NONSMOOTH HYPERSURFACES WITH SMOOTH LEVI CURVATURE

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Abstract. In $\mathbb{R}^3$, Lipschitz continuous viscosity solutions to the K-prescribed Levi curvature equation are smooth and strictly pseudoconvex if K is smooth and strictly positive, see [2]. We show here that in $\mathbb{R}^{2n+1}$, $n > 1$, a similar result does not hold; that is, we prove the existence in $\mathbb{C}^{n+1}$, $n > 1$, of nonsmooth pseudoconvex hypersurfaces with smooth Levi curvature.

1. Introduction

This paper concerns the existence of non-smooth viscosity solutions to the Levi curvature equation. The equation was first introduced by Bedford and Gaveau in [1] and recently it has received attention in [6], [3], [11], [2], [9], [8], and [4].

The set up is as follows: $\Omega \subset \mathbb{R}^{2n+1}$ is an open set, and $K : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function. We denote by $\xi = (x, y, t)$ points of $\mathbb{R}^{2n+1}$, with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. A function $u : \Omega \to \mathbb{R}$ is a classical solution to the K-prescribed Levi curvature if $u \in C^2(\Omega)$ and satisfies the pde

$$\mathcal{L}(K, u) = 0, \tag{1.1}$$

with $\mathcal{L}$ defined in (2.1). In this case it is said that the Levi curvature of the graph of u equals K. A notion of viscosity solution to this equation was introduced in [11] and [4] yielding the notions of generalized Levi curvature and generalized pseudo-convexity, see Subsection 2.2. Of course, this raises the question of the regularity of viscosity solutions. Indeed, if $n = 1$, and K is smooth and strictly positive, then it is proved in [2] that every Lipschitz continuous viscosity solution to the equation (1.1) is pseudo-convex and smooth, and therefore it solves the equation in the classical sense. The purpose of this paper is to show that the picture is different in higher dimensions. Indeed, we show here that when $n \geq 2$, viscosity solutions may not be regular when K is positive and smooth, and therefore the

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one-dimensional pseudo-convex hypoellipticity result from [2] no longer holds in higher dimensions. Precisely, we prove the following theorem which is the main result of the paper.

Theorem 1. *Suppose $n \geq 2$, and $K \in C^\infty(B_1 \times \mathbb{R})$ is a bounded and strictly positive function satisfying either one of the following conditions*

(H1) *$s \mapsto K(\xi, s)$ is strictly increasing for every fixed ξ, or*
(H2) *K is independent of ξ and nondecreasing in s.*

Then there exists $r \in (0,1)$ and a pseudoconvex viscosity solution u to the equation

$$\mathcal{L}(K, u) = 0 \quad in \quad B_r, \tag{1.2}$$

such that $u \in Lip(\bar{B}_r)$ with $u \notin C^1(B_r)$ if $n = 2$, and $u \notin C^{1,\beta}$ for any $\beta > 1 - \frac{2}{n}$ when $n > 2$. B_r denotes the Euclidean ball in $\mathbb{R}^{2n+1}$ centered at the origin with radius r.

The proof of this theorem uses Pogorelov's counterexamples, see [10] or [5, Section 5.5], and its extensions developed by Urbas in [12] to show existence of viscosity non classical solutions to real curvature and Hessian equations. A principal tool used to carry out the proof of our theorem is the comparison principle proved in [4, Theorems 4.1 and 4.2].

Our paper is organized as follows. Section 2 contains preliminaries about viscosity solutions for the Levi equation. In Section 3, we show existence of solutions in small balls, which is used in Section 4 to prove the main theorem.

2. Preliminaries and definitions

We recall some definitions and basic results needed in the paper.

2.1. **The prescribed Levi curvature equation.** Let $\Omega \subset \mathbb{R}^{2n+1}$ be open. The function $u : \Omega \to \mathbb{R}$ is a classical solution to the K-prescribed Levi curvature equation if $u \in C^2(\Omega)$ and satisfies the pde

$$\mathcal{L}(K, u) := \det A(Du, D^2u) - K(\xi, u)\, F(Du) = 0, \tag{2.1}$$

for all $\xi \in \Omega$, where Du and D^2u denote the gradient and Hessian of u in all the variables x, y, t, respectively,

$$F(Du) = 2^n \frac{(1 + |Du|^2)^{\frac{n+2}{2}}}{1 + (\partial_t u)^2},$$

and the matrix $A(Du, D^2u)$ is an $n \times n$ matrix defined as follows. For $\ell = 1, \cdots, n$, let

$$a_\ell := a_\ell(Du) = \frac{\partial_{y_\ell} u - \partial_{x_\ell} u\, \partial_t u}{1 + (\partial_t u)^2}, \quad b_\ell := b_\ell(Du) = \frac{-\partial_{x_\ell} u - \partial_{y_\ell} u\, \partial_t u}{1 + (\partial_t u)^2}, \tag{2.2}$$

and let a be the column vector with components a_ℓ, and b the column vector with components b_ℓ. Let Σ be the $n \times (2n+1)$ complex matrix defined by

$$\Sigma = (I_n, -iI_n, a - ib),$$

where I_n is the $n \times n$ identity matrix. Then the matrix $A(Du, D^2u)$ is defined by

$$A(Du, D^2u) := \Sigma\, D^2u\, \bar{\Sigma}^T. \tag{2.3}$$

Equation (2.1) geometrically means that the hypersurface M_u, graph of the solution u, has Levi curvature equal K, agreeing to let

$$M_u := \{z \in \mathbb{C}^{n+1} \ : \ z = (x + iy, t + iu(x, y, t)), (x, y, t) \in \Omega\}.$$

2.2. **Viscosity solutions to the prescribed Levi curvature equation.** Given u and ϕ real functions in Ω, we say that ϕ *touches* u from below (above) at $\xi_0 \in \Omega$ if $u(\xi_0) = \phi(\xi_0)$ and $\phi(\xi) \leq (\geq) u(\xi)$ for all ξ is some neighborhood of ξ_0.

The function $u : \Omega \to \mathbb{R}$ is a *viscosity subsolution* to the equation (2.1) if u is bounded upper semicontinuous in Ω, and for every $\phi \in C^2(\Omega)$ and for every $\xi_0 \in \Omega$ such that ϕ touches u from above at ξ_0 we have $A(D\phi, D^2\phi)(\xi_0) \geq 0$ and $\mathcal{L}(K, \phi)(\xi_0) \geq 0$.

Analogously, the function $u : \Omega \to \mathbb{R}$ is a *viscosity supersolution* to the equation (2.1) if u is bounded lower semicontinuous in Ω, and for every $\phi \in C^2(\Omega)$ and for every $\xi_0 \in \Omega$ such that ϕ touches u from below at ξ_0 and $A(D\phi, D^2\phi)(\xi_0) \geq 0$ we have $\mathcal{L}(K, \phi)(\xi_0) \leq 0$.

A function $u : \Omega \to \mathbb{R}$ is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

We remark that these definitions are equivalent to the ones given in [11] and [4], and imply that the epigraph of a viscosity solution u is a pseudoconvex domain of $\mathbb{C}^{n+1}$ in generalized sense, see [4, Section 2]. We express this property by simply saying that u is pseudoconvex.

2.3. **A Comparison Principle.** The following comparison result plays a crucial role in the proof of Theorem 1.

Let u be a viscosity subsolution and let v be a viscosity supersolution to the equation (2.1) *in a bounded open set $\Omega \subset \mathbb{R}^{2n+1}$. If K satisfies the assumptions in Theorem 1 and*

$$\limsup_{\xi\to\xi_0} u(\xi) \le \liminf_{\xi\to\xi_0} v(\xi) \quad \textit{for every} \quad \xi_0 \in \partial\Omega,$$

then $u \le v$ in Ω. This Comparison Principle is proved in [4, Theorem2 4.1 and 4.2].

3. A preliminary existence result

In this section we prove the existence of a Lipschitz-continuous viscosity solution to a Dirichlet problem for $\mathcal{L}(K,.)$ on the ball B_r, for r sufficiently small. The boundary data will be the restriction to ∂B_r of a convex function φ in $\overline{B_r}$ satisfying the equation $\det A(D\varphi, D^2\varphi) = 0$ in B_r. The crucial point of this preliminary existence result is the dependence of the gradient of the solution only on the gradient of φ. The proof is a refinement of the one of Lemma 5.1 in [4].

Throughout the section we assume that the curvature function K satisfies the assumptions in Theorem 1, and

$$\sup_{B_1\times\mathbb{R}} K < \frac{1}{2R^n}, \tag{3.1}$$

for some $0 < R < 1$. Geometrically, this condition means that

$$\sup_{B_1\times\mathbb{R}} K < \inf_{q\in\partial C_R} K^{(n)}_{\partial C_R}(q), \tag{3.2}$$

where C_R is the cylinder of $\mathbb{C}^{n+1}$

$$C_R := \{(x+iy, t+i\tau) \,:\, (x,y,t) \in B_R, \tau \in \mathbb{R}\},$$

and $K^{(n)}_{\partial C_R}(q)$ denotes the Levi-curvature of the boundary of C_R at the point q.

Indeed, since the real function

$$f(q) = |x|^2 + |y|^2 + t^2 - R^2, \;\; q = (x+iy, t+i\tau),$$

is a defining function of C_R, using (5) from [8] we easily get

$$K^{(n)}_{\partial C_R}(q) = \frac{R^2+t^2}{2R^{n+2}}, \;\; \forall\; q = (x+iy, t+i\tau) \in \partial C_R.$$

As a consequence: $\inf_{q\in\partial C_R} K^{(n)}_{\partial C_R}(q) = \frac{1}{2R^n}$, showing that (3.1) is equivalent to (3.2).

To prove our existence result, Proposition 3, we show the following lemma, which provides a strict subsolution to (2.1).

Lemma 2. *Let $0 < r < R < 1$ and let $\varphi \in C^2(B_1)$ be a convex function. For each $\lambda > 0$ define*

$$u_\lambda(\xi) := \varphi(\xi) - \lambda d(\xi), \quad \xi \in B_1,$$

where $d(\xi) := r^2 - |\xi|^2$. Then, there exists $\lambda^* > 0$, only depending on r and $\sup_{B_R} |D\varphi|$, such that

$$\mathcal{L}(K, u_\lambda) > 0 \text{ in } B_r, \quad \text{for every } \lambda > \lambda^*. \tag{3.3}$$

Proof. If $p \in \mathbb{R}^{2n+1}$ and X is a $(2n+1) \times (2n+1)$ complex matrix, we define

$$A(p, X) := \Sigma(p)\ X\ \overline{\Sigma}^T(p)$$

where $\Sigma(p)$ is the $n \times (2n+1)$ complex matrix

$$\Sigma(p) = (I_n, -iI_n, a(p) - ib(p)), \quad I_n = n \times n \text{ identity matrix},$$

and $a(p)$ and $b(p)$ are the column vectors with respective components,

$$a_l(p) = \frac{p_{n+l} - p_l\, p_{2n+1}}{1 + (p_{2n+1})^2}, \text{ and } b_l = \frac{-p_l - p_{n+l}\, p_{2n+1}}{1 + (p_{2n+1})^2}, \quad l = 1, \dots, n.$$

Since φ is convex, $D^2 u_\lambda = D^2\varphi - \lambda D^2 d \ \geq\ -\lambda D^2 d \ =\ 2\lambda I_{2n+1}$. As a consequence:

$$\begin{aligned}
\frac{\det A(Du_\lambda, D^2 u_\lambda)}{F(Du_\lambda)} \ &\geq\ (2\lambda)^n \frac{\det A(Du_\lambda, I_{2n+1})}{F(Du_\lambda)} \\
&= (2\lambda)^n\ \frac{\det\left(\Sigma(Du_\lambda)\, \overline{\Sigma(Du_\lambda)}^T\right)}{F(Du_\lambda)} \\
&= (2\lambda)^n\, 2^{n-1}\ \frac{2 + |a(Du_\lambda)|^2 + |b(Du_\lambda)|^2}{F(Du_\lambda)} \\
&\geq\ (2\lambda)^n\, 2^{n-1}\ \frac{1 + |Du_\lambda|^2}{1 + (\partial_t u_\lambda)^2}\ \frac{1}{F(Du_\lambda)}\ =\ \frac{1}{2}\ \frac{(2\lambda)^n}{(1 + |Du_\lambda|^2)^{\frac{n}{2}}}.
\end{aligned}$$

Then, since $Du_\lambda(\xi) = 2\lambda\left(\dfrac{D\varphi}{2\lambda} + \xi\right)$, letting $c = \sup_{B_R} |D\varphi|$, we have:

$$\frac{\det A(Du_\lambda, D^2 u_\lambda)}{F(Du_\lambda)} \ \geq\ \frac{1}{2}\ \frac{1}{((\frac{1}{2\lambda})^2 + (r + \frac{c}{2\lambda})^2)^{\frac{n}{2}}}.$$

This inequality, together with condition (3.1), easily implies the existence of a positive constant $\lambda^* = \lambda^*(r, c)$ such that (3.3) holds. □

Using the previous lemma and arguing as in [4, Proposition 5.1], we obtain the main result of this section.

Proposition 3. *If $0 < r < R < 1$, and $\varphi \in C^2(B_1)$ is a convex function such that $\mathcal{L}(K,\varphi) \le 0$ in B_1, then the Dirichlet problem*

$$\mathcal{L}(K,u) = 0, \text{ in } B_r, \qquad u = \varphi \text{ on } \partial B_r \tag{3.4}$$

has a viscosity solution $u \in Lip\,(\overline{B_r})$ satisfying

$$\|u\|_{L^\infty(\overline{B_r})} + \|u\|_{Lip\,(\overline{B_r})} \le C, \tag{3.5}$$

where C only depends on r, $\|\varphi\|_{L^\infty(\overline{B_R})}$, $\|D\varphi\|_{L^\infty(\overline{B_R})}$, and $\|DK\|_{L^\infty(\overline{B_R})}$.

Proof. Let $u_\lambda = \varphi - \lambda d$ be the function given by the previous lemma with $\lambda > \lambda^*$. Then $u_\lambda \in C^2(\overline{B_r})$ and is a classical subsolution to $\mathcal{L}(K,u) = 0$ in B_r. Moreover, $u_\lambda = \varphi$ on ∂B_r. On the other hand, since $\mathcal{L}(K,\varphi) \le 0$ in B_1, φ is a classical supersolution to $\mathcal{L}(K,u) = 0$ in B_r.

Then, by [4, Theorems 1.1 and 1.2], the Dirichlet problem (3.4) has a viscosity solution $u \in C(\overline{B_r})$ satisfying $u_\lambda \le u \le \varphi$ in $\overline{B_r}$. Hence $\sup_{B_r}|u| \le \sup_{B_r}|\varphi| + \lambda r$. On the other hand, by Lemma 2, $\sup_{B_r}|Du_\lambda|$ can be bounded by a constant only depending on r and $\sup_{B_R}|D\varphi|$. By the interior gradient estimates in [4, Proposition 5.1], we can conclude that $u \in Lip(\overline{B_r})$ with $\|u\|_{Lip(\overline{B_r})}$ only depending on r, $\|\varphi\|_{L^\infty(\overline{B_R})}$, $\|D\varphi\|_{L^\infty(\overline{B_R})}$, and $\|DK\|_{L^\infty(\overline{B_R})}$.

□

4. PROOF OF THEOREM 1

Throughout this section we denote by $x = (x_1, x')$, $x' = (x_2, \dots, x_n)$, $y = (y_1, y')$, $y' = (y_2, \dots, y_n)$ with $x, y \in \mathbb{R}^n$, and $\xi = (x, y, t)$ is a point in $\mathbb{R}^{2n+1}$.

As in the previous section, we fix $R \in]0,1[$ such that

$$\sup_{B_1\times\mathbb{R}} K < \frac{1}{2R^n} \tag{4.1}$$

For $0 \le \sigma < 1$ and $0 < r < R$ we define

$$w_\sigma(x,y) = w_\sigma(x_1,x',y_1,y') := (r^2 + x_1^2 + y_1^2)(\sigma + |x'|^2 + |y'|^2)^\alpha, \quad \alpha = 1 - \frac{1}{n}, \tag{4.2}$$

and

$$\psi_\sigma(\xi) = \psi_\sigma(x,y,t) := Mw_\sigma(x,y), \quad \phi_\sigma(\xi) = \phi_\sigma(x,y,t) := 2M(\sigma + |x'|^2 + |y'|^2)^\alpha,$$

with M a positive constant that will be determined later. We have

$$\psi_0 \le \psi_\sigma \le \phi_\sigma, \quad \text{in } B_1.$$

Since $n \geq 2$, the exponent $\alpha = 1 - \frac{1}{n} \geq \frac{1}{2}$ and so that ϕ_σ is convex in $\mathbb{R}^{2n+1}$. Moreover, ϕ_σ is smooth for $\sigma > 0$, and independent of x_1, y_1 and t. From (2.3) we then obtain

$$A(D\phi_\sigma, D^2\phi_\sigma) = \Sigma \begin{pmatrix} D^2_{xx}\phi_\sigma & D^2_{xy}\phi_\sigma & 0 \\ D^2_{yx}\phi_\sigma & D^2_{yy}\phi_\sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{\Sigma}^T = \left(D^2_{x_j x_m}\phi_\sigma + D^2_{y_j y_m}\phi_\sigma + i(D^2_{x_j y_m} - D^2_{y_j x_m})\phi_\sigma \right)_{j,m=1}^n,$$

and since the first column and the first row of this matrix are null vectors we get $\det A(D\phi_\sigma, D^2\phi_\sigma) = 0$.

Therefore:

$$\mathcal{L}(K, \phi_\sigma) = -K(\,\cdot\,, \phi_\sigma)F(D\phi_\sigma) < 0 \quad \text{in } B_1, \quad \forall\, \sigma \in]0,1[. \tag{4.3}$$

Thus, applying Proposition 3, the Dirichlet problem

$$\mathcal{L}(K, u) = 0 \quad \text{in } B_r, \qquad u = \phi_\sigma \quad \text{on } \partial B_r,$$

with $\sigma \in]0,1[$ and $0 < r < R$, has a viscosity solution u_σ such that

$$\| u_\sigma \|_{L^\infty(\overline{B_r})} + \| u_\sigma \|_{Lip(\overline{B_r})} \leq C(r, \sigma, M)$$

with $C(r, \sigma, M)$ depending on σ only through $C(\phi_\sigma) := \| \phi_\sigma \|_{L^\infty(\overline{B_r})} + \| D\phi_\sigma \|_{L^\infty(\overline{B_r})}$. On the other hand, an elementary computation shows that $C(\phi_\sigma) \leq 8M$. Then, we can choose $C(r, \sigma, M)$ independent of σ, and so

$$\| u_\sigma \|_{L^\infty(\overline{B_r})} + \| u_\sigma \|_{Lip(\overline{B_r})} \leq C(r, M). \tag{4.4}$$

Now we claim that, if $0 < r \ll R$, we can fix $M = M(r)$ such that

$$\mathcal{L}(K, \psi_\sigma) > 0 \quad \text{in } B_r, \quad \forall\, \sigma \in]0, r^2[. \tag{4.5}$$

Assuming this claim for a moment, we can use the Comparison Principle of Section 2.3 to compare u_σ with ψ_σ and ϕ_σ. Indeed, by (4.3) and (4.5), ϕ_σ and ψ_σ are, respectively, classical supersolution and subsolution to $\mathcal{L}(K, u) = 0$ in B_r. On the other hand $\psi_\sigma \leq \phi_\sigma$ in B_1, in particular, on ∂B_r. Thus, by the Comparison Principle,

$$\psi_\sigma \leq u_\sigma \leq \phi_\sigma \quad \text{in } B_r, \ \forall \sigma \in]0, r^2[. \tag{4.6}$$

The uniform estimate (4.4) implies the existence of a sequence $\sigma_j \searrow 0$ such that $(u_{\sigma_j})_{j \in \mathbb{N}}$ uniformly converges to a viscosity solution $u \in Lip(\overline{B_r})$ to the Dirichlet problem

$$\mathcal{L}(K, u) = 0 \quad \text{in } B_r, \quad u = \phi_0 \text{ on } \partial B_r;$$

the proof of this fact is very similar to the one given in [7, p. 1241].

Moreover, from (4.6), we get

$$\psi_0 \le u \le \phi_0 \quad \text{in } B_r.$$

In particular

$$Mr^2|x_2|^{2\alpha} \le u(0, x_2, 0, \dots, 0) \le 2M|x_2|^{2\alpha}. \tag{4.7}$$

These inequalities imply:

$$u \notin C^1, \quad \text{if } 2\alpha = 1 \ \ (\text{i.e. } n = 2)$$

and

$$u \notin C^{1,\beta}, \ \text{for every } \beta > 2\alpha - 1 = 1 - \frac{2}{n} \quad \text{if } 2\alpha > 1 \ \ (\text{i.e. } n > 2).$$

The first statement is trivial. For the second one we only have to remark that if $2\alpha > 1$, then $\partial_{x_2} u(0, 0, \dots, 0) = 0 = u(0, 0, \dots, 0)$ so that, if u would be $C^{1,\beta}$, with $\beta > 2\alpha - 1$, we would have $u(0, x_2, \dots, 0) \le C|x_2|^{1+\beta}$ for a suitable $C > 0$ and for every x_2 sufficiently small. Hence, by the first inequality in (4.7), it would be $\beta \le 2\alpha - 1$, a contradiction.

To complete the proof of the theorem, we are left with the proof of Claim (4.5). Direct computations show that

$$\begin{aligned} |Dw_\sigma|^2 = 4\Big(& (x_1^2 + y_1^2)(\sigma + |x'|^2 + |y'|^2)^{2\alpha} \\ & + \alpha^2(|x'|^2 + |y'|^2)(r^2 + x_1^2 + y_1^2)^2(\sigma + |x'|^2 + |y'|^2)^{2(\alpha-1)}\Big) \end{aligned}$$

and

$$\det A(Dw_\sigma, D^2 w_\sigma) = 2^{2n} f_\sigma \tag{4.8}$$

with

$$f_\sigma = \alpha^n (r^2 + x_1^2 + y_1^2)^{n-2} \frac{r^2(\alpha^{-1}\sigma + |x'|^2 + |y'|^2) + \alpha^{-1}\sigma(|x'|^2 + |y'|^2)}{(\sigma + |x'|^2 + |y'|^2)}.$$

For convenience of the reader we include the proof of (4.8). Since w_σ is independent of t we have

$$\begin{aligned} A(Dw_\sigma, D^2 w_\sigma) =& \Sigma \begin{pmatrix} D^2_{xx} w_\sigma & D^2_{xy} w_\sigma & 0 \\ D^2_{yx} w_\sigma & D^2_{yy} w_\sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{\Sigma}^T \\ =& \Big(D^2_{x_j x_m} w_\sigma + D^2_{y_j y_m} w_\sigma + i(D^2_{x_j y_m} - D^2_{y_j x_m}) w_\sigma \Big)_{j,m=1}^n \\ =& 4(\sigma + |z'|^2)^{\alpha-1} \begin{pmatrix} (\sigma + |z'|^2) & \alpha z_1 \bar{z}_j \\ \alpha \bar{z}_1 z_m & (r^2 + |z_1|^2)\Big(\alpha\delta_{jk} + \alpha(\alpha-1)\frac{\bar{z}_j z_m}{\sigma + |z'|^2}\Big) \end{pmatrix}_{j,m=2}^n \end{aligned}$$

where $z_j = x_j + iy_j, z' = x' + iy'$ (we use complex coordinates to make the formulas compact). Since $(\alpha - 1)n = -1$, we get

$$\begin{aligned}
\det A(Dw_\sigma, D^2 w_\sigma) &= 4^n(\sigma + |z'|^2)^{-1} \det \begin{pmatrix} (\sigma + |z'|^2) & \alpha z_1 \bar{z}_j \\ \alpha \bar{z}_1 z_m & (r^2 + |z_1|^2)\left(\alpha\delta_{jk} + \alpha(\alpha-1)\frac{\bar{z}_j z_m}{\sigma + |z'|^2}\right) \end{pmatrix}_{j,m=2}^n \\
&= 4^n \det \begin{pmatrix} 1 & \alpha z_1 \frac{\bar{z}_j}{(\sigma+|z'|^2)^{1/2}} \\ \alpha \bar{z}_1 \frac{z_m}{(\sigma+|z'|^2)^{1/2}} & (r^2 + |z_1|^2)\left(\alpha\delta_{jk} + \alpha(\alpha-1)\frac{\bar{z}_j z_m}{\sigma + |z'|^2}\right) \end{pmatrix}_{j,m=2}^n \\
&= 4^n \det \begin{pmatrix} 1 & 0 \\ \alpha \bar{z}_1 \frac{z_m}{(\sigma+|z'|^2)^{1/2}} & (r^2 + |z_1|^2)\left(\alpha\delta_{jk} + \alpha(\alpha-1)\frac{\bar{z}_j z_m}{\sigma + |z'|^2}\right) - \alpha^2 |z_1|^2 \frac{\bar{z}_j z_m}{\sigma + |z'|^2} \end{pmatrix}_{j,m=2}^n \\
&= 4^n \det \left((r^2 + |z_1|^2)\left(\alpha\delta_{jk} + \alpha(\alpha-1)\frac{\bar{z}_j z_m}{\sigma + |z'|^2}\right) - \alpha^2 |z_1|^2 \frac{\bar{z}_j z_m}{\sigma + |z'|^2} \right)_{j,m=2}^n \\
&:= 4^n \det \Gamma,
\end{aligned}$$

where Γ is a $(n-1)\times(n-1)$ Hermitian matrix. It is easy to see that $\lambda_1 = \alpha(r^2 + |z_1|^2)$ is an eigenvalue of Γ with multiplicity $n-2$. Now, $\operatorname{trace}\Gamma = (n-2)\lambda_1 + \lambda_2$ with

$$\begin{aligned}
\lambda_2 &= (r^2 + |z_1|^2)\left(\alpha + \alpha(\alpha-1)\frac{|z'|^2}{\sigma + |z'|^2}\right) - \alpha^2 |z_1|^2 \frac{|z'|^2}{\sigma + |z'|^2} \\
&= \alpha^2 \frac{r^2\left(\frac{\sigma}{\alpha} + |z'|^2\right) + \frac{\sigma}{\alpha}|z'|^2}{\sigma + |z'|^2}
\end{aligned}$$

Thus, $\det\Gamma = \lambda_1^{n-2}\lambda_2 = \alpha^n (r^2 + |z_1|^2)^{n-2} \frac{r^2\left(\frac{\sigma}{\alpha} + |z'|^2\right) + \frac{\sigma}{\alpha}|z'|^2}{\sigma + |z'|^2} = f_\sigma$, which completes the proof of (4.8).

Then, for every $\sigma \in]0, r^2[$,

$$|Dw_\sigma|^2 \le 2^{2\alpha+3} r^{4\alpha+2} \quad \text{in } B_r, \tag{4.9}$$

and

$$f_\sigma \ge \alpha^n r^{2(n-1)} \quad \text{in } B_r. \tag{4.10}$$

On the other hand, from (4.8) and (2.3), keeping in mind that $\psi_\sigma = Mw_\sigma$ is independent of t, we get

$$\frac{\det A(D\psi_\sigma, D^2\psi_\sigma)}{F(D\psi_\sigma)} = \frac{(2M)^n f_\sigma}{(1 + M^2 |Dw_\sigma|^2)^{\frac{n+2}{2}}}.$$

Therefore, from (4.10) and (4.9), we obtain

$$\frac{\det A(D\psi_\sigma, D^2\psi_\sigma)}{F(D\psi_\sigma)} \ge \frac{(2M\alpha)^n r^{2(n-1)}}{(1 + 2^{2\alpha+3} r^{4\alpha+2} M^2)^{\frac{n+2}{2}}} \quad \text{in } B_r. \tag{4.11}$$

Choosing $M = 2^{-\alpha-(3/2)} r^{-2\alpha-1}$, the right hand side of (4.11) equals

$$\frac{(2\alpha)^n\, 2^{-(3/2)n+1} r^{-n}}{2^{\frac{n+2}{2}}} = \frac{C(n)}{r^n} > \frac{1}{2R^n}, \quad \text{if } r < (2C(n))^{1/n} R.$$

Then from (4.1) we obtain

$$\mathcal{L}(K, \psi_\sigma) = \det A(D\psi_\sigma, D^2\psi_\sigma) \; - \; K(\cdot, \psi_\sigma) F(D\psi_\sigma) \; > \; F(D\psi_\sigma)\left(\frac{1}{2R^n} - K(\cdot, \psi_\sigma)\right) > 0.$$

This proves claim (4.5) and completes the proof of the theorem.

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