(z)$ such that the convex $(n-1)$-surface $\partial\Omega \cap B_R(z)$ has a one-to-one orthogonal projection $\Pi_\alpha : \partial\Omega \cap B_R(z) \longrightarrow \alpha$. Moreover, the unit normal $\vec{n}$ to α in the direction of the halfspace where $\bar{\Omega}$ lies passes through interior points of Ω. Denote by $\mathcal{S}_R(z)$ the set $\Pi_\alpha(\partial\Omega \cap B_R(z))$. Let $\xi_1, \cdots, \xi_{n-1}, \xi_n$ be the Cartesian coordinates introduced in the following way: z is the origin, the axes $\xi_1, \cdots, \xi_{n-1}$ lie in the plane α, and the axis ξ_n is directed along the interior normal $\vec{n}$ to $\partial\Omega$ at the point z. Clearly, the convex surface $\partial\Omega \cap B_R(z)$ is the graph of some convex function $\varphi(\xi_1, \cdots, \xi_{n-1})$ defined on $\mathcal{S}_R(z)$. Obviously, $\varphi(0, \cdots, 0) = 0$, and $\varphi(\xi_1, \cdots, \xi_{n-1}) \geq 0$, $\forall(\xi_1, \cdots, \xi_{n-1}) \in \mathcal{S}_R(z)$. We will say that $\partial\Omega$ has a parabolic support of order $\tau \geq 0$ at the point $z \in \partial\Omega$, if there exists $r_z \in (0, R)$ and $b(z) > 0$ such that

$$\varphi(\xi_1, \cdots, \xi_{n-1}) \geq b(z)(\xi_1^2 + \cdots + \xi_{n-1}^2)^{\frac{\tau+2}{2}} \qquad \forall(\xi_1, \cdots, \xi_{n-1}) \in \mathcal{S}_{r_z}(z),$$

i.e., the convex $(n-1)$-surface $\partial\Omega \cap B_{r_z}(z)$ can be touched from outside by the $(n-1)$-dimensional paraboloid $\xi_n = b(z)(\xi_1^2 + \cdots + \xi_{n-1}^2)^{\frac{\tau+2}{2}}$ of order $\frac{\tau+2}{2}$ at z.

Definition 6.16 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set and $\tau \geq 0$. We say $\partial\Omega$ has a parabolic support of order not more than τ if at every boundary point z of Ω, $\partial\Omega$ has a parabolic support of some order $\tau_z \in [0, \tau]$.*

We will also consider the following condition for the Borel measure μ:
(G) There exist $a > 0$ and $\epsilon > 0$ such that

$$\mu(E) \leq a|E| \quad \text{for all Borel sets } E \subset \Omega \setminus \overline{\Omega_\epsilon}, \tag{69}$$

where $\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$.

We can now state the following theorem whose second part is due to Bakelman (see [2, Theorem 11.4]).

Theorem 6.17 *Let $c(x) = \dfrac{|x|^p}{p}$ with $1 < p < \infty$, and $R(y) = c_0 \det D^2 c^*(-y)$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open and strictly convex set, $\psi \in C(\partial\Omega)$, and μ is a finite Borel measure in Ω. Consider the Dirichlet problem*

$$\omega(R, u, .) = \mu \text{ in } \Omega, \text{and } u = \psi \text{ on } \partial\Omega. \tag{70}$$

We have

(i) *If $p < 2 + \dfrac{1}{n-1}$, then (70) has a convex weak solution $u \in C(\bar{\Omega})$.*

(ii) *If $2 + \dfrac{1}{n-1} \leq p$, and in addition μ satisfies the assumption (G) and $\partial\Omega$ has a parabolic support of order no more than τ for some nonnegative τ satisfying $\dfrac{n}{2}\dfrac{p-2}{p-1} < \dfrac{n+\tau+1}{\tau+2}$, then (70) has a convex weak solution $u \in C(\bar{\Omega})$.*

Proof First observe that $R(y) = c_0(q-1)|y|^{n(q-2)} = c_0(q-1)|y|^{-2\frac{n(2-q)}{2}}$, where $q > 1$ is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Hence, $\int_{\mathbb{R}^n} R(y)\, \mathrm{d}y = c_0(q-1) \int_{\mathbb{R}^n} |y|^{n(q-2)}\, \mathrm{d}y = +\infty$. Proof of (i). Since μ is a finite Borel measure, there exists a sequence of measures μ_j converging weakly to μ such that each μ_j is a finite combination of delta masses with positive coefficients and $\{\mu_j(\Omega)\}$ is bounded by some positive constant B. For each j, by using [1, Theorem 2] we can find $u_j \in C(\bar{\Omega})$ which is the convex weak

solution to $\omega(R, u_j, .) = \mu_j$ in Ω and $u_j = \psi$ on $\partial\Omega$. If we let $W \in C(\bar{\Omega})$ be the convex weak solution to $\det D^2 W = 0$ in Ω and $W = \psi$ on $\partial\Omega$, then $u_j \le W$ on $\bar{\Omega}$. We now prove that $\{u_j\}$ is also uniformly bounded from below in Ω. Indeed, let $\xi \in \partial\Omega$ and $\epsilon > 0$. There exists $\delta > 0$ such that $|\psi(x) - \psi(\xi)| < \epsilon$ for $|x - \xi| < \delta$, $x \in \partial\Omega$. Let $A \cdot x + b = 0$ be the equation of the supporting hyperplane to Ω at ξ and assume that $\Omega \subset \{x : l(x) \ge 0\}$, where $l(x) := A \cdot x + b$. Since Ω is strictly convex, there is $\eta > 0$ such that $\{x \in \bar{\Omega} : l(x) \le \eta\} \subset B_\delta(\xi)$. Let $M = \min\{\psi(x) : x \in \partial\Omega, l(x) \ge \eta\}$ and consider the function $a(x) = \psi(\xi) - \epsilon - \kappa l(x)$, where κ is a constant satisfying $\kappa \ge \max\{\frac{\psi(\xi)-\epsilon-M}{\eta}, 0\}$. Hence $a(\xi) = \psi(\xi) - \epsilon$ and it is easy to see that $a(x) \le \psi(x)$ on $\partial\Omega$. Set $v_j(x) = u_j(x) - a(x)$. Then $v_j \in C(\bar{\Omega})$ is convex and $v_j \ge 0$ on $\partial\Omega$. By using Proposition 6.15 and noting $2 - q < 1/n$ we see that there exists $\alpha > 0$ depending only on q such that

$$v_j(x) \ge -\left[C(n, q, \Omega)\mathrm{dist}(x, \partial\Omega)^\alpha \omega(R, v_j, \Omega)\right]^{1/n(q-1)} \text{ on } \Omega.$$

But we have by the definition of v_j

$$\begin{aligned}
\omega(R, v_j, \Omega) &= c_0(q-1) \int\limits_{\partial u_j(\Omega)} |y + \kappa A|^{n(q-2)}\, \mathrm{d}y \\
&= c_0(q-1) \int\limits_{\partial u_j(\Omega)\cap\{|y|\ge 2\kappa|A|\}} |y + \kappa A|^{n(q-2)}\, \mathrm{d}y \\
&\quad + c_0(q-1) \int\limits_{\partial u_j(\Omega)\cap\{|y|\le 2\kappa|A|\}} |y + \kappa A|^{n(q-2)}\, \mathrm{d}y \\
&\le \max\left\{\left(\frac{3}{2}\right)^{n(q-2)}, \left(\frac{1}{2}\right)^{n(q-2)}\right\} c_0(q-1) \int\limits_{\partial u_j(\Omega)} |y|^{n(q-2)}\, \mathrm{d}y \\
&\quad + c_0(q-1) \int\limits_{B(0,3\kappa|A|)} |z|^{n(q-2)}\, \mathrm{d}z \\
&= \max\left\{\left(\frac{3}{2}\right)^{n(q-2)}, \left(\frac{1}{2}\right)^{n(q-2)}\right\} \omega(R, u_j, \Omega) + C(n, q)(\kappa|A|)^{n(q-1)} \\
&\le C(n, q, \kappa, |A|, B).
\end{aligned}$$

Therefore, we obtain

$$W(x) \ge u_j(x) \ge a(x) - C(n, q, \kappa, |A|, B, \Omega)\mathrm{dist}(x, \partial\Omega)^{\alpha/n(q-1)} \text{ on } \Omega. \tag{71}$$

This shows that $\{u_j\}$ is uniformly bounded on Ω and hence we can extract a subsequence still denoted by $\{u_j\}$ which converges locally uniformly on Ω to some convex function $u \in C(\Omega)$. This gives $\omega(R, u, .) = \mu$ on Ω. Moreover, (71) also implies that for every $\xi \in \partial\Omega$ we have $\lim_{\Omega \ni x \to \xi} u(x) = \psi(\xi)$. Thus the proof of (i) is completed. On the other hand, (ii) follows from [1, Theorem 6] or [2, Theorem 11.4] with $\lambda = 0$. □

Since convex functions are c-convex, Lemma 6.14 together with Theorem 6.17 provides the c-convex subsolution w needed in Corollary 6.13 when g is bounded from below by some positive constant c_0 and the cost function $c(x) \approx \frac{1}{p}|x|^p$, $1 < p < \infty$.

We remark that in these cases in fact we have $\omega_c(g,w)(E) \geq c_0|E| + \mu(E)$. Therefore, we do not really need the technical assumption (60) in Corollary 6.13 to ensure the existence of a c-convex weak solution. The reason is that in that proof one can simply take $\mu_j(E) = \mu(E \cap \Omega_j) + \frac{c_0}{j}|E|$. We then also have $\mu_j \rightharpoonup \mu$ weakly, $\mu_j \leq \omega_c(g,w)$, $\mu_j(\Omega) < \omega_c(g,w)(\Omega) \leq \int_{\mathbb{R}^n} g$, and $\mathrm{spt}(\mu_j) = \bar{\Omega}$. The last fact allows us to use Theorem 5.5 to obtain (62).

We conclude our paper mentioning that some related results concerning C^2 estimates for solutions to the optimal transport equation (2) have been considered in the recent paper [12] under the restrictive assumption that the cost function c satisfies a condition involving its fourth derivatives namely condition (A3) in that paper.

Acknowledgement We would like to thank very much Luigi Ambrosio for reading carefully the paper and suggestions.

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