REFRACTORS IN ANISOTROPIC MEDIA ASSOCIATED WITH NORMS

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Abstract. We show existence of interfaces between two anisotropic materials so that light is refracted in accordance with a given pattern of energy. To do this we formulate a vector Snell law for anisotropic media when the wave fronts are given by norms for which the corresponding unit spheres are strictly convex.

Contents

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1. Introduction

Anisotropic materials are those whose optical properties vary according to the direction of propagation of light. Typical examples are crystals, where the refractive index depends on the direction of the incident light, see [BW59, Chapter XV], [LL84, Chapter XI] and [Som54, Chapter IV]. Important research was done on this subject because of it multiple applications, see the fundamental work [KK65], and [YY84], [Sch07] for more recent applications and references. Mathematically, in these materials wave fronts satisfy the Fresnel partial differential equation which in the particular case of isotropic materials is the eikonal equation. A difficulty with anisotropic materials in the geometrical optics regime is that incident rays may be refracted into two rays: an ordinary ray and an extraordinary ray. This is the phenomenon of bi-refringence, observed experimentally in crystals, and is a consequence from the fact that the Fresnel equation splits in general as the product of two surfaces, see (6.8).

The main purpose of this paper is to show existence of interfaces between two homogenous and anisotropic materials so that light is refracted in accordance with a given pattern of energy. As a main step to achieve this, we give a formulation of a vector Snell's law in anisotropic materials, Equation (3.2), when the wave fronts are given by norms in $\mathbb{R}^n$[1] which has independent interest. More precisely, suppose $N_i(x)$, $i = 1, 2$, are norms in $\mathbb{R}^n$, $\Sigma_i = \{x : N_i(x) = 1\}$, $\Omega_i \subset \Sigma_i$ are domains, $f > 0$ is an integrable function on Ω_1, and μ is a Radon measure in Ω_2 with $\int_{\Omega_1} f(x)\,dx = \mu(\Omega_2)$. We have two anisotropic media I and II such that the wave fronts in I are given by N_1 and the wave fronts in II given by N_2. Light rays emanate from the origin, located in medium I, with intensity $f(x)$ for each $x \in \Omega_1$. We seek a surface $\mathcal{S}$ separating media I and II so that all rays emanating from the origin and with directions in Ω_1 are refracted into rays with directions in Ω_2 and

[1]There is no mathematical objection to consider this problem in n dimensions but the physical problem is three dimensional.

the conservation of energy condition

$$\int_{\tau(E)} f(x)\,dx = \mu(E)$$

holds for each Borel set $E \subset \Omega_2$ where $\tau(E) = \{x \in \Omega_1 : x \text{ is refracted into } E\}$. This is called the refractor problem, and when media I and II are homogeneous and isotropic it is solved in [GH09] using optimal mass transport and in [Gut14] with a different method. A main difficulty to solve this problem is to lay down the mathematical formulation of the physical laws and constraints in anisotropic media to our setting with norms.

To place our results in perspective we mention the following. A similar problem for norms but for reflection was studied in [CH09]. Once the Snell law and physical constraints for anisotropic media are formulated and proved in Section 3, our existence results use the abstract method developed in [GH14], where existence results for the near field refractor problem in homogenous and isotropic media are obtained. Further results on geometric optics problems for refraction in homogeneous and isotropic media have been studied in [GM13], [LGM17], and [Kar16]. The isotropic case is more computable than the anisotropic one which is abstract and usually involves further insights from optics and convex analysis. The mathematical literature for these problems in anisotropic media is lacking, and this work is a contribution to fill this lack.

The organization of the paper is as follows. Section 2 recalls a few results on norms and convexity that will be used later. The Snell law in anisotropic media is obtained in Section 3 as a consequence of Fermat's principle of least time. The discussion on the physical constraints for refraction in anisotropic media is in Section 3.1. Section 4 introduces and analyzes the surfaces refracting all rays into a fixed direction which are next used in Section 5 to show the existence Theorems 5.6 and 5.7. Section 6 introduces and analyzes, in a somewhat new way, the Fresnel pde for the wave fronts in general materials non homogenous and anisotropic. In Section 6.1 we apply the results from the previous sections to materials having

permittivity and permeability coefficients ϵ and μ that are constant matrices with one a constant multiple of the other. Finally, in Section 7 we relate our problem to optimal mass transport.

2. Preliminaries on Norms and Convexity

Consider a norm $N(x) = \|x\|$ in $\mathbb{R}^n$, and let $\Sigma = \{x \in \mathbb{R}^n : N(x) = 1\}$ be the unit sphere in the norm such that Σ is a strictly convex surface. As usual, S^{n-1} denotes the unit sphere in the Euclidean norm in $\mathbb{R}^n$.

Given a vector $\nu \in S^{n-1}$, the support function of Σ is defined by

$$\varphi(\nu) = \sup_{x\in\Sigma} x \cdot \nu.$$

Clearly φ is strictly positive ($\varphi(\nu) \geq 1/\|\nu\|$). Since Σ is compact, there is $x_0 \in \Sigma$ such that $\varphi(\nu) = x_0 \cdot \nu$, and since Σ is strictly convex x_0 is unique. The hyperplane Π_ν with equation $\{x : \ell(x) = x \cdot \nu - \varphi(\nu) = 0\}$ is a supporting hyperplane to Σ at x_0. That is, $\ell(x) \leq 0$ for all $x \in \Sigma$ and $\ell(x_0) = 0$. We then have a map $\nu \in S^{n-1} \mapsto x_0 \in \Sigma$

$$\Phi : S^{n-1} \to \Sigma, \qquad \Phi(\nu) = x_0.$$

This map assigns to each vector $\nu \in S^{n-1}$ a unique point $\Phi(\nu) \in \Sigma$ (uniqueness follows from the strict convexity of Σ) so that Π_ν is a supporting hyperplane to Σ at $\Phi(\nu)$. In addition, also from the strict convexity of Σ, and consequently the uniqueness of maximizer of φ, we have $\Phi(-\nu) = -\Phi(\nu)$. Φ is called the support map.

The dual norm of N, denoted by $N^*(x) = \|x\|^*$ is defined as follows. Since $\mathbb{R}^n$ is finite dimensional, each linear functional $\ell : \mathbb{R}^n \to \mathbb{R}$ can be represented as $\ell(x) = y \cdot x$ for a unique $y \in \mathbb{R}^n$. Given $y \in \mathbb{R}^n$, the dual norm of N is then $N^*(y) = \sup_{x\in\Sigma} |x \cdot y|$, and writing $y = \lambda\, \nu$ with $\nu \in S^{n-1}$, we get that $N^*(y) = \lambda\, \varphi(\nu)$. Hence the dual norm sphere of Σ is $\Sigma^* = \{y \in \mathbb{R}^n : N^*(y) = 1\} = \{\nu/\varphi(\nu) : \nu \in S^{n-1}\}$. We recall the following.

Lemma 2.1. [CH09, Lemma 2.3] *For each $x \in \Sigma$ and $\nu^* = \nu/\varphi(\nu)$ with $\nu \in S^{n-1}$ we have*

(a) $|x \cdot v^*| \leq 1$; *and*

(b) $x \cdot v^* = 1$ *if and only if* $x = \Phi(v)$.

Since Σ is a convex surface, for each $x \in \Sigma$, there is a supporting hyperplane to Σ at x and let $\nu(x)$ be the outer unit normal to such a supporting hyperplane. If Σ is such that at each x one can pick a supporting hyperplane with normal $\nu(x)$ in such a way that $\nu(x)$ is continuous for all $x \in \Sigma$, that is, Σ *has a continuous normal field,* then from the proof of [CGH08, Theorem 6.2], Σ has a unique tangent plane at each point; that is, Σ is differentiable.

With the notation from [CH09], the Minkowski functional of Σ defined by $M_\Sigma(x) = \inf\{r > 0 : x \in r\,B\}$ satisfies $M_\Sigma(x) = N(x)$, where $B = \{x \in \mathbb{R}^n : N(x) \leq 1\}$. It is proved in [CH09, Lemma 2.4] that Σ has a continuous normal field $\nu(x)$ if and only if $N \in C^1(\mathbb{R}^n \setminus \{0\})$; and $p(x) := \nabla N(x) = \nu(x)/\varphi(\nu(x))$. We also recall [CH09, Lemma 2.5] saying that if Σ is C^1, then Σ^* is strictly convex. Also, if Σ is strictly convex, then $\Sigma^* \in C^1$, [CH09, Lemma 2.7]. Therefore, if Σ is strictly convex and C^1, then $p : \Sigma \to \Sigma^*$ is an homeomorphism and $p^* \circ p = Id$; $p^* = \nabla N^*$.

3. A vector Snell's law for anisotropic media

We have two homogenous and anisotropic media I and II so that the surfaces for the wave fronts are given by a norm N_1 in medium I, and given by a norm N_2 in medium II.[2] We are assuming that the norms $N_i \in C^1(\mathbb{R}^n \setminus \{0\})$ and the corresponding unit spheres Σ_i are strictly convex, $i = 1, 2$. Suppose further that media I and II are separated by a plane having normal ν from medium I to medium II as in Figure 1.

We begin stating Fermat's principle of least time with respect to the norms N_1 and N_2. Given points $X \in I$ and $Y \in II$, then Fermat's principle states that the

[2] A wave front is a surface in 3d space described by $\psi(x) = c\,t$ where ψ is a function, c is the speed of light in vacuum, and t is time. This means the points x on the wave front that are traveling for a time t are located on the surface $\psi(x) = c\,t$; see [KK65, Chapter II, Sec. 2].

(minimal) optical path from X to Y through the plane Π is the path XP_0Y[3] where $P_0 \in \Pi$ is the unique point (due to strict convexity) such that

$$\min\{N_1(P-X)+N_2(Y-P) : P \in \Pi\} = N_1(P_0-X)+N_2(Y-P_0). \tag{3.1}$$

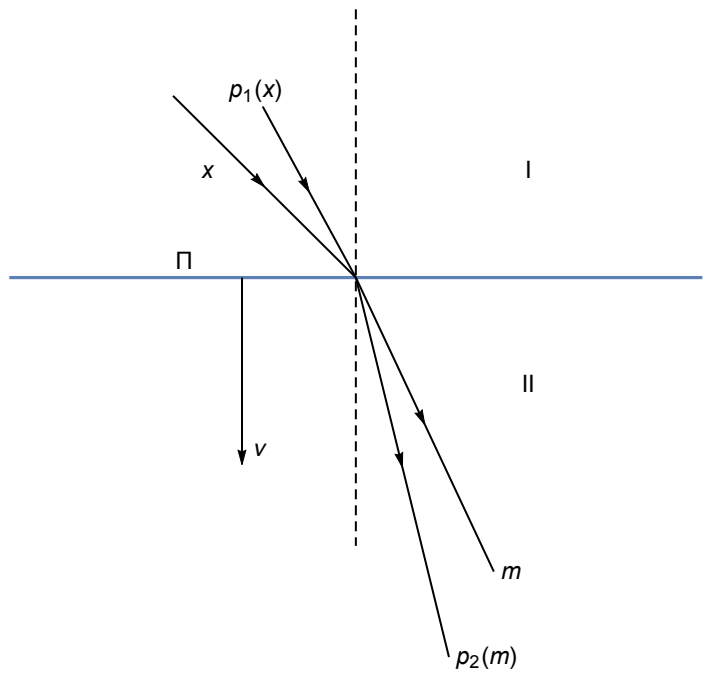

FIGURE 1. Snell's law

We then formulate the Snell law in anisotropic media as follows:

If an incident ray traveling in medium I with direction $x \in \Sigma_1$ and with $x \cdot \nu > 0$, strikes the plane Π at some point P_0, then this ray is refracted into medium II in the direction $m \in \Sigma_2$ if

$$p_2(m) - p_1(x) \parallel \nu, \tag{3.2}$$

with $p_i = \nabla N_i$, $i = 1, 2$; see Figure 1. In addition, for each $x \in \Sigma_1$ with $x \cdot \nu > 0$, there is at most one $m \in \Sigma_2$ satisfying (3.2) *with $m \cdot \nu \geq 0$.*

[3]In homogenous media light travels in straight lines. This follows from [KK65, Equation (3.97)] because the function H there is independent of x, y, z.

We shall prove that (3.2) is equivalent to Fermat's principle (3.1). In fact, let us first see that (3.1) implies that

$$p_2(Y - P_0) - p_1(P_0 - X) \parallel \nu \tag{3.3}$$

where ν is the normal to Π. In fact, for $P \in \Pi$ we can write $P = P_0 + \sum_{i=1}^{n-1} t_i\, e_i$ where $e_1, \cdots, e_{n-1}$ is a basis for Π. From (3.1)

$$\begin{aligned}
&\frac{\partial}{\partial t_j}\left(N_1\left(P_0 + \sum_{i=1}^{n-1} t_i\, e_i - X\right) + N_2\left(Y - P_0 - \sum_{i=1}^{n-1} t_i\, e_i\right)\right)\\
&= p_1\left(P_0 + \sum_{i=1}^{n-1} t_i\, e_i - X\right)\cdot e_j - p_2\left(Y - P_0 - \sum_{i=1}^{n-1} t_i\, e_i\right)\cdot e_j = 0
\end{aligned}$$

when $t_j = 0$, for $j = 1, \cdots, n-1$ so (3.3) follows. Also since p_i are homogenous of degree zero, from (3.3) we then obtain (3.2) with $m = \dfrac{Y - P_0}{N_2(Y - P_0)}$ and $x = \dfrac{P_0 - X}{N_1(P_0 - X)}$.

Vice versa, from (3.2) we deduce (3.1). In fact, let us fix $X \in I$ and $Y \in II$, and consider a path XP_0Y with $P_0 \in \Pi$. The ray from X to P_0 has direction $x = \dfrac{P_0 - X}{N_1(P_0 - x)}$, and the ray from P_0 to Y has direction $m = \dfrac{Y - P_0}{N_2(Y - P_0)}$. If x is refracted into m, then from (3.2), $p_2\left(\dfrac{Y - P_0}{N_2(Y - P_0)}\right) - p_1\left(\dfrac{P_0 - X}{N_1(P_0 - x)}\right) \parallel \nu$, and since p_i are homogeneous of degree zero

$$p_2\,(Y - P_0) - p_1\,(P_0 - X) \parallel \nu. \tag{3.4}$$

Consider the functional $F(P) = N_1(P - X) + N_2(Y - P)$ for $P \in \Pi$. Write $P \in \Pi$ as $P = P_0 + \sum_{i=1}^{n-1} t_i\, e_i$ where $e_1, \cdots, e_{n-1}$ is a basis for Π. We can write $F(P) = F(t_1, \cdots, t_{n-1})$. As before and from (3.4)

$$\frac{\partial}{\partial t_j} F(t_1, \cdots, t_{n-1}) = p_1\left(P_0 + \sum_{i=1}^{n-1} t_i\, e_i - X\right)\cdot e_j - p_2\left(Y - P_0 - \sum_{i=1}^{n-1} t_i\, e_i\right)\cdot e_j = 0$$

for $j = 1, \cdots, n-1$ and $t_1 = \cdots = t_{n-1} = 0$. On the other hand, since the functional F on Π is strictly convex, there are unique $t_1^0, \cdots, t_{n-1}^0$ such that the minimum of F is attained at $t_1^0, \cdots, t_{n-1}^0$. Therefore, $t_1^0 = \cdots = t_{n-1}^0 = 0$ and the point P_0 is the unique point in Π minimizing F, obtaining Fermat's principle.

Finally, let us show that there is at most one $m \in \Sigma_2$ satisfying (3.2). Indeed, from (3.2) there is $\lambda \in \mathbb{R}$ with $p_2(m) = p_1(x) + \lambda\, \nu$. That is, if ℓ is the line passing through $p_1(x)$ with direction ν, then $p_2(m) \in \ell \cap \Sigma_2^*$. So if there were $m_1 \neq m_2$ satisfying (3.2), then $p_2(m_i) \in \ell \cap \Sigma_2^*$, $i = 1, 2$. An outer normal to Σ_2^* at $p_2(m_i)$ equals to $\nabla N_2^*(p_2(m_i)) = p_2^*(p_2(m_i)) = m_i$ because $p_2 : \Sigma_2 \to \Sigma_2^*$ is a homeomorphism since Σ_2 is C^1 and strictly convex. And also $p_2(m_1) \neq p_2(m_2)$. Hence m_1 is an outer normal to the support plane of the strictly convex surface Σ_2^* at $p_2(m_1)$ and the segment $\overline{p_2(m_1)p_2(m_2)}$ lies within Σ_2^*. Thus $(p_2(m_2) - p_2(m_1)) \cdot m_1 < 0$. Similarly, $(p_2(m_1) - p_2(m_2)) \cdot m_2 < 0$. From (3.2), $p_2(m_2) - p_2(m_1)$ is parallel to ν, so we conclude that $m_1 \cdot \nu$ and $m_2 \cdot \nu$ must have different signs. Therefore, only one of the two m_i satisfies $m_i \cdot \nu \geq 0$ as desired.

Remark 3.1. Physically, the norm $N(x) = 1$ represents the location of the points x after traveling for a time t, with $c\, t = 1$, from the origin into the given medium. For example, if we are in an homogenous and isotropic medium with refractive index n, then the wave propagates from the origin with velocity $v = c/n$. So if x satisfies $N(x) = 1$, then the Euclidean distance from O to x must satisfy $|x|/t = v$. Since $t = 1/c$, we obtain $|x| = v/c = 1/n$ so $N(x) = n\,|x|$. Therefore, if medium I has refractive index n_1 and medium II has refractive index n_2, then $N_1(x) = n_1\,|x|$ and $N_2(x) = n_2\,|x|$. We then have $p_i(x) = \nabla N_i(x) = n_i\, \dfrac{x}{|x|}$, $i = 1, 2$, $x \neq 0$, and so from (3.2) we recover the standard vector Snell law: the unit incident direction x is refracted into the unit direction m when $n_1\, x - n_2\, m \parallel \nu$, see [GH09, Formula (2.1)].

3.1. **Physical constraints for refraction.** Since the incident ray x is in medium I we must have $x \cdot \nu > 0$, where ν is the normal to the hyperplane separating I and II, ν having direction from medium I to medium II. Similarly, since the refracted ray m is in medium II, we also have $m \cdot \nu \geq 0$; Figure 1.

We analyze here the meaning of these two physical constraints $x \cdot \nu > 0$ and $m \cdot \nu \geq 0$ in the following two cases.

Case 1: $p_1(x) - p_2(m) = \lambda\, \nu$ with $\lambda > 0$. Then $1 - x \cdot p_2(m) = x \cdot (p_1(x) - p_2(m)) = \lambda\, x \cdot \nu > 0$, and $m \cdot p_1(x) - 1 = m \cdot (p_1(x) - p_2(m)) = \lambda\, m \cdot \nu \geq 0$. Therefore in this case

the physical constraints for refraction are

$$x \cdot p_2(m) < 1 \text{ and } m \cdot p_1(x) \geq 1. \tag{3.5}$$

Case 2: $p_2(m) - p_1(x) = \lambda\, \nu$ with $\lambda > 0$. Then $x \cdot p_2(m) - 1 = x \cdot (p_2(m) - p_1(x)) = \lambda\, x \cdot \nu > 0$, and $1 - m \cdot p_1(x) = m \cdot (p_2(m) - p_1(x)) = \lambda\, m \cdot \nu \geq 0$. Therefore in this case the physical constraints for refraction are

$$x \cdot p_2(m) > 1 \text{ and } m \cdot p_1(x) \leq 1. \tag{3.6}$$

Therefore if an incident ray with direction $x \in \Sigma_1$ is refracted into the direction $m \in \Sigma_2$, then one and only one of physical constraints (3.5) and (3.6) hold.

To illustrate these Cases 1 and 2, we show the following.

Lemma 3.2. *We have the following:*

(a) If Σ_1 is strictly enclosed by Σ_2, that is, $\kappa := \sup_{N_1(x)=1} N_2(x) < 1$, and $x \in \Sigma_1$ is refracted into $m \in \Sigma_2$, then (3.5) *holds.*

(b) If Σ_2 is strictly enclosed by Σ_1, that is, $\kappa^* := \inf_{N_1(x)=1} N_2(x) > 1$, and $x \in \Sigma_1$ is refracted into $m \in \Sigma_2$, then (3.6) *holds.*

Proof. If $x \in \Sigma_1$, then $p_1(x) \in \Sigma_1^*$ and from Lemma 2.1(b) $x \cdot p_1(x) = 1$. Then

$$\begin{aligned} x \cdot (p_1(x) - p_2(m)) &= 1 - x \cdot p_2(m) \\ &= 1 - N_2(x)\, \frac{x}{N_2(x)} \cdot p_2(m) \geq 1 - N_2(x) > 0 \qquad \text{from Lemma 2.1(a).} \end{aligned} \tag{3.7}$$

So we are in Case 1 above and (3.5) follows.

To show (b)

$$\begin{aligned} m \cdot (p_2(m) - p_1(x)) &= 1 - m \cdot p_1(x) \\ &= 1 - N_1(m)\, \frac{m}{N_1(m)} \cdot p_1(x) \geq 1 - N_1(m) \qquad \text{from Lemma 2.1(a)} \\ &\geq 1 - \frac{1}{\kappa^*} N_2(m) = 1 - \frac{1}{\kappa^*} > 0. \end{aligned}$$

So we are in Case 2 above and (3.6) follows.

□

Remark 3.3. Notice that, as explained before, if medium I has refractive index n_1 and medium II has refractive index n_2, then $N_1(x) = n_1\,|x|$ and $N_2(x) = n_2\,|x|$; and $p_i(x) = n_i\,\dfrac{x}{|x|}$, $i = 1, 2$. Hence, if $n_1 > n_2$ we are in case (a) of Lemma 3.2, and condition (3.5) reads $x \cdot m \geq n_2/n_1$ for x, m unit vectors. If $n_1 < n_2$ then we are in case (b) of Lemma 3.2, and condition (3.6) reads $x \cdot m \geq n_1/n_2$ for x, m unit vectors. Therefore, when media I and II are homogenous and isotropic we recover the physical constraints showed in [GH09, Lemma 2.1].

Remark 3.4. Regarding Lemma 3.2, we show an example of surfaces Σ_1, Σ_2 that can cross each other but the physical constraints (3.5) and (3.6) hold. Indeed, let $N_1(x) = \sqrt{(1+\epsilon)^2\,x_1^2 + x_2^2 + x_3^2}$, $N_2(x) = \sqrt{x_1^2 + (1+\epsilon)^2\,x_2^2 + x_3^2}$, and $\Sigma_i = \{x \in \mathbb{R}^3 : N_i(x) = 1\}$, $i = 1, 2$; $\epsilon > 0$. We show that there exist $x \in \Sigma_1$ and $m \in \Sigma_2$ such that (3.5) holds and also there exist $x \in \Sigma_1$ and $m \in \Sigma_2$ such that (3.6) holds. In this case, if $N_1(x) = 1$ and $N_2(m) = 1$ then condition (3.5) means

$$(x_1, x_2, x_3) \cdot (m_1, (1+\epsilon)^2\,m_2, m_3) < 1 \quad \text{and} \quad (m_1, m_2, m_3) \cdot ((1+\epsilon)^2\,x_1, x_2, x_3) \geq 1.$$

If $x', m' \in S^2$, and let $x = \left(x_1'/(1+\epsilon), x_2', x_3'\right)$ and $m = \left(m_1', m_2'/(1+\epsilon), m_3'\right)$, then the last two inequalities are equivalent to

$$\frac{1}{1+\epsilon}\,x_1'\,m_1' + (1+\epsilon)\,x_2'\,m_2' + x_3'\,m_3' < 1 \quad \text{and} \quad (1+\epsilon)\,x_1'\,m_1' + \frac{1}{1+\epsilon}\,x_2'\,m_2' + x_3'\,m_3' \geq 1.$$

If we pick $0 < \delta < 1$ with $\dfrac{1}{1+\epsilon} < (1-\delta)^2 < 1+\epsilon$, then the points $x' = \left(1-\delta, 0, \sqrt{1-(1-\delta)^2}\right)$ and $m' = \left(1-\delta, \sqrt{1-(1-\delta)^2}, 0\right)$ belong to S^2 and satisfy the last two inequalities.

Similarly, that (3.6) has solutions means

$$\frac{1}{1+\epsilon}\,x_1'\,m_1' + (1+\epsilon)\,x_2'\,m_2' + x_3'\,m_3' > 1 \quad \text{and} \quad (1+\epsilon)\,x_1'\,m_1' + \frac{1}{1+\epsilon}\,x_2'\,m_2' + x_3'\,m_3' \leq 1,$$

which are solved by the points $x' = \left(\sqrt{1-(1-\delta)^2}, 1-\delta, 0\right)$ and $m' = \left(0, 1-\delta, \sqrt{1-(1-\delta)^2}\right)$.

4. Uniformly refracting surfaces

In this section, we shall describe the surfaces separating two anisotropic materials I and II, like in Section 3, so that rays emanating from a point source, the origin, located in medium I are refracted in medium II into a fixed direction $m \in \Sigma_2$. These surfaces will have the form

$$\{X \in \mathbb{R}^n : N_1(X) = p_2(m) \cdot X + b\}, \tag{4.1}$$

where $b \in \mathbb{R}$. If we write $X = \rho(x)\, x$ for $N_1(x) = 1$, then the polar radius

$$\rho(x) = \frac{b}{1 - x \cdot p_2(m)} \qquad \text{for } x \in \Sigma_1.$$

To show that these surfaces do the desired refraction job, as in Section 3.1 we distinguish two cases.

Case I: Let $m \in \Sigma_2$ be such that the set

$$\{x \in \Sigma_1 : x \cdot p_2(m) < 1 \text{ and } m \cdot p_1(x) \geq 1\}, \tag{4.2}$$

has non empty interior in the topology of Σ_1. In this case, the desired surface is

$$S_I(m,b) = \left\{\rho(x)x : \rho(x) = \frac{b}{1 - x \cdot p_2(m)},\ x \in \Sigma_1 \text{ with } x \cdot p_2(m) < 1 \text{ and } m \cdot p_1(x) \geq 1\right\} \tag{4.3}$$

with $b > 0$. In fact, to verify that each ray having direction $x \in \Sigma_1$ with $x \cdot p_2(m) < 1$ and $m \cdot p_1(x) \geq 1$ is refracted by $S_I(m,b)$ into m, we need to verify that (3.2) holds, and that the physical constraints $x \cdot \nu > 0$ and $m \cdot \nu \geq 0$ are met with ν a normal from medium I to II. From (4.1), a normal vector to $S_I(m,b)$ toward medium II at a point X is $\nu = p_1(x) - p_2(m)$ with $x = X/N_1(X)$; ν is not necessarily unit, so (3.2) holds. From Lemma 2.1

$$x \cdot \nu = x \cdot (p_1(x) - p_2(m)) = 1 - x \cdot p_2(m) > 0. \tag{4.4}$$

Also $m \cdot \nu = m \cdot (p_1(x) - p_2(m)) = m \cdot p_1(x) - 1 \geq 0$ by the definition of S_I.

Case II: let $m \in \Sigma_2$ be such that

$$\{x \in \Sigma_1 : x \cdot p_2(m) > 1 \text{ and } m \cdot p_1(x) < 1\} \neq \emptyset. \tag{4.5}$$

In this case, the desired surface is

(4.6)

$$S_{II}(m,b) = \left\{ \rho(x)x : \rho(x) = \frac{b}{x \cdot p_2(m) - 1},\ x \in \Sigma_1 \text{ with } m \cdot p_1(x) < 1 \text{ and } x \cdot p_2(m) > 1 \right\}$$

with $b > 0$. In fact and once again, to verify that each ray with direction $x \in \Sigma_1$ such that $m \cdot p_1(x) < 1$ and $x \cdot p_2(m) > 1$ is refracted by $S_{II}(m,b)$ into m, we need to verify that (3.2) holds, and the physical constraints $x \cdot \nu > 0$ and $m \cdot \nu \geq 0$ are met with ν a normal toward medium II. From the definition of S_{II}, a normal -not necessarily unit- to S_{II} at a point X is $\nu = p_2(m) - p_1(x)$ with $x = X/N_1(X)$; and so (3.2) holds. Also $x \cdot \nu = x \cdot (p_2(m) - p_1(x)) = x \cdot p_2(m) - 1 > 0$ by the definition of S_{II}, and thus ν is a normal toward medium II. From Lemma 2.1

$$m \cdot \nu = m \cdot (p_2(m) - p_1(x)) = 1 - m \cdot p_1(x) > 0$$

by the definition of S_{II}.

Remark 4.1. If medium I is homogeneous and isotropic with refractive index n_1, then $N_1(x) = n_1 |x|$. Also, if II is also similar with refractive index n_2, then $N_2(x) = n_2 |x|$. In this case, condition (4.2) is equivalent to $n_1 > n_2$, and the surface $S_I(m,b)$ is a half ellipsoid of revolution with axis m, recovering the surfaces from [GH09, Formula (2.8)]. Similarly, condition (4.5) is equivalent to $n_1 < n_2$, and $S_{II}(m,b)$ is one of the branches of a hyperboloid of two sheets as in [GH09, Formula (2.9)].

Remark 4.2. For the uniformly refracting surfaces S_I and S_{II}, each refracted ray cannot intersect the surface at more than one point, that is, each refracted ray is entirely contained in medium II. We show this for S_{II}. From (4.6) we can re-write S_{II} as $\{X : N_1(X) = X \cdot p_2(m) - b\}$, with $b > 0$. Let B be the unbounded convex body enclosed by S_{II}, i.e., $B = \{X : N_1(X) - X \cdot p_2(m) + b < 0\}$. Then B encloses medium II, while the origin O and medium I are outside B. Consider a ray emanating from O with direction x_0 striking S_{II} at a point X_0 that is then refracted into a ray with

direction m. Suppose by contradiction that this refracted ray strikes S_{II} at another point X_1. Then we observe that

(i) The segment $\overline{X_0X_1}$ has direction m and since B is convex it lies inside B;

(ii) $X_1 = X_0 + t\,m$ for some $t > 0$, and $X_1 + s\,m$ is outside B for any $s > 0$, since B is convex and segment $\overline{X_0X_1}$ traverses B;

(iii) A ray from O striking S_{II} at X_1 is refracted into medium II inside B in the direction m, i.e., $X_1 + s\,m$ is inside B for small $s > 0$. Notice that the half open segment $[O, X_1)$ is contained outside B because if $X_1 \in S_{II}$, then tX_1 is outside B for all $0 \leq t < 1$, that is, $N_1(tX_1) - tX_1 \cdot p_2(m) + b > 0$. In fact, since N_1 is homogeneous of degree one, $N_1(tX_1) - tX_1 \cdot p_2(m) + b = t\,(N_1(X_1) - X_1 \cdot p_2(m) + b) + (1-t)b = (1-t)b > 0$ as $X_1 \in S_{II}$ and $b > 0$.

Thus, (ii) and (iii) are contradictory.

Remark 4.3. Given the origin O in medium I and a point P in medium II, we seek an interface surface $\mathcal{S}$ between these media so that all rays from O are refracted into P according to the Snell law (3.2). If $X(t)$ is a curve on $\mathcal{S}$, the vector $\dfrac{X(t)}{N_1(X(t))}$ must be refracted into the vector $\dfrac{P - X(t)}{N_2(P - X(t))}$, so from (3.2) we must have

$$X'(t) \cdot \left(p_1\left(\frac{X(t)}{N_1(X(t))}\right) - p_2\left(\frac{P - X(t)}{N_2(P - X(t))}\right)\right) = 0.$$

Since p_i are homogenous of degree zero, it follows that $X'(t)\cdot(p_1\,(X(t)) - p_2\,(P - X(t))) = 0$, and therefore $\dfrac{d}{dt}\,(N_1(X(t)) + N_2(P - X(t))) = 0$. The surface $\mathcal{S}$ is then given by the equation

$$N_1(X) + N_2(X - P) = \text{constant}. \tag{4.7}$$

In the homogenous and isotropic case, $N_i(X) = n_i\,|X|$ and so (4.7) yields Descartes ovals as in [GH14, Sec. 4].

5. Solution to the refractor problem

Using the uniformly refracting surfaces introduced in Section 4, we state and solve here the refraction problem we are interested in.

We are given two closed connected domains $\Omega_1 \subset \Sigma_1$, $\Omega_2 \subset \Sigma_2$, a non negative function $f \in L^1(\Omega_1)$, and a Radon measure μ in Ω_2 satisfying the following conditions:

(H.a) the surface measure of the relative boundary of Ω_1 is zero (as a subset of Σ_1);

(H.b) $\inf_{x\in\Omega_1, m\in\Omega_2} m\cdot p_1(x) \geq 1$ and there exists $\kappa > 0$ such that $\sup_{x\in\Omega_1, m\in\Omega_2} x\cdot p_2(m) \leq \kappa < 1$;

(H.c) Σ_1 and Σ_2 are C^1 and strictly convex.

The second inequality in **(H.b)** implies that (4.2) holds for all vectors $m \in \Omega_2$. Refractors are then defined as follows.

Definition 5.1. *The surface $S = \{\rho(x)\, x : x \in \Omega_1\}$, with $\rho \in C(\Omega_1)$, $\rho > 0$, is a refractor from Ω_1 to Ω_2, if for each $x_0 \in \Omega_1$ there exist $m \in \Omega_2$ and $b > 0$ such that the surface $S_I(m, b)$ supports S at x_0, that is,*

$$\rho(x) \leq \frac{b}{1 - x \cdot p_2(m)} \quad \textit{for all } x \in \Omega_1 \textit{ with equality at } x = x_0.$$

The refractor mapping associated with the refractor S is the set valued function

$$\mathcal{R}_S(x_0) = \{m \in \Omega_2 : \textit{there exists } S_I(m, b) \textit{ supporting } S \textit{ at } x_0\}. \tag{5.1}$$

We have the following lemma.

Lemma 5.2. *If a refractor S is parametrized by $\rho(x)\, x$, then ρ is Lipschitz continuous in Ω_1.*

Proof. Let $x_0, x \in \Omega_1$ and $S_I(m,b)$ supporting S at x_0. Then

$$\begin{aligned}\rho(x) - \rho(x_0) &\le \frac{b}{1 - x\cdot p_2(m)} - \frac{b}{1 - x_0\cdot p_2(m)} = b\,\frac{(x-x_0)\cdot p_2(m)}{(1 - x\cdot p_2(m))\,(1 - x_0\cdot p_2(m))}\\ &= \rho(x_0)\,\frac{(x-x_0)\cdot p_2(m)}{1 - x\cdot p_2(m)} \le \max_{\Omega_1}\rho\,\frac{|x-x_0|\,|p_2(m)|}{1-\kappa} \qquad \text{from } \textbf{(H.b)} \text{ above}\\ &\le C\,|x - x_0|.\end{aligned}$$

Reversing the roles of x and x_0 we obtain the lemma. □

We recall $C_S(\Omega_1,\Omega_2)$ denotes the class of set-valued maps $\Phi : \Omega_1 \to \Omega_2$ that are single valued for a.e. $x \in \Omega_1$, with respect to $f\,dx$, that are continuous in Ω_1, and $\Phi(\Omega_1) = \Omega_2$. Continuity of Φ at $x_0 \in \Omega_1$ means that if $x_k \to x_0$ and $y_k \in \Phi(x_k)$, then there is a subsequence y_{k_j} and $y_0 \in \Phi(x_0)$ such that $y_{k_j} \to y_0$.

Lemma 5.3. *If S is a refractor from Ω_1 to Ω_2, then the refractor map $\mathcal{R}_S \in C_S(\Omega_1,\Omega_2)$.*

Proof. Let $m \in \Omega_2$ and define $b = \max_{x\in\Omega_1}(\rho(x)\,(1 - x\cdot p_2(m))$. From condition **(H.b)** above, $b \ge \max_{x\in\Omega_1}(\rho(x)\,(1-\kappa)) > 0$. Also, there is $x_0 \in \Omega_1$ with $b = \rho(x_0)\,(1 - x_0\cdot p_2(m))$ and so $m \in \mathcal{R}_S(x_0)$, showing that $\mathcal{R}_S(\Omega_1) = \Omega_2$.

Next, let us show that $\mathcal{R}_S(x)$ is single valued for a.e. $x \in \Omega_1$. In fact, if at $x_0 \in \Omega_1$ there exist $m_1 \ne m_2 \in \Omega_2$ with $m_i \in \mathcal{R}_S(x_0)$, $i = 1,2$, then x_0 is a singular point to the surface S. Otherwise, since $S_I(m_i,b_i)$, $i = 1,2$ support S at x_0, they would have the same tangent plane at $x_0\rho(x_0)$. Therefore, by the Snell law and since there is at most one m satisfying (3.2), we obtain $m_1 = m_2$. From Lemma 5.2, S is Lipschitz, and since $|\partial\Omega_1| = 0$, we obtain that $\mathcal{R}_S(x)$ is single valued a.e. in Ω_1.

It remains to show that $\mathcal{R}_S$ is continuous. Let $x_i \to x_0 \in \Omega_1$ and let $m_i \in \mathcal{R}_S(x_i)$. Hence $\rho(x) \le \dfrac{b_i}{1 - x\cdot p_2(m_i)}$ for all $x \in \Omega_1$ with equality at $x = x_i$. As before, $b_i = \rho(x_i)\,(1 - x_i\cdot p_2(m_i)) \ge (1-\kappa)\,\min_{\Omega_1}\rho$, and $b_i = \rho(x_i)\left(1 - N_2(x_i)\,\dfrac{x_i}{N_2(x_i)}\cdot p_2(m_i)\right) \le \max_{\Omega_1}\rho\,\left(1 + \sup_{N_1(x)=1} N_2(x)\right)$ from Lemma 2.1(a). We have $m_i \in \Omega_2 \subset \Sigma_2$ and $p_2 \in C(\Sigma_2)$. By compactness there are subsequences $m_{i_k} \to m_0 \in \Omega_2$ and $b_{i_k} \to b_0 > 0$ so that $\rho(x) \le \dfrac{b_0}{1 - x\cdot p_2(m_0)}$ for all $x \in \Omega_1$ with equality at $x = x_0$. This completes the proof to the lemma.

□

We obtain from Lemma 5.3 and [GH14, Lemma 2.1] that if S is a refractor from Ω_1 to Ω_2, then the set function

$$\mathcal{M}_{S,f}(E) = \int_{\mathcal{R}_S^{-1}(E)} f(x)\,dx \tag{5.2}$$

is a Borel measure in Ω_2, that *is called the refractor measure*.

Given $\mathcal{F} \subset C(\Omega_1)$ and a map $\mathcal{T} : \mathcal{F} \to C_S(\Omega_1, \Omega_2)$, we say $\mathcal{T}$ is continuous at $\phi \in \mathcal{F}$ if whenever $\phi_j \in \mathcal{F}$, $\phi_j \to \phi$ uniformly in Ω_1, $x_0 \in \Omega_1$ and $y_j \in \mathcal{T}(\phi_j)(x_0)$, then there exists a subsequence $y_{j_\ell} \to y_0$ with $y_0 \in \mathcal{T}(\phi)(x_0)$. If we let

$$\mathcal{F} = \{\rho \in C(\Omega_1) : S_\rho \text{ is a refractor from } \Omega_1 \text{ to } \Omega_2 \text{ parametrized by } \rho(x)x\} \tag{5.3}$$

then we have the following lemma.

Lemma 5.4. *The mapping $\mathcal{T} : \mathcal{F} \to C_S(\Omega_1, \Omega_2)$ defined by $\mathcal{T}(\rho) = \mathcal{R}_{S_\rho}$ is continuous at each $\rho \in \mathcal{F}$.*

Proof. Let $\rho_j, \rho \in \mathcal{F}$ with $\rho_j \to \rho$ uniformly in Ω_1, $x_0 \in \Omega_1$ and $m_j \in \mathcal{R}_{S_{\rho_j}}(x_0)$. Hence $\rho_j(x) \le \dfrac{b_j}{1 - x \cdot p_2(m_j)}$ for all $x \in \Omega_1$ with equality at $x = x_0$. As in the last part of the proof of Lemma 5.3, b_j are bounded away from 0 and ∞. Therefore there exist subsequences $b_{j_\ell} \to b$ and $m_{j_\ell} \to m$ with $\rho(x) \le \dfrac{b}{1 - x \cdot p_2(m)}$ for all $x \in \Omega_1$ with equality at $x = x_0$. Thus $m \in \mathcal{R}_{S_\rho}(x_0)$ and we are done. □

As a consequence of Lemma 5.4 we obtain from [GH14, Lemma 2.3] that

$$\text{if } \rho_j \to \rho \text{ uniformly in } \Omega_1, \text{ then } \mathcal{M}_{S_{\rho_j},f} \to \mathcal{M}_{S_\rho,f} \text{ weakly.}$$

It is easy to verify that

(A1) if S_{ρ_1} and S_{ρ_2} are refractors from Ω_1 to Ω_2, then $S_{\rho_1 \wedge \rho_2}$ is a refractor from Ω_1 to Ω_2 with $\rho_1 \wedge \rho_2 = \min\{\rho_1, \rho_2\}$;

(A2) if $\rho_1(x_0) \le \rho_2(x_0)$, then $\mathcal{R}_{S_{\rho_1}}(x_0) \subset \mathcal{R}_{S_{\rho_1 \wedge \rho_2}}(x_0)$;

(A3) we let $h_{b,m}(x) = \dfrac{b}{1 - x \cdot p_2(m)}$, and we have

$$\{h_{b,m} : m \in \Omega_2, 0 < b < \infty\} \subset \mathcal{F},$$

with $\mathcal{F}$ defined by (5.3) (notice that from **(H.b)** $h_{b,m}(x)$ is well defined for $x \in \Omega_1$ and $m \in \Omega_2$). In addition we have the following

(a) $m \in \mathcal{R}_{S_{h_{b,m}}}(x)$ for all $x \in \Omega_1$, from Section 4, Case I;

(b) $h_{b_1,m} \leq h_{b_2,m}$ for $b_1 \leq b_2$;

(c) $h_{b,m} \to 0$ uniformly in Ω_1 as $b \to 0$;

(d) $h_{b,m} \to h_{b_0,m}$ uniformly in Ω_1 as $b \to b_0$.

We then introduce the following definition.

Definition 5.5. *Let $f \in L^1(\Omega_1)$ and let μ be a Radon measure in Ω_2 with $\int_{\Omega_1} f\,dx = \mu(\Omega_2)$. The refractor S from Ω_1 to Ω_2 is a weak solution of the refractor problem if*

$$\mathcal{M}_{S,f}(E) = \mu(E)$$

for each Borel set $E \subset \Omega_2$, where $\mathcal{M}_{S,f}$ is the refractor measure defined by (5.2).

With the above set up, we obtain the following theorems showing solvability of the refractor problem for anisotropic media under assumptions **(H.a)**, **(H.b)**, and **(H.c)** above. We first show solvability when the measure μ is discrete.

Theorem 5.6. *Let $f \in L^1(\Omega_1)$ with $f > 0$ a.e., $m_1, \cdots, m_N \in \Omega_2$ be distinct points, and $g_1, \cdots, g_N$ positive numbers satisfying $\int_{\Omega_1} f\,dx = \sum_{i=1}^N g_i$.*

Then for each $0 < b_1 < \infty$ there exist unique positive $b_2, \cdots, b_N$ such that

$$S = \{\rho(x)x : x \in \Omega_1\} \text{ with } \rho(x) = \min_{1 \leq i \leq N} h_{b_i,m_i}(x),$$

is a weak solution to the refractor problem. In addition, $\mathcal{M}_{S,f}(\{m_i\}) = g_i$ for $1 \leq i \leq N$.

Proof. Existence follows from [GH14, Theorem 2.5], for which we need to verify that the assumptions of that theorem are met. In fact, we need to show that we can choose positive numbers $b_2^0, \cdots, b_N^0$ such that $\rho_0(x) = \min_{1 \leq i \leq N} h_{b_i^0,m_i}(x)$ such that

$\mathcal{M}_{S_{\rho_0},f}(m_i) \le g_i$ for $2 \le i \le N$, where $b_1^0 = b_1$. We have $h_{b_1^0,m_1}(x) = \dfrac{b_1^0}{1 - x \cdot p_2(m_1)} \le \dfrac{b_1}{1-\kappa}$ from **(H.b)**. Also $h_{b_i^0,m_i}(x) = \dfrac{b_i^0}{1 - x \cdot p_2(m_i)} \ge \dfrac{b_i^0}{1 + \sup_{N_1(x)} N_2(x)}$ for $2 \le i \le N$ from Lemma 2.1(b). Therefore choosing $b_2^0, \cdots, b_N^0$ suitable so that $\rho_0(x) = h_{b_1^0,m_1}(x)$, the needed assumptions are met and the existence follows. The uniqueness follows from [GH14, Theorem 2.7] since $f > 0$ a.e. □

We are now ready to prove the following existence theorem for a general Radon measure μ.

Theorem 5.7. *Let $f \in L^1(\Omega_1)$ with $f > 0$ a.e, and let μ be a Radon measure in Ω_2 such that $\int_{\Omega_1} f(x)\,dx = \mu(\Omega_2)$. Then for each $x_0 \in \Omega_1$ and $R_0 > 0$, there exists $\mathcal{S}$ weak solution to the refractor problem passing through the point $X_0 = R_0\, x_0$.*

Proof. Let $\mu_\ell = \sum_{i=1}^N g_i\, \delta_{m_i}$ be a sequence of discrete measures with $\mu_\ell \to \mu$ weakly and $\mu_\ell(\Omega_2) = \mu(\Omega_2)$ for $\ell = 1, 2, \cdots$. From Theorem 5.6 and for the measure μ_ℓ, there exists a refractor $S_{\rho_\ell^*}$ parametrized by ρ_ℓ^*. Notice that $S_{C_\ell\, \rho_\ell^*}$ is also a solution to the same refractor problem since $\mathcal{R}_{C_\ell\, \rho_\ell^*} = \mathcal{R}_{\rho_\ell^*}$ for each positive constant C_ℓ. Then pick C_ℓ so that $C_\ell\, \rho_\ell^*(x_0) = R_0$. Now we use the existence result [GH14, Theorem 2.8], and in order to do that we need to verify that the hypotheses (i) and (ii) of that theorem hold in the present case. To verify (i) we show that if $R_1 \in \text{Range}\,(h_{b,m})$, then

$$R_1\, \frac{1-\kappa}{1 + \sup_{N_1(x)} N_2(x)} \le h_{b,m} \le R_1\, \frac{1 + \sup_{N_1(x)} N_2(x)}{1-\kappa}.$$

In fact, there exists $x_1 \in \Omega_1$ with $R_1 = h_{b,m}(x_1) = \dfrac{b}{1 - x_1 \cdot p_2(m)}$, so $h_{b,m}(x) = \dfrac{b}{1 - x \cdot p_2(m)} = R_1\, \dfrac{1 - x_1 \cdot p_2(m)}{1 - x \cdot p_2(m)}$, and the desired inequalities follow from Lemma 2.1(b) and **(H.b)**. The verification of (ii), that is, the family $\{\rho \in \mathcal{F} : C_0 \le \rho \le C_1\}$ is compact in $C(\Omega_1)$, follows from Lemma 5.2 and the proof of Lemma 5.4. □

Remark 5.8. In the same way we can state and solve the refractor problem under **(H.a)**, **(H.c)**, and with **(H.b)** replaced by the condition

(H.b′) $\sup_{x\in\Omega_1, m\in\Omega_2} m\cdot p_1(x) < 1$ and there exists $\delta > 0$ such that $\inf_{x\in\Omega_1, m\in\Omega_2} x\cdot p_2(m) = 1+\delta$, using instead the uniformly refracting surfaces $S_{II}(m,b)$ defined by (4.6). Now the functions $h_{b,m}$ are defined by $h_{b,m}(x) = \dfrac{b}{x\cdot p_2(m)-1}$ and the properties (A1)-(A3) defined after Lemma 5.4 must be changed in accordance with properties (A1′)-(A3′) in [GH14, Section 2.2]. All lemmas in this section then hold true with obvious changes. For the existence of solutions we now need to use [GH14, Theorems 2.9 and 2.11].

In Theorem 5.7, uniqueness up to dilations, follows from optimal mass transport, see Section 7.

6. Propagation of light in anisotropic materials

We begin this section with some background on the propagation of light in anisotropic materials. Let us assume we have a material whose permittivity and permeability are given by positive definite and symmetric matrices $\epsilon(x,y,z)$ and $\mu(x,y,z)$, respectively. Assuming we are in the geometric optics regime, i.e., the wave length of the radiation is very small compared with the objects considered, it is known [KK65, Chap. III, Sect. 4] that the function $\psi = \psi(x,y,z)$ defining the wave fronts $\psi(x,y,z)$=constant, satisfies the following first order pde, the Fresnel differential equation:

$$\det\begin{pmatrix} \epsilon & R \\ -R & \mu \end{pmatrix} = 0, \tag{6.1}$$

where R is the 3×3 skew-symmetric matrix

$$R = \begin{pmatrix} 0 & -\psi_z & \psi_y \\ \psi_z & 0 & -\psi_x \\ -\psi_y & \psi_x & 0 \end{pmatrix}.$$

We can re write Fresnel's equation in a simpler form using the following Schur's determinant identity: *if A is an $n\times n$ invertible matrix, B is $n\times m$, C is $m\times n$ and D*

is $m \times m$, then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \, \det\left(D - C\,A^{-1}\,B\right) = \det D \, \det\left(A - B\,D^{-1}\,C\right),$$

of course for the last identity D is invertible. We then get

$$\det\begin{pmatrix} \epsilon & R \\ -R & \mu \end{pmatrix} = \det \epsilon \, \det\left(\mu + R\,\epsilon^{-1}\,R\right) = \det \mu \, \det\left(\epsilon + R\,\mu^{-1}\,R\right)$$

and since ϵ, μ are positive definite, (6.1) is equivalent to either

$$\det\left(\mu + R\,\epsilon^{-1}\,R\right) = 0, \tag{6.2}$$

or

$$\det\left(\epsilon + R\,\mu^{-1}\,R\right) = 0. \tag{6.3}$$

Letting

$$\tau = \mu^{-1/2}\epsilon\mu^{-1/2}, \tag{6.4}$$

τ is symmetric and positive definite, so there is an orthogonal matrix O and a diagonal matrix D such that

$$\tau = ODO^t.$$

For a column vector $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ define

$$\mathrm{Skew}(v) = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

Given a 3×3 matrix B we have the formula

$$B^t\,\mathrm{Skew}(B\,v)\,B = \det B\,\mathrm{Skew}(v).$$

We then re write (6.3) as follows:

$$\det\left(\epsilon + R\,\mu^{-1}\,R\right) = \det \mu \, \det\left(\mu^{-1/2}\epsilon\mu^{-1/2} + \left(\mu^{-1/2}R\mu^{-1/2}\right)\left(\mu^{-1/2}R\mu^{-1/2}\right)\right).$$

Also since $\mu^{-1/2}$ is symmetric, we have

$$\mu^{-1/2}R\mu^{-1/2} = \mu^{-1/2}\mathrm{Skew}(\nabla\psi)\mu^{-1/2} = \mu^{-1/2}\mathrm{Skew}\left(\mu^{-1/2}\mu^{1/2}\nabla\psi\right)\mu^{-1/2}$$
$$= \det\left(\mu^{-1/2}\right)\mathrm{Skew}\left(\mu^{1/2}\nabla\psi\right).$$

Also

$$\begin{aligned}
&\mu^{-1/2}\epsilon\mu^{-1/2} + \left(\mu^{-1/2}R\mu^{-1/2}\right)\left(\mu^{-1/2}R\mu^{-1/2}\right)\\
&= ODO^t + OO^t\left(\mu^{-1/2}R\mu^{-1/2}\right)OO^t\left(\mu^{-1/2}R\mu^{-1/2}\right)OO^t\\
&= O\left(D + O^t\left(\mu^{-1/2}R\mu^{-1/2}\right)OO^t\left(\mu^{-1/2}R\mu^{-1/2}\right)O\right)O^t.
\end{aligned}$$

We have

$$\begin{aligned}
\bar{R} := O^t\left(\mu^{-1/2}R\mu^{-1/2}\right)O &= \det\left(\mu^{-1/2}\right)O^t\,\mathrm{Skew}\left(\mu^{1/2}\nabla\psi\right)O\\
&= \det\left(\mu^{-1/2}\right)O^t\,\mathrm{Skew}\left(O\,O^t\mu^{1/2}\nabla\psi\right)O\\
&= \det\left(\mu^{-1/2}\right)\det O\,\mathrm{Skew}\left(O^t\mu^{1/2}\nabla\psi\right),
\end{aligned}$$

so

$$\begin{aligned}
D + \bar{R}\bar{R} &= D + \left(\det\left(\mu^{-1/2}\right)\right)^2(\det O)^2\,\mathrm{Skew}\left(O^t\mu^{1/2}\nabla\psi\right)\mathrm{Skew}\left(O^t\mu^{1/2}\nabla\psi\right)\\
&= D + \left(\det\left(\mu^{-1/2}\right)\right)^2\mathrm{Skew}\left(O^t\mu^{1/2}\nabla\psi\right)\mathrm{Skew}\left(O^t\mu^{1/2}\nabla\psi\right)\\
&= D + \mathrm{Skew}\left(O^t\frac{\mu^{1/2}}{\det\left(\mu^{1/2}\right)}\nabla\psi\right)\mathrm{Skew}\left(O^t\frac{\mu^{1/2}}{\det\left(\mu^{1/2}\right)}\nabla\psi\right).
\end{aligned}$$

Therefore

$$\det\left(\epsilon + R\,\mu^{-1}\,R\right) = \det\mu\,\det\left(D + \mathrm{Skew}\left(O^t\frac{\mu^{1/2}}{\det\left(\mu^{1/2}\right)}\nabla\psi\right)\mathrm{Skew}\left(O^t\frac{\mu^{1/2}}{\det\left(\mu^{1/2}\right)}\nabla\psi\right)\right)$$
$$= \det\mu\,\det\left(D + \mathrm{Skew}(p)\,\mathrm{Skew}(p)\right) = 0, \tag{6.5}$$

where

$$p = \begin{pmatrix} p_1\\ p_2\\ p_3\end{pmatrix} := O^t\frac{\mu^{1/2}}{\det\left(\mu^{1/2}\right)}\nabla\psi.$$

Notice that this calculation is done at a fixed point (x, y, z) since the matrices ϵ and μ depend on the point (x, y, z); therefore the matrices D and O depend also on (x, y, z). Next we have

$$\mathrm{Skew}(p)\,\mathrm{Skew}(p) = p\,p^t - p^t\,p\,Id = p \otimes p - (p \cdot p)\,Id,$$

so by (6.5) the Fresnel equation for the wave fronts (6.3) is then

$$0 = \det\left(D + \mathrm{Skew}(p)\,\mathrm{Skew}(p)\right) = \det\left(D + p \otimes p - (p \cdot p)\,Id\right).$$

To write this equation in a more convenient form, set

$$D = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix},$$

(a matrix depending on (x, y, z)), so

$$\begin{aligned} 0 &= \det\left(D + p \otimes p - (p \cdot p)\,Id\right) \\ &= \det \begin{pmatrix} \tau_1 + p_1^2 - |p|^2 & p_1 p_2 & p_1 p_3 \\ p_2 p_1 & \tau_2 + p_2^2 - |p|^2 & p_2 p_3 \\ p_3 p_1 & p_3 p_2 & \tau_3 + p_3^2 - |p|^2 \end{pmatrix} \\ &= \det \begin{pmatrix} \tau_1 - p_2^2 - p_3^2 & p_1 p_2 & p_1 p_3 \\ p_2 p_1 & \tau_2 - p_1^2 - p_3^2 & p_2 p_3 \\ p_3 p_1 & p_3 p_2 & \tau_3 - p_1^2 - p_2^2 \end{pmatrix}. \end{aligned}$$

Let us now define for an arbitrary vector (p_1, p_2, p_3) the following functions, *which depend on the point* (x, y, z) *since* τ_i *depend on* (x, y, z)

$$\Phi(p_1, p_2, p_3) = \frac{1}{2}\left(\frac{1}{\tau_2} + \frac{1}{\tau_3}\right)p_1^2 + \frac{1}{2}\left(\frac{1}{\tau_1} + \frac{1}{\tau_3}\right)p_2^2 + \frac{1}{2}\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)p_3^2,$$

and

$$\Psi(p_1, p_2, p_3) = (p_1^2 + p_2^2 + p_3^2)\left(\frac{1}{\tau_2\tau_3}p_1^2 + \frac{1}{\tau_1\tau_3}p_2^2 + \frac{1}{\tau_1\tau_2}p_3^2\right).$$

It is easy to check that

(6.6)
$$\frac{1}{\tau_1\tau_2\tau_3}\det\begin{bmatrix}\tau_1-p_2^2-p_3^2 & p_1p_2 & p_1p_3\\ p_2p_1 & \tau_2-p_1^2-p_3^2 & p_2p_3\\ p_3p_1 & p_3p_2 & \tau_3-p_1^2-p_2^2\end{bmatrix}=1-2\Phi(p_1,p_2,p_3)+\Psi(p_1,p_2,p_3).$$

Now write

$$1-2\Phi+\Psi=1-2\Phi+\Phi^2-\Phi^2+\Psi=(1-\Phi)^2-(\Phi^2-\Psi).$$

Next notice that

$$\Phi^2\geq\Psi, \tag{6.7}$$

which follows using Lagrange multipliers since $\Phi^2-\Psi$ is homogenous of degree four. So we can write

$$1-2\Phi+\Psi=\left(1-\Phi-\sqrt{\Phi^2-\Psi}\right)\left(1-\Phi+\sqrt{\Phi^2-\Psi}\right).$$

We then obtain that the Fresnel equation of wave fronts (6.3) can be split as the following two equations

$$1-\Phi-\sqrt{\Phi^2-\Psi}=0 \qquad \text{or} \qquad 1-\Phi+\sqrt{\Phi^2-\Psi}=0, \tag{6.8}$$

see also [KK65, (3.48c)]. Each of these equations describes a three dimensional surface that depends of the point (x,y,z) chosen at the beginning; see Figure 2. That is, in this way each point (x,y,z) in the space has associated a pair of surfaces, one enclosing the other. The inner surface is convex and the outer surface is neither convex nor concave. We have shown that the vector

$$p=\begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}:=O^t\frac{\mu^{1/2}}{\det(\mu^{1/2})}\nabla\psi, \tag{6.9}$$

belongs to one of the surfaces, with all quantities calculated at (x,y,z), and the matrix O is orthogonal and diagonalizes the matrix τ. In other words, we have shown that the gradient $\nabla\psi(x,y,z)$ of the wave front ψ=constant, when multiplied

by the matrix $\dfrac{\mu^{1/2}}{\det(\mu^{1/2})}$ and conveniently rotated by O^t, belongs to one of the surfaces described by the equations (6.8).

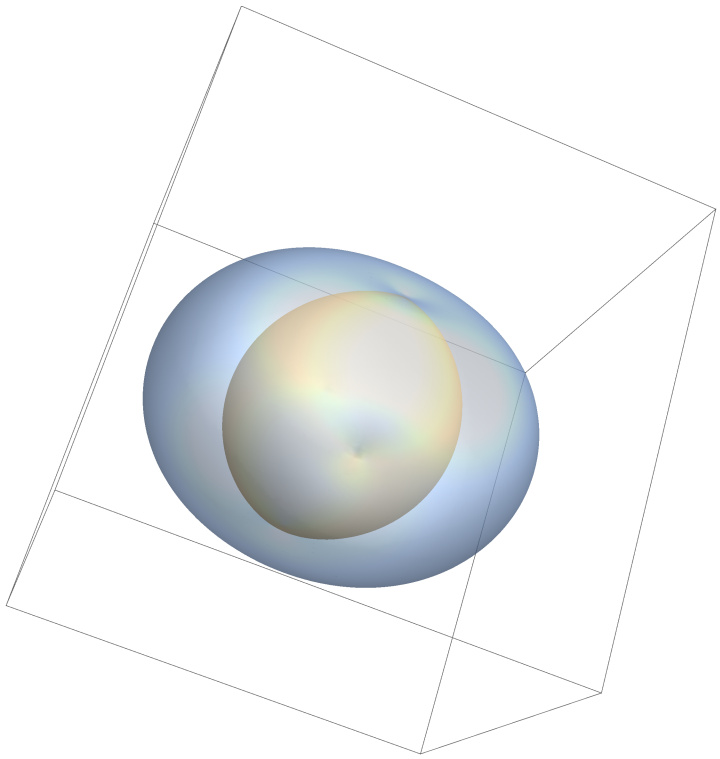

FIGURE 2. Fresnel surfaces when $\tau_1 = 1, \tau_2 = 2, \tau_3 = 3$; $\mu = Id$

Notice that when the permittivity matrix is $\epsilon\, Id$ and the permeability matrix is $\mu\, Id$, where ϵ and μ are scalar functions depending only on position, then we recover the eikonal equation $|\nabla\psi|^2 = \epsilon\,\mu$. In fact, in this case the matrix $\tau = (\epsilon/\mu)\, Id$, so $\tau_i = \epsilon/\mu$ for $i = 1, 2, 3$, $O = Id$,

$$\Phi(p_1, p_2, p_3) = \frac{\mu}{\epsilon}(p_1^2 + p_2^2 + p_3^2),$$

and

$$\Psi(p_1, p_2, p_3) = \left(\frac{\mu}{\epsilon}\right)^2 (p_1^2 + p_2^2 + p_3^2)^2.$$

So $\Phi^2 = \Psi$ and both surfaces in (6.8) are identically equal to

$$1 - \Phi(p_1, p_2, p_3) = 1 - \frac{\mu}{\epsilon}(p_1^2 + p_2^2 + p_3^2) = 0.$$

So the vector p in (6.9) satisfies the last equation and $p = \dfrac{1}{\mu}\nabla\psi$, therefore $|\nabla\psi|^2 = \epsilon\,\mu$.

6.1. **Case considered for the application of our results.** For the application of our results from Sections 3–5 we consider materials having permittivity and permeability tensors ϵ and μ that are positive definite symmetric constant matrices with $\mu = a\,\epsilon$, where a is a positive number. These are homogeneous materials that when ϵ is not the identity matrix are anisotropic. We will associate with such a material a norm as follows. From the calculations above, the Fresnel equation in this case is as follows. From (6.4) we get $\tau = \dfrac{1}{a}\,Id$, so

$$\Phi(p) = a\,|p|^2, \qquad \Psi(p) = a^2\,|p|^4.$$

Obviously, $\Phi^2 = \Psi$ and so the Fresnel equation is $1 - \Phi = 0$, i.e., $|p|^2 = 1/a$ and therefore it has only one sheet. Then from (6.9) the vector $\dfrac{\mu^{1/2}}{\det\mu^{1/2}}\,\nabla\psi$ satisfies the equation

$$\left|\frac{\sqrt{a}\,\mu^{1/2}}{\det\mu^{1/2}}\,\nabla\psi\right|^2 = 1.$$

The last expression induces the following dual norm

$$N^*(p) = \left|\frac{\sqrt{a}\,\mu^{1/2}}{\det\mu^{1/2}}\,p\right|.$$

The norm N^* is the dual to the norm given by

$$N(x) = \sup_{N^*(p)=1} |x\cdot p| = \frac{\det\mu^{1/2}}{\sqrt{a}}\,\left|\mu^{-1/2}\,x\right|,$$

which is the norm we associate to the material. Notice that if μ is the identity matrix, then $\epsilon = \dfrac{1}{a}Id$, the material is isotropic and has index of refraction $n = 1/\sqrt{a}$. The norm obtained this way is then $N(x) = n\,|x|$, in agreement with the physical explanation for isotropic media given after (3.2).

Now, if $N(x) = |A\,x|$ with A a constant matrix, then $\nabla N(x) = \dfrac{1}{N(x)}\,A^tAx$. Therefore, having two materials I and II so that the wave fronts are given by norms $N_1(x) = |A_1x|$ and $N_2(x) = |A_2x|$, respectively, the Snell law (3.2) takes the following form: Each incident ray traveling in medium I with direction $x \in \Sigma_1$, i.e., $N_1(x) = 1$,

with $x \cdot \nu \geq 0$ and striking the plane Π at some point P_0 is refracted to medium II into a direction $m \in \Sigma_2$, i.e., $N_2(m) = 1$, if

$$A_2^t A_2 m - A_1^t A_1 x \parallel \nu,$$

where ν is the unit normal at P_0 from medium I to medium II.

In our application we have materials I and II having constant tensors $(\epsilon_1, a_1\,\epsilon_1)$ and $(\epsilon_2, a_2\,\epsilon_2)$, respectively, and therefore the associated norms to I and II are

$$N_1(x) = \frac{\det \mu_1^{1/2}}{\sqrt{a_1}} \left|\mu_1^{-1/2}\, x\right|, \qquad N_2(m) = \frac{\det \mu_2^{1/2}}{\sqrt{a_2}} \left|\mu_2^{-1/2}\, m\right|,$$

respectively. If we let

$$A_1 = \frac{\det \mu_1^{1/2}}{\sqrt{a_1}}\, \mu_1^{-1/2}, \qquad A_2 = \frac{\det \mu_2^{1/2}}{\sqrt{a_2}}\, \mu_2^{-1/2},$$

then $N_1(x) = |A_1\, x|$ and $N_2(m) = |A_2\, m|$. To apply the results of the previous sections to this case, we let

$$\kappa = \sup_{|A_1\, x|=1} |A_2\, x| = \sup_{|z|=1} \left|A_2\, A_1^{-1} z\right| = \left\|A_2\, A_1^{-1}\right\|$$

the norm of the matrix $A_2\, A_1^{-1}$ induced by the standard Euclidean norm $|\cdot|$. [4] Hence when $\left\|A_2\, A_1^{-1}\right\| < 1$ from Lemma 3.2 (a), we have that the results from Section 5 are applicable to this case. On the other hand, if κ^* is as in Lemma 3.2 (b), setting $A = A_2 A_1^{-1}$ we get

$$\sqrt{\text{minimum eigenvalue of } A^t A} = \inf_{|z|=1} \left|A_2\, A_1^{-1} z\right| > 1,$$

and the results from Remark 5.8 are applicable in this case. We can also write

$$A_2\, A_1^{-1} = \frac{\det \mu_2^{1/2}}{\sqrt{a_2}}\, \mu_2^{-1/2}\, \frac{\sqrt{a_1}}{\det \mu_1^{1/2}}\, \mu_1^{1/2} = \sqrt{\frac{a_1}{a_2}}\, \frac{\det \mu_2^{1/2}}{\det \mu_1^{1/2}}\, \mu_2^{-1/2}\, \mu_1^{1/2}.$$

[4] That is, κ is the spectral norm of the matrix $A := A_2\, A_1^{-1}$, i.e.,

$$\kappa = \sqrt{\text{maximum eigenvalue of } (A^t\, A)}$$

.

Once again notice that if μ_i is the identity matrix, then $\epsilon_i = \dfrac{1}{a_i} Id$, the materials are isotropic and have index of refraction $n_i = \sqrt{\epsilon_i \, \mu_i} = 1/\sqrt{a_i}$. The norms are then $N_i(x) = n_i \, |x|$, $i = 1, 2$, and $\kappa = \sqrt{a_1/a_2} = n_2/n_1$ in agreement with the physical explanation for isotropic media given after (3.2).

Finally, we remark that for the materials considered light rays travel in straight lines and they do not exhibit bi refringence, that is, each incident ray is refracted only into one ray. The last property is because the Fresnel equation has only one sheet. That rays travel in straight lines follows from Fermat's principle of least time explained in Section 3. Indeed, let X, Y be two points in space, $\gamma(\theta) = (1-\theta)\, X + \theta\, Y$, $0 \le \theta \le 1$, and let $\phi(\theta)$ be any curve from X to Y. Then the optical length T for each curve satisfies $T(\gamma) = \int_0^1 \|\gamma'(\theta)\| \, d\theta = \|X - Y\|$, and $T(\phi) = \int_0^1 \|\phi'(\theta)\| \, d\theta \ge \left\| \int_0^1 \phi'(\theta) \, d\theta \right\| = \|X - Y\|$.

For general anisotropic materials when ϵ is not a multiple of μ, the Fresnel equation has two sheets, see Figure 2, and as mentioned before bi-refringence occurs. This is the case for crystals, that is, when ϵ is a diagonal constant matrix and $\mu = Id$.

7. Connection with optimal mass transport

The setting up, analysis, and results from the previous sections allow us to cast the refraction problem in optimal transport terms. However, the method used in Section 5 to prove existence of solutions relies more on a deeper insight of the physical and geometric features of the refractor problem.

To apply the optimal mass transport approach, we use the abstract set up in [GH09, Section 3.2 and 3.3] and from Definition 5.1 introduce the cost function

$$c(x, m) = \log\left(\frac{1}{1 - x \cdot p_2(m)}\right)$$

for $x \in \Omega_1 \subset \Sigma_1$, $m \in \Omega_2 \subset \Sigma_2$, keeping in mind **(H.a)**, **(H.b)**, and **(H.c)**. With [GH09, Definition 3.9] of c-concavity, we have that $\mathcal{S} = \{\rho(x)x : x \in \Omega_1\}$ is a refractor in the sense of Definition 5.1 above if and only if $\log \rho$ is c-concave. From

the definition of c-normal mapping $\mathcal{N}_{c,\phi}$ given in [GH09, Definition 3.10], and the definition of refractor mapping $\mathcal{R}_S$ given by (5.1), we have that $\mathcal{R}_S = \mathcal{N}_{c,\log\rho}$. One can easily check that $\mathcal{S}$ is a weak solution of the refractor problem if and only if $\log\rho$ is c-concave and $\mathcal{N}_{c,\log\rho}$ is a measure preserving map in the sense of [GH09, Equation (3.9)] from $f(x)\,dx$ to μ. Hence existence and uniqueness up to dilations of the refractor problem follows as in [GH09, Theorem 3.15].

From Remark 5.8 and using the cost function $c(x,m) = \log(x \cdot p_2(m) - 1)$, we obtain similar results under **(H.a)**, **(H.b')**, and **(H.c)**.

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