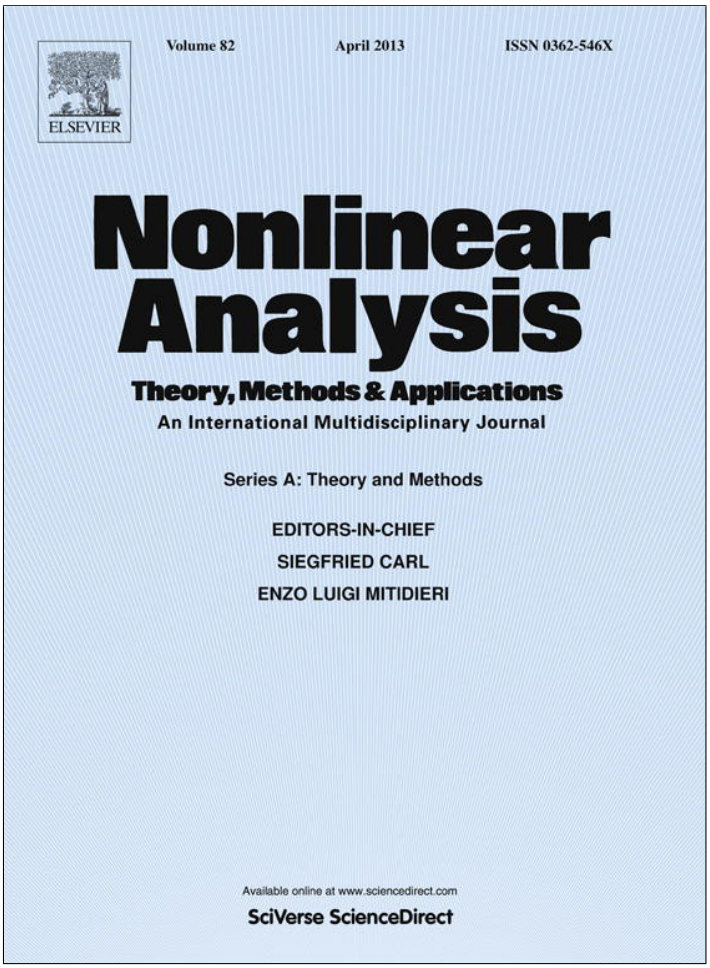

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The refractor problem with loss of energy

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1. Introduction

When radiation strikes a surface separating two homogeneous media I and II with different refractive indices, part of the radiation is transmitted through media II and another part is reflected back into media I. It is even true that at normal incidence a small percentage of radiation is internally reflected. The amount of radiation transmitted and reflected depends on the angle of incidence, and it can be calculated using the Maxwell equations of electrodynamics and is explicitly given by the Fresnel formulas; see the classic fundamental book by Born and Wolf [1, Section 1.5]. Indeed, if media I and II have refractive indices n_1 and n_2 respectively, and if an incident wave propagates with unit direction x and the transmitted wave has unit direction m, then the percentage of internally reflected energy can be conveniently written for our purposes as

$$r(x) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{x\cdot m} - (1+\kappa^2)\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[1 - 2\kappa\, x\cdot m + \kappa^2\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\right)$$

where $\kappa = n_2/n_1$. Therefore, the percentage of energy transmitted is $t(x) = 1 - r(x)$. This expression for r follows from the classical Fresnel formulas and Snell's law; see Section 4. Here $I_\perp$ and $I_\parallel$ are the coefficients of the amplitude of the incident wave, which might depend on x in a continuous way. It is important to notice that from Snell's law, $x - \kappa m = \lambda\, \nu$, where ν is the unit normal to the surface at the striking point and $\lambda > 0$. This implies that the function $r(x)$ is a function only depending on x and the normal ν.

In this paper we consider the problem of constructing a surface interface of media I and II in such a way that both the incident radiation and the radiation we want to transmit are prescribed, and the splitting of energy described before is taken into account. Indeed, we propose the following new model. Suppose we have $f \in L^1(\Omega)$ and $g \in L^1(\Omega^*)$, both Ω, Ω^* are domains in the sphere in $\mathbb{R}^3$, the space with physical significance for our problem. The question is to find a surface $\mathcal{R}$ parameterized by $\{\rho(x)x : x \in \Omega\}$ that separates media I and II such that each ray emanating from a point source, the origin, in the direction $x \in \Omega$ with intensity $f(x)$ is refracted into a direction $m \in \Omega^*$ and received with intensity $g(m)$. From the Fresnel formulas a surface $\mathcal{R}$ is only able to transmit in the direction x an amount of energy equal to

$$f(x)t_{\mathcal{R}}(x)$$

where $t_{\mathcal{R}} = 1 - r_{\mathcal{R}}$, since the amount $f(x)r_{\mathcal{R}}(x)$ is reflected back. As we said, the function $t_{\mathcal{R}}(x)$ depends of course on the surface $\mathcal{R}$ but only through x and the unit normal vector $\nu = \nu(x)$ at the striking point; see (4.7). Since we will be seeking for

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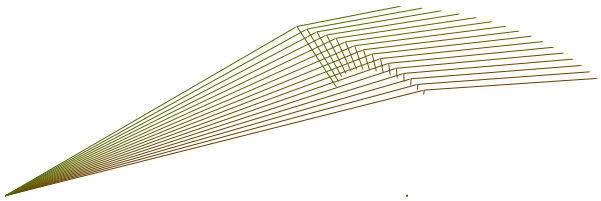

Fig. 1. Refracted and reflected vectors.

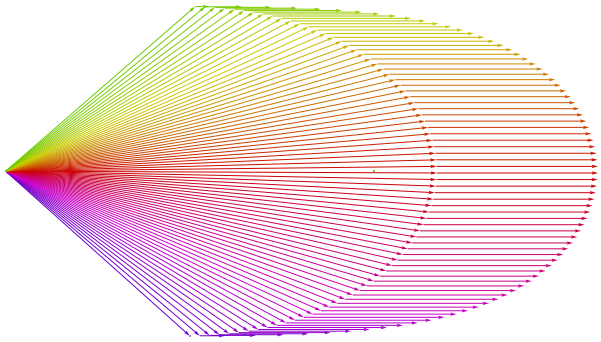

Fig. 2. Refracted vectors for an ellipse refracting into a fixed direction.

refracting surfaces $\mathcal{R}$, which, in particular, are convex or concave, the normal vector $\nu(x)$ exists for almost every direction x. Also $t_{\mathcal{R}}(x) = G(x, \nu(x))$, with a function $G(x, x')$ continuous in $\Omega \times \Omega^*$ and so $t_{\mathcal{R}}$ is defined for a.e. direction x. We then propose the following model: the refracting surface $\mathcal{R}$ is a solution to our problem if

$$\int_{\mathcal{T}_{\mathcal{R}}(F)} f(x)\, t_{\mathcal{R}}(x)\, dx \geq \int_F g(m)\, dm \tag{1.1}$$

for each Borel subset $F \subset \Omega^*$. Here $\mathcal{T}_{\mathcal{R}}(F)$ is the collection of all directions $x \in \Omega$ that are refracted into a direction in the set F; see Section 3. We prove that if $\mathcal{R}$ is a refractor, then the function $t_{\mathcal{R}}(x)$ is continuous relative to the set $\Omega \setminus S$, where S is the set of directions where ρ is not differentiable, i.e., $|S| = 0$; see Proposition 5.3. Therefore $t_{\mathcal{R}}(x)$ is measurable and so (1.1) is well defined. Since a fraction of the energy is used in internal reflection, to be able to transmit and receive $g(m)$ a little extra energy will be needed at the outset. A refractor $\mathcal{R}$ will be admissible to transmit the amount g if

$$\int_{\Omega} f(x)\, t_{\mathcal{R}}(x)\, dx \geq \int_{\Omega^*} g(m)\, dm. \tag{1.2}$$

Since a priori we only know f, g and not $\mathcal{R}$, we do not know if this is satisfied. In order to make sure this is the case, it can be proved that, for example, if $n_2/n_1 = \kappa < 1$, then $r(x) \leq C_\epsilon < 1$ for all $x \in \Omega$ such that $x \cdot m \geq \kappa + \epsilon$,[2] where $\epsilon > 0$ and with C_ϵ independent of $\mathcal{R}$, see Section 4.1. So if we assume that the input energy is sufficiently larger than the output energy, then (1.2) holds. More precisely, if

$$\int_{\Omega} f(x)\, dx \geq \frac{1}{1 - C_\epsilon} \int_{\Omega^*} g(m)\, dm,$$

then (1.2) holds.

Fig. 1 represents an arc of ellipse separating glass and air, $\kappa = 2/3$, where the refracted and reflected directions are multiplied by the Fresnel coefficients $t(x)$ and $r(x)$ respectively. Fig. 2 represents all the refracted vectors in an ellipse having the uniform refraction property, i.e. all rays are refracted into a fix direction, where the refracted vectors are multiplied by the Fresnel coefficient $t(x)$. Notice that the size of the refracted vectors close to the critical angle, i.e., $x \cdot m = \kappa$ tend to zero.

With this model we solve our problem, that is, we show existence of solutions, even for Radon measures μ instead of g. The basic geometry of the refractors is described in [6] and depends on κ. Indeed, the surfaces having the uniform refracting property are semi ellipsoids if $\kappa < 1$ and one sheet of hyperboloids of two sheets if $\kappa > 1$; see Lemma 2.2. A difficulty in our case is the presence of the coefficient $t_{\mathcal{R}}(x)$ in (1.1). This prevents us from using the optimal transportation methods used in [6]. The route used now is to solve first the problem when the right hand side is a linear combination of delta functions and then proceed by approximation. To carry out this we need to understand how the Fresnel coefficients, which are discussed and estimated in Section 4, evolve when a sequence of refractors converge; see Section 5. When the measure μ in the target is discrete, refractors always overshoot energy in one direction, Theorems 6.2 and 6.9. When μ is any Radon measure, it will

[2] We recall that the physical constraint for refraction is that $x \cdot m \geq \kappa$ for $\kappa < 1$, see Lemma 2.1.

be shown in Section 6.2 that refractors transmit more energy in any a priori chosen direction $m_o \in \overline{\Omega^*}$ which lies in the support of μ, and there is one refractor overshooting the least amount of energy in the direction of m_o; see Theorem 6.11, and Section 6.3.

Finally, we show in Section 8 that the surface solution to the problem satisfies an inequality involving a fully nonlinear pde of Monge–Ampère type.

To place our results in perspective, we enumerate some related results in this area. The refractor problem assuming energy conservation, i.e., $t_{\mathcal{R}}(x) = 1$ in (1.1), was considered for the first time in [6] for the far field case, and in [5] and [7] for the near field case. For reflectors also assuming energy conservation, see [4,10,3,2] for the far field problem, and [8], and [9] for the near field.

We believe this paper is the first contribution to construct a refractor taking into account the energy used in internal reflection.

2. Preliminaries

2.1. Reflection and transmission

Let Γ be a surface in $\mathbb{R}^n$ that separates two homogeneous and isotropic media I and II where radiation propagates with velocities v_1 and v_2, respectively. The refractive index of media I is $n_1 = c/v_1$, where c is the velocity of propagation of light in vacuum; similarly for media II, $n_2 = c/v_2$.

If a light ray with direction $x \in S^{n-1}$ travels through I and strikes Γ at the point P, then this ray is split into two rays with directions:

(a) $r =$ ray reflected inside media I (internally reflected),
(b) $m =$ ray refracted or transmitted inside media II;

all are unit vectors. The internally reflected ray satisfies $x - r = 2(x \cdot \nu)\,\nu$, where ν is the normal unit to Γ at P in the direction of medium II, and $\cdot$ is the Euclidean inner product. For the transmitted ray m we have

$$n_1\ (x \times \nu) = n_2\ (m \times \nu)\,.$$

So all vectors x, ν, r and m are coplanar. If $\theta_1 =$ angle between x and ν, the incidence angle, and $\theta_2 =$ angle between m and ν, the refraction angle, then we get the familiar expression of Snell's law of refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$. So the vector $n_1 x - n_2 m$ is parallel to the normal ν. That is, if $\kappa = n_2/n_1$, then we have Snell's law in vector form

$$x - \kappa\, m = \lambda \nu, \tag{2.1}$$

for $\lambda = \Phi(x \cdot \nu)$; $\Phi(t) = t - \kappa\sqrt{1 - \kappa^{-2}(1 - t^2)}$.

We recall the following physical conditions for refraction, [6, Lemma 2.1].

Lemma 2.1. *Let n_1 and n_2 be the indices of refraction of two homogeneous media I and II, respectively, and $\kappa = n_2/n_1$. Then a light ray in medium I with direction $x \in S^{n-1}$ is refracted by some surface into a light ray with direction $m \in S^{n-1}$ in medium II if and only if $m \cdot x \geq \kappa$, when $\kappa < 1$; and if and only if $m \cdot x \geq 1/\kappa$, when $\kappa > 1$.*

It is important to determine the surfaces that have uniform refracting property, that is, those that refract all rays emanating from the origin into a fixed direction; see [6, Section 2.2]. These surfaces geometrically depend of the value of κ. For $\kappa < 1$ and $b > 0$, let $E(m, b)$ be given by

$$E(m, b) = \left\{ \rho(x)x : \rho(x) = \frac{b}{1 - \kappa\, m \cdot x},\ x \in S^{n-1},\ x \cdot m \geq \kappa \right\}. \tag{2.2}$$

This surface describes a semi-ellipsoid with axis m and foci O and $\frac{2\kappa b}{1-\kappa^2} m$. If on the other hand, $\kappa > 1$, define for $b > 0$

$$H(m, b) = \left\{ \rho(x)x : \rho(x) = \frac{b}{\kappa\, m \cdot x - 1},\ x \in S^{n-1},\ x \cdot m \geq 1/\kappa \right\}. \tag{2.3}$$

Then $H(m, b)$, is the sheet with opening in direction m of a hyperboloid of revolution of two sheets with axis m and foci at O and $\frac{2\kappa b}{1-\kappa^2} m$.

The uniform refraction property is then given by the following lemma.

Lemma 2.2. *Let n_1 and n_2 be the indices of refraction of two homogeneous media I and II, respectively, and $\kappa = n_2/n_1$. Assume that the origin O is inside medium I, and $E(m, b)$, $H(m, b)$ are defined by (2.2) and (2.3), respectively. We have:*

(i) *If $\kappa < 1$ and $E(m, b)$ is the interface of media I and II, then $E(m, b)$ refracts all rays emitted from O into rays in medium II with direction m.*
(ii) *If $\kappa > 1$ and $H(m, b)$ separates media I and II, then $H(m, b)$ refracts all rays emitted from O into rays in medium II with direction m.*

Fig. 3 illustrates the uniform refraction property in an ellipse with $\kappa = 2/3$, that is, the ellipse is made of glass and outside is air.

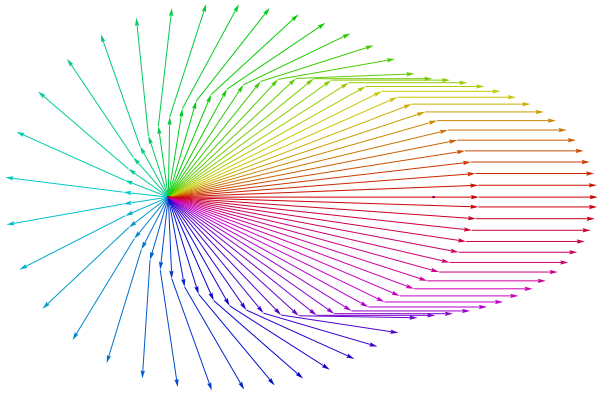

Fig. 3. Refracting property in the ellipse.

3. Refractor mappings and properties

Let Ω and Ω^* be two domains of the unit sphere S^{n-1} of $\mathbb{R}^n$ with $|\partial\Omega| = 0$.

3.1. Case $\kappa < 1$

Suppose medium I is denser than medium II, so $\kappa = n_2/n_1 < 1$. Suppose that Ω and Ω^* satisfy the property that

$$\inf_{m\in\Omega^*, x\in\Omega} m \cdot x \geq \kappa.$$

Definition 3.1. A surface $\mathcal{R}$ in $\mathbf{R}^n$ parameterized for $x \in \Omega$ by $\rho(x)x$, with $\rho(x)$ a positive function, is a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ if for any $x_o \in \overline{\Omega}$ there exists a semi-ellipsoid $E(m, b)$ with $m \in \overline{\Omega^*}$ such that $\rho(x_o) = \frac{b}{1-\kappa m\cdot x_o}$ and $\rho(x) \leq \frac{b}{1-\kappa m\cdot x}$ for all $x \in \overline{\Omega}$. We call $E(m, b)$ a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_o)x_o$ or simply at x_o.

From the definition, refractors are concave and therefore continuous.

Definition 3.2. Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined for $x_o \in \overline{\Omega}$ by

$$\mathcal{N}_{\mathcal{R}}(x_o) = \{m \in \overline{\Omega^*} : E(m, b) \text{ supports } \mathcal{R} \text{ at } \rho(x_o)x_o \text{ for some } b > 0\}.$$

Given $m_o \in \overline{\Omega^*}$ the tracing map of $\mathcal{R}$ is defined by

$$\mathcal{T}_{\mathcal{R}}(m_o) = \{x \in \overline{\Omega} : m_o \in \mathcal{N}_{\mathcal{R}}(x)\}.$$

We now prove some basic properties about the refractor and tracing mapping; see [6, Section 3.1]. Note that $\mathcal{T}_{\mathcal{R}}(\overline{\Omega^*}) = \overline{\Omega}$.

Lemma 3.3. *Any refractor is globally Lipschitz on $\overline{\Omega}$.*

Lemma 3.4. *If $m \in \overline{\Omega^*}$, then $\mathcal{T}_{\mathcal{R}}(m)$ is closed in $\overline{\Omega}$.*

Lemma 3.5. *We have*

(i) $[\mathcal{T}_{\mathcal{R}}(F)]^c \subset \mathcal{T}_{\mathcal{R}}(F^c)$ *for all $F \subset \overline{\Omega^*}$, with equality except for a set of measure zero.*
(ii) *The set $\mathcal{C} = \{F \subset \overline{\Omega^*} : \mathcal{T}_{\mathcal{R}}(F)$ is Lebesgue measurable$\}$ is a σ-algebra containing all Borel sets in $\overline{\Omega^*}$.*

The following lemma is not proved in [6] and it will be used later.

Lemma 3.6. *Let $\mathcal{R}_j = \{\rho_j(x)x : x \in \overline{\Omega}\}$, $j \geq 1$ be refractors from $\overline{\Omega}$ to $\overline{\Omega^*}$. Suppose that $0 < a_1 \leq \rho_j \leq a_2$ and $\rho_j \to \rho$ pointwise in $\overline{\Omega}$. Then:*

(i) $\mathcal{R} := \{\rho(x)x : x \in \overline{\Omega}\}$ *is a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$.*
(ii) *For any compact set $K \subset \overline{\Omega^*}$*

$$\limsup_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(K) \subset \mathcal{T}_{\mathcal{R}}(K).$$

(iii) *For any open set $G \subset \overline{\Omega^*}$,*

$$\mathcal{T}_{\mathcal{R}}(G) \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S,$$

where S is the singular set of $\mathcal{R}$.

Proof. (i) Fix $x_o \in \overline{\Omega}$. Then there exist $m_j \in \overline{\Omega^*}$ and $b_j > 0$ such that $E(m_j, b_j)$ supports $\mathcal{R}_j$ at $\rho(x_o)x_o$ and thus

$$\rho_j(x_o) = \frac{b_j}{1-\kappa m_j \cdot x_o} \quad \text{and} \quad \rho_j(x) \le \frac{b_j}{1-\kappa m_j \cdot x}$$

for all $x \in \overline{\Omega}$. Consequently

$$\frac{b_j}{1-\kappa m_j \cdot x_o} \le a_2 \quad \text{and} \quad a_1 \le \frac{b_j}{1-\kappa m_j \cdot x}$$

for all j and therefore

$$a_1(1-\kappa) \le b_j \le a_2$$

for all j. If need be by passing to a subsequence we obtain m_o and b_o such that $m_j \to m_o \in \overline{\Omega^*}$ and $b_j \to b_o$. We claim $E(m_o, b_o)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$. Indeed

$$\rho(x_o) = \lim_j \rho_j(x_o) = \lim_j \frac{b_j}{1-\kappa m_j \cdot x_o} = \frac{b_o}{1-\kappa m_o \cdot x_o}$$

and

$$\rho(x) = \lim_j \rho_j(x) \le \lim_j \frac{b_j}{1-\kappa m_j \cdot x} = \frac{b_o}{1-\kappa m_o \cdot x}$$

for all $x \in \overline{\Omega}$. Thus $\mathcal{R}$ is a refractor.

(ii) Let $x_o \in \limsup \mathcal{T}_{\mathcal{R}_j}(K)$. Without loss of generality assume that $x_o \in \mathcal{T}_{\mathcal{R}_j}(K)$ for all $j \ge 1$. Then there exist $m_j \in \mathcal{N}_{\mathcal{R}_j}(x_o) \cap K$ and b_j such that

$$\rho_j(x_o) = \frac{b_j}{1-\kappa m_j \cdot x_o} \quad \text{and} \quad \rho_j(x) \le \frac{b_j}{1-\kappa m_j \cdot x}$$

for all $x \in \overline{\Omega}$. As in the proof of (i) we may assume that $m_j \to m_o \in K$ and $b_j \to b_o$ and conclude that $E(m_o, b_o)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$, proving that $x_o \in \mathcal{T}_{\mathcal{R}}(m_o)$. Hence $x_o \in \mathcal{T}_{\mathcal{R}}(K)$.

(iii) Let G be an open subset of $\overline{\Omega^*}$. By (ii) $\limsup \mathcal{T}_{\mathcal{R}_j}(G^c) \subset \mathcal{T}_{\mathcal{R}}(G^c)$ as G^c is compact. Also

$$\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c \subset \limsup_{j\to\infty}\{[\mathcal{T}_{\mathcal{R}_j}(G)]^c \cup [\mathcal{T}_{\mathcal{R}_j}(G) \cap \mathcal{T}_{\mathcal{R}_j}(G^c)]\} \tag{3.1}$$

and by Lemma 3.5 the right hand side of (3.1) is equal to $\limsup_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G^c)$. By (ii) we will then have

$$\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c \subset \mathcal{T}_{\mathcal{R}}(G^c) = \{[\mathcal{T}_{\mathcal{R}}(G)]^c \cup [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]\}.$$

Taking complements we obtain

$$\left\{\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c\right\}^c \supset [\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c.$$

Consequently

$$\liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \supset [\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c$$

and thus

$$\left[[\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c\right] \cup S \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S.$$

But $\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c) \subset S$. Thus

$$\mathcal{T}_{\mathcal{R}}(G) \subset \mathcal{T}_{\mathcal{R}}(G) \cup S \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S$$

as required. □

Lemma 3.7. *Let $f \in L^1(\overline{\Omega})$. The set function defined by*

$$G_{\mathcal{R}}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f(x)\,dx$$

is a Borel measure in $\overline{\Omega^*}$.

Lemma 3.8. *Let $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ such that $\inf_{x\in\overline{\Omega}} \rho(x) = 1$. Then there is a constant C, depending only on κ, such that*

$$\sup_{x\in\overline{\Omega}} \rho(x) \le C.$$

Proof. Suppose $\inf_{x\in\overline{\Omega}} \rho(x)$ is attained at $x_o \in \overline{\Omega}$, and let $E(m, b_o)$ be a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_o)x_o$. Then

$$1 = \rho(x_o) = \frac{b_o}{1 - \kappa m_o \cdot x_o} \quad \text{and} \quad \rho(x) \le \frac{b_o}{1 - \kappa m_o \cdot x} \quad \forall\, x \in \overline{\Omega}.$$

From the first equation we get that $b_o = 1 - \kappa m_o \cdot x_o \le 1 + \kappa$ and using this in the inequality we obtain

$$\rho(x) \le \frac{1+\kappa}{1-\kappa} \quad \text{for all } x \in \overline{\Omega}$$

and this proves the lemma. □

3.2. Case $\kappa > 1$

We have in this case that medium II is denser than medium I.

Suppose that Ω and Ω^* satisfy the physical property that

$$\inf_{m\in\Omega^*, x\in\Omega} m \cdot x \ge 1/\kappa + \epsilon \tag{3.2}$$

for some $\epsilon > 0$.

The set up when $\kappa > 1$ is similar to the case $\kappa < 1$, a main difference is to use the semi-hyperboloids $H(m, b)$, defined by (2.3) in place of the semi-ellipsoids $E(m, b)$.

Definition 3.9. A surface $\mathcal{R}$ in $\mathbb{R}^n$ parameterized for $x \in \Omega$ by $\rho(x)x$, with $\rho(x)$ a positive function, is a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ for the case $\kappa > 1$ if for any $x_o \in \overline{\Omega}$ there exists a semi-hyperboloid $H(m, b),\ m \in \Omega^*$ such that $\rho(x_o) = \frac{b}{\kappa m\cdot x_o - 1}$ and $\rho(x) \ge \frac{b}{\kappa m\cdot x - 1}$ for all $x \in \bar{\Omega}$. We call such $H(m, b)$ a supporting semi-hyperboloid to $\mathcal{R}$ at $\rho(x_o)x_o$ or simply at x_o.

With this definition refractors are convex and therefore continuous. The refractor mapping of $\mathcal{R}$ and the tracing mapping of $\mathcal{R}$ are also defined similarly.

Definition 3.10. Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined by for $x_o \in \overline{\Omega}$

$$\mathcal{N}_{\mathcal{R}}(x_o) = \{m \in \overline{\Omega^*} : H(m, b) \textit{ supports } \mathcal{R} \textit{ at } \rho(x_o)x_o \text{ for some } b > 0\}.$$

Given $m_o \in \overline{\Omega^*}$ the tracing mapping of $\mathcal{R}$ is defined by

$$\mathcal{T}_{\mathcal{R}}(m_o) = \{x \in \overline{\Omega} : m_o \in \mathcal{N}_{\mathcal{R}}(x)\}.$$

Similar statements to those mentioned in Section 3.1, with similar proofs, also hold in this case.

4. Fresnel formulas

Electromagnetic radiation can be represented as a plane wave of the form

$$\mathbf{E}(\mathbf{r}, t) = A \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot x}{v}\right) + \delta\right),$$

where A is a constant vector in $\mathbb{R}^3$ called amplitude vector, $\mathbf{r}$ is the vector position in $\mathbb{R}^3$, t represents the time, v is the velocity of propagation in the media, x is the unit direction of propagation, ω is the angular frequency, and δ is the phase; see [1, Chapter 1]. It follows from the Maxwell equations that the field $\mathbf{E}(\mathbf{r}, t)$ is perpendicular to the direction of propagation, that is, $\mathbf{E}(\mathbf{r}, t)\cdot x = 0$. A plane wave emanates from the origin with unit direction x, the incident wave, traveling for a while in media I and strikes a surface Γ separating media I and II at point P. Then the wave is split into two waves: a transmitted (or refracted) wave propagating on media II and a reflected wave propagated back into media I. Suppose ν is the normal to Γ at P in the direction of media II. Consider Π the plane containing the vectors x and ν which is called the plane of incidence. Let θ_i be the angle between these vectors (the incident angle), and take a right hand orthonormal system of coordinates $\{\alpha, \beta, \nu\}$ with origin at P such that the vector α lies on Π. We can write $x = \sin\theta_i\,\alpha + \cos\theta_i\,\nu$ and $A = a\,\alpha + b\,\beta + c\,\nu$ and since $A \cdot x = 0$, we have that

$$A = -c\cos\theta_i\,\alpha + b\,\beta + c\sin\theta_i\,\nu.$$

It is customary to call $b = I_\perp$, the perpendicular component because the vector $b\beta$ is perpendicular to the plane of incidence, and $c = I_\parallel$, the parallel component; see [1, Section 1.5]. In addition, since the amplitude vector may depend on the direction of propagation, we have that

$$A(x) = -I_\parallel(x)\cos\theta_i\,\alpha + I_\perp(x)\,\beta + I_\parallel(x)\sin\theta_i\,\nu,$$

so in general the incident plane wave has the form

$$\mathbf{E}^i(\mathbf{r},t) = A(x)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot x}{v_1}\right) + \delta\right),$$

where v_1 is the velocity of propagation in media I. From the Maxwell equations this field gives rise to a magnetic field $\mathbf{B}^i(\mathbf{r},t)$ that is also perpendicular to the direction x of propagation and is also perpendicular to $\mathbf{E}^i(\mathbf{r},t)$, and having the form

$$\mathbf{B}^i(\mathbf{r},t) = \frac{1}{v_1}\left(-I_\perp(x)\cos\theta_i\,\alpha - I_\parallel(x)\,\beta + I_\perp(x)\sin\theta_i\,\nu\right)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot x}{v_1}\right) + \delta\right).$$

Let us now introduce m, the direction of propagation of the transmitted wave, and let θ_t be the angle between the normal ν and m. Similarly, if $\mathbf{s}$ is the direction of propagation of the reflected wave, then θ_r is the angle between the normal ν and $\mathbf{s}$. We have from the Snell law of reflection that $\mathbf{s} = \sin\theta_r\,\alpha + \cos\theta_r\,\nu = \sin\theta_i\,\alpha - \cos\theta_i\,\nu$. Then the corresponding electric and magnetic fields corresponding to transmission are

$$\mathbf{E}^t(\mathbf{r},t) = \left(-T_\parallel\cos\theta_t\,\alpha + T_\perp\beta + T_\parallel\sin\theta_t\,\nu\right)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot m}{v_2}\right)\right) = \mathbf{E}_0^t\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot m}{v_2}\right)\right)$$

$$\mathbf{B}^t(\mathbf{r},t) = \frac{1}{v_2}\left(-T_\perp\cos\theta_t\,\alpha - T_\parallel\,\beta + T_\perp\sin\theta_t\,\nu\right)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot m}{v_2}\right)\right) = \mathbf{B}_0^t\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot m}{v_2}\right)\right);$$

and similarly the fields corresponding to reflection are

$$\mathbf{E}^r(\mathbf{r},t) = \left(-R_\parallel\cos\theta_r\,\alpha + R_\perp\,\beta + R_\parallel\sin\theta_r\,\nu\right)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot \mathbf{s}}{v_1}\right)\right) = \mathbf{E}_0^r\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot \mathbf{s}}{v_1}\right)\right)$$

$$\mathbf{B}^r(\mathbf{r},t) = \frac{1}{v_1}\left(-R_\perp\cos\theta_r\,\alpha - R_\parallel\,\beta + R_\perp\sin\theta_r\,\nu\right)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot \mathbf{s}}{v_1}\right)\right) = \mathbf{B}_0^r\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot \mathbf{s}}{v_1}\right)\right).$$

Taking into account the Maxwell equations in integral form and the continuity of the tangential components of the electric and magnetic fields at the interface surface Γ, one gets (see [1, Section 1.5.2]) the Fresnel equations expressing the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave:

$$T_\parallel = \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}\,I_\parallel$$

$$T_\perp = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}\,I_\perp$$

$$R_\parallel = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}\,I_\parallel$$

$$R_\perp = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}\,I_\perp,$$

where $n_1 = c/v_1$ and $n_2 = c/v_2$.

It is convenient for our analysis to rewrite these equations in terms of the directions x and m. Indeed, we have $\cos\theta_i = x\cdot\nu$ and $\cos\theta_t = m\cdot\nu$; and set $\kappa = n_2/n_1$. From the Snell law, $x - \kappa m = \lambda\nu$, so the Fresnel equations take the form

$$T_\parallel = \frac{2\,x\cdot\nu}{\kappa\,x\cdot\nu + m\cdot\nu}\,I_\parallel = \frac{2\,x\cdot(x-\kappa\,m)}{(\kappa\,x+m)\cdot(x-\kappa\,m)}\,I_\parallel$$

$$T_\perp = \frac{2\,x\cdot\nu}{x\cdot\nu + \kappa\,m\cdot\nu}\,I_\perp = \frac{2\,x\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\,I_\perp$$

$$R_\parallel = \frac{\kappa\,x\cdot\nu - m\cdot\nu}{\kappa\,x\cdot\nu + m\cdot\nu}\,I_\parallel = \frac{(\kappa\,x-m)\cdot(x-\kappa\,m)}{(\kappa\,x+m)\cdot(x-\kappa\,m)}\,I_\parallel$$

$$R_\perp = \frac{x\cdot\nu - \kappa\,m\cdot\nu}{x\cdot\nu + \kappa\,m\cdot\nu}\,I_\perp = \frac{(x-\kappa\,m)\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\,I_\perp.$$

Notice that the denominators of the perpendicular components are the same and likewise for the parallel components.

The reflection and transmission coefficients, that is, the percentage of energy transmitted and reflected, are given by

$$\mathfrak{R} = \frac{J^r}{J^i} = \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2, \quad \text{and} \quad \mathcal{T} = \frac{J^t}{J^i} = \frac{n_2\cos\theta_t}{n_1\cos\theta_i}\left(\frac{|\mathbf{E}_0^t|}{|\mathbf{E}_0^i|}\right)^2;$$

see [1, Section 1.5.3]. By conservation of energy or by direct verification $\mathfrak{R} + \mathcal{T} = 1$.

We then have from Fresnel's equations that

$$|\mathbf{E}_0^r|^2 = R_\parallel^2 + R_\perp^2 = \left[\frac{(\kappa\, x - m)\cdot(x - \kappa\, m)}{(\kappa\, x + m)\cdot(x - \kappa\, m)}\right]^2 I_\parallel^2 + \left[\frac{(x - \kappa\, m)\cdot(x - \kappa\, m)}{(x + \kappa\, m)\cdot(x - \kappa\, m)}\right]^2 I_\perp^2,$$

and so

$$\begin{aligned}\mathfrak{R} &= \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2 = \frac{R_\parallel^2 + R_\perp^2}{I_\parallel^2 + I_\perp^2}\\ &= \left[\frac{(\kappa\, x - m)\cdot(x - \kappa\, m)}{(\kappa\, x + m)\cdot(x - \kappa\, m)}\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[\frac{(x - \kappa\, m)\cdot(x - \kappa\, m)}{(x + \kappa\, m)\cdot(x - \kappa\, m)}\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\\ &= \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{x\cdot m} - (1+\kappa^2)\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[1 - 2\kappa\, x\cdot m + \kappa^2\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\right)\end{aligned}$$

which is a function only of $x \cdot m$, and also of x if the perpendicular and parallel components depend on x. In principle the coefficients $I_\parallel$ and $I_\perp$ might depend on the direction x, in other words, for each direction x we would have a wave that changes its amplitude with the direction of propagation. The energy of the incident wave would be $f(x) = |\mathbf{E}_0^i|^2 = I_\parallel(x)^2 + I_\perp(x)^2$. Notice that if the incidence is normal, that is, $x = m$, then $\mathfrak{R} = \left(\frac{1-\kappa}{1+\kappa}\right)^2$ which shows that even for radiation normal to the interface some energy is used in reflection. For example, if we go from air to glass, $n_1 = 1$ and $n_2 = 1.5$, we have $\kappa = 1.5$ so $\mathfrak{R} = .04$, which means that 4% of the energy is lost in internal reflection when incident radiation is normal to the interface.

Let us introduce the following notation

$$\phi(s) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{s} - (1+\kappa^2)\right]^2 \alpha + \left[1 - 2\kappa\, s + \kappa^2\right]^2 \beta\right), \tag{4.1}$$

where $\alpha = \frac{I_\parallel(x)^2}{I_{\parallel(x)}^2 + I_\perp(x)^2}$, and $\beta = \frac{I_\perp(x)^2}{I_\parallel(x)^2 + I_\perp(x)^2}$. We further assume α and β depend on x continuously.[3] We then have

$$\phi(x\cdot m) = \mathfrak{R}. \tag{4.2}$$

We notice that from the Snell's law

$$x - \kappa\, m = \lambda\nu$$

with

$$\lambda = \Psi(x\cdot\nu) \quad \text{where } \Psi(t) = t - \kappa\sqrt{1 - \kappa^{-2}(1-t^2)}. \tag{4.3}$$

So we can write

$$x\cdot m = \frac{1}{\kappa}\,(1 - \lambda\, x\cdot\nu) = \frac{1}{\kappa}\,(1 - \Psi(x\cdot\nu)\, x\cdot\nu)\,.$$

Consequently, the reflection coefficient $\mathfrak{R}$, and therefore the transmission coefficient $\mathcal{T}$ are both functions of $x \cdot \nu$, that is, the direction of propagation and the normal to the interface surface, that is,

$$\mathfrak{R} = \phi\left(\frac{1}{\kappa}\,(1 - \Psi(x\cdot\nu)\, x\cdot\nu)\right); \qquad \mathcal{T} = 1 - \phi\left(\frac{1}{\kappa}\,(1 - \Psi(x\cdot\nu)\, x\cdot\nu)\right). \tag{4.4}$$

4.1. Estimation of the Fresnel coefficients

For later purposes we need to estimate the function ϕ. We have $0 \le \alpha, \beta \le 1$ and $\alpha + \beta = 1$. Set

$$g(t) = \left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2, \qquad h(t) = \left[1 - 2\kappa\, t + \kappa^2\right]^2,$$

so $\phi(t) = \frac{1}{(1-\kappa^2)^2}\,(g(t)\,\alpha + h(t)\,\beta)$.

[3] Later on for the application in Theorems 6.2 and 7.5 to show that refractors overshoot in one direction, we will assume $I_\perp, I_\parallel$ are constants.

Case $\kappa < 1$. Suppose $\kappa + \epsilon \le t \le 1$. We have $g'(t) = -4\kappa\left[\frac{2\kappa}{t} - (1+\kappa^2)\right]\frac{1}{t^2}$, so $g'(t) > 0$ for $t > \frac{2\kappa}{1+\kappa^2}$, and $g'(t) < 0$ for $t < \frac{2\kappa}{1+\kappa^2}$. Since $\kappa < 1$, we have $\kappa + \epsilon < \frac{2\kappa}{1+\kappa^2} < 1$ for $\epsilon > 0$ small. Therefore, g decreases in the interval $[\kappa+\epsilon, \frac{2\kappa}{1+\kappa^2}]$, and g increases in the interval $[\frac{2\kappa}{1+\kappa^2}, 1]$. Hence

$$\max_{[\kappa+\epsilon,1]} g(t) = \max\{g(\kappa+\epsilon), g(1)\}.$$

We have that $g(1) = (1-\kappa)^4$, and $g(\kappa+\epsilon) > g(1)$ for ϵ small, so

$$\max_{[\kappa+\epsilon,1]} g(t) = g(\kappa+\epsilon).$$

On the other hand, $h'(t) = -4\kappa\left[1 - 2\kappa\, t + \kappa^2\right]$, and so $h'(t) > 0$ for $t > \frac{1+\kappa^2}{2\kappa}$ and $h'(t) < 0$ for $t < \frac{1+\kappa^2}{2\kappa}$. Since $\frac{1+\kappa^2}{2\kappa} > 1$, the function h is decreasing in the interval $[\kappa+\epsilon, 1]$ and so

$$\max_{[\kappa+\epsilon,1]} h(t) = h(\kappa+\epsilon).$$

Therefore we obtain that

$$\max_{[\kappa+\epsilon,1]} \phi(t) \le \frac{1}{(1-\kappa^2)^2}\left(\alpha\, g(\kappa+\epsilon) + \beta\, h(\kappa+\epsilon)\right).$$

It is easy to see that

$$g(\kappa+\epsilon) < (1-\kappa^2)^2, \quad \text{and} \quad h(\kappa+\epsilon) < (1-\kappa^2)^2$$

and so we obtain the bound

$$\max_{[\kappa+\epsilon,1]} \phi(t) \le C_\epsilon < 1,$$

with

$$C_\epsilon = \max\left\{\frac{g(\kappa+\epsilon)}{(1-\kappa^2)^2}, \frac{h(\kappa+\epsilon)}{(1-\kappa^2)^2}\right\} \tag{4.5}$$

independent of α and β. We notice also that $C_\epsilon \to 1$ as $\epsilon \to 0^+$, and $C_\epsilon \to \left(\frac{1-\kappa}{1+\kappa}\right)^2$ as $\epsilon \to (1-\kappa)^-$. Also notice that the function ϕ in (4.1) is in general not decreasing in the interval $[\kappa+\epsilon, 1]$, that is, one can choose α close to one and β close to zero with $\alpha + \beta = 1$, so that this is the case.

Case $\kappa > 1$. For ϵ small we have $\frac{1}{\kappa} + \epsilon < \frac{2\kappa}{1+\kappa^2} < 1$, so as before, g decreases in the interval $[\frac{1}{\kappa}+\epsilon, \frac{2\kappa}{1+\kappa^2}]$, and g increases in the interval $[\frac{2\kappa}{1+\kappa^2}, 1]$. Hence

$$\max_{[(1/\kappa)+\epsilon,1]} g(t) = \max\left\{g\left(\frac{1}{\kappa}+\epsilon\right), g(1)\right\}.$$

Since now $\kappa > 1$ we have that $g(1) < g\left(\frac{1}{\kappa}+\epsilon\right)$, for ϵ small, and so

$$\max_{[(1/\kappa)+\epsilon,1]} g(t) = g((1/\kappa)+\epsilon).$$

Since we always have $\frac{1+\kappa^2}{2\kappa} > 1$, the function h is decreasing in the interval $[(1/\kappa)+\epsilon, 1]$ and so

$$\max_{[(1/\kappa)+\epsilon,1]} h(t) = h((1/\kappa)+\epsilon).$$

Therefore we obtain that

$$\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \le \frac{1}{(1-\kappa^2)^2}\left(\alpha\, g((1/\kappa)+\epsilon) + \beta\, h((1/\kappa)+\epsilon)\right).$$

It is clear that $g((1/\kappa)+\epsilon) < (1-\kappa^2)^2$, and $h((1/\kappa)+\epsilon) < (1-\kappa^2)^2$ when $0 < \epsilon < 1 - (1/\kappa)$. So we obtain the bound

$$\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \le C_\epsilon < 1,$$

with

$$C_\epsilon = \max\left\{\frac{g((1/\kappa)+\epsilon)}{(1-\kappa^2)^2}, \frac{h((1/\kappa)+\epsilon)}{(1-\kappa^2)^2}\right\} \tag{4.6}$$

independent of α and β.

Definition 4.1. A refractor $\mathcal{R}$ is a weak solution of the refractor problem with emitting illumination intensity f on $\overline{\Omega}$ and prescribed refracted illumination intensity μ on $\overline{\Omega^*}$ if

$$\int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x)t_{\mathcal{R}}(x)dx \geq \mu(\omega)$$

for every Borel subset ω of $\overline{\Omega^*}$. Here

$$t_{\mathcal{R}}(x) = 1 - \phi\left(\frac{1}{\kappa}\left(1 - \Psi(x\cdot\nu)\, x\cdot\nu\right)\right), \tag{4.7}$$

where ϕ and Ψ are given by (4.1) and (4.3) respectively, and ν is the outer unit normal to $\mathcal{R}$ at a.e. $x \in \Omega$. Also $\mathcal{T}_{\mathcal{R}}$ is defined with ellipsoids if $\kappa < 1$ and with semi-hyperboloids if $\kappa > 1$.

5. Properties of the ellipsoids

Suppose $\mathbf{b} = (b_1, \ldots, b_N)$ has positive coordinates, $m_1, \ldots, m_N$ are different points in the sphere S^{n-1}, and $\Omega \subset S^{n-1}$ with $\inf_{x\in\overline{\Omega}, 1\leq j\leq N} x\cdot m_j \geq \kappa$. We let

$$\rho(x) = \min_{1\leq i\leq N} \frac{b_i}{1-\kappa x\cdot m_i},$$

and $\mathcal{R} = \mathcal{R}(\mathbf{b}) = \{\rho(x)x : x \in \overline{\Omega}\}$.

Lemma 5.1. *If $x_0 \in \mathcal{T}_{\mathcal{R}}(m_j)$ and x_0 is not a singular point to $\mathcal{R}$, then the semi-ellipsoid $E(m_j, b_j)$ supports $\mathcal{R}$ at the point x_0.*

Proof. Since $m_j \in \mathcal{N}_{\mathcal{R}}(x_0)$, there exists a supporting semi-ellipsoid $E(m_j, b)$ to $\mathcal{R}$ at the point x_0 for some $b > 0$. We prove that $b = b_j$. Indeed, $\rho(x) \leq \frac{b}{1-\kappa x\cdot m_j}$ for all $x \in \bar{\Omega}$ with equality at $x = x_0$. Hence $\frac{b}{1-\kappa x_0\cdot m_j} \leq \frac{b_j}{1-\kappa x_0\cdot m_j}$, and so $b \leq b_j$. If $b < b_j$, then

$$\rho(x) \leq \frac{b}{1-\kappa x\cdot m_j} < \frac{b_j}{1-\kappa x\cdot m_j}, \quad \forall x \in \bar{\Omega},$$

so $\rho(x) = \min_{k\neq j} \frac{b_k}{1-\kappa m_k\cdot x}$. This implies that there exists $k \neq j$ such that $\frac{b_k}{1-\kappa m_k\cdot x}$ is a supporting ellipsoid at x_0. This means that at x_0, there are two supporting ellipsoids to $\mathcal{R}$ with different axes m_k, m_j, and therefore x_0 is a singular point, a contradiction. □

If at a point x_0, the ellipsoids $E(m_j, b_j)$ and $E(m_k, b_k)$ support $\mathcal{R}(\mathbf{b})$ with $m_k \neq m_j$, then x_0 is a singular point, and therefore the points supported by two ellipsoids with different axes form a set of measure zero.

Lemma 5.2. *Let $E(m_k, b_k)$ be a sequence of semi ellipsoids with $m_k \to m$ and $b_k \to b$, as $k \to \infty$. If $z_k \in E(m_k, b_k)$ with $z_k \to z_0$ as $k \to \infty$, then $z_0 \in E(m, b)$, and the normal $\nu_k(z_k)$ to the ellipsoid $E(m_k, b_k)$ at z_k satisfies $\nu_k(z_k) \to \nu(z_0)$ the normal to the ellipsoid $E(m, b)$ at the point z_0.*

Proof. Indeed, the equation of $E(m_k, b_k)$ in rectangular coordinates is $|z| - \kappa m_k\cdot z = b_k$, then the normal vector at z is $\nu_k(z) = \frac{z}{|z|} - \kappa m_k$, and so $\nu_k(z_k) = \frac{z_k}{|z_k|} - \kappa m_k \to \frac{z_0}{|z_0|} - \kappa m$ which is the normal to $E(m, b)$ at z_0. The normal at z written in polar coordinates, with $z = \rho_k(x)x$ has the form $\nu_k(x) = x - \kappa m_k$ so $\nu_k(x) \to x - \kappa m = \nu(x)$, the normal to the ellipsoid $E(m, b)$. □

Proposition 5.3. *Let $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$. Let S be the singular set of ρ. Then the Fresnel coefficient $t_{\mathcal{R}}(x)$ is continuous relative to the set $\overline{\Omega} \setminus S$, and therefore is a Lebesgue measurable function in Ω.*

Proof. We shall prove that $r_{\mathcal{R}}$ is continuous. Suppose $C_1 \leq \rho(x) \leq C_2$, with C_i, $i = 1, 2$, positive constants. From (4.4), $r_{\mathcal{R}}(x)$ is a function $\phi(x) = F(x, \nu(x))$ which is defined in $\overline{\Omega} \setminus S$ where $F(x, m)$ is a continuous function in $\overline{\Omega} \times \overline{\Omega^*}$.

To prove that $r_{\mathcal{R}}(x)$ is continuous we shall prove that it is both lower and upper semicontinuous relative to the set $\overline{\Omega} \setminus S$. For the lower semicontinuity relative to the set $\overline{\Omega} \setminus S$, we shall prove that the set

$$E_\alpha = \{x \in \overline{\Omega} \setminus S : \phi(x) \leq \alpha\}$$

is closed relative to $\overline{\Omega} \setminus S$, for all α, that is, if $x_j, x_0 \in \overline{\Omega} \setminus S$, with $x_j \to x_0$ and $x_j \in E_\alpha$, then $x_0 \in E_\alpha$. First we claim that if $x_j \to x_0$ with $x_j, x_0 \in \overline{\Omega} \setminus S$, then there exists a subsequence x_{j_l} such that $\nu(x_{j_l}) \to \nu(x_0)$, as $l \to \infty$. Let $E(m_j, b_j)$ be a supporting ellipsoid to the refractor at $\rho(x_j)x_j$. So

$$\rho(x) \leq \frac{b_j}{1-\kappa m_j\cdot x}$$

with equality at $x = x_j$. Hence

$$C_1(1-\kappa) \le C_1(1-\kappa m_j \cdot x_j) \le b_j \le C_2(1-\kappa m_j \cdot x_j) \le C_2(1-\kappa^2).$$

Therefore, there is a subsequence $b_{j_l} \to b_0 > 0$ and $m_{j_l} \to m_0$, as $l \to \infty$. Then by Lemma 5.2 we have the claim.

Now if $x_j \in E_\alpha$, then $\phi(x_j) = F(x_j, \nu(x_j)) \le \alpha$, but from the claim, there exists a subsequence x_{j_l} such that $\nu(x_{j_l}) \to \nu(x_0)$ as $l \to \infty$. Then from the continuity of F we are done. □

Lemma 5.4. *Suppose $\mathcal{R}_j$ and $\mathcal{R}$ are refractors with defining functions $\rho_j(x)$ and $\rho(x)$ and corresponding Fresnel coefficients ϕ_j and ϕ, respectively. Suppose $\rho_j \to \rho$ pointwise in $\overline{\Omega}$ with $C_1 \le \rho_j(x) \le C_2$ in $\overline{\Omega}$ for some positive constants C_1 and C_2. Let S be the union of all singular points of all the refractors $\mathcal{R}_j$ and $\mathcal{R}$. Then for each $y \notin S$ there is subsequence $\phi_{j_\ell}(y) \to \phi(y)$ as $\ell \to \infty$.*

Proof. Given $y \notin S$ and j, there exist $b_j > 0$ and $m_j \in \overline{\Omega^*}$ such that $\rho_j(z) \le \frac{b_j}{1-\kappa m_j \cdot z}$ for all $z \in \overline{\Omega}$ with equality at $z = y$. We then have that

$$C_1 \le \frac{b_j}{1-\kappa m_j \cdot y} \le C_2$$

and so

$$C_1(1-\kappa) \le C_1(1-\kappa m_j \cdot y) \le b_j \le C_2(1-\kappa m_j \cdot y) \le C_2(1-\kappa^2),$$

that is, b_j is bounded away from zero and infinity. Therefore there exist subsequences $b_{j_\ell} \to b > 0$, $m_{j_\ell} \to m \in \bar{\Omega}^*$, and so the semi-ellipsoid $E(m, b)$ supports $\mathcal{R}$ at y, so $y \in \mathcal{T}_{\mathcal{R}}(m)$. If $y \notin S$, then the normal $\nu_{j_\ell}(y)$ to the ellipsoid $E(m_{j_\ell}, b_{j_\ell})$ equals the normal to the refractor $\mathcal{R}_{j_\ell}$ at y, and the normal $\nu(y)$ to the ellipsoid $E(m, b)$ equals the normal to the refractor $\mathcal{R}$ at y. Since $E(m_{j_\ell}, b_{j_\ell})$ tends to $E(m, b)$ as $\ell \to \infty$, it follows that $\nu_{j_\ell}(y) \to \nu(y)$ for $y \notin S$ as $\ell \to \infty$ from Lemma 5.2. Consequently $\phi_{j_\ell}(y) \to \phi(y)$ for $y \notin S$. □

The following theorem is needed in the proof of Theorem 6.11.

Theorem 5.5. *Assume the hypotheses and notation of Lemma 5.4. Let $F \subset \overline{\Omega}^*$ compact, $F_j = \mathcal{T}_{\mathcal{R}_j}(F)$, and ϕ_j are the Fresnel coefficients of $\mathcal{R}_j$. Then*

$$\limsup_{j\to\infty} \chi_{F_j}(y)\phi_j(y) = \phi(y) \limsup_{j\to\infty} \chi_{F_j}(y), \tag{5.1}$$

$$\limsup_{j\to\infty} \chi_{F_j}(y)\phi_j(y) \le \chi_{\mathcal{T}_{\mathcal{R}}(F)}(y)\phi(y), \tag{5.2}$$

$$\limsup_{j\to\infty} \int_{\overline{\Omega}} \chi_{F_j}(x)\phi_j(x)f(x)\,dx, \le \int_{\mathcal{T}_{\mathcal{R}}(F)} \phi(x)f(x)\,dx, \tag{5.3}$$

for $y \notin S$, the union of the singular sets of all ρ_j and ρ. If F is any set contained in $\overline{\Omega^*}$, then we also have

$$\liminf_{j\to\infty} \chi_{F_j}(y)\phi_j(y) = \phi(y) \liminf_{j\to\infty} \chi_{F_j}(y), \tag{5.4}$$

for all $y \notin S$.

Proof. Let $F^* = \limsup_{j\to\infty} F_j = \bigcap_{k=1}^\infty \bigcup_{j=k}^\infty F_j$. If $y \notin F^*$, then $y \in \bigcup_{k=1}^\infty \bigcap_{j=k}^\infty F_j^c$, that is, $y \notin F_j$ for all $j \ge k$ and hence both sides of (5.1) are zero.

Fix $y \notin S$. If $y \in F^*$, then the right hand side of (5.1) equals $\phi(y)$. If $\limsup_{j\to\infty} \chi_{F_j}(y)\phi_j(y) = B(y) > 0$, then there is a subsequence F_{j_ℓ} such that $\lim_{\ell\to\infty} \chi_{F_{j_\ell}}(y)\phi_{j_\ell}(y) = B(y) > 0$. Hence $y \in F_{j_\ell}$ for all ℓ sufficiently large. This means $y \in \mathcal{T}_{\mathcal{R}_{j_\ell}}(F)$ and so $y \in \mathcal{T}_{\mathcal{R}_{j_\ell}}(m_{j_\ell})$ for some $m_{j_\ell} \in F$ for all ℓ sufficiently large. By compactness there is a subsequence of m_{j_ℓ} converging to some $m \in F$, and therefore $y \in \mathcal{T}_{\mathcal{R}}(F)$ and $\chi_{F_{j_\ell}}(y) \le \chi_{\mathcal{T}_{\mathcal{R}}(F)}(y)$ for ℓ large. Now from Lemma 5.4, there is a subsequence (depending on y) such that $\phi_{j_{\ell_k}}(y) \to \phi(y)$ as $k \to \infty$. Consequently, $B(y) = \lim_{\ell\to\infty} \chi_{F_{j_{\ell_k}}}(y)\phi_{j_{\ell_k}}(y) = \phi(y)$, so we obtain that

$$\limsup_{j\to\infty} \chi_{F_j}(y)\phi_j(y) \le \chi_{\mathcal{T}_{\mathcal{R}}(F)}(y)\phi(y),$$

and (5.1) for $y \in F^*$. On the other hand, if $y \in F^*$ and $\limsup_{j\to\infty} \chi_{F_j}(y)\phi_j(y) = 0$, then $\lim_{\ell\to\infty} \chi_{F_{j_\ell}}(y)\phi_{j_\ell}(y) = 0$ for a subsequence, and $y \in F_{j_\ell}$ for all ℓ. Therefore $\lim_{\ell\to\infty} \phi_{j_\ell}(y) = 0$. Once again from Lemma 5.4 $\lim_{k\to\infty} \phi_{j_{\ell k}}(y) = \phi(y) = 0$ for a subsequence. This completes the proof of (5.1). Inequality (5.3) follows from (5.2) from reverse Fatou lemma.

It remains to prove (5.4). Let $F_* = \liminf_{j\to\infty} F_j = \bigcup_{k=1}^\infty \bigcap_{j=k}^\infty F_j$. If $y \notin F_*$, then the right hand side of (5.4) is zero; and $y \in \bigcap_{k=1}^\infty \bigcup_{j=k}^\infty F_j^c$, so there exists a subsequence F_{j_ℓ} with $\chi_{F_{j_\ell}}(y) = 0$ for all ℓ. Therefore $\lim_{\ell\to\infty} \chi_{F_{j_\ell}}(y)\phi_{j_\ell}(y) = 0$ and consequently $\liminf_{j\to\infty} \chi_{F_j}(y)\phi_j(y) = 0$. So both sides of (5.4) are zero. Suppose now that $y \in F_*$. There are two

possibilities $\liminf_{j\to\infty} \chi_{F_j}(y)\phi_j(y) = 0$ or $\liminf_{j\to\infty} \chi_{F_j}(y)\phi_j(y) > 0$. In the first case, there is a subsequence such that $\lim_{\ell\to\infty} \chi_{F_{j_\ell}}(y)\phi_{j_\ell}(y) = 0$, and since $y \in F_*$, $\chi_{F_j}(y) = 1$ for j large and so $\lim_{\ell\to\infty} \phi_{j_\ell}(y) = 0$. From Lemma 5.4, $\phi_{j_{\ell_k}}(y) \to \phi(y)$ as $k \to \infty$ and so $\phi(y) = 0$. In case $\liminf_{j\to\infty} \chi_{F_j}(y)\phi_j(y) = B(y) > 0$, we have $\liminf_{j\to\infty} \phi_j(y) = B(y) > 0$ and so $\lim_{\ell\to\infty} \phi_{j_\ell}(y) = B(y)$ and once again from Lemma 5.4, $\phi_{j_{\ell_k}}(y) \to \phi(y) = B(y)$ as $k \to \infty$ and the proof is complete. □

Remark 5.6 (*Invariance by dilations*)**.** Suppose that $\mathcal{R}$ is a refractor weak solution in the sense of Definition 4.1 with intensities f, μ and defined by $\rho(x)x$ for $x \in \overline{\Omega}$. Then for each $\alpha > 0$, the refractor $\alpha\mathcal{R}$ defined by $\alpha\,\rho(x)x$ for $x \in \overline{\Omega}$ is a weak solution in the sense of Definition 4.1 with the same intensities. In fact, $E(m, b)$ is a supporting ellipsoid to $\mathcal{R}$ at the point y if and only if $E(m, \alpha\, b)$ is a supporting ellipsoid to $\alpha\mathcal{R}$ at the point y. This means that $\mathcal{T}_{\mathcal{R}}(m) = \mathcal{T}_{\alpha\mathcal{R}}(m)$ for each $m \in \overline{\Omega^*}$. Also $\mathcal{T}_{\alpha\mathcal{R}} = \mathcal{T}_{\mathcal{R}}$.

6. Existence of solutions when $\kappa < 1$

We prove existence first in the discrete case and then pass to the limit.

6.1. Existence of weak solutions when μ is discrete

Let $m_1, m_2, \ldots, m_N$ be distinct points in $\overline{\Omega^*}$. For $\mathbf{b} = (b_1, \ldots, b_N)$ with each $b_i > 0$, we denote by $\mathcal{R}(\mathbf{b})$ the refractor defined by

$$\mathcal{R}(\mathbf{b}) = \left\{\rho(x)x : x \in \overline{\Omega},\ \rho(x) = \min_{1\le i\le N} \frac{b_i}{1-\kappa m_i\cdot x}\right\}. \tag{6.1}$$

We have the following proposition.

Proposition 6.1. *Let S be the singular set of the refractor $\mathcal{R} = \mathcal{R}(\mathbf{b})$, then*

$$\mathcal{N}_{\mathcal{R}}(\Omega \setminus S) \subset \{m_1, \ldots, m_N\}, \quad \text{for } N \ge 1, \tag{6.2}$$

and

$$\tag{6.3}$$

$$\mathcal{N}_{\mathcal{R}}(\Omega \setminus S) = \{m_1\}, \quad \text{for } N = 1. \tag{6.4}$$

Proof. Let $x_0 \notin S$ and let $\frac{b}{1-\kappa\, m\cdot x}$ be a supporting ellipsoid at x_0. There exists j such that $\rho(x_0) = \frac{b_j}{1-\kappa\, m_j\cdot x_0}$ and $\frac{b_j}{1-\kappa\, m_j\cdot x}$ supports ρ. Since x_0 is not a singular point of ρ, it follows that $m = m_j$.

(6.4) follows immediately form (6.2) since $S = \emptyset$ and $\mathcal{N}_{\mathcal{R}}(\Omega) \ne \emptyset$. □

The following is the main theorem in this section and states existence of solutions to the refractor problem when μ is a discrete measure. *Assuming that the amplitudes $I_\parallel$, $I_\perp$ of the incoming wave are constants*, the solution in Theorem 6.2 overshoots energy in the direction m_1. In other words, the amount of energy send by the refractor to the direction m_1 exceeds the prescribed value. We will prove later on in Theorem 6.9, that there is a refractor that overshoots the least amount of energy.

Theorem 6.2. *Let $f \in L^1(\overline{\Omega})$ with $\inf_{\overline{\Omega}} f > 0$, $m_1, \ldots, m_N \in \Omega^*$, distinct points with $N \ge 2$, $g_1, \ldots, g_N > 0$, and $\mu = \sum_{i=1}^N g_i\,\delta_{m_i}$. Suppose that $\inf_{x\in\Omega, 1\le j\le N} x\cdot m_j \ge \kappa + \epsilon$, and*

$$\int_{\overline{\Omega}} f(x)\,dx \ge \frac{1}{1-C_\epsilon}\,\mu(\overline{\Omega^*}), \tag{6.5}$$

where C_ϵ is given in (4.5). Then there exists $\mathbf{b}_o = (b_1, \ldots, b_N) \in \mathbb{R}^N$, $b_i > 0$ and a refractor $\mathcal{R}_o = \mathcal{R}(\mathbf{b}_o)$ such that

$$\int_{\mathcal{T}_{\mathcal{R}_o}(m_i)} f(x)t_{\mathcal{R}_o}(x)dx = g_i$$

for $i = 2, \ldots, N$, and

$$\int_{\mathcal{T}_{\mathcal{R}_o}(m_1)} f(x)t_{\mathcal{R}_o}(x)dx > g_1.$$

Therefore, $\mathcal{R}_o$ is a weak solution to the refractor problem with intensities f and μ; that is, $\int_{\mathcal{T}_{\mathcal{R}_o}(\omega)} f(x)t_{\mathcal{R}_o}(x)dx \ge \mu(\omega)$ for each Borel set $\omega \subset \Omega^*$, and satisfies

$$\int_{\mathcal{T}_{\mathcal{R}_o}(\omega)} f(x)t_{\mathcal{R}_o}(x)dx = \mu(\omega), \tag{6.6}$$

for each Borel set $\omega \subset \Omega^*$ with $m_1 \notin \omega$.

Remark 6.3. If $N = 1$, then the problem might be overdetermined. We have that $\mathcal{R}(\mathbf{b})$ equals the ellipsoid $E(m_1, b_1)$ for some $b_1 > 0$ all over $\overline{\Omega}$. Clearly this predetermines the value of $t_{\mathcal{R}}(x)$ and so the value of $\int_{\overline{\Omega}} f(x)t_{\mathcal{R}}(x)dx$, and therefore we must have

$$\int_{\overline{\Omega}} f(x)t_{\mathcal{R}}(x)dx > g_1. \tag{6.7}$$

Thus it will not be reasonable to expect existence of a solution for values g_1 which do not satisfy (6.7).

To prove Theorem 6.2, we first establish the following two lemmas and Theorem 6.6.

Lemma 6.4. *Let W be the collection of $\mathbf{b} = (1, b_2, \ldots, b_N)$ with $b_i > 0$ such that $\mathcal{R}(\mathbf{b})$ satisfies*

$$G_{\mathcal{R}(\mathbf{b})}(m_i) = \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b})}(x)dx \le g_i \quad \text{for } i = 2, \ldots, N$$

where the g_i's and f are as in Theorem 6.2. Then

i. $W \neq \emptyset$.
ii. *If $\mathbf{b} = (1, b_2, \ldots, b_N) \in W$ then*

$$\frac{1}{1+\kappa} \le b_i \tag{6.8}$$

for $i = 2, \ldots, N$.

Proof. (i.) If for some $i \neq 1$, $E(m_i, b)$, is a supporting semi-ellipsoid to $\mathcal{R}(\mathbf{b})$ at $\rho(x)x$ then

$$\frac{b}{1-\kappa^2} \le \frac{b}{1-\kappa m_i \cdot x} = \rho(x) \le \frac{1}{1-\kappa m_1 \cdot x} \le \frac{1}{1-\kappa}$$

and so $b \le 1+\kappa$. Therefore, if $b_i > 1+\kappa$ for all $2 \le i \le N$, then $E(m_i, b_i)$ cannot be a supporting semi-ellipsoid to $\mathcal{R}(\mathbf{b})$ at any $x \in \overline{\Omega}$, and therefore, from Lemma 5.1, the set $\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)$ is contained in the set of singular points of $\mathcal{R}(\mathbf{b})$ for $i = 2, \ldots, N$, and consequently $G_{\mathcal{R}(\mathbf{b})}(m_i) = 0 < g_i$. Hence taking $\mathbf{b} = (1, b_2, \ldots, b_N)$ with $b_i > 1+\kappa$ for $2 \le i \le N$, yields $\mathbf{b} \in W$.

(ii.) First notice that if at a point x_0, the ellipsoids $E(m_j, b_j)$ and $E(m_k, b_k)$ support $\mathcal{R}(\mathbf{b})$ with $m_k \neq m_j$, then x_0 is a singular point, and therefore the points supported by two ellipsoids with different axes form a set of measure zero. From this, we have that if $\mathbf{b} \in W$, then $g_1 \le G_{\mathcal{R}(\mathbf{b})}(m_1)$. Indeed,

$$\begin{aligned}
\sum_{i=1}^{N} G_{\mathcal{R}(\mathbf{b})}(m_i) &= \sum_{i=1}^{N} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b})}(x)dx \\
&= \int_{\cup_{i=1}^{N} \mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b})}(x)dx \\
&= \int_{\overline{\Omega}} f(x)t_{\mathcal{R}(\mathbf{b})}(x)dx \\
&\ge (1 - C_\epsilon) \int_{\overline{\Omega}} f(x)dx \\
&\ge \mu(\overline{\Omega^*}) = \sum_{i=1}^{N} g_i,
\end{aligned}$$

and so we have that

$$[g_1 - G_{\mathcal{R}(\mathbf{b})}(m_1)] + \sum_{i=2}^{N} \left[g_i - G_{\mathcal{R}(\mathbf{b})}(m_i)\right] \le 0.$$

If $\mathbf{b} \in W$,

$$\sum_{i=2}^{N} g_i - G_{\mathcal{R}(\mathbf{b})}(m_i) \ge 0.$$

Thus $g_1 \le G_{\mathcal{R}(\mathbf{b})}(m_1)$.

Suppose that $\mathcal{R}(\mathbf{b}) = \{\rho(x)x : x \in \overline{\Omega}\}$. We shall prove that there exists a point $\rho(x_o)x_o$ such that $\rho(x_o)x_o \in \mathcal{R}(b) \cap E(m_1, 1)$ and $\rho(x_o)x_o \notin E(m_i, b_i)$ for all $i \geq 2$. Otherwise, $\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_1) \subset S$ where S is the singular set of ρ. But then,

$$G_{\mathcal{R}(\mathbf{b})}(m_1) = \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_1)} f(x)t_{\mathcal{R}(\mathbf{b})}(x)dx \leq \int_S f(x)t_{\mathcal{R}(\mathbf{b})}(x)dx = 0$$

contradicting the fact that $g_1 > 0$. Then

$$\rho(x_o) = \frac{1}{1 - \kappa m_1 \cdot x_o} < \frac{b_i}{1 - \kappa m_i \cdot x_o}$$

for $i = 2, \ldots, N$, from which we conclude that

$$\frac{1}{1+\kappa} < b_i$$

for all $i = 2, \ldots, N$. □

Lemma 6.5. *Let $\mathbf{b}_j = (b_1^j, \ldots, b_N^j)$ and $\mathbf{b}_o = (b_1^o, \ldots, b_N^o)$ with $\mathbf{b}_j \to \mathbf{b}_o$ in $\mathbf{R}^{\mathbf{N}}$ as $j \to \infty$. Suppose $\mathcal{R}_j = \mathcal{R}(\mathbf{b}_j) = \{\rho_j(x)x : x \in \overline{\Omega}\}$ and $\mathcal{R}_o = \mathcal{R}(\mathbf{b}_o) = \{\rho(x)x : x \in \overline{\Omega}\}$. Then, $\rho_j \to \rho$ uniformly on $\overline{\Omega}$.*

Proof. If $x_o \in \overline{\Omega}$,

$$\begin{aligned}\rho_j(x_o) - \rho(x_o) &= \rho_j(x_o) - \frac{b_l}{1 - \kappa m_l \cdot x_o} \quad \text{for some } l,\\ &\leq \frac{b_l^j}{1 - \kappa m_l \cdot x_o} - \frac{b_l}{1 - \kappa m_l \cdot x_o} \leq \frac{\|\mathbf{b}^j - \mathbf{b}\|}{1-\kappa},\end{aligned}$$

thus $\rho_j \to \rho$ uniformly on $\overline{\Omega}$. □

Theorem 6.6. *Let $\delta > 0$. Then $G_{\mathcal{R}(\mathbf{b})}(m_i)$ are continuous in the region $R_\delta = \{b_i : b_i \geq \delta,\ i = 2, \ldots, N\}$, for all $1 \leq i \leq N$.*

Proof. Let $\mathbf{b}_j, j \geq 1$ be a sequence in R_δ converging to $\mathbf{b}_o$. Suppose also that $\mathcal{R}(\mathbf{b}_j) = \{\rho_j(x)x : x \in \overline{\Omega}\}$ and $\mathcal{R}(\mathbf{b}_o) = \{\rho_o(x)x : x \in \overline{\Omega}\}$. By Lemma 6.5, $\rho_j \to \rho$ uniformly on $\overline{\Omega}$. Moreover for any $x \in \overline{\Omega}$ and $j \geq 1$,

$$\frac{\delta}{1+\kappa} \leq \frac{b_l^j}{1 - \kappa m_l \cdot x} = \rho_j(x)$$

for some $l \in \{1, 2, \ldots, N\}$. Also

$$\rho_j(x) = \min_{1 \leq i \leq N} \frac{b_i^j}{1 - \kappa m_i \cdot x} \leq \frac{1}{1 - \kappa m_1 \cdot x} \leq \frac{1}{1-\kappa}.$$

We thus obtain $a_1, a_2 > 0$ such that $0 < a_1 \leq \rho_j(x) \leq a_2$ for all j and for all $x \in \overline{\Omega}$. Let $G \subset \overline{\Omega^*}$ be a neighborhood of m_i such that $m_l \notin G$ for $l \neq i$. If $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(G)$ and $x_o \notin S$, there exists a unique $m \in G$ and $b > 0$ such that

$$\rho_j(x_o) = \frac{b}{1 - \kappa m \cdot x_o} \quad \text{and} \quad \rho_j(x) \leq \frac{b}{1 - \kappa m \cdot x} \quad \text{for all } x \in \overline{\Omega}.$$

But by definition of $\mathcal{R}(\mathbf{b}_j)$, $m = m_l$ for some $l = 1, \ldots, N$. Thus $m = m_i$. From this we conclude that $\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(G) \subset \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i) \cup S$. Combining this with Lemma 3.6(iii.) and the fact that S has measure zero, we obtain

$$\begin{aligned}\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx &\leq \int_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i) \cup S} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx\\ &\leq \int_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx + \int_S f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx\\ &= \int_{\overline{\Omega}} \chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) f(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx. \qquad (6.9)\end{aligned}$$

By applying (5.4), to (6.9) we get,

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \leq \int_{\overline{\Omega}} \chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) f(x)dx. \qquad (6.10)$$

It is also true that

$$\chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) = \liminf \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x).$$

Using this in (6.10), we obtain

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \le \int_{\overline{\Omega}} \liminf \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)f(x)dx$$

from which we deduce by Fatou's lemma that

$$\begin{aligned}\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx &\le \liminf \int_{\overline{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)f(x)dx \\ &= \liminf \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x)f(x)dx.\end{aligned}$$

To complete the proof we shall prove that

$$\limsup \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x)f(x)dx \le \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx. \tag{6.11}$$

First notice that

$$\begin{aligned}\limsup \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x)f(x)dx &= \limsup \int_{\overline{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)f(x)dx \\ &\le \int_{\overline{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)f(x)dx\end{aligned}$$

where the last inequality is due to reverse Fatou's Lemma. By (5.1) and the fact that $\limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) = \chi_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)$, we have

$$\begin{aligned}\int_{\overline{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)f(x)dx &= \int_{\overline{\Omega}} \chi_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \\ &= \int_{\overline{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \\ &= \int_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx.\end{aligned}$$

But then by Lemma 3.6(ii.),

$$\int_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \le \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} f(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx$$

from which we conclude (6.11) and therefore concluding the proof of the theorem. □

We shall now complete the proof of Theorem 6.2.

Proof of Theorem 6.2. Fix $\tilde{\mathbf{b}} = (1, \tilde{b}_2, \ldots, \tilde{b}_N) \in W$ and consider

$$\widetilde{W} = \{\mathbf{b} = (1, b_2, \ldots, b_N) \in W : b_i \le \tilde{b}_i \quad \text{for } i = 2, \ldots, N\}.$$

Then from Lemma 6.4(ii.) and Theorem 6.6, $\widetilde{W}$ is compact. Consider the map

$$d : \widetilde{W} \to \mathbf{R}$$

given by

$$d(\mathbf{b}) = \sum_{i=1}^{N} b_i$$

where $\mathbf{b} = (1, b_2, \ldots, b_N)$. Let $\mathbf{b}^* = (1, b_2^*, \ldots, b_N^*)$ be such that

$$\mathbf{b}^* = arg \min_{\mathbf{b} \in \widetilde{W}} d(\mathbf{b}).$$

$\mathcal{R}(b^*)$ is the refractor we are looking for. We first show that $\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_j)} f(x)t_{\mathcal{R}(b^*)}(x) = g_j$, for $j = 2, \dots, N$. Suppose the contrary and without loss of generality that

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_2)} f(x)t_{\mathcal{R}(b^*)}(x) < g_2.$$

Take $0 < \lambda < 1$ and consider $\mathbf{b}^*_\lambda = (1, \lambda b^*_2, \dots, b^*_N)$. If $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i)$, x_o is not a singular point of $\mathcal{R}(\mathbf{b}^*_\lambda)$, and ρ is the defining function of $\mathcal{R}(\mathbf{b}^*)$, then, from Lemma 5.1, the ellipsoid $E(m_i, b^*_i)$ supports both $\mathcal{R}(\mathbf{b}^*_\lambda)$ and $\mathcal{R}(\mathbf{b}^*)$ at x_o, when $i \neq 2$, and in particular, from the Snell law, the normals at x_0 to both $\mathcal{R}(\mathbf{b}^*_\lambda)$ and $\mathcal{R}(\mathbf{b}^*)$ are the same. Hence $\mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i) \setminus (\text{singular set of} \mathcal{R}(\mathbf{b}^*_\lambda)) \subset \mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)$, and $t_{\mathcal{R}(\mathbf{b}^*_\lambda)}(x) = t_{\mathcal{R}(\mathbf{b}^*)}(x)$ for all $x \in \mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i)$ outside the singular set of $\mathcal{R}(\mathbf{b}^*_\lambda)$, $i \neq 2$. Therefore

$$\begin{aligned}\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b}^*_\lambda)}(x)dx &= \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b}^*)}(x)dx \\ &\leq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)} f(x)t_{\mathcal{R}(\mathbf{b}^*)}(x)dx \leq f_i, \quad i \neq 2.\end{aligned}$$

Then by using Theorem 6.6, we choose λ sufficiently close 1 so that $G_{\mathbf{b}^*_\lambda}(m_2) < g_2$, concluding that $\mathbf{b}^*_\lambda \in \widetilde{W}$. But this is a contradiction as it implies $d(\mathbf{b}^*_\lambda) < d(\mathbf{b}^*)$.

Second, we show that $\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_1)} f(x)t_{\mathcal{R}(b^*)}(x) > g_1$. From the proof of Lemma 6.4 (i), $\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_1)} f(x)t_{\mathcal{R}(b^*)}(x) \geq g_1$. Suppose by contradiction that the equality holds. Hence

$$\int_{\overline{\Omega}} f(x)t_{\mathcal{R}(\mathbf{b}^*)}(x)dx = \sum_{i=1}^N G_{\mathcal{R}(\mathbf{b}^*)}(m_i) = \sum_{i=1}^N g_i \leq (1 - C_\epsilon) \int_{\overline{\Omega}} f(x)dx$$

obtaining

$$\int_{\overline{\Omega}} f(x)[(1 - C_\epsilon) - t_{\mathcal{R}(\mathbf{b}^*)}(x)]\, dx \geq 0.$$

From (4.4), we have $t_{\mathcal{R}(\mathbf{b}^*)}(x) \geq 1 - C_\epsilon$. Since $\inf_\Omega f > 0$, we obtain $t_{\mathcal{R}(\mathbf{b}^*)}(x) = 1 - C_\epsilon$ a.e. $x \in \overline{\Omega}$. For $x \in E := \mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_1) \setminus S$, from (4.1) and (4.2), we have that $t_{\mathcal{R}(\mathbf{b}^*)}(x) = 1 - \phi(x \cdot m_1)$, so $\phi(x \cdot m_1) = C_\epsilon$ for $x \in E$. We have that $|E| > 0$. We claim that the linear set $E' = \{x \cdot m_1 : x \in E\}$ is infinite. Otherwise, there exist $c_1, \dots, c_K$ such that $E' = \{c_1, \dots, c_K\}$. If $E_j = \{x \in E : x \cdot m_1 = c_j\}$, then $E = \cup_{i=1}^K E_j$. But E_j is contained in S^{n-1} intersected with the plane $\{x \in \mathbb{R}^n : x \cdot m_1 = c_j\}$, so its spherical measure is zero, and therefore $|E| = 0$. This is a contradiction, and the claim is proved.

On the other hand, since the amplitudes $I_\perp$ and $I_\parallel$ of the incoming wave are constant, the function $\phi(s)$ in (4.1) is piecewise strictly monotone, and therefore $\{s : \phi(s) = c\}$ is a finite set for each constant c. So it is impossible that $\phi = C_\epsilon$ on the infinite set E'.

Finally, for $\omega \subset \Omega^*$ Borel set, we have

$$\mathcal{T}_{\mathcal{R}}(\omega) = \bigcup_{j=1}^N \mathcal{T}_{\mathcal{R}}(\omega) \cap \mathcal{T}_{\mathcal{R}}(m_j) = \bigcup_{j=1}^N \mathcal{T}_{\mathcal{R}}(\omega \cap \{m_j\}), \text{ a.e.},$$

and so

$$\begin{aligned}\int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x)t_{\mathcal{R}}(x)\, dx &= \sum_{j=1}^N \int_{\mathcal{T}_{\mathcal{R}}(\omega\cap\{m_j\})} f(x)t_{\mathcal{R}}(x)\, dx \\ &= \sum_{j:m_j\in\omega} \int_{\mathcal{T}_{\mathcal{R}}(\omega\cap\{m_j\})} f(x)t_{\mathcal{R}}(x)\, dx = \sum_{j:m_j\in\omega} \int_{\mathcal{T}_{\mathcal{R}}(m_j)} f(x)t_{\mathcal{R}}(x)\, dx \geq \mu(\omega).\end{aligned}$$

Therefore, if $m_1 \notin \omega$, then we obtain (6.6). □

Remark 6.7. Notice that if ρ_0 is the defining function of the refractor in Theorem 6.2, from the proof of this theorem and Lemma 6.4, it follows that $\rho_0(x) \geq \frac{1}{(1+\kappa)^2}$ for all $x \in \Omega$.

Remark 6.8. Theorem 6.2, with $\mu = \sum_{i=1}^N g_i\, \delta_{m_i}$, yields the existence of a refractor so that $N - 1$ directions m_j receive exactly an amount of energy equal to g_j, while the remaining direction m_k receives an amount of energy bigger than g_k.

We complete this subsection showing that there exists a refractor $\mathcal{R}_o$ as in Theorem 6.2 that overshoots a minimum amount of energy.

Theorem 6.9. *Let $\mathcal{C}$ denote the class of all refractors as in Theorem 6.2. There exists $\mathcal{R}_o \in \mathcal{C}$ satisfying*

$$\int_{\mathcal{T}_{\mathcal{R}_o}(m_i)} f(x)t_{\mathcal{R}_o}(x)dx = g_i$$

for $i = 2, \dots, N$, and

$$\int_{\mathcal{T}_{\mathcal{R}_o}(m_1)} f(x)t_{\mathcal{R}_o}(x)dx = \inf_{\mathcal{R}_{\mathbf{b}}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(m_1)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx. \tag{6.12}$$

Consequently,

$$\int_{\mathcal{T}_{\mathcal{R}_o}(\omega)} f(x)t_{\mathcal{R}_o}(x)dx = \inf_{\mathcal{R}_{\mathbf{b}}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(\omega)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx \tag{6.13}$$

for every Borel subset $\omega \subseteq \overline{\Omega^*}$.

Proof. Identifying the refractors in Theorem 6.2 with vectors, we set

$$\mathcal{C} = \left\{ \mathbf{b} \in \mathbb{R}^N : \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(m_i)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx = g_i, 2 \le i \le N, \ \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(m_1)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx > g_1 \right\}.$$

Let $\mathbf{b} = (b_1, \dots, b_N) \in \mathcal{C}$, $\mathcal{R}_{\mathbf{b}} = \{\rho_{\mathbf{b}}(x)x : x \in \overline{\Omega}\}$, $\alpha_{\mathbf{b}} = \min_{x\in\bar{\Omega}} \rho_{\mathbf{b}}(x)$, and set $\tilde{\mathbf{b}} = \frac{1}{\alpha_{\mathbf{b}}}\mathbf{b}$. From Remark 5.6, the refractor $\mathcal{R}_{\tilde{\mathbf{b}}} = \{\rho_{\tilde{\mathbf{b}}}(x)x : x \in \overline{\Omega}\}$ is a rescaling of the refractor $\mathcal{R}_{\mathbf{b}}$ and therefore refracts in the same way as the refractor $\mathcal{R}_{\mathbf{b}}$, so $\tilde{\mathbf{b}} \in \mathcal{C}$. Moreover, $\min_{x\in\overline{\Omega}} \rho_{\tilde{\mathbf{b}}}(x) = 1$. Then by Lemma 3.8, we have $\rho_{\tilde{\mathbf{b}}}(x) \le C$ for all $x \in \overline{\Omega}$ with a constant C depending only on κ.

Since

$$G_{\mathcal{R}(\tilde{\mathbf{b}})}(m_i) := \int_{\mathcal{T}_{\mathcal{R}_{\tilde{\mathbf{b}}}}(m_i)} f(x)t_{\mathcal{R}_{\tilde{\mathbf{b}}}}(x)dx \ge g_i > 0$$

for all $i = 1, \dots, N$, we have that given $1 \le j \le N$ we can pick a point $x_j \in \mathcal{T}_{\mathcal{R}(\tilde{b})}(m_j) \setminus S$, so that the ellipsoid $E(m_j, \tilde{b}_j)$ supports $\mathcal{R}(\tilde{b})$ at x_j. Since $\rho_{\tilde{\mathbf{b}}}(x) \le C$ that gives $\frac{\tilde{b}_j}{1-\kappa x\cdot m_j} \le C$ from which we deduce that $\tilde{b}_j \le (1+\kappa)C$.

This proves that the set

$$\tilde{\mathcal{C}} = \left\{ \frac{1}{\alpha_{\mathbf{b}}}\mathbf{b} : \mathbf{b} \in \mathcal{C}, \ \alpha_b = \min_{x\in\overline{\Omega}} \rho_{\mathbf{b}}(x) \right\}$$

is bounded from above by a constant depending only on κ.

To prove that this set is bounded from below as well, choose $\tilde{\mathbf{b}} \in \tilde{\mathcal{C}}$ and let $y \in \overline{\Omega}$ and ℓ be such that $1 = \rho_{\tilde{\mathbf{b}}}(y) = \frac{\tilde{b}_\ell}{1-\kappa m_\ell\cdot y}$. Since $\rho_{\tilde{\mathbf{b}}}(y) = \min_{1\le j\le N} \frac{\tilde{b}_j}{1-\kappa m_j\cdot y}$, we have $1 \le \frac{\tilde{b}_j}{1-\kappa m_j\cdot y}$ for all $1 \le j \le N$ and from this we conclude that $1-\kappa \le \tilde{b}_j$ for all j.

Therefore, $\tilde{\mathcal{C}}$ is bounded from above and below by positive constants. Consider now

$$I = \inf_{\tilde{\mathbf{b}}\in\tilde{\mathcal{C}}} \int_{\mathcal{T}_{\mathcal{R}_{\tilde{\mathbf{b}}}}(m_1)} f(x)t_{\mathcal{R}_{\tilde{\mathbf{b}}}}(x)dx.$$

Clearly $g_1 \le I$ and there is a sequence $\tilde{\mathbf{b}}_j \in \tilde{\mathcal{C}}$ such that $G_{\mathcal{R}(\tilde{\mathbf{b}}_j)}(m_1) \to I$. By compactness there is a subsequence of these $\tilde{\mathbf{b}}_j$ that converges to some point $\mathbf{b}_o \in \tilde{\mathcal{C}}$, and by Theorem 6.6, the refractor $\mathcal{R}_o := \mathcal{R}(\tilde{\mathbf{b}}_o)$ is the one that overshoots less energy. Again by (6.5), and the fact that $\inf_\Omega f > 0$, as it was shown at the end of the proof of Theorem 6.2, the refractor $\mathcal{R}(\tilde{\mathbf{b}}_o)$ satisfies $G_{\mathcal{R}(\tilde{\mathbf{b}}_o)}(m_1) > g_1$.

This is useful to prove (6.13). Indeed let $\omega \subseteq \overline{\Omega^*}$ and $m_1 \notin \omega$. Then

$$\int_{\mathcal{T}_{\mathcal{R}_o}(\omega)} f(x)t_{\mathcal{R}_o}(x)dx = \sum_{m_i\in\omega} g_i = \inf_{\mathcal{R}_{\mathbf{b}}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(\omega)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx.$$

If $m_1 \in \omega$,

$$\begin{aligned}
\int_{\mathcal{T}_{\mathcal{R}_0}(\omega)} f(x)t_{\mathcal{R}_0}(x)dx &= \int_{\mathcal{T}_{\mathcal{R}_0}(\omega\setminus m_1)} f(x)t_{\mathcal{R}_0}(x)dx + \int_{\mathcal{T}_{\mathcal{R}_0}(m_1)} f(x)t_{\mathcal{R}_0}(x)dx \\
&= \sum_{m_i\in\omega, i\neq 1} g_i + \inf_{\mathcal{R}_{\mathbf{b}}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(m_1)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx \\
&= \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(\omega\setminus m_1)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx + \inf_{\mathcal{R}_{\mathbf{b}}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(m_1)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx, \quad \forall \mathcal{R}_{\mathbf{b}} \in \mathcal{C} \\
&\leq \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(\omega\setminus m_1)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx + \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(m_1)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx, \quad \forall \mathcal{R}_{\mathbf{b}} \in \mathcal{C} \\
&= \int_{\mathcal{T}_{\mathcal{R}_{\mathbf{b}}}(\omega)} f(x)t_{\mathcal{R}_{\mathbf{b}}}(x)dx, \quad \forall \mathcal{R}_{\mathbf{b}} \in \mathcal{C}.
\end{aligned}$$

Then taking infimum over $\mathcal{R}_{\mathbf{b}} \in \mathcal{C}$ and since $\mathcal{R}_0 \in \mathcal{C}$ we are done. □

Remark 6.10. We mention that for the case considered in [6] when all energy is transmitted, that is, when $t_{\mathcal{R}}(x) = 1$, the energy condition (6.5) is replaced by

$$\int_{\overline{\Omega}} g(x)\, dx = \mu(\overline{\Omega^*}).$$

This yields the definition of solution with equality, in other words, in this case $G_{\mathcal{R}(\mathbf{b})}(m_i) = \int_{\mathcal{T}_{\mathcal{R}}(m_i)} f(x)\, dx = g_i$, for $i = 1, 2, \ldots, N$.

6.2. Existence of weak solutions when μ is a Radon measure

In this section, $\mathcal{R}(\overline{\Omega}, \overline{\Omega^*})$ denotes the collection of all refractors from $\overline{\Omega}$ to $\overline{\Omega^*}$.

Theorem 6.11. *Let $f \in L^1(\overline{\Omega})$ with $\inf_{\overline{\Omega}} f > 0$, and let μ be a Radon measure in Ω^*. Suppose that $\inf_{x\in\Omega; m\in\Omega^*} x \cdot m \geq \kappa + \epsilon$ and*

$$\int_{\overline{\Omega}} f(x)\, dx \geq \frac{1}{1 - C_\epsilon}\, \mu(\overline{\Omega^*}), \tag{6.14}$$

where C_ϵ is given in (4.5). Let $m_o \in supp\ (\mu)$, the support of μ, and

$$\mathcal{C} = \left\{ \mathcal{R} \in \mathcal{R}(\overline{\Omega}, \overline{\Omega^*}) : \mu(\omega) \leq \int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x)t_{\mathcal{R}}(x)dx\ \forall \omega \text{ with equality when } m_o \notin \omega \right\}$$

for all Borel sets $\omega \subset \overline{\Omega^*}$. Then

(i) *$\mathcal{C} \neq \emptyset$, i.e., there exists a refractor solution to the refractor problem in the sense of Definition 4.1, with intensities f and μ;*
(ii) *There exists $\mathcal{R}_o \in \mathcal{C}$ such that*

$$\int_{\mathcal{T}_{\mathcal{R}_0}(m_o)} f(x)t_{\mathcal{R}_0}(x)dx = \inf_{\mathcal{R}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}}(m_o)} f(x)t_{\mathcal{R}}(x)dx$$

and consequently,

$$\int_{\mathcal{T}_{\mathcal{R}_0}(\omega)} f(x)t_{\mathcal{R}_0}(x)dx = \inf_{\mathcal{R}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x)t_{\mathcal{R}}(x)dx \tag{6.15}$$

for every Borel subset $\omega \subseteq \overline{\Omega^*}$.

Proof. To prove part (i), partition the domain Ω^* into a disjoint finite union of Borel sets with non empty interiors and with small diameter, say less than δ, so that m_0 is in the interior of one of them. Notice that the μ-measure of such a set is positive since $m_o \in$ supp (μ). Of all these sets discard the ones that have μ-measure zero. We then label the remaining sets $\omega_1^1, \ldots, \omega_{N_1}^1$ and we may assume $m_0 \in (\omega_1^1)^\circ$ and $\mu(\omega_j^1) > 0$. Next pick $m_i^1 \in \omega_i^1$, so that $m_1^1 = m_0$, and define a measure on $\overline{\Omega^*}$ by

$$\mu_1 = \sum_{i=1}^{N_1} \mu(\omega_i^1)\delta_{m_i^1}.$$

Then

$$\mu_1(\overline{\Omega^*}) = \mu(\overline{\Omega^*}) \leq (1 - C_\epsilon) \int_{\overline{\Omega}} f(x)\, dx.$$

Thus by Theorem 6.2, there exists a refractor

$$\mathcal{R}_1 = \left\{ \rho_1(x)x : \rho_1(x) = \min_{1 \leq i \leq N_1} \frac{b_i}{1 - \kappa m_i^1 \cdot x} \right\}$$

satisfying

$$\mu_1(\omega) \leq \int_{\mathcal{T}_{\mathcal{R}_1}(\omega)} f(x) t_{\mathcal{R}_1}(x) dx$$

with equality if $m_o \notin \omega$.

Next subdivide each ω_j^1 into a finite number of disjoint Borel subsets with non empty interiors and with diameter less than $\delta/2$, and such that m_o belongs to the interior of one of them. Again notice that since $m_o \in \operatorname{supp}(\mu)$, the set in the new subdivision containing m_o has positive μ-measure. Again discard the ones having μ-measure zero and label them $\omega_1^2, \ldots, \omega_{N_2}^2$. We may assume by relabeling that $\omega_1^2 \subset \omega_1^1$ and $m_0 \in (\omega_1^2)^\circ$. Next pick $m_i^2 \in \omega_i^2$, such that $m_1^2 = m_o$ and consider the measure μ_2 on $\overline{\Omega^*}$ defined by:

$$\mu_2 = \sum_{i=1}^{N_2} \mu(\omega_i^2) \delta_{m_i^2}.$$

Then

$$\mu_2(\overline{\Omega^*}) = \mu(\overline{\Omega^*}) \leq (1 - C_\epsilon) \int_{\overline{\Omega}} f(x)\, dx.$$

Once again by Theorem 6.2, there exists a refractor

$$\mathcal{R}_2 = \left\{ \rho_2(x)x : \rho_2(x) = \min_{1 \leq i \leq N_2} \frac{b_i}{1 - \kappa m_i^2 \cdot x} \right\}$$

satisfying

$$\mu_2(\omega) \leq \int_{\mathcal{T}_{\mathcal{R}_2}(\omega)} f(x) t_{\mathcal{R}_2}(x) dx$$

with equality if $m_o \notin \omega$.

By this way for each $\ell = 1, 2, \ldots$, we obtain a finite disjoint sequence of Borel sets ω_j^ℓ, $1 \leq j \leq N_\ell$, with non empty interiors, diameters less that $\delta/2^\ell$ and $\mu(\omega_j^\ell) > 0$ such that $m_o \in (\omega_1^\ell)^\circ$, $\omega_1^{\ell+1} \subset \omega_1^\ell$, and pick $m_j^\ell \in \omega_j^\ell$ with $m_1^\ell = m_o$, for all ℓ and j. The corresponding measures on $\overline{\Omega^*}$ are given by

$$\mu_\ell = \sum_{i=1}^{N_\ell} \mu(\omega_i^\ell) \delta_{m_i^\ell}$$

satisfying

$$\mu_\ell(\overline{\Omega^*}) = \mu(\overline{\Omega^*}) \leq (1 - C_\epsilon) \int_{\overline{\Omega}} f(x)\, dx.$$

We then have a corresponding sequence of refractor solutions given by

$$\mathcal{R}_\ell = \left\{ \rho_\ell(x)x : \rho_\ell(x) = \min_{1 \leq i \leq N_\ell} \frac{b_i}{1 - \kappa m_i^\ell \cdot x} \right\}$$

and satisfying

$$\mu_\ell(\omega) \leq \int_{\mathcal{T}_{\mathcal{R}_\ell}(\omega)} f(x) t_{\mathcal{R}_\ell}(x) dx$$

with equality if $m_o \notin \omega$.

From Remark 6.7 we can normalize $\mathcal{R}_l$ so that $\inf_{\overline{\Omega}} \rho_l(x) = 1$. Then by Lemma 3.8 there exists C such that

$$\sup_{x\in\overline{\Omega}} \rho_\ell(x) \le C$$

for all $\ell \ge 1$.

Also if $x_o, x_1 \in \overline{\Omega}$ and $E(m_o, b_o)$ is a supporting semi ellipsoid to $\mathcal{R}_\ell$ at $\rho_\ell(x_o)x_o$ then for $x_1 \in \overline{\Omega}$ we have

$$\begin{aligned}\rho_\ell(x_1) - \rho_\ell(x_o) &\le \frac{b_o}{1-\kappa m_o\cdot x_1} - \frac{b_o}{1-\kappa m_o\cdot x_o}\\ &= \frac{\kappa m_o}{1-\kappa m_o\cdot x_1}\frac{b_o}{1-\kappa m_o\cdot x_o}\|x_1 - x_o\|\\ &\le \frac{C}{1-\kappa}\|x_1 - x_o\|.\end{aligned}$$

By changing the roles of x_o and x_1 we conclude that

$$|\rho_\ell(x_1) - \rho_\ell(x_o)| \le \frac{C}{1-\kappa}\|x_1 - x_o\| \quad \text{for all } \ell.$$

Thus $\{\rho_\ell\}$ is an equicontinuous family which is bounded uniformly. Then by Arzelá–Ascoli theorem, if need be by taking a subsequence, we have that $\rho_\ell \to \rho$ uniformly on $\overline{\Omega}$. By Lemma 3.6(i), $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ is a refractor.

Set

$$G_{\mathcal{R}_\ell}(\omega) := \int_{\mathcal{T}_{\mathcal{R}_\ell}(\omega)} f(x)t_{\mathcal{R}_\ell}(x)dx, \quad \text{and} \quad G_{\mathcal{R}}(\omega) := \int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x)t_{\mathcal{R}}(x)dx.$$

We shall prove that $G_{\mathcal{R}_\ell}$ converges weakly to $G_{\mathcal{R}}$. To that end we first prove that for any closed subset F of $\overline{\Omega^*}$

$$\limsup G_{\mathcal{R}_\ell}(F) \le G_{\mathcal{R}}(F). \tag{6.16}$$

Indeed, from Theorem 5.5 we get that

$$\begin{aligned}\limsup \tilde{G}_{\mathcal{R}_\ell}(F) &= \limsup \int_{\mathcal{T}_{\mathcal{R}_\ell}(F)} f(x)t_{\mathcal{R}_\ell}(x)dx\\ &\le \int_{\overline{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}_\ell}(F)}(x)f(x)t_{\mathcal{R}_\ell}(x)dx \le \int_{\mathcal{T}_{\mathcal{R}}(F)} f(x)t_{\mathcal{R}}(x)dx = G_{\mathcal{R}}(F).\end{aligned}$$

Moreover for any open set $G \subset \overline{\Omega^*}$ we have

$$G_{\mathcal{R}}(G) = \int_{\mathcal{T}_{\mathcal{R}}(G)} f(x)t_{\mathcal{R}}(x)dx \le \liminf_{\ell\to\infty} G_{\mathcal{R}_\ell}(G). \tag{6.17}$$

Indeed, from Lemma 3.6 we have

$$\begin{aligned}G_{\mathcal{R}}(G) &= \int_{\mathcal{T}_{\mathcal{R}}(G)} f(x)t_{\mathcal{R}}(x)dx \le \int_{\liminf_{\ell\to\infty}\mathcal{T}_{\mathcal{R}_\ell}(G)} f(x)t_{\mathcal{R}}(x)dx\\ &= \int_{\overline{\Omega^*}} \liminf_{\ell\to\infty} \chi_{\mathcal{T}_{\mathcal{R}_\ell}(G)}(x)\, t_{\mathcal{R}}(x) f(x)dx\\ &= \int_{\overline{\Omega^*}} \liminf_{\ell\to\infty} \chi_{\mathcal{T}_{\mathcal{R}_\ell}(G)}(x)t_{\mathcal{R}_\ell}(x) f(x)dx \quad \text{by (5.4)},\\ &\le \liminf_{\ell\to\infty} \int_{\overline{\Omega^*}} \chi_{\mathcal{T}_{\mathcal{R}_\ell}(G)}(x)t_{\mathcal{R}_\ell}(x) f(x)dx = \liminf_{\ell\to\infty} G_{\mathcal{R}_\ell}(G),\end{aligned}$$

by Fatou's lemma. Consequently $G_{\mathcal{R}_\ell} \to G_{\mathcal{R}}$ weakly.

Since $\mu_\ell(\omega) = G_{\mathcal{R}_\ell}(\omega)$ for every Borel set ω with $m_o \notin \omega$, we obtain that μ_ℓ converges weakly to $G_{\mathcal{R}}$ on $\overline{\Omega^*} \setminus \{m_o\}$. On the other hand, since $\mu_\ell \to \mu$ weakly, we conclude that

$$\mu(\omega) = G_{\mathcal{R}}(\omega)$$

if $m_o \notin \omega$. Moreover for any Borel set $\omega \subset \overline{\Omega^*}$, we have $\mu_\ell(\omega) \le G_{\mathcal{R}_\ell}(\omega)$. Since $\mu_\ell \to \mu$ weakly and $G_{\mathcal{R}_\ell} \to G_{\mathcal{R}}$ weakly, we then conclude that $\mu \le G_{\mathcal{R}}$ and the proof of (i) is complete.

To prove (ii) without loss of generality assume that for all refractors $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\} \in \mathcal{C}$, $\min_{x\in\overline{\Omega}} \rho = 1$. Otherwise we consider $\bar{\rho}(x) = \frac{\rho(x)}{m_\rho}$ where $m_\rho = \min_{x\in\overline{\Omega}} \rho$. Then by Lemma 3.8, for all refractors $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, we have $\rho(x) \le C$ with C depending only on κ.

Let

$$I = \inf\left\{\int_{\mathcal{T}_{\mathcal{R}}(m_0)} f(x)t_{\mathcal{R}}(x)\,dx : \mathcal{R} \in \mathcal{C}\right\}.$$

We want to find a refractor $\mathcal{R}_o = \{\rho_o(x)x : x \in \overline{\Omega}\}$ such that I is attained when $\mathcal{R} = \mathcal{R}_o$. We claim this is the refractor transmitting the least amount of energy in the direction m_0.

By definition of infimum, there is a sequence $\mathcal{R}_k = \{\rho_k(x)x : x \in \overline{\Omega}\}$ of refractors in $\mathcal{C}$ such that

$$I = \lim_{k\to\infty} \int_{\mathcal{T}_{\mathcal{R}_k}(m_0)} f(x)t_{\mathcal{R}_k}(x)\,dx.$$

By Lemma 3.8, we have $1 \le \rho_k \le C(\kappa)$. The ρ_k's are uniformly Lipschitz in $\overline{\Omega}$ because if $\rho \in \mathcal{R}(\overline{\Omega}, \overline{\Omega^*})$, we can write

$$\rho(x) - \rho(x_0) \le \frac{b}{1-\kappa\, m\cdot x} - \frac{b}{1-\kappa\, m\cdot x_0} \le b\,\frac{\kappa}{(1-\kappa)^2}|x - x_0|,$$

with $\rho(x_0) = \frac{b}{1-\kappa\, m\cdot x_0}$. Since ρ is uniformly bounded, b is also uniformly bounded and therefore ρ is uniformly Lipschitz.

By Arzelá–Ascoli there is a subsequence ρ_k converging uniformly to some ρ_o. Since

$$G_{\mathcal{R}_k}(\omega) := \int_{\mathcal{T}_{\mathcal{R}_k}(\omega)} f(x)t_{\mathcal{R}_k}(x)dx = \mu(\omega)$$

for each ω such that $m_o \notin \omega$, and $G_{\mathcal{R}_k}$ converges weakly to $G_{\mathcal{R}_o}$ in $\overline{\Omega^*}$, we have that $G_{\mathcal{R}_o}(\omega) = \mu(\omega)$ for all ω with $m_o \notin \omega$. By Theorem 5.5 we have:

$$I = \limsup_{k\to\infty} \int_{\mathcal{T}_{\mathcal{R}_k}(m_0)} f(x)t_{\mathcal{R}_k}(x)\,dx \le \int_{\mathcal{T}_{\mathcal{R}_o}(m_0)} f(x)\,t_{\mathcal{R}_o}(x)\,dx.$$

Therefore, if $m_o \in \omega$, we have

$$\mu(\omega) \le G_{\mathcal{R}_o}(\omega)$$

since for all k we have $\mu(\omega) \le G_{\mathcal{R}_k}(\omega)$. Therefore $\mathcal{R}_o \in \mathcal{C}$.

We claim that

$$I = \int_{\mathcal{T}_{\mathcal{R}_o}(m_0)} f(x)t_{\mathcal{R}_o}(x)\,dx.$$

Note that to prove our claim, it only remains to prove that

$$I \ge \int_{\mathcal{T}_{\mathcal{R}_o}(m_0)} f(x)t_{\mathcal{R}_o}(x)\,dx. \tag{6.18}$$

Consider an open set G containing m_o. From Lemma 3.6(iii), we have

$$\mathcal{T}_{\mathcal{R}_o}(G) \subset \liminf_{k\to\infty} \mathcal{T}_{\mathcal{R}_k}(G) \cup S,$$

where S is the singular set of $\mathcal{R}_o$.

Then

$$\begin{aligned}
\int_{\mathcal{T}_{\mathcal{R}_o}(G)} f(x)\,t_{\mathcal{R}_o}(x)\,dx &\le \int_{\liminf_{k\to\infty}\mathcal{T}_{\mathcal{R}_k}(G)\cup S} f(x)\,t_{\mathcal{R}_o}(x)\,dx \\
&= \int_{\liminf_{k\to\infty}\mathcal{T}_{\mathcal{R}_k}(G)} f(x)\,t_{\mathcal{R}_o}(x)\,dx \\
&= \int_{\overline{\Omega}} \chi_{\liminf_{k\to\infty}\mathcal{T}_{\mathcal{R}_k}(G)}(x)\,f(x)\,t_{\mathcal{R}_o}(x)\,dx \\
&\quad \text{by (5.4)} \\
&= \int_{\overline{\Omega}} \chi_{\liminf_{k\to\infty}\mathcal{T}_{\mathcal{R}_k}(G)}(x)\,t_{\mathcal{R}_k}(x)\,f(x)\,dx
\end{aligned}$$

since

$$\chi_{\liminf_{k\to\infty}\mathcal{T}_{\mathcal{R}_k}(G)}(x) = \liminf_{k\to\infty}\chi_{\mathcal{T}_{\mathcal{R}_k}(G)}(x) \quad \text{we get}$$
$$= \int_{\overline{\Omega}} \liminf_{k\to\infty}\chi_{\mathcal{T}_{\mathcal{R}_k}(G)}(x)\, t_{\mathcal{R}_k}(x) f(x)\, dx$$

which by Fatou's Lemma gives

$$\begin{aligned}
&\le \liminf_{k\to\infty}\int_{\overline{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}_k}(G)}(x)\, t_{\mathcal{R}_k}(x) f(x)\, dx\\
&= \liminf_{k\to\infty}\int_{\mathcal{T}_{\mathcal{R}_k}(G)} t_{\mathcal{R}_k}(x) f(x)\, dx\\
&= \liminf_{k\to\infty}\left(\int_{\mathcal{T}_{\mathcal{R}_k}(G\setminus m_0)} t_{\mathcal{R}_k}(x) f(x)\, dx + \int_{\mathcal{T}_{\mathcal{R}_k}(m_0)} t_{\mathcal{R}_k}(x) f(x)\, dx\right)\\
&= \liminf_{k\to\infty}\left(\mu(G\setminus m_0) + \int_{\mathcal{T}_{\mathcal{R}_k}(m_0)} t_{\mathcal{R}_k}(x) f(x)\, dx\right)\\
&= \mu(G\setminus m_0) + I,
\end{aligned}$$

for all open sets G containing m_0. Since the measure μ is a Radon measure and hence outer regular; i.e. for any measurable set E

$$\mu(E) = \inf\{\mu(G) : E \subset G, G \text{ open}\},$$

we have from $\mu(G\setminus m_o) = \mu(G) - \mu(m_o)$ that

$$\inf\{\mu(G\setminus m_0) : G \text{ open } m_o \in G\} = \inf\{\mu(G) : G \text{ open } m_o \in G\} - \mu(m_o) = 0.$$

Therefore we obtain (6.18).

We can now proceed as in the last part of the proof of Theorem 6.9 to obtain (6.15). □

6.3. Discussion about overshooting

In this subsection, we will discuss the question of overshooting to the direction $m_0 \in \operatorname{supp}\mu$ for refractors in the class $\mathcal{C}$ defined in Theorem 6.11. Let $\mathcal{R}$ be any refractor in $\mathcal{C}$, and recall (6.14).

Case $\mu(m_0) > 0$.

We shall prove in this case that for each open set $G \subset \overline{\Omega^*}$, with $m_0 \in G$, we have

$$\int_{\mathcal{T}_{\mathcal{R}}(G)} f(x)\, t_{\mathcal{R}}(x)\, dx > \mu(G). \tag{6.19}$$

Notice that since $\mathcal{R} \in \mathcal{C}$, we have equality (6.19) for each G with $m_0 \notin G$. Suppose by contradiction there exists an open set G, with $m_0 \in G$, such that

$$\int_{\mathcal{T}_{\mathcal{R}}(G)} f(x)\, t_{\mathcal{R}}(x)\, dx = \mu(G).$$

Under this assumption, we are going to prove that $t_{\mathcal{R}}(x)$ is constant a.e., regardless of $\mu(m_0) > 0$. We have $\mathcal{T}_{\mathcal{R}}(\overline{\Omega^*}) = \overline{\Omega}$, and $\mathcal{T}_{\mathcal{R}}(\overline{\Omega^*}) = \mathcal{T}_{\mathcal{R}}(\overline{\Omega^*}\setminus G) \cup \mathcal{T}_{\mathcal{R}}(G)$ where in the union the sets are disjoint a.e. Then

$$\begin{aligned}
\int_{\overline{\Omega}} f(x)\, t_{\mathcal{R}}(x)\, dx &= \int_{\mathcal{T}_{\mathcal{R}}(\overline{\Omega^*})} f(x)\, t_{\mathcal{R}}(x)\, dx\\
&= \int_{\mathcal{T}_{\mathcal{R}}(\overline{\Omega^*}\setminus G)} f(x)\, t_{\mathcal{R}}(x)\, dx + \int_{\mathcal{T}_{\mathcal{R}}(G)} f(x)\, t_{\mathcal{R}}(x)\, dx\\
&= \mu(\overline{\Omega^*}\setminus G) + \mu(G) = \mu(\overline{\Omega^*})\\
&\le (1 - C_\epsilon)\int_{\overline{\Omega}} f(x)\, dx
\end{aligned}$$

and so we get[4]

$$\int_{\overline{\Omega}} f(x)\ (t_{\mathcal{R}}(x) - (1 - C_\epsilon))\ dx \le 0. \tag{6.21}$$

From (4.4) and the estimates in Section 4.1, we have $t_{\mathcal{R}}(x) - (1 - C_\epsilon) \ge 0$, and since $\inf f > 0$, (6.21) implies that

$$t_{\mathcal{R}}(x) = 1 - C_\epsilon, \quad \text{for a.e. } x \in \Omega.$$

Since $\int_{\mathcal{T}_{\mathcal{R}}(m_0)} f(x)\, t_{\mathcal{R}}(x)\, dx \ge \mu(m_0) > 0$, the set $E := \mathcal{T}_{\mathcal{R}}(m_0) \setminus S$ has positive measure (S being the singular set of $\mathcal{R}$). If $x \in E$, then we have that $t_{\mathcal{R}}(x) = 1 - \phi(x \cdot m_0)$, so $\phi(x \cdot m_0) = C_\epsilon$ for $x \in E$. Now with the argument at the end of the proof of Theorem 6.2 we obtain a contradiction.

Therefore (6.19) is proved.

Case $\mu(m_0) = 0$ and $\int_{\mathcal{T}_{\mathcal{R}}(m_0)} f(x)\, t_{\mathcal{R}}(x)\, dx > \mu(m_0) = 0$.

In this case we also have that (6.19) holds. Suppose by contradiction this is not true. Again from the above argument $t_{\mathcal{R}}(x) = 1 - C_\epsilon$ a.e. Since the set $\mathcal{T}_{\mathcal{R}}(m_0)$ has positive measure, again from the argument mentioned above we obtain a contradiction. So (6.19) follows.

Case $\mu(m_0) = 0$ and $\int_{\mathcal{T}_{\mathcal{R}}(m_0)} f(x)\, t_{\mathcal{R}}(x)\, dx = \mu(m_0) = 0$.

This implies that the set $\mathcal{T}_{\mathcal{R}}(m_0)$ has measure zero. Let G be any open set containing m_0. We have

$$\begin{aligned}\int_{\mathcal{T}_{\mathcal{R}}(G)} f(x)\, t_{\mathcal{R}}(x)\, dx &= \int_{\mathcal{T}_{\mathcal{R}}(G\setminus m_0)} f(x)\, t_{\mathcal{R}}(x)\, dx + \int_{\mathcal{T}_{\mathcal{R}}(m_0)} f(x)\, t_{\mathcal{R}}(x)\, dx \\ &= \mu(G \setminus m_0) + \mu(m_0) = \mu(G).\end{aligned}$$

This identity also holds for any open set not containing m_0, and so for any open set contained in $\overline{\Omega}$. Since both measures $\int_{\mathcal{T}_{\mathcal{R}}(\cdot)} f(x)\, t_{\mathcal{R}}(x)\, dx$ and μ are outer regular, then they are equal, and therefore in this case the refractor does not overshoot.

7. Existence of solutions when $\kappa > 1$

In this section we prove existence of weak solutions for the refractor problem when $n_2/n_1 = \kappa > 1$. That is, if n_1 and n_2 are the refractive indices of two homogeneous and isotropic media I and II, respectively, then medium II is denser than medium I. The proofs are similar to the case $\kappa < 1$ and we just indicate the differences.

Suppose that $\overline{\Omega}$ and $\overline{\Omega^*}$ are two domains of the unit sphere S^{n-1} of $\mathbf{R}^n$ with the physical property that

$$\inf_{m\in\Omega^*, x\in\Omega} m \cdot x \ge 1/\kappa + \epsilon \tag{7.1}$$

for some $\epsilon > 0$.

The refractor problem in the case $\kappa > 1$ can be solved in a similar manner to the case $\kappa < 1$. The main difference is to use the branch $H(m, b)$, in (2.3), of a semi-hyperboloid of two sheets in place of the semi-ellipsoids $E(m, b)$. Refractors in this case are defined in Definition 3.9.

The following proposition regarding the Fresnel coefficient $t_{\mathcal{R}}(x)$ is proved in a similar way as in Proposition 5.3.

Proposition 7.1. *Let $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ for the case $\kappa > 1$. Let E be the singular set of ρ. Then the Fresnel coefficient $t_{\mathcal{R}}(x)$ is continuous relative to the set $\overline{\Omega} \setminus E$.*

We also have the following lemma similar to Lemma 5.2.

Lemma 7.2. *Let $H(m_l, b_l)$ be a sequence of semi-hyperboloids with $m_l \to m$ and $b_l \to b$, as $l \to \infty$. Let $z_l \in H(m_l, b_l)$ with $z_l \to z_0$ as $l \to \infty$. Then $z_0 \in H(m, b)$, and the normal $\nu_l(z_l)$ to the semi hyperboloid $H(m_l, b_l)$ at z_l satisfies $\nu_l(z_l) \to \nu(z_0)$ the normal to the semi hyperboloid $H(m, b)$ at the point z_0.*

Proof. In rectangular coordinates the equation of $H(m_l, b_l)$ is $\kappa m_l \cdot z - |z| = b_l$. Then the normal vector at z is $\nu_l(z) = \kappa m_l - \frac{z}{|z|}$, and so

$$\nu_l(z_l) = \kappa m_l - \frac{z_l}{|z_l|} \to \kappa m - \frac{z_0}{|z_0|}$$

which is the normal to $H(m, b)$ at z_0. The normal at z written in polar coordinates, i.e., $z = \rho_l(x)x$ has the form $\nu_l(x) = \kappa m_l - x$ so $\nu_l(x) \to \kappa m - x = \nu(x)$, the normal to the hyperboloid $H(m, b)$. □

[4] Notice that if instead of (6.14), we assume the stronger condition

$$(1 - C_\epsilon) \int_{\Omega} f(x)\, dx > \mu(\overline{\Omega^*}), \tag{6.20}$$

then in (6.21) we obtain a strict inequality which yields a contradiction. That is, (6.20) implies (6.19) in any case, i.e., when $\mu(m_0) \ge 0$.

The proof of the following lemma goes verbatim with that of Lemma 5.4.

Lemma 7.3. *Suppose $\mathcal{R}_j$ and $\mathcal{R}$ are refractors with defining functions $\rho_j(x)$ and $\rho(x)$ and corresponding transmission coefficients t_j and t, respectively. Suppose $\rho_j \to \rho$ pointwise on $\overline{\Omega}$ with $C_1 \le \rho_j(x) \le C_2$ in $\overline{\Omega}$ for some positive constants C_1 and C_2. Let S be the union of all singular points of all the refractors $\mathcal{R}_j$ and $\mathcal{R}$. Then for each $y \notin S$ there is subsequence $t_{j_l}(y) \to t(y)$ as $l \to \infty$.*

A statement like in Theorem 5.5 follows from Lemmas 7.2 and 7.3. We can also prove the following lemma about the refractor measure.

Lemma 7.4. *Let $\mathcal{R}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$. Let $f \in L^1(\overline{\Omega})$ with $\inf_{\overline{\Omega}} f > 0$. Define a set function on Borel subsets of $\overline{\Omega^*}$, by*

$$G_{\mathcal{R}}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f(x) t_{\mathcal{R}}(x) dx$$

where dx is the surface measure on S^{n-1} and $t_{\mathcal{R}}(x)$ is as given by (4.7). Then $G_{\mathcal{R}}$ is a finite Borel measure defined on $\mathcal{C}$. $G_{\mathcal{R}}$ is called the refractor measure associated with $\mathcal{R}$ and f.

We now discuss the existence of solution for the case $\kappa > 1$.

7.1. Existence of a weak solution when μ equals sum of delta measures

Let $m_1, m_2, \ldots, m_N, N \ge 2$ be distinct points in $\overline{\Omega^*}$. For $\mathbf{b} = (b_1, \ldots, b_N) \in \mathbf{R}^N$ with each $b_i > 0$, we denote by $\mathcal{R}(\mathbf{b})$ the refractor defined by

$$\mathcal{R}(\mathbf{b}) = \left\{ \rho(x)x : x \in \overline{\Omega}, \rho(x) = \max_{1 \le i \le N} \frac{b_i}{\kappa m_i \cdot x - 1} \right\}. \tag{7.2}$$

We remark that the statements in Lemma 5.1 and Proposition 6.1 also hold when $\kappa > 1$ with $\mathcal{R}(\mathbf{b})$ defined by (7.2).

We now state the existence of weak solution for the case $\kappa > 1$ when refraction happens only in finitely many directions.

Theorem 7.5. *Let $f \in L^1(\overline{\Omega})$ with $\inf_{\overline{\Omega}} f > 0$, and let $m_1, \ldots, m_N, N \ge 2$ be distinct points in $\overline{\Omega^*}$. Let $g_1, \ldots, g_N > 0$ and $\mu = \sum_{i=1}^N g_i \delta_{m_i}$. Suppose that $\inf_{x \in \Omega, 1 \le i \le N} x \cdot m_i \ge \frac{1}{\kappa} + \epsilon$, and*

$$\int_{\overline{\Omega}} f(x) dx \ge \frac{1}{1 - C_\epsilon} \mu(\overline{\Omega^*}), \tag{7.3}$$

where C_ϵ is as in (4.6). Then there exists $\mathbf{b}_o \in \mathbf{R}^N$ and a refractor $\mathcal{R}(\mathbf{b}_o)$ such that

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(m_i)} f(x) t_{\mathcal{R}}(x) dx = g_i$$

for $i = 2, \ldots, N$, and

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(m_1)} f(x) t_{\mathcal{R}}(x) dx > g_1.$$

Therefore, $\mathcal{R}$ is a weak solution to the refractor problem with intensities f and μ, that is, $\int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x) t_{\mathcal{R}}(x) dx \ge \mu(\omega)$ for each Borel set $\omega \subset \Omega^*$, and satisfies

$$\int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x) t_{\mathcal{R}}(x) dx = \mu(\omega),$$

for each Borel set $\omega \subset \Omega^*$ with $m_1 \notin \omega$.

The proof is analogous to the proof of Theorem 6.2 and it follows as a consequence of the following lemmas.

The proof of the following lemma is similar to that of Lemma 6.4.

Lemma 7.6. *Let $W \subset \mathbf{R}^N$ be the set given as $W = \{\mathbf{b} = (1, b_2, \ldots, b_N) : b_i > 0\}$ with the property that for all $\mathbf{b} \in W$, $\mathcal{R}(\mathbf{b})$ satisfies*

$$G_{\mathcal{R}(\mathbf{b})}(m_i) = \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} f(x) t_{\mathcal{R}(\mathbf{b})}(x) dx \le g_i \quad \text{for all } i = 2, \ldots, N,$$

where g_is and f are as in Theorem 7.5, then

i. $W \neq \emptyset$.
ii. *If* $\mathbf{b} = (1, b_2, \ldots, \mathbf{b}_N) \in W$ *then*

$$b_i \leq \frac{\kappa - 1}{\epsilon\kappa} \tag{7.4}$$

for all $i = 2, \ldots, N$.

Proof. (i). If for some $i \neq 1$, $H(m_i, b)$, is a supporting semi-hyperboloid to $\mathcal{R}(\mathbf{b})$ at $\rho(x)x$ then

$$\frac{b}{\kappa\,\epsilon} \geq \frac{b}{\kappa m_i \cdot x - 1} = \rho(x) \geq \frac{1}{\kappa m_1 \cdot x - 1} \geq \frac{1}{\kappa - 1}$$

and so $b \geq \kappa\epsilon/(\kappa - 1)$. Therefore, if $0 < b_i < \kappa\epsilon/(\kappa - 1)$ for all $2 \leq i \leq N$, then $H(m_i, b_i)$ cannot be a supporting semi-hyperboloid to $\mathcal{R}(\mathbf{b})$ at any $x \in \overline{\Omega}$, and therefore, from the analogue of Lemma 5.1 when $\kappa > 1$, the set $\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)$ is contained in the set of singular points of $\mathcal{R}(\mathbf{b})$ for $i = 2, \ldots, N$, and consequently $G_{\mathcal{R}(\mathbf{b})}(m_i) = 0 < g_i$. Hence taking $\mathbf{b} = (1, b_2, \ldots, b_N)$ with $0 < b_i < \kappa\epsilon/(\kappa - 1)$ for $2 \leq i \leq N$, yields $\mathbf{b} \in W$.

(ii). First notice that if at a point x_0, the semi hyperboloids $H(m_j, b_j)$ and $H(m_k, b_k)$ support $\mathcal{R}(\mathbf{b})$ with $m_k \neq m_j$, then x_0 is a singular point, and therefore the points supported by two semi hyperboloids with different axes form a set of measure zero. From this, we have that if $\mathbf{b} \in W$, and $g_1 \leq G_{\mathcal{R}(\mathbf{b})}(m_1)$. Indeed,

$$\begin{aligned}
\sum_{i=1}^{N} G_{\mathcal{R}(\mathbf{b})}(m_i) &= \sum_{i=1}^{N} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} f(x) t_{\mathcal{R}(\mathbf{b})}(x)dx \\
&= \int_{\cup_{i=1}^{N} \mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} f(x) t_{\mathcal{R}(\mathbf{b})}(x)dx \\
&= \int_{\overline{\Omega}} f(x) t_{\mathcal{R}(\mathbf{b})}(x)dx \\
&\geq (1 - C_\epsilon) \int_{\bar{\Omega}} f(x)dx \\
&\geq \mu(\bar{\Omega}^*) = \sum_{i=1}^{N} g_i,
\end{aligned}$$

and so we have that

$$[g_1 - G_{\mathcal{R}(\mathbf{b})}(m_1)] + \sum_{i=2}^{N} \left[g_i - G_{\mathcal{R}(\mathbf{b})}(m_i)\right] \leq 0.$$

If $\mathbf{b} \in W$,

$$\sum_{i=2}^{N} g_i - G_{\mathcal{R}(\mathbf{b})}(m_i) \geq 0.$$

Thus $g_1 \leq G_{\mathcal{R}(\mathbf{b})}(m_1)$.

Suppose that $\mathcal{R}(\mathbf{b}) = \{\rho(x)x : x \in \overline{\Omega}\}$. We shall prove that there exists a point $\rho(x_o)x_o$ such that $\rho(x_o)x_o \in \mathcal{R}(\mathbf{b}) \cap H(m_1, 1)$ and $\rho(x_o)x_o \notin H(m_i, b_i)$ for all $i \geq 2$. Otherwise, $\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_1) \subset S$ where S is the singular set of ρ. But then,

$$G_{\mathcal{R}(\mathbf{b})}(m_1) = \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_1)} f(x) t_{\mathcal{R}(\mathbf{b})}(x)dx \leq \int_S f(x) t_{\mathcal{R}(\mathbf{b})}(x)dx = 0$$

contradicting the fact that $g_1 > 0$. Then

$$\rho(x_o) = \frac{1}{\kappa m_1 \cdot x_o - 1} > \frac{b_i}{\kappa m_i \cdot x_o - 1}$$

for $i = 2, \ldots, N$, from which we conclude that

$$b_i < \frac{\kappa - 1}{\epsilon\kappa}$$

for all $i = 2, \ldots, N$. □

We also have the following property similar to Theorem 6.6.

Theorem 7.7. *Let $\delta > 0$. Then $G_{\mathcal{R}(\mathbf{b})}(m_i)$ are continuous in the region $R_\delta = \{\mathbf{b} = (1, b_2, \dots, b_N) : 0 < b_i \le \delta,\ i = 2, \dots, N\}$, for all $1 \le i \le N$.*

Proof. Let $\mathbf{b}_j,\ j \ge 1$ be a sequence in R_δ converging to $\mathbf{b}_o \in R_\delta$. Let $\mathcal{R}(\mathbf{b}_j) = \{\rho_j(x)x : x \in \overline{\Omega}\}$, and $\mathcal{R}(\mathbf{b}_o) = \{\rho(x)x : x \in \overline{\Omega}\}$. From the analogue of Lemma 6.5 for $\kappa > 1$, we have $\rho_j \to \rho$ uniformly on $\overline{\Omega}$. Moreover, for any $x \in \overline{\Omega}$ and $j \ge 1$,

$$\rho_j(x) = \frac{b_l^j}{\kappa m_l \cdot x - 1} \le \max\left\{\frac{\delta}{\epsilon\kappa}, \frac{1}{\kappa\epsilon}\right\}$$

for some $l \in \{1, 2, \dots, N\}$ and also

$$\frac{1}{\kappa - 1} \le \frac{1}{\kappa m_1 \cdot x - 1} \le \rho_j(x) = \max_{1 \le i \le N} \frac{b_i^j}{\kappa m_i \cdot x - 1}.$$

We thus obtain $a_1, a_2 > 0$ such that $0 < a_1 \le \rho_j(x) \le a_2$. Let $G \subset \overline{\Omega^*}$ be a neighborhood of m_i such that $m_l \notin G$ for $l \ne i$. If $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(G)$ and $x_o \notin S$, there exists a unique $m \in G$ and $b > 0$ such that

$$\rho_j(x_o) = \frac{b}{\kappa m \cdot x_o - 1} \quad \text{and} \quad \rho_j(x) \ge \frac{b}{\kappa m \cdot x - 1} \quad \text{for all } x \in \overline{\Omega}.$$

But by definition of $\mathcal{R}(\mathbf{b}_j)$, $m = m_l$ for some $l = 1, \dots, N$. Thus $m = m_i$. From this we conclude that $\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(G) \subset \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i) \cup S$. Combining this with Lemma 3.6(iii) and the fact that S has measure zero, we obtain

$$\begin{aligned}
&\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} f(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \\
&\le \int_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i) \cup S} f(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \\
&\le \int_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} f(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx + \int_S f(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \\
&= \int_{\overline{\Omega}} \chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) f(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx.
\end{aligned} \tag{7.5}$$

By applying (5.4), to (7.5) we get,

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \le \int_{\overline{\Omega}} \chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)dx. \tag{7.6}$$

It is also true that

$$\chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) = \liminf \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x).$$

Using this in (7.6), we obtain

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \le \int_{\overline{\Omega}} \liminf \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)dx$$

from which we deduce by Fatou's lemma that

$$\begin{aligned}
\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx &\le \liminf \int_{\overline{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)dx \\
&= \liminf \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)dx.
\end{aligned}$$

To complete the proof we shall prove that

$$\limsup \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)dx \le \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)dx. \tag{7.7}$$

First notice that

$$\begin{aligned}
\limsup \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)dx &= \limsup \int_{\overline{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)dx \\
&\le \int_{\overline{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)dx
\end{aligned}$$

where the last inequality is due to reverse Fatou's Lemma. By (5.1) and the fact that $\limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) = \chi_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)$, we have

$$\begin{aligned}\int_{\overline{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x)\, dx &= \int_{\overline{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) g(x)\, dx \\ &= \int_{\overline{\Omega}} \chi_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) g(x)\, dx \\ &= \int_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)\, dx.\end{aligned}$$

But then by Lemma 3.6(ii),

$$\int_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)\, dx \le \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x)\, dx$$

from which we conclude Eq. (7.7) and therefore concluding the proof of the theorem. □

We shall now prove Theorem 7.5.

Proof of Theorem 7.5. Let $\tau > 0$ be small and

$$\widetilde{W} = \{\mathbf{b} = (1, b_2, \ldots, b_N) \in W : b_i \ge \tau \text{ for } i = 2, \ldots, N\}.$$

Then from Lemma 7.6(ii) and Theorem 7.7, $\widetilde{W}$ is compact. Consider the map

$$d : \widetilde{W} \to \mathbf{R}$$

given by

$$d(\mathbf{b}) = \sum_{i=1}^{N} b_i$$

where $\mathbf{b} = (1, b_2, \ldots, b_N)$. Let $\mathbf{b}^* = (1, b_2^*, \ldots, b_N^*)$ be such that

$$\mathbf{b}^* = arg \max_{\mathbf{b} \in \widetilde{W}} d(\mathbf{b}).$$

$\mathcal{R}(b^*)$ is the refractor we are looking for. We first show that $\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_j)} g(x) t_{\mathcal{R}(b^*)}(x) = f_j$, for $j = 2, \ldots, N$. Suppose the contrary and without loss of generality that

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_2)} g(x) t_{\mathcal{R}(b^*)}(x) < f_2.$$

Take $\lambda > 1$ and consider $\mathbf{b}_\lambda^* = (1, \lambda b_2^*, \ldots, b_N^*)$. If $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}_\lambda^*)}(m_i)$, x_o is not a singular point of $\mathcal{R}(\mathbf{b}_\lambda^*)$, then, from the analogue of Lemma 5.1 for $\kappa > 1$, the semi hyperboloid $H(m_i, b_i^*)$ supports both $\mathcal{R}(\mathbf{b}_\lambda^*)$ and $\mathcal{R}(\mathbf{b}^*)$ at x_o, when $i \ne 2$, and in particular, from the Snell law, the normals at x_0 to both $\mathcal{R}(\mathbf{b}_\lambda^*)$ and $\mathcal{R}(\mathbf{b}^*)$ are the same. Hence $\mathcal{T}_{\mathcal{R}(\mathbf{b}_\lambda^*)}(m_i) \setminus (\text{singular set of } \mathcal{R}(\mathbf{b}_\lambda^*)) \subset \mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)$, and $t_{\mathcal{R}(\mathbf{b}_\lambda^*)}(x) = t_{\mathcal{R}(\mathbf{b}^*)}(x)$ for all $x \in \mathcal{T}_{\mathcal{R}(\mathbf{b}_\lambda^*)}(m_i)$ outside the singular set of $\mathcal{R}(\mathbf{b}_\lambda^*)$, $i \ne 2$. Therefore

$$\begin{aligned}\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_\lambda^*)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}_\lambda^*)}(x) dx &= \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_\lambda^*)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}^*)}(x) dx \\ &\le \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}^*)}(x) dx \le f_i, \quad i \ne 2.\end{aligned}$$

Then by using Theorem 7.7, we choose λ sufficiently close 1 so that $G_{\mathbf{b}_\lambda^*}(m_2) < f_2$, concluding that $\mathbf{b}_\lambda^* \in \widetilde{W}$. But this is a contradiction as it implies $d(\mathbf{b}_\lambda^*) > d(\mathbf{b}^*)$.

To prove that $\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_1)} g(x) t_{\mathcal{R}(b^*)}(x) > f_1$, we proceed exactly as in the proof of Theorem 6.2. □

Remark 7.8. Similarly to Remarks 6.8 and 6.10, we have the same observations when $\kappa > 1$.

Remark 7.9. For $\kappa > 1$ and as in Theorem 6.9, there is a refractor overshooting a minimal amount of energy.

7.2. Existence of weak solution when μ is a finite Radon measure

To prove the existence theorem for the case when μ is any Radon measure we first state the following lemma which is the counterpart of Lemma 3.8.

Lemma 7.10. *Let $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ be any refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ such that $\inf_{x\in\overline{\Omega}} \rho(x) = 1$. Then there is a constant C such that*

$$\sup_{x\in\overline{\Omega}} \rho(x) \le C.$$

Proof. Let $H(m_o, b_o)$ be the supporting hyperboloid of $\mathcal{R}$ at $\rho(x_o)x_o$ where x_o is given by

$$\rho(x_o) = \sup_{x\in\overline{\Omega}} \rho(x).$$

Then

$$\rho(x_o) = \frac{b_o}{\kappa m_o \cdot x_o - 1} \quad \text{and} \quad \rho(x) \ge \frac{b_o}{\kappa m_o \cdot x - 1} \quad \forall x \in \bar{\Omega}.$$

Since

$$\frac{b_o}{\kappa - 1} \le \frac{b_o}{\kappa m_o \cdot x - 1}$$

for all $x \in \overline{\Omega}$,

$$\frac{b_o}{\kappa - 1} \le \inf_{x\in\bar{\Omega}} \frac{b_o}{\kappa m_o \cdot x - 1} \le \inf_{x\in\overline{\Omega}} \rho(x) = 1.$$

Thus $b_o < \kappa - 1$ and

$$\rho(x_o) < \frac{\kappa - 1}{\epsilon\kappa}$$

as required. □

We now state and prove the main existence theorem.

Theorem 7.11. *Let $f \in L^1(\overline{\Omega})$ with $\inf_{\overline{\Omega}} f > 0$, and let μ be a Radon measure in Ω^*. Suppose that $\inf_{x\in\Omega; m\in\Omega^*} x \cdot m \ge \frac{1}{\kappa} + \epsilon$ and*

$$\int_{\overline{\Omega}} f(x)\, dx \ge \frac{1}{1 - C_\epsilon}\, \mu(\overline{\Omega^*}), \tag{7.8}$$

where C_ϵ is given in (4.6). Let $m_o \in supp\ (\mu)$, the support of μ, and

$$\mathcal{C} = \left\{ \mathcal{R} \in \mathcal{R}(\overline{\Omega}, \overline{\Omega}^*) : \mu(\omega) \le \int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x) t_{\mathcal{R}}(x) dx \forall \omega \text{ with equality when } m_o \notin \omega \right\}$$

for all Borel sets $\omega \subset \overline{\Omega^*}$. Then

(i) *$\mathcal{C} \ne \emptyset$, i.e., there exists a refractor solution to the refractor problem, in the sense of Definition 4.1, with intensities f and μ;*
(ii) *There exists $\mathcal{R}_o \in \mathcal{C}$ such that*

$$\int_{\mathcal{T}_{\mathcal{R}_o}(m_o)} f(x) t_{\mathcal{R}_o}(x) dx = \inf_{\mathcal{R}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}}(m_o)} f(x) t_{\mathcal{R}}(x) dx$$

and consequently,

$$\int_{\mathcal{T}_{\mathcal{R}_o}(\omega)} f(x) t_{\mathcal{R}_o}(x) dx = \inf_{\mathcal{R}\in\mathcal{C}} \int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x) t_{\mathcal{R}}(x) dx$$

for every Borel subset $\omega \subseteq \overline{\Omega^*}$.

Proof. The proof proceeds in the same way as the proof of Theorem 6.11 using now Lemma 7.10 instead of Lemma 3.8. The limiting process follows from the analogue of Theorem 5.5 which now follows from Lemmas 7.2 and 7.3. □

Remark 7.12. A discussion similar to the one in Section 6.3 applies for $\kappa > 1$ to the refractors in Theorem 7.11.

8. The differential equation for the problem

Let X denote a point in the sphere S^{n-1}, and set $X = (x, x_n)$ with $x = (x_1, \dots, x_{n-1})$. Let $\mathcal{R} = \{\rho(X)X : X \in \bar{\Omega}\}$ be a solution of the refractor problem from $\bar{\Omega}$ to $\bar{\Omega}^*$ with emitting illumination intensity g and prescribed refracted illumination intensity f. Let $\mathcal{U} = \{x = (x_1, \dots, x_{n-1}) : (x, \sqrt{1-|x|^2}) \in \overline{\Omega}\}$ be the orthogonal projection of $\overline{\Omega}$. If we assume $\bar{\Omega}$ is a subset of upper unit sphere $S^{n-1}_+ = S^{n-1} \cap \{x_n > 0\}$ then we can identify $\overline{\Omega}$ with $\mathcal{U}$. Moreover we can consider ρ as a function of x, with $x \in \mathcal{U}$. For the purpose of deriving the partial differential equation, we assume throughout this chapter that ρ is C^2 smooth.

Let $Y \in S^{n-1}$ be the refracted direction of the ray X by the surface $\rho(X)X$. Recall, from Snell's law (2.1) that

$$Y = \frac{1}{\kappa}\left(X - \Phi(X \cdot \nu)\nu\right), \tag{8.1}$$

where

$$\Phi(t) = t - \kappa\sqrt{1 - \kappa^{-2}(1-t^2)}, \tag{8.2}$$

and ν is the outward unit normal to the refractor at the point $\rho(X)X$. We denote by T the map $X \mapsto Y$ and we considered it defined in $\mathcal{U}$; that is, $T : U \to \overline{\Omega}^*$.

Since $Y = (y_1, \dots, y_n) \in S^{n-1}$, we have $Y \cdot \partial_k Y = 0$ for $1 \le k \le n-1$. Therefore the vectors $\partial_k Y$, $1 \le k \le n-1$ are in the tangent plane to the sphere at the point Y. Let $u_0 \in \mathcal{U}$. The tangent plane to the sphere at $Y(u_0)$ is the collection of points

$$P(u) = Y(u_0) + (\partial_1 Y(u_0), \dots, \partial_{n-1} Y(u_0))(u - u_0)$$

$u \in \mathbf{R}^{n-1}$. Let R be the $(n-1)$-dimensional box given by $R = [s_1, t_1] \times \cdots \times [s_{n-1}, t_{n-1}]$. Let $P_0 = P(s_1, \dots, s_{n-1})$, $P_1 = P(t_1, s_2, \dots, s_{n-1})$, $P_2 = P(s_1, t_2, s_3, \dots, s_{n-1})$, $P_j = P(s_1, \dots, s_{j-1}, t_j, s_{j+1}, \dots, s_{n-1})$, and $P_{n-1} = P(s_1, \dots, s_{n-2}, t_{n-1})$. We have that the vectors $\overrightarrow{P_0P_j}$ satisfy $\overrightarrow{P_0P_j} = (t_j - s_j)\,\partial_j Y(u_0)$ for $1 \le j \le n-1$. Notice that the $(n-1)$-dimensional volume of $T(R)$ is approximately the $(n-1)$-dimensional volume of the box B on the tangent plane. Recall that the volume of the box generated by n vectors in $\mathbf{R}^n$ is given by the determinant of the matrix whose columns are the given vectors. Since $Y(u_0)$ is perpendicular to the tangent plane at this point and $|Y| = 1$, the $(n-1)$-dimensional volume of the box B is equal to the n-dimensional volume of the box B' generated by B and Y. We then obtain that the $(n-1)$ dimensional volume of B is given by

$$|B| = (t_1 - s_1)\cdots(t_{n-1} - s_{n-1})\,|\det J| \tag{8.3}$$

where J is the matrix (which we call Jacobian matrix of T) given by

$$J = \begin{bmatrix} \partial_1 y_1 & \cdots & \partial_{n-1} y_1 & y_1 \\ \partial_1 y_2 & \cdots & \partial_{n-1} y_2 & y_2 \\ & \cdots & & \\ \partial_1 y_{n-1} & \cdots & \partial_{n-1} y_{n-1} & y_{n-1} \\ \partial_1 y_n & \cdots & \partial_{n-1} y_n & y_n \end{bmatrix}.$$

Since $\partial_k y_n = -\frac{1}{y_n}(y_1\partial_k y_1 + \cdots + y_{n-1}\partial_k y_{n-1})$ for $1 \le k \le n-1$, replacing these values in the last row of J and using the fact that the determinant is multilinear in the rows yields

$$\det J = \frac{1}{y_n}\det\begin{bmatrix} \partial_1 y_1 & \cdots & \partial_{n-1} y_1 \\ \partial_1 y_2 & \cdots & \partial_{n-1} y_2 \\ & \cdots & \\ \partial_1 y_{n-1} & \cdots & \partial_{n-1} y_{n-1} \end{bmatrix}. \tag{8.4}$$

If we denote the matrix in (8.4) by Dy then $Dy = (\partial_j y_i)$, $1 \le i, j \le n-1$ and

$$\det J = \frac{1}{y_n}\det Dy.$$

If $dS_{\overline{\Omega^*}}$ and $dS_{\mathcal{U}}$ denote the area elements corresponding to $\overline{\Omega^*}$ and the volume element corresponding to $\mathcal{U}$ respectively then

$$|\det J| = \frac{dS_{\overline{\Omega^*}}}{dS_{\mathcal{U}}}. \tag{8.5}$$

Also if $dS_{\overline{\Omega}}$ is the area element corresponding to $\overline{\Omega}$ then $\frac{dS_{\overline{\Omega}}}{dS_{\mathcal{U}}} = \frac{1}{\sqrt{1-|x|^2}}$.

Let $x_o \notin S$ where S is the singular set of $\mathcal{R}$ and $m_o = \mathcal{T}_{\mathcal{R}}(x_o) = T(x_o)$ where T is viewed as a map on $\bar{\Omega}$. Let $r > 0$ and $B_r(m_o)$ be the ball centered at m_o with radius r and contained in Ω^*. Then from the energy condition

$$\int_{\mathcal{T}_{\mathcal{R}}(\omega)} f(x)t_{\mathcal{R}}(x)dx \geq \int_{\omega} g(m)dm$$

we have

$$\frac{|\mathcal{T}_{\mathcal{R}}(B_r(m_o))|}{|B_r(m_o)|}\frac{1}{|\mathcal{T}_{\mathcal{R}}(B_r(m_o))|}\int_{\mathcal{T}_{\mathcal{R}}(B_r(m_o))} f(x)t_{\mathcal{R}}(x)dx \geq \frac{1}{|B_r(m_o)|}\int_{B_r(m_o)} g(m)dm.$$

If as $r \to 0$, both $|B_r(m_o)|$ and $|\mathcal{T}_{\mathcal{R}}(B_r(m_o))|$ tend to 0, then by Lebesgue Differentiation Theorem, we deduce that $f(x)t_{\mathcal{R}}(x)dS_{\overline{\Omega}} \geq g(T(x))dS_{\overline{\Omega^*}}$.

Combining this result with (8.5), we obtain

$$|\det J| = \frac{dS_{\overline{\Omega^*}}}{dS_{\mathcal{U}}} \leq \frac{f(x)t_{\mathcal{R}}(x)}{\sqrt{1-|x|^2}g(T(x))}. \tag{8.6}$$

We now find the $|\det Dy|$ explicitly to show this Jacobian is an operator of Monge–Ampère type.

8.1. A Monge–Ampère type operator for ρ

In this section we shall derive the Monge–Ampère equation satisfied by ρ. First, we prove that the normal ν to $\mathcal{R}$ has the following expression obtained in [9] for reflectors, and used there to derive the pde satisfied by $1/\rho$, where ρ is the defining function of the reflector. For convenience of the reader, with give the full proof of the formula for the normal noticing the difference in the sign because we consider the outer normal.

Lemma 8.1. *The unit outer normal ν to the surface $\mathcal{R}$ at the point $\rho(x)X,\ X \in \overline{\Omega}$, is given by*

$$\nu = \frac{-\hat{D}\rho(x) + X\left(\rho(x) + D\rho(x)\cdot x\right)}{\sqrt{\rho^2 + |D\rho|^2 - (D\rho\cdot x)^2}}, \tag{8.7}$$

where $X = (x, \sqrt{1-|x|^2})$ and $\hat{D}\rho(x) = (\partial_1\rho(x), \ldots, \partial_{n-1}\rho(x), 0) = (D\rho(x), 0)$.

In addition, we have

$$X\cdot\nu = \frac{\rho}{\sqrt{\rho^2 - (x\cdot D\rho)^2 + |D\rho|^2}}. \tag{8.8}$$

Proof. First notice that the vectors $\partial_{x_k}((x, x_n)\rho(x))$ are tangential to the graph of the refractor for $k = 1, \ldots, n-1$. Therefore we have

$$\partial_{x_k}((x, x_n)\rho(x))\cdot\nu = 0$$

for $k = 1, \ldots, n-1$. Write $\nu = (\nu', \nu_n)$. Calculating explicitly the derivatives we get,

$$\rho\sum_{i=1}^{n-1}\delta_{ik}\nu_i + \partial_{x_k}\rho\sum_{i=1}^{n-1}x_i\nu_i = \left(\rho\frac{x_k}{\sqrt{1-|x|^2}} - \sqrt{1-|x|^2}\partial_{x_k}\rho\right)\nu_n, \tag{8.9}$$

for $k = 1, \ldots, n-1$.

If η, ξ are row vectors in $\mathbf{R}^n$, the tensor product is the $n\times n$ matrix defined by

$$\xi\otimes\eta = \xi^t\eta,$$

with the multiplication of matrices. Moreover if I is the $n\times n$ identity matrix and C is any constant, then the Sherman–Morrison formula says

$$(I + C\xi\otimes\eta)^{-1} = I - \frac{C\xi\otimes\eta}{1 + C(\xi\cdot\eta)}. \tag{8.10}$$

In matrix form, (8.9) will then become

$$(\rho I + D\rho\otimes x)(\nu')^t = \left(\rho\frac{x^t}{\sqrt{1-|x|^2}} - \sqrt{1-|x|^2}(D\rho)^t\right)\nu_n.$$

From (8.10) we have

$$\begin{aligned}(\rho I + D\rho \otimes x)^{-1} &= \rho^{-1}\left(I + \frac{D\rho}{\rho} \otimes x\right)^{-1}\\ &= \rho^{-1}\left(I - \frac{\frac{D\rho}{\rho} \otimes x}{1 + \frac{D\rho}{\rho} \cdot x}\right)\\ &= \rho^{-1}\left(I - \frac{D\rho \otimes x}{\rho + D\rho \cdot x}\right).\end{aligned}$$

Therefore

$$\begin{aligned}(\nu')^t &= \rho^{-1}\left(I - \frac{D\rho \otimes x}{\rho + D\rho \cdot x}\right)\left(\rho \frac{x^t}{\sqrt{1-|x|^2}} - \sqrt{1-|x|^2}(D\rho)^t\right)\nu_n\\ &= \rho^{-1}\left(\frac{\rho}{\sqrt{1-|x|^2}}x^t - \sqrt{1-|x|^2}(D\rho)^t - \frac{\rho}{\sqrt{1-|x|^2}(\rho + D\rho \cdot x)}(D\rho \otimes x)x^t\right.\\ &\quad \left. + \frac{\sqrt{1-|x|^2}}{\rho + D\rho \cdot x}(D\rho \otimes x)(D\rho)^t\right)\nu_n.\end{aligned}$$

Now observe that for any row vectors ξ, η, γ we have $(\xi \otimes \eta)\gamma^t = (\eta \cdot \gamma)\xi^t$. So $(D\rho \otimes x)x^t = |x|^2(D\rho)^t$ and $(D\rho \otimes x)(D\rho)^t = (x \cdot D\rho)(D\rho)^t$. Therefore

$$\begin{aligned}(\nu')^t &= \rho^{-1}\left(\frac{\rho}{\sqrt{1-|x|^2}}x^t - \left(\sqrt{1-|x|^2} + \frac{|x|^2\rho}{\sqrt{1-|x|^2}(\rho + D\rho \cdot x)} - \frac{\sqrt{1-|x|^2}}{\rho + D\rho \cdot x}(x \cdot D\rho)\right)(D\rho)^t\right)\nu_n\\ &= \rho^{-1}\left(\frac{\rho}{\sqrt{1-|x|^2}}x^t - \frac{\rho}{\sqrt{1-|x|^2}(\rho + D\rho \cdot x)}(D\rho)^t\right)\nu_n\\ &= \frac{1}{\sqrt{1-|x|^2}}\left(x^t - \frac{1}{\rho + D\rho \cdot x}(D\rho)^t\right)\nu_n.\end{aligned}$$

So writing the normal as a row vector we get

$$\nu = (\nu', \nu_n) = \left(\frac{1}{\sqrt{1-|x|^2}}\left(x - \frac{1}{\rho + D\rho \cdot x}D\rho\right), 1\right)\nu_n. \tag{8.11}$$

Using this formula we get that

$$X \cdot \nu = \frac{1}{\sqrt{1-|x|^2}}\left(\frac{\rho}{\rho + D\rho \cdot x}\right)\nu_n.$$

Since ν is the outer normal to the refractor at X, we must have $X \cdot \nu \geq 0$. Therefore ν_n and $\rho + D\rho \cdot x$ must have the same sign. Also since $|\nu'|^2 + \nu_n^2 = 1$, we obtain from (8.11) that

$$\left(\frac{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}{(1-|x|^2)(\rho + D\rho \cdot x)^2}\right)\nu_n^2 = 1.$$

Notice here that $\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2 > 0$ since $|X| = 1$. So

$$\nu_n = \pm\sqrt{\frac{(1-|x|^2)(\rho + D\rho \cdot x)^2}{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}} = \pm|\rho + D\rho \cdot x|\sqrt{\frac{1-|x|^2}{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}}. \tag{8.12}$$

Hence from (8.11)

$$\begin{aligned}\nu &= (\nu', \nu_n)\\ &= \pm|\rho + D\rho \cdot x|\sqrt{\frac{(1-|x|^2)}{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}}\left(\frac{1}{\sqrt{1-|x|^2}}\left(x - \frac{1}{\rho + D\rho \cdot x}D\rho\right), 1\right)\\ &= \pm\frac{|\rho + D\rho \cdot x|}{\rho + D\rho \cdot x}\left(\frac{-D\rho + (\rho + D\rho \cdot x)}{\sqrt{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}}x, \frac{\sqrt{1-|x|^2}(\rho + D\rho \cdot x)}{\sqrt{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}}\right)\end{aligned}$$

and (8.7) follows. Moreover,

$$X \cdot \nu = \pm \frac{|\rho + D\rho \cdot x|}{\rho + D\rho \cdot x} \frac{\rho}{\sqrt{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}}.$$

If $\rho + D\rho \cdot x > 0$, then ν_n is positive and so in (8.12) we need to choose the plus sign. If $\rho + D\rho \cdot x < 0$, then $\nu_n < 0$ and so in (8.12) we need to choose the minus sign. Therefore in any case we obtain (8.8). □

For brevity let us first introduce the functions;

$$h(x, z, p) = \frac{\Phi\left(\frac{z}{\sqrt{z^2+|p|^2-(p\cdot x)^2}}\right)}{\sqrt{z^2 + |p|^2 - (p \cdot x)^2}}, \tag{8.13}$$

where Φ is defined in (8.2), and

$$\begin{aligned} w(x, z, p) &= 1 - \Phi\left(\frac{z}{\sqrt{z^2 + |p|^2 - (p \cdot x)^2}}\right) \frac{z + p \cdot x}{\sqrt{z^2 + |p|^2 - (p \cdot x)^2}} \\ &= 1 - h(x, z, p)(z + p \cdot x). \end{aligned} \tag{8.14}$$

We now prove the main result of this section.

Theorem 8.2. *If ρ is the function defining a refractor $\mathcal{R}$, solution to the refractor problem with intensity $f \in L^1(\bar{\Omega})$ on $\bar{\Omega}$ and $g \in L^1(\bar{\Omega}^*)$ on $\bar{\Omega}^*$, then*

$$|\det(D^2\rho + C^{-1}B)| \le \frac{f(x)t_{\mathcal{R}}(x)\kappa^{n-1}\, w}{g(T(x))\, h^{n-1}\left(1 - h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right) \cdot D_p h\right)}, \tag{8.15}$$

where C^{-1} is given in (8.22), B given by (8.20), h is defined in (8.13), and w defined in (8.14).

Proof. From (8.1), (8.7), (8.8), and the definitions of h and w we get that the components of y in (8.1) can be written as

$$y_i = \frac{1}{\kappa}\left(w(x, \rho(x), D\rho(x))x_i + h(x, \rho(x), D\rho(x))\, \rho_{x_i}\right), \quad 1 \le i \le n-1,$$

and

$$y_n = \frac{1}{\kappa} w \sqrt{1 - |x|^2}. \tag{8.16}$$

Differentiating y_i with respect to x_j, with $1 \le i, j \le n-1$, we get

$$\begin{aligned} \partial_j y_i = \frac{1}{\kappa}\Bigg(& w\,\delta_{ij} + x_i\left(w_{x_j} + w_z\,\rho_{x_j} + \sum_{k=1}^{n-1} w_{p_k}\,\rho_{x_k x_j}\right) + h(x, \rho, D\rho)\rho_{x_i x_j} \\ & + \rho_{x_i}\left(h_{x_j} + h_z\,\rho_{x_j} + \sum_{k=1}^{n-1} h_{p_k}\,\rho_{x_k x_j}\right)\Bigg). \end{aligned}$$

Recall that all the vectors $x, D\rho, D_x w, D_p w, D_z w, D_x h, D_p h$ are regarded as row vectors. The matrix $Dy = (\partial_j y_i)$, $1 \le i, j \le n-1$, can then be written as

$$\begin{aligned} Dy = \frac{1}{\kappa}\Big(& w\,I + x \otimes D_x w + w_z x \otimes D\rho + x \otimes \left((D^2\rho)(D_p w)^t\right)^t \\ & + h(x, \rho, D\rho)\, D^2\rho + D\rho \otimes D_x h + h_z\, D\rho \otimes D\rho + D\rho \otimes \left((D^2\rho)(D_p h)^t\right)^t\Big). \end{aligned}$$

Note that if u, v are both row vectors and A is an $n \times n$ symmetric matrix, then $u \otimes \left(Av^t\right)^t = (u \otimes v)A$, and we obtain the formula

$$\begin{aligned} Dy = \frac{1}{\kappa}\big(& w\,I + x \otimes D_x w + w_z x \otimes D\rho + D\rho \otimes D_x h + h_z\, D\rho \otimes D\rho \\ & + \left(x \otimes D_p w\right) D^2\rho + h(x, \rho, D\rho)\, D^2\rho + \left(D\rho \otimes D_p h\right) D^2\rho\big). \end{aligned}$$

Let

$$C(x) = \left(x \otimes D_p w\right) + h(x, \rho, D\rho)\, I + \left(D\rho \otimes D_p h\right) \tag{8.17}$$

$$B(x) = w\,I + x \otimes D_x w + w_z x \otimes D\rho + D\rho \otimes D_x h + h_z\, D\rho \otimes D\rho. \tag{8.18}$$

So

$$Dy = \frac{1}{\kappa}\left[B(x) + C(x)\,D^2\rho\right]. \tag{8.19}$$

We have $D_x w = -D_x h\,(z + p\cdot x) - h\,p$, $w_z = -h_z\,(z + p\cdot x) - h$, and $D_p w = -D_p h\,(z + p\cdot x) - h\,x$. So we can write

$$\begin{aligned}
C(x) &= -(\rho + D\rho\cdot x)\left(x\otimes D_p h\right) - h(x\otimes x) + h(x,\rho,D\rho)\,I + \left(D\rho\otimes D_p h\right)\\
&= \left((-(\rho + D\rho\cdot x)x + D\rho)\otimes D_p h\right) - h(x\otimes x) + h(x,\rho,D\rho)\,I\\
&= \left((-(\rho + D\rho\cdot x)x + D\rho)\otimes D_p h\right) - h\left((x\otimes x) - I\right)\\
&= h\left(\left(h^{-1}\left(-(\rho + D\rho\cdot x)x + D\rho\right)\otimes D_p h\right) + \left(((-x)\otimes x) + I\right)\right)\\
&= h\left(\mathcal{M}_1 + \mathcal{M}_2\right)\\
&= h\mathcal{M}_2\left(I + \mathcal{M}_2^{-1}\mathcal{M}_1\right),
\end{aligned}$$

with

$$\mathcal{M}_1 = -h^{-1}\left((\rho + D\rho\cdot x)x - D\rho\right)\otimes D_p h, \qquad \mathcal{M}_2 = ((-x)\otimes x) + I;$$

and

$$\begin{aligned}
B(x) &= wI - x\otimes\left(D_x h(\rho + D\rho\cdot x) + hD\rho\right) - \left(h_z(\rho + D\rho\cdot x) + h\right)x\otimes D\rho + D\rho\otimes D_x h + h_z D\rho\otimes D\rho.\\
&= \left(1 - (\rho + D\rho\cdot x)h\right)I - \left((\rho + D\rho\cdot x)x - D\rho\right)\otimes D_x h\\
&\quad - x\otimes\left(\left(2h + h_z(\rho + D\rho\cdot x)\right)D\rho\right) + h_z\,D\rho\otimes D\rho.
\end{aligned} \tag{8.20}$$

From the Sherman-Morrison formula we have that if $\mathcal{M} = I + \xi^t\eta$ where ξ and η are any vectors, then

$$\det\,\mathcal{M} = 1 + \xi\cdot\eta, \qquad \mathcal{M}^{-1} = I - \frac{\xi^t\eta}{1 + \xi\cdot\eta}. \tag{8.21}$$

We can calculate explicitly the inverse of $C(x)$ noticing it is the product of two matrices having the form of $\mathcal{M}$. We have

$$C^{-1} = \frac{1}{h}\left(I + \mathcal{M}_2^{-1}\mathcal{M}_1\right)^{-1}\mathcal{M}_2^{-1}, \quad \text{and} \quad \mathcal{M}_2^{-1} = I + \frac{x\otimes x}{1 - |x|^2}.$$ [5]

If we set $v = h^{-1}\left((\rho + D\rho\cdot x)x - D\rho\right)$, then $I + \mathcal{M}_2^{-1}\mathcal{M}_1 = I - \mathcal{M}_2^{-1}(v\otimes D_p h) = I + \left(-\mathcal{M}_2^{-1}v^t\right)D_p h$. Therefore

$$\left(I + \mathcal{M}_2^{-1}\mathcal{M}_1\right)^{-1} = I + \frac{\left(\mathcal{M}_2^{-1}v^t\right)D_p h}{1 - \left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h} := \mathcal{N},$$

and so

$$C^{-1} = \frac{1}{h}\,\mathcal{N}\left(I + \frac{x\otimes x}{1 - |x|^2}\right). \tag{8.22}$$

Let us calculate this matrix more explicitly. We have

$$\begin{aligned}
\mathcal{M}_2^{-1}v^t &= v^t + \frac{1}{1 - |x|^2}x^t x v^t\\
&= h^{-1}\left[(\rho + D\rho\cdot x)x^t - (D\rho)^t + \frac{1}{1 - |x|^2}x^t x\left((\rho + D\rho\cdot x)x^t - (D\rho)^t\right)\right]\\
&= h^{-1}\left[(\rho + D\rho\cdot x)x^t - (D\rho)^t + \frac{1}{1 - |x|^2}\left(|x|^2(\rho + D\rho\cdot x)x^t - (D\rho\cdot x)x^t\right)\right]\\
&= h^{-1}\left[\frac{\rho}{1 - |x|^2}x^t - (D\rho)^t\right].
\end{aligned}$$

So

$$\mathcal{N} = I + \frac{h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right)\otimes D_p h}{1 - h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right)\cdot D_p h}, \tag{8.23}$$

[5] Notice that this is the first fundamental form of the upper sphere parameterized by $X(x_1, x_2, \ldots, x_{n-1}) = \left(x_1, \ldots, x_{n-1}, \sqrt{1 - |x|^2}\right)$. The coefficients of this form are $g_{ij} = \partial_i X\cdot\partial_j X = \delta_{ij} + \frac{1}{1-|x|^2}x_i x_j$, $1 \le i, j \le n - 1$.

and $\det \mathcal{N} = \frac{1}{1-h^{-1}\left(\frac{\rho}{1-|x|^2}x-D\rho\right)\cdot D_p h}$. Therefore from (8.22)

$$\det C = h^{n-1}(1-|x|^2)\left(1-h^{-1}\left(\frac{\rho}{1-|x|^2}x-D\rho\right)\cdot D_p h\right). \tag{8.24}$$

Notice that for row vectors α, β, ξ, η, we have $(\alpha\otimes\beta)(\xi\otimes\eta) = (\beta\cdot\xi)(\alpha\otimes\eta)$. Then

$$\begin{aligned}
C^{-1} &= \frac{1}{h}\left[I+\frac{x\otimes x}{1-|x|^2}+\frac{\left(\mathcal{M}_2^{-1}v^t\right)D_p h}{1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h}+\left(\frac{\left(\mathcal{M}_2^{-1}v^t\right)D_p h}{1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h}\right)\left(\frac{x\otimes x}{1-|x|^2}\right)\right]\\
&= \frac{1}{h}\left[I+\frac{x\otimes x}{1-|x|^2}+\frac{\left(\mathcal{M}_2^{-1}v^t\right)D_p h}{1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h}+\frac{1}{\left(1-|x|^2\right)\left(1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h\right)}\left(\left(\mathcal{M}_2^{-1}v^t\right)D_p h\, x^t x\right)\right]\\
&= \frac{1}{h}\left[I+\frac{x\otimes x}{1-|x|^2}+\frac{\left(\mathcal{M}_2^{-1}v^t\right)D_p h}{1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h}+\frac{x\cdot D_p h}{\left(1-|x|^2\right)\left(1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h\right)}\left(\left(\mathcal{M}_2^{-1}v^t\right)^t\otimes x\right)\right]\\
&= \frac{1}{h}\left[I+\frac{x\otimes x}{1-|x|^2}+\frac{1}{1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h}\left(\mathcal{M}_2^{-1}v^t\right)\left(D_p h+\frac{x\cdot D_p h}{\left(1-|x|^2\right)}x\right)\right]\\
&= \frac{1}{h}\left[I+\frac{x\otimes x}{1-|x|^2}+\frac{1}{1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h}\left(h^{-1}\left[\frac{\rho}{1-|x|^2}x^t-(D\rho)^t\right]\right)\left(D_p h+\frac{x\cdot D_p h}{\left(1-|x|^2\right)}x\right)\right].
\end{aligned}$$

Let

$$\mathcal{A} = \frac{1}{1-\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h}\left(h^{-1}\left[\frac{\rho}{1-|x|^2}x^t-(D\rho)^t\right]\right)\left(D_p h+\frac{x\cdot D_p h}{\left(1-|x|^2\right)}x\right).$$

We have $\left(\mathcal{M}_2^{-1}v^t\right)^t\cdot D_p h = h^{-1}\left[\frac{\rho}{1-|x|^2}\left(x\cdot D_p h\right)-D\rho\cdot D_p h\right]$. So

$$\mathcal{A} = \frac{1}{h-\left[\frac{\rho}{1-|x|^2}\left(x\cdot D_p h\right)-D\rho\cdot D_p h\right]}\left(\frac{\rho}{1-|x|^2}x-D\rho\right)\otimes\left(D_p h+\frac{x\cdot D_p h}{\left(1-|x|^2\right)}x\right),$$

and

$$C^{-1} = \frac{1}{h}\left[I+\frac{x\otimes x}{1-|x|^2}+\mathcal{A}\right].$$

From (8.19) $Dy = \frac{1}{\kappa}C\left(C^{-1}B(x)+D^2\rho\right)$, and so

$$\det Dy = \frac{1}{\kappa^{n-1}}\det C\ \det\left(C^{-1}B+D^2\rho\right). \tag{8.25}$$

Combining (8.4), (8.6), (8.16) and (8.25), we obtain

$$\left|\frac{1}{\kappa^{n-1}}\det C\ \det\left(C^{-1}B+D^2\rho\right)\right| \le \frac{f(x)t_{\mathcal{R}}(x)w}{\kappa g(T(x))}.$$

Finally from (8.24) we get

$$|\det(D^2\rho+C^{-1}B)| \le \frac{f(x)t_{\mathcal{R}}(x)\kappa^{n-2}\,w}{g(T(x))\,h^{n-1}\left(1-h^{-1}\left(\frac{\rho}{1-|x|^2}x-D\rho\right)\cdot D_p h\right)}. \quad \square$$

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