INTERIOR GRADIENT ESTIMATES FOR SOLUTIONS TO THE LINEARIZED MONGE–AMPÈRE EQUATION

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Abstract. Let ϕ be a convex function on a convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$. The corresponding linearized Monge–Ampère equation is

$$\operatorname{trace}(\Phi D^2 u) = f,$$

where $\Phi := \det D^2\phi \, (D^2\phi)^{-1}$ is the matrix of cofactors of $D^2\phi$. We establish interior Hölder estimates for derivatives of solutions to such equation when the function f on the right hand side belongs to $L^p(\Omega)$ for some $p > n$. The function ϕ is assumed to be such that $\phi \in C(\bar{\Omega})$ with $\phi = 0$ on $\partial\Omega$ and the Monge–Ampère measure $\det D^2\phi$ is given by a density $g \in C(\Omega)$ which is bounded away from zero and infinity.

1. Introduction

Let Ω be a convex domain in $\mathbb{R}^n$ and $\phi \in C(\Omega)$ be a convex function satisfying

$$\lambda \leq \det D^2\phi \leq \Lambda \qquad \text{in } \Omega, \tag{1.1}$$

where $0 < \lambda < \Lambda < \infty$. Given a function $u(x)$, we form $\det D^2(\phi + tu)$, and it is easy to see that

$$\det D^2(\phi + tu) = \det D^2\phi + t \operatorname{trace}(\Phi D^2 u) + \cdots + t^n \det D^2 u,$$

with $\Phi := \det D^2\phi \, (D^2\phi)^{-1}$ is the matrix of cofactors of $D^2\phi$. The coefficient of t in this expansion is called the *linearization of the Monge-Ampère equation* (1.1) at ϕ and will be denoted by

$$\mathcal{L}_\phi u := \operatorname{trace}(\Phi D^2 u) = \sum_{i,j} \Phi_{ij} D_{ij} u = \sum_{i,j} D_i \left(\Phi_{ij} D_j u\right) = \operatorname{div}(\Phi D u).$$

The third equality is due to the fact that $\Phi = (\Phi_{ij})$ is divergence free. Thus $\mathcal{L}_\phi$ is both a non divergence and divergence second order operator. As $D^2\phi$ is positive semi-definite, the matrix of cofactors Φ is also positive semi-definite and consequently, $\mathcal{L}_\phi$ is an elliptic partial differential operator, possibly degenerate. The operator $\mathcal{L}_\phi$ appears in several

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applications including affine differential geometry [TW00, Tru01, TW08], complex geometry [Don05] and fluid mechanics [Bre91], [NCP91], [Loe06].

The linearized Monge–Ampère equation was studied by Caffarelli and Gutiérrez in [CG97] where the authors showed that nonnegative solutions to $\mathcal{L}_\phi u = 0$ satisfy a uniform Harnack's inequality. This important property implies uniform Hölder continuity of solutions. Recently, Gutiérrez and Tournier [GT06] studied the L^p integrability of second derivatives of solutions to $\mathcal{L}_\phi u = f$. They proved that for any domain $\Omega' \Subset \Omega$, there exist $\delta > 0$ and $C > 0$ depending only on n, λ, Λ and $\mathrm{dist}(\Omega', \partial\Omega)$ such that

$$\|D^2 u\|_{L^\delta(\Omega')} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}).$$

In another direction, Savin [Sav10] investigated the Liouville property for solutions of the linearized operator in two dimensions. By using the Harnack's inequality of Caffarelli and Gutiérrez in certain nondegenerate directions he was able to prove that global Lipschitz solutions to $\mathcal{L}_\phi u = 0$ in $\mathbb{R}^2$ must be linear.

The purpose in this paper is to study interior Schauder estimates for solutions to the equation $\mathcal{L}_\phi u = f$. To obtain $C^{2,\alpha}_{loc}$ estimates for the solution u, it is reasonable to expect that one has to assume further that $\det D^2\phi$ is locally Hölder continuous. However under this hypothesis, the second derivatives of u are indeed locally Hölder continuous since the operator $\mathcal{L}_\phi$ becomes uniformly elliptic thanks to Caffarelli $C^{2,\alpha}_{loc}$ estimate in [Caf90] for the solution ϕ of the Monge–Ampère equation. Thus the remaining interesting question is to investigate $C^{1,\alpha}_{loc}$ estimates for the solution u and this is the subject of the current article. We establish interior Hölder estimates for derivatives of solutions to the equation $\mathcal{L}_\phi u = f$ having the form

$$\|u\|_{C^{1,\alpha'}(\Omega')} \leq C\Big(\|u\|_{L^\infty(\Omega)} + [f]^n_{\alpha,\Omega}\Big),$$

for any $\alpha' \in (0, \alpha)$ and under the assumption that $\det D^2\phi$ is continuous (see Theorem 4.7). We stress that under this condition, the linearized operator is in general not uniformly elliptic and this is the main difficulty of the problem. Our estimates depend on $D^2\phi$ only through its determinant, not on the maximum and minimum of its eigenvalues. In order to handle the degeneracy of $\mathcal{L}_\phi$, we use the idea in [CG97] by working with sections of solutions to the Monge-Ampère equation. The role of these sections in our analysis is the same as that of Euclidean balls in the theory of uniformly elliptic equations.

Our proof resides in a perturbation argument which is an adaptation to our context the perturbation method in [Caf89, CC95] where fully nonlinear uniformly elliptic equations are considered. The idea is to compare solutions of the Monge-Ampère equation $\det D^2\phi = g$ with solutions of the good Monge-Ampère equation $\det D^2 w = 1$. It is simple to estimate the supremum norm of $\phi - w$, however it is much harder to estimate

the difference of their corresponding cofactor matrices which is relevant to the linearized Monge–Ampère operator and allows us to compare solutions of $\mathcal{L}_\phi u = f$ to solutions of $\mathcal{L}_w h = 0$ with the same Dirichlet boundary data. We achieve these estimates by using a compactness argument, the weak maximum principle, Caffarelli $W^{2,p}$-estimate for solutions of the Monge–Ampère equation and the $W^{2,\delta}$-estimate proved in [GT06] for solutions of the linearized equation (see Lemma 4.2, Lemma 3.4 and Lemma 3.5). The estimate obtained for $\|u - h\|_{L^\infty}$ in principle depends on the modulus of continuity of the boundary data but fortunately this can be controlled uniformly and universally thanks to Caffarelli-Gutiérrez interior Hölder estimate for the solution u. The next step in deriving the desired gradient estimate is to iterate the comparison process by rescaling the solution accordingly and as a consequence u gets closer to a linear polynomial when we restrict to a smaller section of ϕ. In order to conclude that u is in $C^{1,\alpha}$, the last step is to show that the section of ϕ is more round (almost like a ball) when its height gets smaller. To this end, we study in Lemma 3.2 and Lemma 3.3 the shape of sections of solutions to the Monge-Ampère equation by using various available results.

The paper is organized as follows. In Section 2 we mention preliminary results for the Monge-Ampère and linearized Monge-Ampère equations. In Subsection 3.1 we study the eccentricity of sections, and then in Subsection 3.2 establish a convergence result for the cofactor matrices. In Subsection 4.1 we prove an approximation lemma which plays a crucial role in the paper. Finally, the gradient estimates are derived in Subsection 4.2.

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2. Preliminary results

2.1. **Monge-Ampère equation.** In this subsection we list the results about sections and normalization that are relevant for what follows. Given a function $\phi : \Omega \to \mathbb{R}$, $\partial\phi$ denotes the subdifferential of ϕ. The Monge-Ampère measure associated with ϕ is $\mu(E) := |\partial\phi(E)|$, for all Borel subsets $E \subset \Omega$. In case ϕ is convex and $\phi \in C^2(\Omega)$, we have

$$|\partial\phi(E)| = \int_E \det D^2\phi(y)\, dy.$$

A *normalized convex domain* is a convex domain $\Omega \subseteq \mathbb{R}^n$ such that $B_1(0) \subseteq \Omega \subseteq B_n(0)$. We remark that if S is any convex set with nonempty interior, there exists an ellipsoid E satisfying $E \subseteq S \subseteq nE$ and hence, there is an affine transformation T such that $B_1(0) \subseteq T(S) \subseteq B_n(0)$. A *section of a convex function* $\phi \in C^1(\Omega)$ centered at $\bar{x}$ and with height t is

defined by

$$S_t(\phi, \bar{x}) = \Big\{x \in \Omega : \phi(x) < \phi(\bar{x}) + D\phi(\bar{x}) \cdot (x - \bar{x}) + t\Big\}. \tag{2.1}$$

The next three results about sections hold under the assumption:

(**H**) *Ω is a normalized convex domain and $\phi \in C(\bar{\Omega})$ is a convex function such that*

$$\lambda \leq \det D^2\phi \leq \Lambda \text{ in } \Omega \quad \text{and} \quad \phi = 0 \text{ on } \partial\Omega.$$

Lemma 2.1. [Gut01, Theorem 3.3.8] *For any $\Omega' \Subset \Omega$, there exist positive constants h_0, C_1, C_2 and b such that for $x_0 \in \Omega'$ and $0 < h \leq h_0$*

$$B_{C_1 h}(x_0) \subset S_h(\phi, x_0) \subset B_{C_2 h^b}(x_0),$$

where $b = b(n, \lambda, \Lambda)$ and h_0, C_1, C_2 depend only on n, λ, Λ and $dist(\Omega', \partial\Omega)$.

Lemma 2.2. [Gut01, Theorem 3.3.7] *There exists $\theta > 1$ such that if $x \in S_t(\phi, y)$ then $S_t(\phi, y) \subset S_{\theta t}(\phi, x)$.*

Lemma 2.3. [Gut01, Corollary 3.2.4] *There exist constants C and C' depending only on n, λ and Λ such that $\quad C\, t^{n/2} \leq |S_t(\phi, x)| \leq C'\, t^{n/2} \quad$ whenever $S_t(\phi, x) \Subset \Omega$.*

Lemma 2.3 says that the Lebesgue measure of any section depends essentially on the parameter t and is comparable to the Lebesgue measure of an Euclidean ball of radius $\sqrt{t}$. However, a section may look like an ellipsoid in which the ratio between the longest axes and the shortest axes goes to infinity as the parameter t goes to 0. In other words, the eccentricity of a section is not bounded by a constant depending only on λ, Λ and n.

2.2. **The linearized Monge-Ampère operator.** Throughout this paper we always assume that Ω and ϕ satisfy (**H**) unless otherwise stated and we will work with strong solutions in the Sobolev space $W^{2,n}_{loc}(\Omega)$ of the linearized Monge-Ampère equation. That is, the equation $\mathcal{L}_\phi u = f$ in Ω is interpreted in the almost everywhere sense in Ω. Notice that since ϕ is strictly convex by the assumption (**H**), the Hessian $D^2\phi$ is defined almost everywhere as a positive semi-definite matrix and so is the cofactor matrix Φ. All the estimates proved in the paper depend only on the structure and they are independent of the regularity. The following Alexandroff-Bakelman-Pucci maximum principle will be used later and can be found in [GT83, Theorem 9.1] (see also [CC95, Theorem 3.6]).

Theorem 2.4 (ABP estimate)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and $f \in L^n(\Omega)$. Assume that the matrix $A = [a^{ij}]$ is measurable and positive almost everywhere in Ω and $u \in W^{2,n}_{loc}(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$a^{ij} u_{ij} \geq f \quad \textit{almost everywhere in} \quad \Omega.$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C_n \, diam\,(\Omega)\, \|f/(\det A)^{1/n}\|_{L^n(\Omega)}.$$

One of the important properties of the linearized Monge-Ampère operator is that its nonnegative solutions satisfy Harnack's inequality, a result proved by Caffarelli and Gutiérrez in [CG97]. Accordingly they obtain the following fundamental oscillation estimate, which we formulate here for the inhomogeneous equation as in [TW08].

Theorem 2.5. *Let $u \in W^{2,n}_{loc}(\Omega)$ be a solution of $\mathcal{L}_\phi u = f$ in Ω. Then for any section $S_h(\phi, x_0) \Subset \Omega$, we have the estimate*

$$\operatorname*{osc}_{S_\rho(\phi,x_0)} u \leq C\left(\frac{\rho}{h}\right)^\alpha \Big\{ \operatorname*{osc}_{S_h(\phi,x_0)} u + \rho^{\frac{1}{2}} \|f\|_{L^n(S_h(\phi,x_0))} \Big\} \qquad \textit{for all} \quad \rho < h,$$

where $C > 0$ and $\alpha > 0$ depend only on n and Λ/λ, and $\operatorname{osc}_E u := \max_E u - \min_E u$.

Theorem 2.5 implies the Hölder estimate for solutions. However, the constants in this case depend on the norm of the affine transformation used to normalize the section. Indeed, it follows from the arguments in [CG97, pp. 456-457] that

$$|u(x) - u(y)| \leq C\|A\|^\beta |x-y|^\beta \Big\{ \|u\|_{L^\infty(S_{2h}(\phi,x_0))} + (2h)^{\frac{1}{2}} \|f\|_{L^n(S_{2h}(\phi,x_0))} \Big\} \quad \forall x, y \in S_h(\phi, x_0)$$

where C is a universal constant and $Tx = A(x - x_0) + y_0$ is the affine transformation normalizing $S_{2\theta h}(\phi, x_0)$, i.e., $B_1(0) \subset T(S_{2\theta h}(\phi, x_0)) \subset B_n(0)$ (θ is the engulfing constant given by Lemma 2.2).

In the ideal situation when one knows that $S_{2\theta h}(\phi, x_0) \approx B_{C\sqrt{h}}(x_0)$, then $\|A\| \lessapprox h^{-1/2}$. However one does not have this under the condition (**H**) for ϕ. We will need the above Hölder estimate in the proof of Theorem 4.5 where all sections under consideration have the property as in Lemma 2.1. But for such section $S_h(\phi, x_0)$, we get $\|A\| \leq Ch^{-1}$ since $AB_{Ch}(0) + y_0 \subset B_n(0)$. Thus in that case, we have

$$|u(x) - u(y)| \leq C^* h^{-\beta} |x-y|^\beta \Big\{ \|u\|_{L^\infty(S_{2h}(\phi,x_0))} + (2h)^{\frac{1}{2}} \|f\|_{L^n(S_{2h}(\phi,x_0))} \Big\} \quad \forall x, y \in S_h(\phi, x_0) \tag{2.2}$$

where C^* is a universal constant.

As a consequence, we obtain the following Hölder estimate.

Corollary 2.6. *Assume that Ω and ϕ satisfy* (**H**). *If $u \in W^{2,n}_{loc}(B_1)$ is a solution of $\mathcal{L}_\phi u = f$ in B_1, then there exist constants $0 < \beta < 1$ and $C > 0$ depending only on n, λ, Λ such that*

$$|u(x) - u(y)| \leq C|x-y|^\beta \Big\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)} \Big\} \quad \textit{for any} \quad x, y \in B_{\frac{1}{2}}. \tag{2.3}$$

2.3. **A classical regularity theorem.** In this subsection we assume that w is the convex solution of the equation

$$\begin{cases} \det D^2 w = 1 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.4}$$

where Ω is a normalized convex domain. It follows from Pogorelov's estimate that the operator $\mathcal{L}_w u$ is uniformly elliptic in the interior of Ω and hence its solutions have all the usual regularity properties. We recall the classical $C^{1,1}$ interior estimate for linear uniformly elliptic equations (see for example [GT83, Theorem 6.2] or [HL97, Theorem 5.20]).

Theorem 2.7. *Let $B_{\frac{6}{5}} \subset \Omega \subset B_n$ be a normalized domain. Then for any $\varphi \in C(\partial B_1)$ there exists a solution $h \in C^2(B_1) \cap C(\bar{B}_1)$ of $\mathcal{L}_w h = 0$ in B_1 and $h = \varphi$ on ∂B_1 such that*

$$\|h\|_{C^{1,1}(B_{\frac{1}{2}})} \leq c_e \|\varphi\|_{L^\infty(\partial B_1)},$$

where the constant $c_e > 0$ depends only on n.

3. Properties of solutions to the Monge-Ampère equation

3.1. **Geometry of sections.** We begin with a lemma which gives estimates of the third derivatives of solutions to the Monge-Ampère equation in terms of the eccentricity of the boundary of the domain. This result will be used in this subsection to discuss the geometry of sections of solutions to the Monge-Ampère equation. Some related results in this direction appeared in [Hua06] and [Hua09].

Lemma 3.1. *Let Ω be an open convex set such that $B_{R_1}(0) \subset \Omega \subset B_{R_2}(0)$, with $1 \leq R_1 < R_2 \leq n$, and suppose w is a smooth solution to $\det D^2 w = 1$ in Ω and $w = 0$ on $\partial\Omega$. Then for any domain $\Omega' \Subset \Omega$, there exists a positive constant C^*, depending only on n and $dist(\Omega', \partial\Omega)$, such that*

$$\|D^3 w\|_{L^\infty(\Omega')} \leq C^* (R_2^2 - R_1^2).$$

Proof. Let $P(x) := \frac{1}{2}|x|^2 - \frac{1}{4}(R_1^2 + R_2^2)$. Then it is clear that

$$P(x) - \frac{1}{4}(R_2^2 - R_1^2) \leq 0 \leq P(x) + \frac{1}{4}(R_2^2 - R_1^2) \quad \text{for all} \quad x \in \partial\Omega.$$

Hence by the comparison principle

$$\|w - P\|_{L^\infty(\Omega)} \leq \frac{1}{4}(R_2^2 - R_1^2).$$

Since w is smooth, the function $v = w - P$ satisfies the linear equation

$$\text{trace}\left(A(x) D^2 v(x)\right) = 0,$$

with

$$A(x) := \int_0^1 \left(tD^2w(x) + (1-t)I\right)^{-1} \det\left(tD^2w(x) + (1-t)I\right) dt.$$

From Pogorelov's estimates [Gut01, formula (4.2.6)], it follows that the matrix $A(x)$ is uniformly elliptic on each sub domain $\Omega' \Subset \Omega$. Hence from interior Schauder estimates [GT83, Corollary 6.3], we have

$$\|w - P\|_{C^2(\Omega')} \leq C\,\|w - P\|_{L^\infty(\Omega)} \leq C\,(R_2^2 - R_1^2), \tag{3.5}$$

where C depends only on the dimension n and $\operatorname{dist}(\Omega', \partial\Omega)$. Next, differentiating the equation $\det D^2w(x) = 1$ we obtain that $v_i := D_i(w - P)$ satisfy the linearized equation

$$\operatorname{trace}(W(x)D^2v_i(x)) = 0,$$

for $i = 1, \dots, n$, where W is the matrix of cofactors of D^2w. Once again, by Pogorelov's estimates, the matrix W is uniformly elliptic on each sub domain $\Omega' \Subset \Omega$; and from interior Schauder estimates [GT83, Corollary 6.3] and (3.5) we finally get

$$\|D^2w_i\|_{L^\infty(\Omega')} = \|D^2v_i\|_{L^\infty(\Omega')} \leq C\,\|v_i\|_{L^\infty(\Omega'')} \leq C\,(R_2^2 - R_1^2),$$

where $\Omega' \Subset \Omega'' \Subset \Omega$. This completes the proof of the lemma. □

In the following I denotes the identity matrix and $N_\delta(E) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) < \delta\}$ is the $\delta-$neighborhood of the set E with respect to the Euclidean distance. Also for a strictly convex function v defined on Ω and $t > 0$, $S_t(v)$ denotes the section of v centered at its minimum point and with height t. That is,

$$S_t(v) := \{x \in \Omega : v(x) \leq \min_\Omega v + t\}.$$

Lemma 3.2. *Suppose $B_1 \subset \Omega \subset B_n$ is a normalized convex domain. Then there exist constants $\mu_0 > 0$, $\tau_0 > 0$ and a positive definite matrix $M = A^tA$ and $p \in \mathbb{R}^n$ satisfying*

$$\det M = 1, \qquad 0 < c_1 I \leq M \leq c_2 I, \qquad \text{and} \qquad |p| \leq c,$$

such that if $u \in C(\bar{\Omega})$ is a strictly convex function in Ω with

$$\begin{cases} 1 - \epsilon \leq \det D^2u \leq 1 + \epsilon & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then for $0 < \mu \leq \mu_0$ and $\varepsilon \leq \tau_0\mu^2$, we have

$$B_{\left(1 - C(\mu^{1/2} + \mu^{-1}\epsilon^{1/2})\right)\sqrt{2}}(0) \subset \mu^{-1/2}TS_\mu(u) \subset B_{\left(1 + C(\mu^{1/2} + \mu^{-1}\epsilon^{1/2})\right)\sqrt{2}}(0), \tag{3.6}$$

and

$$\left|u(x) - \left(u(x_0) + p \cdot (x - x_0) + \frac{1}{2}\langle M(x - x_0), (x - x_0)\rangle\right)\right| \leq C(\mu^{3/2} + \varepsilon) \quad \text{in} \quad S_\mu(u), \tag{3.7}$$

where $x_0 \in \Omega$ is the minimum point of u and $Tx := A(x - x_0)$.

Proof. Let $w(x)$ be the smooth convex solution to the equation $\det D^2 w(x) = 1$ in Ω and $w = 0$ on $\partial\Omega$. Then from the comparison principle

$$|u(x) - w(x)| \le c(n)\epsilon \quad \forall x \in \Omega, \tag{3.8}$$

and consequently,

$$|u(x_0) - w(x_1)| \le c(n)\epsilon, \tag{3.9}$$

where $x_1 \in \Omega$ is the minimum point of w. We have by [Gut01, Proposition 3.2.3] that $|u(x_0)| \approx c_n$ and $|w(x_1)| \approx c_n$, and hence from Aleksandrov's maximum principle [Gut01, Theorem 1.4.2] we get $\operatorname{dist}(x_i, \partial\Omega) \ge c_n$. Let $\Omega_0 := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge c_n\}$. Then

$$\frac{2}{C_2^2} I \le D^2 w(x) \le \frac{2}{C_1^2} I \quad \text{for all} \quad x \in \Omega_0 \tag{3.10}$$

by Pogorelov's estimate [Gut01, formula (4.2.6)]. We now claim that

$$|Dw(x_0)| \le c\varepsilon^{1/2}, \tag{3.11}$$

where c is a universal constant. Indeed, from the Taylor formula

$$w(x_0) - u(x_0) = w(x_1) - u(x_0) + \frac{1}{2}\langle D^2 w(\xi)(x_0 - x_1), x_0 - x_1\rangle,$$

with ξ an intermediate point between x_0 and x_1, and since x_0 and x_1 are away from the boundary, by (3.10) we get that $|x_0 - x_1| \le c\,\epsilon^{1/2}$. Hence writing

$$D_i w(x_0) = D_i w(x_0) - D_i w(x_1) = -\int_0^1 D(D_i w)(x_0 + t(x_1 - x_0)) \cdot (x_1 - x_0)\, dt$$

and using once again (3.10) we obtain (3.11).

Next, Aleksandrov's maximum principle yields $S_\mu(w, x_1) \subset \Omega_0$, for $0 < \mu \le c_n$. Moreover, it follows from (3.8), (3.9) and (3.11) that $S_\mu(w, x_0) \subset S_{c\epsilon^{1/2}+\mu}(w, x_1)$. Therefore,

$$S_\mu(w, x_0) \subset \Omega_0, \quad \text{for} \quad \epsilon < \tau_n \mu^2 \quad \text{and} \quad \mu \le \mu_n. \tag{3.12}$$

We claim that there exists a universal constant $C_3 > 0$ such that

$$S_{\mu - C_3\varepsilon^{1/2}}(w, x_0) \subset S_\mu(u) \subset S_{\mu + C_3\varepsilon^{1/2}}(w, x_0). \tag{3.13}$$

Indeed, let $C_3 = 2[nc + c(n)]$ where c is the constant in (3.11). If $x \in S_{\mu - C_3\varepsilon^{1/2}}(w, x_0)$, then

$$\begin{aligned} u(x) &\le w(x) + c(n)\varepsilon \le w(x_0) + Dw(x_0) \cdot (x - x_0) + \mu - C_3\varepsilon^{1/2} + c(n)\varepsilon \\ &\le u(x_0) + Dw(x_0) \cdot (x - x_0) + \mu - C_3\varepsilon^{1/2} + 2c(n)\varepsilon \\ &\le u(x_0) + 2n\, c\, \varepsilon^{1/2} + \mu - C_3\varepsilon^{1/2} + 2c(n)\varepsilon \le u(x_0) + \mu. \end{aligned}$$

On the other hand, if $x \in S_\mu(u)$ then

$$\begin{aligned} w(x) &\le u(x) + c(n)\varepsilon \le u(x_0) + \mu + c(n)\varepsilon \\ &\le w(x_0) + Dw(x_0)\cdot(x - x_0) + \mu + 2c(n)\varepsilon - Dw(x_0)\cdot(x - x_0) \\ &\le w(x_0) + Dw(x_0)\cdot(x - x_0) + \mu + C_3\varepsilon^{1/2}. \end{aligned}$$

Hence the claim (3.13) is proved.

We next claim that there exists $\mu_0 > 0$ such that if $\mu \le \mu_0$ and $\gamma \le \frac{3}{4}\mu$, then

$$\partial S_{\mu+\gamma}(w, x_0) \subset N_{\frac{C\gamma}{\sqrt{\mu}}}(\partial S_\mu(w, x_0)) \quad \text{and} \quad \partial S_{\mu-\gamma}(w, x_0) \subset N_{\frac{C\gamma}{\sqrt{\mu}}}(\partial S_\mu(w, x_0)). \tag{3.14}$$

In order to prove this, we first show that

$$B_{C_1\sqrt{\mu}}(x_0) \subset S_\mu(w, x_0) \subset B_{C_2\sqrt{\mu}}(x_0) \tag{3.15}$$

for $0 < \mu \le \mu_0$ and C_i the constants in (3.10). Keeping in mind (3.12) and (3.10), if $x \in S_\mu(w, x_0)$, then by using Taylor formula we have

$$\begin{aligned} \mu &\ge w(x) - w(x_0) - Dw(x_0)\cdot(x - x_0) = \int_0^1 [Dw(x_0 + t(x - x_0)) - Dw(x_0)]\cdot[x - x_0]\, dt \\ &= \int_0^1 t \int_0^1 \langle D^2 w(x_0 + \theta t(x - x_0))\cdot(x - x_0), (x - x_0)\rangle\, d\theta\, dt \\ &\ge \int_0^1 t \int_0^1 \frac{2}{C_2^2}|x - x_0|^2\, d\theta\, dt = \frac{1}{C_2^2}|x - x_0|^2, \end{aligned}$$

which yields $x \in B_{C_2\sqrt{\mu}}(x_0)$. Similarly, and assuming $B_{C_1\sqrt{\mu}}(x_0) \subset \Omega_0$ for $\mu \le \mu_0$, if $x \in B_{C_1\sqrt{\mu}}(x_0)$, then

$$\begin{aligned} w(x) - w(x_0) - Dw(x_0)\cdot(x - x_0) &= \int_0^1 t \int_0^1 \langle D^2 w(x_0 + \theta t(x - x_0))\cdot(x - x_0), x - x_0\rangle d\theta dt \\ &\le \int_0^1 t \int_0^1 \frac{2}{C_1^2}|x - x_0|^2\, d\theta\, dt = \frac{1}{C_1^2}|x - x_0|^2 \le \mu, \end{aligned}$$

that is, $x \in S_\mu(w, x_0)$ and hence (3.15) is proved. To prove (3.14), let $x \in \partial S_{\mu+\gamma}(w, x_0)$ and let z_1 be the intersecting point of $\partial S_\mu(w, x_0)$ and the segment connecting x_0 and x. We have for some ξ in the segment joining x and z_1

$$\begin{aligned} \gamma &= w(x) - w(z_1) - Dw(x_0)\cdot(x - z_1) = Dw(\xi)\cdot(x - z_1) - Dw(x_0)\cdot(x - z_1) \\ &= [Dw(\xi) - Dw(x_0)]\cdot[\xi - x_0]\frac{|x - z_1|}{|\xi - x_0|} \\ &= \frac{|x - z_1|}{|\xi - x_0|}\int_0^1 \langle D^2 w(x_0 + \theta(\xi - x_0))\cdot(\xi - x_0), (\xi - x_0)\rangle\, d\theta \ge C|x - z_1||\xi - x_0|. \end{aligned}$$

But as $\xi \notin S_\mu(w, x_0)$ and $B_{C_1\sqrt{\mu}}(x_0) \subset S_\mu(w, x_0)$ by (3.15), we get $|\xi - x_0| \geq C_1\sqrt{\mu}$. Therefore, we obtain $|x - z_1| \leq C\frac{\gamma}{\sqrt{\mu}}$ and thus we have established the first relation in (3.14). Similarly, let $x \in \partial S_{\mu-\gamma}(w, x_0)$ and let z_1 be the intersecting point of $\partial S_\mu(w, x_0)$ and the ray starting from x_0 and go through x. We have for some $\xi \in \overline{x\, z_1}$

$$\begin{aligned}\gamma &= w(z_1) - w(x) - Dw(x_0)\cdot(z_1 - x) = [Dw(\xi) - Dw(x_0)]\cdot[\xi - x_0]\frac{|z_1 - x|}{|\xi - x_0|}\\ &= \frac{|z_1 - x|}{|\xi - x_0|}\int_0^1 \langle D^2 w(x_0 + \theta(\xi - x_0))\cdot(\xi - x_0), (\xi - x_0)\rangle\, d\theta \geq C|z_1 - x||\xi - x_0|.\end{aligned}$$

On the other hand, as $\xi \notin S_{\mu-\gamma}(w, x_0)$ and $B_{\frac{C_1\sqrt{\mu}}{2}}(x_0) \subset B_{C_1\sqrt{\mu-\gamma}}(x_0) \subset S_{\mu-\gamma}(w, x_0)$ by (3.15) and the fact that $\gamma \leq \frac{3}{4}\mu$, we get $|\xi - x_0| \geq \frac{C_1\sqrt{\mu}}{2}$. Therefore, we obtain $|x - z_1| \leq C\frac{\gamma}{\sqrt{\mu}}$ yielding the second relation in (3.14).

Let $M = D^2 w(x_0)$ and $E = \{x : \frac{1}{2}\langle D^2 w(x_0)(x - x_0), (x - x_0)\rangle \leq 1\}$. We next compare $S_\mu(w, x_0)$ with ellipsoids and claim that

$$\partial S_\mu(w, x_0) \subset N_{C\mu}(\partial\mu^{1/2}E), \tag{3.16}$$

for some structural constant C and all $0 < \mu \leq \mu_0$. Here the dilation is with respect to x_0. To prove this it is sufficient to show

$$\partial S_\mu(w, x_0) \subset (1 + C\sqrt{\mu})\sqrt{\mu}E \setminus (1 - C\sqrt{\mu})\sqrt{\mu}E \tag{3.17}$$

since it is easy to see that $(1 + C\sqrt{\mu})\sqrt{\mu}E \setminus (1 - C\sqrt{\mu})\sqrt{\mu}E \subset N_{C\mu}(\partial\mu^{1/2}E)$. If x is in the set $\partial((1 + C\sqrt{\mu})\sqrt{\mu}E)$, then by the Taylor formula

$$\begin{aligned}w(x) - w(x_0) - Dw(x_0)\cdot(x - x_0)&\\ = \frac{1}{2}\langle D^2 w(x_0)(x - x_0), (x - x_0)\rangle &+ O(|D^3 w(\xi)||x - x_0|^3)\\ \geq (1 + C\sqrt{\mu})^2\mu &- K|x - x_0|^3,\end{aligned} \tag{3.18}$$

with K is a structural constant (see Lemma 3.1). Let us write $x = x_0 + (1 + C\sqrt{\mu})\sqrt{\mu}(y - x_0)$ for some $y \in \partial E$. Hence, as $1 = \frac{1}{2}\langle D^2 w(x_0)(y - x_0), (y - x_0)\rangle \geq c|y - x_0|^2$ we get $|y - x_0| \leq C'$. Now pick C such that $C > K(2C')^3$ (the same constant C will be used to prove the remaining case) and adjust μ_0 if needed such that $C\sqrt{\mu} \leq 1$. Then we have

$$|x - x_0| = (1 + C\sqrt{\mu})\sqrt{\mu}|y - x_0| \leq C'(1 + C\sqrt{\mu})\sqrt{\mu} \leq 2C'\sqrt{\mu}$$

and therefore from (3.18) we obtain

$$w(x) - w(x_0) - Dw(x_0)\cdot(x - x_0) \geq (1 + C\sqrt{\mu})^2\mu - K(2C')^3\sqrt{\mu}\mu > \mu.$$

Thus, $x \notin S_\mu(w, x_0)$ and hence $\partial S_\mu(w, x_0) \subset (1 + C\sqrt{\mu})\sqrt{\mu}E$. Similarly, if $x \in (1 - C\sqrt{\mu})\sqrt{\mu}E$, then

$$\begin{aligned} w(x) - w(x_0) - Dw(x_0) \cdot (x - x_0) &= \frac{1}{2}\langle D^2 w(x_0)(x - x_0), (x - x_0)\rangle + O(|D^3 w(\xi)||x - x_0|^3) \\ &\le (1 - C\sqrt{\mu})^2\mu + K|x - x_0|^3. \end{aligned}$$

Write $x = x_0 + (1 - C\sqrt{\mu})\sqrt{\mu}(y - x_0)$ for some $y \in \partial E$, and as before we get $|y - x_0| \le C'$. It follows that $|x - x_0| = (1 - C\sqrt{\mu})\sqrt{\mu}|y - x_0| \le C'(1 - C\sqrt{\mu})\sqrt{\mu} \le C'\sqrt{\mu}$ and therefore since $C > KC'^3$ and by combining with the above estimate we obtain

$$w(x) - w(x_0) - Dw(x_0) \cdot (x - x_0) \le (1 - C\sqrt{\mu})^2\mu + KC'^3\sqrt{\mu}\mu \le (1 - C\sqrt{\mu})\mu + KC'^3\sqrt{\mu}\mu \le \mu.$$

Thus, $x \in S_\mu(w, x_0)$ giving $(1 - C\sqrt{\mu})\sqrt{\mu}E \subset S_\mu(w, x_0)$, or $\partial S_\mu(w, x_0) \subset [(1 - C\sqrt{\mu})\sqrt{\mu}E]^c$ which completes the proof of (3.17) and so the claim (3.16) holds.

From (3.13) and (3.14) we obtain $\partial S_\mu(u) \subset N_{C\mu^{-1/2}\varepsilon^{1/2}}(\partial S_\mu(w, x_0))$, which together with (3.16) yields

$$\partial S_\mu(u) \subset N_{C(\mu + \mu^{-1/2}\varepsilon^{1/2})}(\partial \mu^{1/2} E). \tag{3.19}$$

Observe that $\partial \mu^{1/2} E = x_0 + A^{-1}\Big(\partial B_{\sqrt{2\mu}}(0)\Big)$, where $M = A^t A$. Also if we let $T = A(x - x_0)$, then

$$T\Big(N_\delta(\partial \mu^{1/2} E)\Big) \subset B_{\sqrt{2\mu} + \|A\|\delta}(0) \quad \text{and} \quad T\Big(N_\delta(\partial \mu^{1/2} E)\Big) \subset \Big(B_{\sqrt{2\mu} - \|A\|\delta}(0)\Big)^c.$$

Hence taking $\delta = C(\mu + \mu^{-1/2}\varepsilon^{1/2})$ from (3.19) we get (3.6).

We finally prove (3.7). The second inclusion in (3.6), the fact that $\|A\|$ is bounded, and $\epsilon \le \tau_0 \mu^2$, yield that $S_\mu(u) \subset B_{C\mu^{1/2}}(x_0)$. Thus by letting $p := Dw(x_0)$, we get

$$\begin{aligned} &\Big|u(x) - \Big(u(x_0) + p \cdot (x - x_0) + \frac{1}{2}\langle M(x - x_0), (x - x_0)\rangle\Big)\Big| \\ &\le |u(x) - w(x)| + |w(x) - w(x_0) - p \cdot (x - x_0) - \frac{1}{2}\langle M(x - x_0), (x - x_0)\rangle| + |w(x_0) - u(x_0)| \\ &\le 2c\epsilon + \|D^3 w\|_{L^\infty(B_{C\mu^{1/2}}(x_0))}|x - x_0|^3 \le C(\epsilon + \mu^{3/2}), \qquad \text{from (3.8)}, \end{aligned}$$

and (3.7) is proved.

□

If $\partial\Omega$ is close to $\partial B_{\sqrt{2}}$, then we can get better estimates for M and $\partial S_\mu(u)$ as follows.

Lemma 3.3. *Suppose $B_{(1-\sigma)\sqrt{2}}(0) \subset \Omega \subset B_{(1+\sigma)\sqrt{2}}(0)$ is a convex domain, $0 < \sigma \le 1/4$. There exist $\mu_0 > 0$, $\tau_0 > 0$ which are independent of σ, a positive definite matrix $M = A^t A$, and $p \in \mathbb{R}^n$ with*

$$\det M = 1, \qquad (1 - C\sigma)I \le M \le (1 + C\sigma)I, \qquad \textit{and} \qquad |p - x_0| \le C\sigma,$$

such that if $u \in C(\bar{\Omega})$ is a strictly convex function in Ω satisfying

$$\begin{cases} 1-\epsilon \le \det D^2 u \le 1+\epsilon & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then for $0 < \mu \le \mu_0$ and $\varepsilon \le \tau_0 \mu^2$ we have

$$B_{\left(1-C(\sigma\mu^{1/2}+\mu^{-1}\epsilon^{1/2})\right)\sqrt{2}}(0) \subset \mu^{-1/2} T S_\mu(u) \subset B_{\left(1+C(\sigma\mu^{1/2}+\mu^{-1}\epsilon^{1/2})\right)\sqrt{2}}(0), \tag{3.20}$$

and

$$\left| u(x) - \left(u(x_0) + p\cdot(x-x_0) + \frac{1}{2}\langle M(x-x_0),(x-x_0)\rangle \right) \right| \le C(\sigma\mu^{3/2}+\varepsilon) \quad \text{in } S_\mu(u), \tag{3.21}$$

where $x_0 \in \Omega$ is the minimum point of u and $Tx := A(x-x_0)$.

Proof. It is similar to that of Lemma 3.2. With the same notation, observe that the inclusions (3.13) and (3.14) still hold. The only difference with Lemma 3.2 is that since now $\partial\Omega \subset N_{\sigma\sqrt{2}}(\partial B_{\sqrt{2}})$, we get the following improvement of (3.16):

$$\partial S_\mu(w, x_0) \subset N_{C\sigma\mu}(\partial \mu^{1/2} E). \tag{3.22}$$

Indeed, as in the proof of Lemma 3.2 it is enough to show that

$$\partial S_\mu(w, x_0) \subset (1 + C\sqrt{\sigma\mu})\sqrt{\mu}E \setminus (1 - C\sqrt{\sigma\mu})\sqrt{\mu}E.$$

But this follows by the same arguments as in Lemma 3.2 except the estimate $|D^3 w(\xi)| \le K$ used there is replaced by $|D^3 w(\xi)| \le C^*\sigma$, which is due to Lemma 3.1.

From (3.13) and (3.14) we get $\partial S_\mu(u) \subset N_{C\mu^{-1/2}\varepsilon^{1/2}}(\partial S_\mu(w, x_0))$. This together with (3.22) gives $\partial S_\mu(u) \subset N_{C(\sigma\mu+\mu^{-1/2}\varepsilon^{1/2})}(\partial\mu^{1/2}E)$ yielding (3.20) as in Lemma 3.2.

As a consequence of the estimate (3.5) in the proof of Lemma 3.1, we get

$$\|D^2 w(x_0) - I\| \le C\sigma \quad \text{and} \quad \|Dw(x_0) - x_0\| \le C\sigma$$

giving the stated estimates for $M := D^2 w(x_0)$ and $p := Dw(x_0)$. Finally thanks to the second inclusion in (3.20), the fact $\|A\|$ is bounded and $\epsilon \le \tau_0\mu^2$, we now have $S_\mu(u) \subset B_{C(\mu^{1/2}+\sigma\mu)}(x_0)$. Therefore the estimate (3.21) follows from the arguments in Lemma 3.2 by using the improvement $\|D^3 w\|_{L^\infty(B_{C\mu^{1/2}}(x_0))} \le C^*\sigma$ obtained from Lemma 3.1. □

3.2. **Convergence of cofactors.** In this subsection we prove a result concerning the convergence of cofactor matrices in L^p when the determinants of their corresponding Hessians converge uniformly. The next lemma is an important ingredient in that proof.

Lemma 3.4. *Let $B_{\frac{6}{5}} \subset \Omega_k \subset B_n$ be a sequence of normalized convex domains converging in the Hausdorff metric to a normalized convex domain $B_{\frac{6}{5}} \subset \Omega \subset B_n$. For each $k \in \mathbb{N}$, let $\phi_k \in C(\bar{\Omega}_k)$ be a convex function satisfying*

$$\begin{cases} 1 - \frac{1}{k} \leq \det D^2\phi_k \leq 1 + \frac{1}{k} & \text{in} \quad \Omega_k \\ \phi_k = 0 & \text{on} \quad \partial\Omega_k. \end{cases}$$

Suppose that $\{\phi_k\}$ converges uniformly on compact subsets of Ω to a convex function $\phi \in C(\bar{\Omega})$ which is a solution of $\det D^2\phi = 1$ in Ω and $\phi = 0$ on $\partial\Omega$. Then for any $0 < p < \infty$, we have

$$\lim_{k\to\infty} \|D^2\phi_k - D^2\phi\|_{L^p(B_1)} = 0.$$

Proof. We first show there exists $0 < \delta < 1$ such that

$$\lim_{k\to\infty} \|D^2\phi_k - D^2\phi\|_{L^\delta(B_1)} = 0. \tag{3.23}$$

Note that (3.23) and Chebyshev's inequality imply that $D^2\phi_k \to D^2\phi$ in measure, and hence up to a subsequence $D^2\phi_k(x) \longrightarrow D^2\phi(x)$ for almost every $x \in B_1$.

Let $\epsilon > 0$ be an arbitrary small constant, and let $\Omega_\epsilon := \{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) > \epsilon\}$. We have $\operatorname{dist}(x, \partial\Omega) = \epsilon$ for all $x \in \partial\Omega_\epsilon$. Moreover $\operatorname{dist}(x, \partial\Omega_k) \longrightarrow \operatorname{dist}(x, \partial\Omega)$ uniformly on the compact set $\partial\Omega_\epsilon$ since $\{\Omega_k\}$ converges to Ω in the Hausdorff metric. Therefore, there is a number $k_\epsilon \in \mathbb{N}$ such that for all $k \geq k_\epsilon$,

$$\operatorname{dist}(x, \partial\Omega_k) \leq 2\epsilon \quad \forall x \in \partial\Omega_\epsilon.$$

Thus by using Aleksandrov's estimate (see [Gut01, Theorem 1.4.2]) we get $-C_n(2\epsilon)^{1/n} \leq \phi_k(x) \leq 0$ and $-C_n\epsilon^{1/n} \leq \phi(x) \leq 0$ for all $x \in \partial\Omega_\epsilon$. Consequently, $\max_{\partial\Omega_\epsilon} |\phi_k - \phi| \leq C_n\epsilon^{1/n}$. This together with the maximum principle (see [Hua09, Lemma 3.1]) gives

$$\|\phi_k - \phi\|_{L^\infty(\Omega_\epsilon)} \leq C_n\epsilon^{1/n} + \frac{c_n}{k}. \tag{3.24}$$

Consider the operator $\mathcal{M}u := (\det D^2u)^{1/n}$ and its linearized operator

$$\hat{L}_u v := \frac{1}{n}(\det D^2u)^{1/n}\operatorname{trace}((D^2u)^{-1}D^2v).$$

Also the linearized operator of $\det D^2u$ is denoted by

$$\mathcal{L}_u v := \operatorname{trace}((\det D^2u)(D^2u)^{-1}D^2v).$$

Let $v_k := \phi_k - \phi$. Since $\mathcal{M}$ is concave, we obtain

$$(1 - \frac{1}{k})^{1/n} - 1 \leq \mathcal{M}\phi_k - \mathcal{M}\phi \leq \hat{L}_\phi v_k \quad \text{in} \quad \Omega_\epsilon.$$

Because $\hat{L}_\phi$ is uniformly elliptic in $B_{\frac{11}{10}}$, by one-sided $W^{2,\delta}$-estimates in [CC95, Lemma 7.8], there exists $0 < \delta_1 < 1$ such that

$$\Big(\frac{1}{|B_1|}\int_{B_1} |(\Delta^2_{he} v_k)^+|^{\delta_1}\Big)^{1/\delta_1} \le C\left\{\|v_k\|_{L^\infty(\Omega_\epsilon)} + \Big[1-(1-\frac{1}{k})^{1/n}\Big]\right\}, \tag{3.25}$$

where

$$\Delta^2_{he} v_k(x) := \frac{v_k(x+he)+v_k(x-he)-2v_k(x)}{h^2}.$$

We also have

$$\hat{L}_{\phi_k} v_k \le \mathcal{M}\phi_k - \mathcal{M}\phi \le \Big(1+\frac{1}{k}\Big)^{1/n} - 1 \quad \text{in} \quad \Omega_\epsilon.$$

It follows that

$$\mathcal{L}_{\phi_k} v_k = n(\det D^2\phi_k)^{\frac{n-1}{n}} \hat{L}_{\phi_k} v_k \le C_n\Big[(1+\frac{1}{k})^{1/n} - 1\Big] \quad \text{in} \quad \Omega_\epsilon.$$

Hence by one-sided $W^{2,\delta}$-estimates in [GT06], there exists $0 < \delta_2 < 1$ such that

$$\Big(\frac{1}{|B_1|}\int_{B_1} |(\Delta^2_{he} v_k)^-|^{\delta_2}\Big)^{1/\delta_2} \le C\left\{\|v_k\|_{L^\infty(\Omega_\epsilon)} + C_n\Big[(1+\frac{1}{k})^{1/n} - 1\Big]\right\}. \tag{3.26}$$

Take $\delta := \min\{\delta_1, \delta_2\}$. Then it follows from (3.25) and (3.26) that

$$\|D^2 v_k\|_{L^\delta(B_1)} \le C\left\{\|v_k\|_{L^\infty(\Omega_\epsilon)} + \Big[1-(1-\frac{1}{k})^{1/n}\Big] + C_n\Big[(1+\frac{1}{k})^{1/n} - 1\Big]\right\}$$

which together with (3.24) gives

$$\|D^2 v_k\|_{L^\delta(B_1)} \le C\left\{C_n\epsilon^{1/n} + \frac{c_n}{k} + \Big[1-(1-\frac{1}{k})^{1/n}\Big] + C_n\Big[(1+\frac{1}{k})^{1/n} - 1\Big]\right\}$$

for $k \ge k_\epsilon$. Thus $\limsup_{k\to\infty} \|D^2 v_k\|_{L^\delta(B_1)} \le C\epsilon^{1/n}$ for all $\epsilon > 0$ small yielding (3.23).

Next, let $\bar{p} > 1$ be such that $\bar{p} > p$. Then by Caffarelli's $W^{2,p}$-estimate (see [Gut01, Theorem 6.4.1]) and the uniform boundedness of $\{D\phi_k\}$ on compact subsets of Ω, there are $k_0 \in \mathbb{N}$ and $C > 0$ depending only on $\bar{p}$ and the dimension n such that

$$\|\phi_k\|_{W^{2,\bar{p}}(B_1)} \le C \qquad \text{for all} \quad k \ge k_0. \tag{3.27}$$

This and the assumption $\phi_k \to \phi$ uniformly on B_1 imply together with (3.23) that there exists a subsequence, still denoted by $\{\phi_k\}$, such that

$$\phi_k \rightharpoonup \phi \quad \text{weakly in } W^{2,\bar{p}}(B_1) \quad \text{and} \quad D^2\phi_k(x) \to D^2\phi(x) \quad \text{for a.e.} \quad x \in B_1.$$

Thus if $\epsilon > 0$, then by Egoroff's theorem there is a measurable set $E \subset B_1$ such that $|E| < \epsilon$ and $D^2\phi_k \to D^2\phi$ uniformly on $B_1 \setminus E$. Consequently,

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{B_1} \|D^2\phi_k - D^2\phi\|^p dx \\
&= \limsup_{k\to\infty} \left[\int_{B_1\setminus E} \|D^2\phi_k - D^2\phi\|^p dx + \int_E \|D^2\phi_k - D^2\phi\|^p dx \right] \\
&= \limsup_{k\to\infty} \int_E \|D^2\phi_k - D^2\phi\|^p dx \le |E|^{\frac{\bar p - p}{\bar p}} \limsup_{k\to\infty} \|D^2\phi_k - D^2\phi\|^p_{L^{\bar p}(B_1)} \le C(n,p,\bar p)\epsilon^{\frac{\bar p - p}{\bar p}},
\end{aligned}$$

where we have used (3.27) in the last inequality. Since $\epsilon > 0$ is arbitrary, we conclude that $D^2\phi_k \to D^2\phi$ in $L^p(B_1)$ for a subsequence. Moreover by the uniqueness of ϕ (since ϕ is the unique convex solution of $\det D^2\phi = 1$ in Ω and $\phi = 0$ on $\partial\Omega$), we infer that in fact the whole sequence $\{D^2\phi_k\}$ converges to $D^2\phi$ in $L^p(B_1)$. □

We are now ready to prove the strong convergence of cofactor matrices in L^p.

Lemma 3.5. *Under the assumptions of Lemma 3.4 we have that*

$$\lim_{k\to\infty} \|\Phi_k - \Phi\|_{L^q(B_1)} = 0,$$

for each $q \ge 1$, where Φ_k and Φ are the cofactor matrices of $D^2\phi_k$ and $D^2\phi$ respectively.

Proof. Since $\det D^2\phi = 1$, we have

$$\Phi_k - \Phi = \Big(1 - \frac{1}{\det D^2\phi_k}\Big)\Phi_k - \frac{1}{\det D^2\phi_k}\Phi_k(D^2\phi_k - D^2\phi)\Phi.$$

Moreover it can be shown that

$$\|\Phi\|_{L^\infty(B_1)} \le C_n \|D^2\phi\|^{n-1}_{L^\infty(B_1)} \quad \text{and} \quad \|\Phi_k\|_{L^p(B_1)} \le C_n \|D^2\phi_k\|^{n-1}_{L^{p(n-1)}(B_1)}$$

for any $p > 0$. If $q \ge 1$ and $r > 1$, with $1/r + 1/r' = 1$, then it follows from the above and Hölder's inequality that

$$\begin{aligned}
\|\Phi_k - \Phi\|_{L^q(B_1)} &\le \frac{1}{k-1}\|\Phi_k\|_{L^q(B_1)} + \frac{k}{k-1}\|\Phi\|_{L^\infty(B_1)}\|\Phi_k(D^2\phi_k - D^2\phi)\|_{L^q(B_1)} \\
&\le C_n\Big(\frac{1}{k-1} + \|D^2\phi\|^{n-1}_{L^\infty(B_1)}\|D^2\phi_k - D^2\phi\|_{L^{qr'}(B_1)}\Big)\|\Phi_k\|_{L^{qr}(B_1)} \\
&\le C_n\Big(\frac{1}{k-1} + \|D^2\phi\|^{n-1}_{L^\infty(B_1)}\|D^2\phi_k - D^2\phi\|_{L^{qr'}(B_1)}\Big)\|D^2\phi_k\|^{n-1}_{L^{qr(n-1)}(B_1)}.
\end{aligned}$$

Let us choose $r = n/(n-1)$ so $r' = n$. Then

$$\|\Phi_k - \Phi\|_{L^q(B_1)} \le C_n\Big(\frac{1}{k-1} + \|D^2\phi\|^{n-1}_{L^\infty(B_1)}\|D^2\phi_k - D^2\phi\|_{L^{qn}(B_1)}\Big)\|D^2\phi_k\|^{n-1}_{L^{qn}(B_1)}$$

which together with (3.27) for $\bar p = qn$ and Lemma 3.4 yields the conclusion of the lemma. □

4. Estimates for the first derivatives

4.1. **An approximation lemma.** We assume below that $\phi, w \in C(\bar{\Omega})$ are convex functions satisfying $\frac{1}{2} \leq \det D^2\phi \leq \frac{3}{2}$, $\det D^2 w = 1$ in Ω and $\phi = w = 0$ on $\partial\Omega$, where $B_{\frac{6}{5}} \subset \Omega \subset B_n$ is a normalized convex domain. Also the cofactor matrix of $D^2\phi$ is denoted by Φ and the cofactor matrix of $D^2 w$ is denoted by W. The next lemma allows us to compare explicitly two solutions originating from two different linearized Monge-Ampère equations.

Lemma 4.1. *Let $\rho^* : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function with $\lim_{\epsilon\to 0^+} \rho^*(\epsilon) = 0$. Suppose $v \in W^{2,n}_{loc}(B_1) \cap C(\bar{B}_1)$ is a solution of $\Phi_{ij}D_{ij}v = f$ in B_1 with $|v| \leq 1$ in B_1, and $h \in W^{2,n}_{loc}(B_1) \cap C(\bar{B}_1)$ is a solution of*

$$\begin{cases} \mathrm{W}_{ij}D_{ij}h &= 0 \qquad \text{in } B_1 \\ \qquad\quad h &= v \qquad \text{on } \partial B_1. \end{cases}$$

Assume that v and h have ρ^* as a modulus of continuity in $\bar{B}_1$. Then for any $0 < \tau < 1$, we have

$$\|v - h\|_{L^\infty(B_{1-\tau})} \leq C_n \left\{\rho^*\Big(\|\Phi - \mathrm{W}\|^{1/2}_{L^n(B_1)}\Big) + \|f\|_{L^n(B_1)}\right\}$$

provided that $\|\Phi - \mathrm{W}\|_{L^n(B_1)} \leq \tau^2$. Here C_n is a positive constant depending only on n.

Proof. Observe that we in fact have $h \in C^\infty(B_1) \cap C(\bar{B}_1)$. Also the maximum principle (see Theorem 2.4) implies that $|h| \leq 1$ in B_1. Define $\epsilon := \|\Phi - \mathrm{W}\|_{L^n(B_1)}$.

For any $x \in \partial B_{1-\delta}$, we can take $y \in \partial B_1$ such that $|x - y| = \delta$. Then since $v - h = 0$ on ∂B_1 and by using the assumption, we get

$$|(v-h)(x)| = |(v-h)(x) - (v-h)(y)| \leq |v(x) - v(y)| + |h(x) - h(y)| \leq 2\rho^*(\delta).$$

We hence conclude that

$$\|v - h\|_{L^\infty(\partial B_{1-\delta})} \leq 2\rho^*(\delta) \quad \forall 0 < \delta < 1. \tag{4.28}$$

We claim for any $0 < \delta < 1$

$$\|D^2 h\|_{L^\infty(B_{1-\delta})} \leq C\delta^{-2}\rho^*(\delta). \tag{4.29}$$

Indeed let $x_0 \in B_{1-\delta}$ be arbitrary and take $x_1 \in \partial B_{\delta/2}(x_0)$. We have

$$\mathrm{W}_{ij}D_{ij}(h - h(x_1)) = \mathrm{W}_{ij}D_{ij}h = 0 \quad \text{in} \quad B_1.$$

Hence we can apply interior C^2-estimate (see Theorem 2.7) to $h - h(x_1)$ in $B_{\delta/2}(x_0) \subset B_1$ and obtain

$$\|D^2 h(x_0)\| \leq C\delta^{-2} \sup_{B_{\delta/2}(x_0)} |h - h(x_1)| \leq C\delta^{-2}\rho^*(\delta)$$

giving (4.29).

Note that $v - h \in W^{2,n}_{loc}(B_1)$ is a solution of

$$\Phi_{ij}D_{ij}(v-h) = f - \Phi_{ij}D_{ij}h = f - [\Phi_{ij} - \mathrm{W}_{ij}]D_{ij}h =: F \quad \text{in} \quad B_1.$$

Hence by the ABP estimate from Theorem 2.4 we have with (4.28) and (4.29)

$$\begin{aligned}
&\|v-h\|_{L^\infty(B_{1-\delta})} \le \|v-h\|_{L^\infty(\partial B_{1-\delta})} + C\|F\|_{L^n(B_{1-\delta})} \\
&\le \|v-h\|_{L^\infty(\partial B_{1-\delta})} + C\|[\Phi_{ij} - \mathrm{W}_{ij}]D_{ij}h\|_{L^n(B_{1-\delta})} + C\|f\|_{L^n(B_1)} \\
&\le \|v-h\|_{L^\infty(\partial B_{1-\delta})} + C\|D^2h\|_{L^\infty(B_{1-\delta})}\|\Phi_{ij} - \mathrm{W}_{ij}\|_{L^n(B_1)} + C\|f\|_{L^n(B_1)} \\
&\le C\rho^*(\delta)(1+\delta^{-2}\epsilon) + C\|f\|_{L^n(B_1)}.
\end{aligned}$$

By taking $\delta = \epsilon^{1/2}$ we obtain $\|v-h\|_{L^\infty(B_{1-\epsilon^{1/2}})} \le C\Big\{\rho^*(\epsilon^{1/2}) + \|f\|_{L^n(B_1)}\Big\}$ and the lemma follows because $\epsilon^{1/2} \le \tau$ by the assumption. □

The main result of this subsection is the following approximation lemma which will play an important role in our proof of the $C^{1,\alpha}$ interior estimate.

Lemma 4.2. *Let $\rho : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function with $\lim_{\epsilon\to 0^+}\rho(\epsilon) = 0$. Let $B_{\frac{6}{5}} \subset \Omega \subset B_n$ be a normalized convex domain. Let $\varphi \in C(\partial B_1)$ have ρ as a modulus of continuity on ∂B_1 and satisfy $\|\varphi\|_{L^\infty(\partial B_1)} \le K$ for some positive constant K.*

Then, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n, \rho, K) > 0$ such that if

$$\begin{cases} 1-\delta \le \det D^2\phi \le 1+\delta & \text{in} \quad \Omega \\ \phi = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad \text{and} \qquad \|f\|_{L^n(B_1)} \le \delta$$

then any two solutions v and h in $W^{2,n}_{loc}(B_1)$ of, respectively,

$$\begin{cases} \mathcal{L}_\phi v = f & \text{in} \quad B_1, \\ v = \varphi & \text{on} \quad \partial B_1 \end{cases} \qquad \text{and} \qquad \begin{cases} \mathcal{L}_w h = 0 & \text{in} \quad B_1, \\ h = \varphi & \text{on} \quad \partial B_1 \end{cases}$$

satisfy

$$\|v-h\|_{L^\infty(B_1)} \le \epsilon.$$

Proof. Suppose by contradiction that it is not true. Then there exist $\epsilon > 0$, $n \in \mathbb{N}$, ρ, $K > 0$, a sequence of normalized convex domains $B_{\frac{6}{5}} \subset \Omega_k \subset B_n$, $\varphi_k \in C(\partial B_1)$ has ρ as a modulus of continuity with $\|\varphi_k\|_{L^\infty(\partial B_1)} \le K$, f_k with $\|f_k\|_{L^n(B_1)} \le \frac{1}{k}$ and a sequence of convex functions $\phi_k, w_k \in C(\bar{\Omega}_k)$ with

$$\begin{cases} 1-\frac{1}{k} \le \det D^2\phi_k \le 1+\frac{1}{k} & \text{in} \quad \Omega_k \\ \phi_k = 0 & \text{on} \quad \partial\Omega_k \end{cases} \qquad \text{and} \qquad \begin{cases} \det D^2 w_k = 1 & \text{in} \quad \Omega_k \\ w_k = 0 & \text{on} \quad \partial\Omega_k \end{cases}$$

for which there are solutions v_k and h_k in $W^{2,n}_{loc}(B_1)$ of

$$\begin{cases} \mathcal{L}_{\phi_k} v_k = f_k & \text{in} \quad B_1, \\ v_k = \varphi_k & \text{on} \quad \partial B_1 \end{cases} \qquad \text{and} \qquad \begin{cases} \mathcal{L}_{w_k} h_k = 0 & \text{in} \quad B_1, \\ h_k = \varphi_k & \text{on} \quad \partial B_1 \end{cases}$$

such that

$$\|v_k - h_k\|_{L^\infty(B_1)} \geq \epsilon \quad \text{for all} \quad k. \tag{4.30}$$

By Blaschke selection theorem, there exists a subsequence of Ω_k, still denoted by Ω_k, such that Ω_k converges in the Hausdorff metric to a normalized convex domain $B_{\frac{6}{5}} \subset \Omega \subset B_n$. Also by [Gut01, Lemma 5.3.1] we have up to a subsequence $\phi_k \to \phi$ and $w_k \to w$ uniformly on compact subsets of Ω, where $\phi, w \in C(\bar{\Omega})$ are both convex solutions to the equation

$$\begin{cases} \det D^2 w = 1 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Thus $\phi \equiv w$ by the uniqueness of convex solutions to the Monge-Ampère equation.

Since all φ_k's have the same modulus of continuity ρ on ∂B_1, $\|\varphi_k\|_{L^\infty(\partial B_1)} \leq K$ for all k and $\|f_k\|_{L^n(B_1)} \to 0$ as $k \to \infty$, we can use Lemma 4.3 below to conclude that there exists a nondecreasing continuous function ρ^* in $(0, \infty)$ with $\lim_{\epsilon \to 0^+} \rho^*(\epsilon) = 0$ depending only on n, K and ρ such that for all k

$$|v_k(x) - v_k(y)| \leq \rho^*(|x - y|) \quad \text{for all} \quad x, y \in \bar{B}_1. \tag{4.31}$$

Hence $\{v_k\}$ is an equicontinuous (and uniformly bounded by the ABP estimate) sequence of functions in $\bar{B}_1$. Therefore, by taking a subsequence, we may assume that $v_k \to v_\infty$ uniformly on $\bar{B}_1$ as $k \to \infty$ for some function $v_\infty \in C(\bar{B}_1)$.

Similarly to v_k, by taking a further subsequence, we may assume that

$$h_k \to h_\infty \quad \text{uniformly on} \quad \bar{B}_1 \quad \text{as} \quad k \to \infty$$

for some function $h_\infty \in C(\bar{B}_1)$.

Next we show that $v_\infty \equiv h_\infty$ in B_1. Indeed, it follows from Lemma 3.5 that $\Phi_k \longrightarrow \Phi$ in $L^n(B_1)$ and $\mathrm{W}_k \longrightarrow \mathrm{W}$ in $L^n(B_1)$, where W_k is the cofactor matrix of $D^2 w_k$ and W is the cofactor matrix of $D^2 w$. Since $w \equiv \phi$, this implies that

$$\lim_{k \to \infty} \|\Phi_k - \mathrm{W}_k\|_{L^n(B_1)} = 0. \tag{4.32}$$

Therefore, for any $0 < \tau < 1$ we infer from Lemma 4.1 that

$$\|v_k - h_k\|_{L^\infty(B_{1-\tau})} \leq C_n \left\{ \rho^*\Big(\|\Phi_k - \mathrm{W}_k\|^{1/2}_{L^n(B_1)} \Big) + \|f_k\|_{L^n(B_1)} \right\}$$

for all k sufficiently large. This together with (4.32) and the fact $\|f_k\|_{L^n(B_1)} \to 0$ yields $v_\infty \equiv h_\infty$ in $B_{1-\tau}$. Due to the arbitrariness of $0 < \tau < 1$ we then conclude that $v_\infty \equiv h_\infty$ in B_1, which is a contradiction with (4.30). □

In the proof above we used the following lemma which is a simple modification of [CC95, Proposition 4.14]. We include a proof here for the shake of completeness.

Lemma 4.3. *Let $\rho : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function with $\lim_{\epsilon\to 0^+}\rho(\epsilon) = 0$. Let $B_1 \subset \Omega \subset B_n$ be a normalized convex domain and $u \in W^{2,n}_{loc}(B_1) \cap C(\bar{B}_1)$ be a solution to $\mathcal{L}_\phi u = f$ in B_1 for some convex function $\phi \in C(\bar{\Omega})$ satisfying*

$$\begin{cases} \frac{1}{2} \le \det D^2\phi \le \frac{3}{2} & \text{in} \quad \Omega \\ \phi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Assume that $\varphi := u|_{\partial B_1}$ has ρ as a modulus of continuity on ∂B_1 and K is a positive constant such that $\|\varphi\|_{L^\infty(\partial B_1)} \le K$ and $\|f\|_{L^n(B_1)} \le K$.

Then there exists a nondecreasing continuous function ρ^* in $(0,\infty)$ with $\lim_{\epsilon\to 0^+}\rho^*(\epsilon) = 0$ and depending only on n, K and ρ such that

$$|u(x) - u(y)| \le \rho^*(|x-y|) \quad \text{for all} \quad x, y \in \bar{B}_1.$$

Proof. Let $\epsilon > 0$. We need to prove that

$$|u(x) - u(y)| \le \epsilon \quad \text{for any} \quad x, y \in \bar{B}_1 \quad \text{satisfying} \quad |x-y| \le \delta,$$

where δ depends only ϵ, n, K and ρ. By the interior Hölder estimate (2.3), it is enough to bound $|u(x) - u(x_0)|$ for $x \in B_1$ and $x_0 \in \partial B_1$. Hence let us fix $x_0 \in \partial B_1$; we may assume that $B_1 = B_1((0,\dots,0,1))$ and $x_0 = 0 \in \partial B_1$.

We now take $\delta_1 > 0$ depending only on ϵ and ρ such that

$$|u(x) - u(0)| = |\varphi(x) - \varphi(0)| \le \rho(|x|) \le \epsilon \tag{4.33}$$

for any $x \in \partial B_1$ satisfying $|x| \le \delta_1$. Note that, by the ABP estimate from Theorem 2.4 and the assumption $\det D^2\phi \ge \frac{1}{2}$,

$$\begin{aligned} |u(x) - u(0) \pm \epsilon| &\le 2\sup_{B_1}|u| + \epsilon \le 2\|\varphi\|_{L^\infty(\partial B_1)} + C_n \operatorname{diam}(B_1)\|f\|_{L^n(B_1)} + \epsilon \\ &\le 2K + C_nK + \epsilon =: C_1 \qquad \forall x \in B_1. \end{aligned} \tag{4.34}$$

We consider the functions

$$h_\pm(x) := u(x) - u(0) \pm \epsilon \pm C_1\big(\inf\{y_n : \ y \in \bar{B}_1 \cap \partial B_{\delta_2}(0)\}\big)^{-1} x_n$$

in the region $A := B_1 \cap B_{\delta_2}(0)$, where $\delta_2 \le \delta_1$ will be chosen later. It then follows from (4.33) and (4.34) that

$$h_- \le 0 \quad \text{on} \quad \partial A \quad \text{and} \quad h_+ \ge 0 \quad \text{on} \quad \partial A.$$

Moreover $\mathcal{L}_\phi h_\pm = \mathcal{L}_\phi u = f$ in A. Hence the ABP estimate (applied in A) gives

$$h_- \le C_n \mathrm{diam}\,(A)\|f\|_{L^n(A)} \le C_n K\delta_2 \qquad \text{in} \quad A,$$
$$h_+ \ge -C_n \mathrm{diam}\,(A)\|f\|_{L^n(A)} \ge -C_n K\delta_2 \quad \text{in} \quad A.$$

If we take $\delta_2 \le \delta_1$ such that $C_n K\delta_2 \le \epsilon$, we conclude that

$$|u(x) - u(0)| \le 2\epsilon + C_1\big(\inf\{y_n : \ y \in \bar{B}_1 \cap \partial B_{\delta_2}(0)\}\big)^{-1} x_n \quad \text{for all} \quad x \in A.$$

Note that

$$\inf\{y_n : \ y \in \bar{B}_1 \cap \partial B_{\delta_2}(0)\} = \inf\{y_n : \ y \in \partial B_1, \ |y| = \delta_2\} = \frac{\delta_2^2}{2}.$$

It follows that

$$|u(x) - u(0)| \le 2\epsilon + \frac{2C_1}{\delta_2^2} x_n = 2\epsilon + Cx_n \quad \text{for all} \quad x \in A = B_1 \cap B_{\delta_2}(0),$$

for a constant C depending only on ϵ, n, K and ρ. Hence

$$|u(x) - u(0)| \le 3\epsilon \quad \text{for all} \quad x \in B_1 \cap B_\delta(0),$$

for $\delta := \min\{\frac{\epsilon}{C}, \delta_2\}$. □

Remark 4.4. For the purpose of proving $C^{1,\alpha}$ interior estimates in the next subsection, one could take the function h in Lemma 4.1 and Lemma 4.2 to be a solution of $\mathcal{L}_w h = 0$ in $B_{3/4}$ and $h = v$ on $\partial B_{3/4}$ (v is still a solution of $\mathcal{L}_\phi v = f$ in B_1). Although in this case one can only conclude that v is close to the good function h in the supremum norm on $B_{1/2}$, one can avoid the direct dependence on ρ^* and ρ in Lemma 4.1 and Lemma 4.2 by using the fact that both v and h have Hölder modulus of continuity in $B_{3/4}$ which is a consequence of Corollary 2.6. However our statements of Lemma 4.1 and Lemma 4.2 are more intrinsic and might be useful for some other purposes.

4.2. **Interior $C^{1,\alpha}$ estimate for the solution.** We are ready to prove the two main theorems of this paper. The first result requires $\det D^2\phi$ is near 1, but no continuity of $\det D^2\phi$ is needed. The second result is obtained as a direct consequence of the first one but holds for general $\lambda \le \det D^2\phi \le \Lambda$ provided that $\det D^2\phi$ is continuous. Recall that $S_t(\phi)$ denotes the section of the convex function ϕ at its minimum point. Also for convenience, we assume the minimum point of ϕ is the origin in the next theorem.

Theorem 4.5. *Assume that $0 < \alpha' < \alpha < 1$, $r_0 > 0$ and $C_1 > 0$. Then there exists $\theta = \theta(n, \alpha, \alpha', r_0) > 0$ such that if $B_1(0) \subset \Omega \subset B_n(0)$ is a normalized convex domain, $\phi \in C(\bar{\Omega})$ is a convex solution of*

$$\begin{cases} 1 - \theta \le \det D^2\phi & \le 1 + \theta \ \textit{in}\ \Omega \\ \phi & = 0 \quad \textit{on}\ \partial\Omega \end{cases}$$

and

$$\left(\frac{1}{|S_r(\phi)|}\int_{S_r(\phi)}|f|^n\,dx\right)^{\frac{1}{n}} \le C_1 r^{\frac{\alpha-1}{2}} \quad \textit{for all} \quad S_r(\phi)\Subset\Omega \textit{ with } r\le r_0,$$

then any solution $u\in W^{2,n}_{loc}(\Omega)$ of $\mathcal{L}_\phi u = f$ in Ω is $C^{1,\alpha'}$ at the minimum point (the origin) of ϕ. More precisely, there is an affine function $l(x)$ such that

$$r^{-(1+\alpha')}\|u-l\|_{L^\infty(B_r(0))} + |l(0)| + \|Dl(0)\| \le C\Big\{\|u\|_{L^\infty(\Omega)} + C_1\Big\} \qquad \forall r\le \mu^*,$$

where C and μ^* are positive constants depending only on n, α, α' and r_0.

Proof.
1. Let $K := \|u\|_{L^\infty(\Omega)} + \theta^{-1}C_1$. We consider $v(x) := \frac{u(x)}{K}$ and ϕ unchanged. Then $\mathcal{L}_\phi v(x) = f^*(x) := \frac{f(x)}{K}$ in Ω, and $\|v\|_{L^\infty(\Omega)}\le 1$. Moreover

$$\begin{aligned}\left(\frac{1}{|S_r(\phi)|}\int_{S_r(\phi)}|f^*(x)|^n dx\right)^{\frac{1}{n}} &= \frac{1}{K}\left(\frac{1}{|S_r(\phi)|}\int_{S_r(\phi)}|f(x)|^n dx\right)^{\frac{1}{n}}\\ &\le \frac{C_1 r^{\frac{\alpha-1}{2}}}{K} \le \frac{C_1 r^{\frac{\alpha-1}{2}}}{\theta^{-1}C_1} = \theta r^{\frac{\alpha-1}{2}} \quad \forall S_r(\phi)\Subset\Omega \text{ with } r\le r_0.\end{aligned}$$

It follows that we may (and do) assume that

$$\begin{cases} 1-\theta \le \det D^2\phi \le 1+\theta \text{ in } \Omega\\ \qquad\qquad \phi = 0 \quad \text{on } \partial\Omega,\end{cases}$$

$$\Big(\frac{1}{|S_r(\phi)|}\int_{S_r(\phi)}|f|^n dx\Big)^{\frac{1}{n}} \le \theta r^{\frac{\alpha-1}{2}} \quad \text{for all} \quad S_r(\phi)\Subset\Omega \text{ with } r\le r_0,$$

$$\mathcal{L}_\phi u(x) = f(x) \quad \text{in} \quad \Omega \qquad \text{and} \qquad \|u\|_{L^\infty(\Omega)}\le 1.$$

We need to prove that

$$\sup_{0<r\le\mu^*}\Big(r^{-(1+\alpha')}\|u-l\|_{L^\infty(B_r(0))}\Big) + |l(0)| + \|Dl(0)\| \le C \tag{4.35}$$

for an affine function $l(x)$, with θ, μ^* and C depending only on n, α, α' and r_0.
2. Claim: There exist $0<\mu<1$ depending only on n, α and r_0, a sequence of positive definite matrices A_k with $\det A_k = 1$ and a sequence of affine functions $l_k(x) = a_k + b_k\cdot x$ such that for all $k = 1,2,3,\dots$

(1) $\|A_{k-1}A_k^{-1}\| \le \frac{1}{\sqrt{c_1}}$, $\quad \|A_k\| \le \sqrt{c_2(1+C\delta_0)(1+C\delta_1)\cdots(1+C\delta_{k-1})}$;
(2) $B_{(1-\delta_k)\sqrt{2}}(0) \subset \mu^{\frac{-k}{2}}A_k S_{\mu^k}(\phi) \subset B_{(1+\delta_k)\sqrt{2}}(0)$;
(3) $\|u-l_{k-1}\|_{L^\infty(S_{\mu^k}(\phi))} \le \mu^{\frac{k-1}{2}(1+\alpha)}$;
(4) $|a_k - a_{k-1}| + \mu^{\frac{k}{2}}\|(A_k^{-1})^t\cdot(b_k-b_{k-1})\| \le 2c_e\mu^{\frac{k-1}{2}(1+\alpha)}$;

(5) $\dfrac{\left|(u-l_{k-1})(\mu^{\frac{k}{2}}A_k^{-1}x)-(u-l_{k-1})(\mu^{\frac{k}{2}}A_k^{-1}y)\right|}{\mu^{\frac{k-1}{2}(1+\alpha)}} \leq 2C^*(\sqrt{c_1\mu})^{-\beta}|x-y|^\beta$, for all points $x, y \in \mu^{\frac{-k}{2}}A_kS_{\mu^k}(\phi)$,

where

$$A_0 := I, \quad l_0(x) := 0, \quad \delta_0 := 0; \quad \delta_1 := C(\mu^{1/2}+\mu^{-1}\theta^{1/2}) < 1-\frac{6}{5\sqrt{2}}, \quad \text{and}$$
$$\delta_k := C(\delta_{k-1}\mu^{1/2}+\mu^{-1}\theta^{1/2}) \quad \text{for } k \geq 2.$$

Also C^*, C, c_e, c_1, c_2 and β are universal constants (c_e is the constant in Lemma 2.7; C^* and β are the constants given in the local Hölder estimate (2.2); c_1 and c_2 are given by Lemma 3.2 and C is given by Lemma 3.3).

3. Proof of the claim:

Let $\mu_0 > 0$ and $\tau_0 > 0$ be the universal small constants given by Lemma 3.2. Let $0 < \mu \leq \mu_0$ be fixed such that $\mu \leq r_0$, $(2\mu)^{\frac{1+\alpha}{2}} \leq 1$, $C_2\sqrt{3\mu} \leq 1/2$ and $6c_eC_2^2\mu^{\frac{1-\alpha}{2}} \leq 1$, where C_2 is the universal constant in Pogorelov's estimate (3.10). Let us next determine the constant θ. We take $\theta := \min\{\frac{1}{C_n}, \tau_0\mu^2, \delta\}$, where C_n is the n root of the constant C' in Lemma 2.3 corresponding to $\lambda = 1/2$ and $\Lambda = 3/2$ and δ is the constant in Lemma 4.2 corresponding to $\rho(s) = 2C^*(\sqrt{c_1\mu})^{-\beta}s^\beta$, $K = 1$ and $\epsilon = 3c_eC_2^2\mu$. By taking θ even smaller if necessary, we assume that $\delta_1 = C(\mu^{1/2}+\mu^{-1}\theta^{1/2}) < 1-\frac{6}{5\sqrt{2}}$.

k=1: Applying Lemma 3.2 we obtain a positive definite matrix $M = A^tA$ with $\det A = \det M = 1$, $c_1I \leq M \leq c_2I$ such that if we take $A_1 := A$ then

$$B_{(1-\delta_1)\sqrt{2}}(0) \subset \mu^{\frac{-1}{2}}A_1S_\mu(\phi) \subset B_{(1+\delta_1)\sqrt{2}}(0), \quad \text{with} \quad \delta_1 := C(\mu^{1/2}+\mu^{-1}\theta^{1/2}).$$

Then (1) and (2) hold obviously since $\|A_1^{-1}\| \leq 1/\sqrt{c_1}$ and $\|A_1\| \leq \sqrt{c_2}$. Also (3) is satisfied as $l_0 \equiv 0$ and $\|u\|_{L^\infty(\Omega)} \leq 1$. We next verify (5). By the Hölder estimate (2.2),

$$|u(x_1)-u(y_1)| \leq C^*\mu^{-\beta}|x_1-y_1|^\beta\Big\{\|u\|_{L^\infty(S_{2\mu}(\phi))}+(2\mu)^{\frac{1}{2}}\|f\|_{L^n(S_{2\mu}(\phi))}\Big\} \quad \forall x_1, y_1 \in S_\mu(\phi).$$

Hence for any $x, y \in \mu^{\frac{-1}{2}}A_1S_\mu(\phi)$, by taking $x_1 := \mu^{\frac{1}{2}}A_1^{-1}x \in S_\mu(\phi)$ and $y_1 := \mu^{\frac{1}{2}}A_1^{-1}y \in S_\mu(\phi)$ we obtain

$$\begin{aligned}(4.36)\qquad & |u(\mu^{\frac{1}{2}}A_1^{-1}x)-u(\mu^{\frac{1}{2}}A_1^{-1}y)| \\ &\leq C^*\mu^{-\beta}|\mu^{\frac{1}{2}}A_1^{-1}(x-y)|^\beta\Big\{\|u\|_{L^\infty(S_{2\mu}(\phi))}+(2\mu)^{\frac{1}{2}}\|f\|_{L^n(S_{2\mu}(\phi))}\Big\} \\ &\leq C^*\mu^{-\beta}(\mu^{\frac{1}{2}}\|A_1^{-1}\|)^\beta|x-y|^\beta\Big\{1+C_n\theta(2\mu)^{\frac{1+\alpha}{2}}\Big\} \leq 2C^*(\sqrt{c_1\mu})^{-\beta}|x-y|^\beta\end{aligned}$$

giving (5) for $k = 1$ as desired.

k=2: We first construct l_1 and verify (3) for $k = 2$ and (4) for $k = 1$. Then we construct A_2 and verify (1), (2) and (5) for $k = 2$.

+ Constructing $l_1(x)$: Recall that $D\phi(0) = 0$ since the origin is the minimum point of ϕ. Hence $S_\mu(\phi) = \{y \in \Omega : \ \phi(y) - \phi(0) - \mu \le 0\}$. Let

$$\phi^*(y) := \frac{1}{\mu}[\phi(\mu^{\frac{1}{2}}A_1^{-1}y) - \phi(0) - \mu], \qquad y \in \Omega_1^*$$
$$v(y) := (u - l_0)(\mu^{\frac{1}{2}}A_1^{-1}y) = u(\mu^{\frac{1}{2}}A_1^{-1}y), \quad y \in \Omega_1^*,$$

where $\Omega_1^* := \mu^{\frac{-1}{2}}A_1 S_\mu(\phi)$. Then as $\det A_1 = 1$ and since

$$D^2\phi^*(y) = (A_1^{-1})^t D^2\phi(\mu^{\frac{1}{2}}A_1^{-1}y)A_1^{-1} \quad \text{and} \quad D^2 v(y) = \mu\ (A_1^{-1})^t D^2 u(\mu^{\frac{1}{2}}A_1^{-1}y)A_1^{-1},$$

we get

$$\begin{cases} 1 - \theta \le \det D^2\phi^* \le 1 + \theta & \text{in} \quad \Omega_1^* \\ \phi^* = 0 & \text{on} \quad \partial\Omega_1^* \end{cases}$$

and

$$\Phi^*(y) := \det D^2\phi^*(y)\ (D^2\phi^*(y))^{-1} = \frac{\det D^2\phi(\mu^{\frac{1}{2}}A_1^{-1}y)}{(\det A_1)^2} A_1 (D^2\phi(\mu^{\frac{1}{2}}A_1^{-1}y))^{-1} A_1^t$$
$$= A_1 \Phi(\mu^{\frac{1}{2}}A_1^{-1}y)A_1^t.$$

Consequently,

$$\mathcal{L}_{\phi^*} v(y) = \text{trace}(\Phi^*(y)D^2 v(y)) = \mu\ \text{trace}\Big(A_1\Phi(\mu^{\frac{1}{2}}A_1^{-1}y)A_1^t(A_1^{-1})^t D^2 u(\mu^{\frac{1}{2}}A_1^{-1}y)A_1^{-1}\Big)$$
$$= \mu\ \text{trace}\Big(\Phi(\mu^{\frac{1}{2}}A_1^{-1}y)D^2 u(\mu^{\frac{1}{2}}A_1^{-1}y)\Big)$$
$$= \mu\ \mathcal{L}_\phi u(\mu^{\frac{1}{2}}A_1^{-1}y) = \mu\ f(\mu^{\frac{1}{2}}A_1^{-1}y) =: \tilde{f}(y) \qquad \text{in} \quad \Omega_1^*.$$

Notice that

$$\left(\frac{1}{|\Omega_1^*|}\int_{\Omega_1^*} |\tilde{f}(y)|^n dy\right)^{\frac{1}{n}} = \mu\left(\frac{1}{|\mu^{\frac{-1}{2}}A_1 S_\mu(\phi)|}\int_{\mu^{\frac{-1}{2}}A_1 S_\mu(\phi)} |f(\mu^{\frac{1}{2}}A_1^{-1}y)|^n dy\right)^{\frac{1}{n}}$$
$$= \mu\ \left(\frac{1}{|S_\mu(\phi)|}\int_{S_\mu(\phi)} |f(x)|^n dx\right)^{\frac{1}{n}} \le \mu\theta\mu^{\frac{\alpha-1}{2}} = \theta\mu^{\frac{1+\alpha}{2}} \le \theta.$$

We apply Lemma 4.2 with $\phi \rightsquigarrow \phi^*$, $f \rightsquigarrow \tilde{f}$, $\Omega \rightsquigarrow \Omega_1^*$ and $\varphi := v|_{\partial B_1}$. Note that by (3) we have $\|\varphi\|_{L^\infty(\partial B_1)} \le 1$, and also φ has modulus of continuity $\rho(s) = 2C^*(\sqrt{c_1\mu})^{-\beta}s^\beta$ by (5) for $k = 1$. Recall that $\theta \le \delta$, where δ is the constant in Lemma 4.2 corresponding to $\epsilon := 3c_e C_2^2\mu$. Hence if h is the solution of

$$\begin{cases} \mathcal{L}_w h = 0 & \text{in} \quad B_1 \\ h = \varphi = v & \text{on} \quad \partial B_1 \end{cases} \quad \text{where} \quad \begin{cases} \det D^2 w = 1 & \text{in} \quad \Omega_1^* \\ w = 0 & \text{on} \quad \partial\Omega_1^*, \end{cases}$$

then

$$\|v - h\|_{L^\infty(B_1)} \le \epsilon := 3c_e C_2^2\mu.$$

We have $\|h\|_{L^\infty(B_1)} \le 1$ by the maximum principle. Moreover, it follows from the formulas (3.13) and (3.15) that

$$S_{2\mu}(\phi^*) \subset B_{C_2\sqrt{2\mu+C_3\theta^{1/2}}}(0) \subset B_{C_2\sqrt{3\mu}}(0).$$

Thus by letting $\bar{l}(y) := h(0)+Dh(0)\cdot y$ and applying Theorem 2.7 (recall that $C_2\sqrt{3\mu} \le \frac{1}{2}$), we get

$$\|h-\bar{l}\|_{L^\infty(S_{2\mu}(\phi^*))} \le \|h-\bar{l}\|_{L^\infty(B_{C_2\sqrt{3\mu}}(0))} \le 3c_eC_2^2\mu.$$

Therefore,

$$\|v-\bar{l}\|_{L^\infty(S_{2\mu}(\phi^*))} \le \|v-h\|_{L^\infty(S_{2\mu}(\phi^*))} + \|h-\bar{l}\|_{L^\infty(S_{2\mu}(\phi^*))} \le 6c_eC_2^2\mu \le \mu^{\frac{1}{2}(1+\alpha)}. \tag{4.37}$$

Define

$$l_1(x) := l_0(x) + \bar{l}(\mu^{\frac{-1}{2}}A_1x). \tag{4.38}$$

Then since $S_\mu(\phi^*) = \mu^{\frac{-1}{2}}A_1S_{\mu^2}(\phi)$, we obtain from (4.37) for $x \in S_{\mu^2}(\phi)$ that

$$\begin{aligned}|u(x)-l_1(x)| &= |(u-l_0)(x) - \bar{l}(\mu^{\frac{-1}{2}}A_1x)| = |v(\mu^{\frac{-1}{2}}A_1x) - \bar{l}(\mu^{\frac{-1}{2}}A_1x)|\\ &\le \|v-\bar{l}\|_{L^\infty(S_\mu(\phi^*))} \le \mu^{\frac{1}{2}(1+\alpha)}.\end{aligned}$$

Thus (3) for $k = 2$ is verified. Also (4) for $k = 1$ holds because it follows from the definition (4.38) and the definition of $\bar{l}$ that $a_1 = a_0 + h(0)$ and $b_1 = b_0 + \mu^{\frac{-1}{2}}A_1^tDh(0)$. Hence by using Theorem 2.7, we get

$$|a_1-a_0| + \mu^{\frac{1}{2}}\|(A_1^{-1})^t\cdot(b_1-b_0)\| = |h(0)| + \|Dh(0)\| \le 2c_e$$

giving (4) for $k = 1$.

+ Constructing A_2: Applying Lemma 3.3 for ϕ^* and Ω_1^* we obtain a positive definite matrix $M = A^tA$ with $\det M = 1$, $(1-C\delta_1)I \le M \le (1+C\delta_1)I$ such that

$$B_{(1-\delta_2)\sqrt{2}}(0) \subset \mu^{\frac{-1}{2}}AS_\mu(\phi^*) \subset B_{(1+\delta_2)\sqrt{2}}(0), \quad \text{with} \quad \delta_2 := C(\delta_1\mu^{1/2} + \mu^{-1}\theta^{1/2}).$$

Define $A_2 := AA_1$ which implies in particular that A_2 is a positive definite matrix with $\det A_2 = 1$. Then as $S_\mu(\phi^*) = \mu^{\frac{-1}{2}}A_1S_{\mu^2}(\phi)$ we conclude that

$$B_{(1-\delta_2)\sqrt{2}}(0) \subset \mu^{-1}A_2S_{\mu^2}(\phi) \subset B_{(1+\delta_2)\sqrt{2}}(0).$$

Thus (2) and the first part of (1) for $k = 2$ hold obviously since $A_1A_2^{-1} = A^{-1}$ and $\|A^{-1}\| \le \frac{1}{\sqrt{1-C\delta_1}} \le \frac{1}{\sqrt{c_1}}$. Next observe from the definition of A that $(1-C\delta_1)|x|^2 \le |Ax|^2 \le (1+C\delta_1)|x|^2$. Hence

$$|A_2x|^2 = |AA_1x|^2 \le (1+C\delta_1)|A_1x|^2 \le c_2(1+C\delta_1)|x|^2$$

yielding the second part of (1), i.e., $\|A_2\| \leq \sqrt{c_2(1+C\delta_1)}$. It remains to verify (5) for $k=2$. Using the definitions of v and l_1 we have

$$\begin{aligned}(u-l_1)(\mu A_2^{-1}x) &= (u-l_0)(\mu A_2^{-1}x) - \bar{l}(\mu^{\frac{1}{2}}A_1A_2^{-1}x) = v(\mu^{\frac{1}{2}}A_1A_2^{-1}x) - \bar{l}(\mu^{\frac{1}{2}}A_1A_2^{-1}x)\\ &= (v-\bar{l})(\mu^{\frac{1}{2}}A^{-1}x) \qquad\qquad \text{for all} \quad x \in \mu^{-1}A_2S_{\mu^2}(\phi).\end{aligned}$$

Moreover since $\mathcal{L}_{\phi^*}(v-\bar{l})(y) = \mathcal{L}_{\phi^*}v(y) = \tilde{f}(y) = \mu\, f(\mu^{\frac{1}{2}}A_1^{-1}y)$ for $y \in \Omega_1^*$ and

$$\begin{aligned}\|\tilde{f}\|_{L^n(S_{2\mu}(\phi^*))} &= \mu\Big(\int_{\mu^{\frac{-1}{2}}A_1S_{2\mu^2}(\phi)} |f(\mu^{\frac{1}{2}}A_1^{-1}y)|^n\, dy\Big)^{\frac{1}{n}} = \mu^{\frac{1}{2}}\Big(\int_{S_{2\mu^2}(\phi)} |f(x)|^n\, dx\Big)^{\frac{1}{n}}\\ &\leq C_n\theta\mu^{\frac{1}{2}(1+\alpha)}(2\mu)^{\frac{\alpha}{2}},\end{aligned}$$

we get from the Hölder estimate (2.2) and (4.37) that

$$\begin{aligned}|(v-\bar{l})(x_1) - (v-\bar{l})(y_1)| &\leq C^*\mu^{-\beta}|x_1-y_1|^\beta\Big\{\|v-\bar{l}\|_{L^\infty(S_{2\mu}(\phi^*))} + (2\mu)^{\frac{1}{2}}\|\tilde{f}\|_{L^n(S_{2\mu}(\phi^*))}\Big\}\\ &\leq C^*\mu^{-\beta}|x_1-y_1|^\beta\mu^{\frac{1}{2}(1+\alpha)}\Big\{1 + C_n\theta(2\mu)^{\frac{1+\alpha}{2}}\Big\}\\ &\leq 2C^*\mu^{-\beta}|x_1-y_1|^\beta\mu^{\frac{1}{2}(1+\alpha)} \qquad\qquad \forall x_1, y_1 \in S_\mu(\phi^*).\end{aligned}$$

Therefore for any $x, y \in \mu^{-1}A_2S_{\mu^2}(\phi)$, by taking $x_1 := \mu^{\frac{1}{2}}A^{-1}x = \mu^{\frac{1}{2}}A_1A_2^{-1}x \in \mu^{\frac{-1}{2}}A_1S_{\mu^2}(\phi) = S_\mu(\phi^*)$ and $y_1 := \mu^{\frac{1}{2}}A^{-1}y \in S_\mu(\phi^*)$ we obtain

$$\begin{aligned}&\frac{\big|(u-l_1)(\mu A_2^{-1}x) - (u-l_1)(\mu A_2^{-1}y)\big|}{\mu^{\frac{1}{2}(1+\alpha)}} = \frac{\big|(v-\bar{l})(\mu^{\frac{1}{2}}A^{-1}x) - (v-\bar{l})(\mu^{\frac{1}{2}}A^{-1}y)\big|}{\mu^{\frac{1}{2}(1+\alpha)}}\\ &\leq 2C^*\mu^{-\beta}|\mu^{\frac{1}{2}}A^{-1}(x-y)|^\beta \leq 2C^*\mu^{-\beta}(\mu^{\frac{1}{2}}\|A^{-1}\|)^\beta|x-y|^\beta \leq 2C^*(\sqrt{c_1\mu})^{-\beta}|x-y|^\beta\end{aligned}$$

giving (5) for $k=2$ as desired.

Suppose the claim holds up to $\mathbf{k = i \geq 2}$ and we want to prove that it also holds for $\mathbf{k = i+1}$. We first construct $l_i(x)$ and verify (3) for $k = i+1$ and (4) for $k = i$. Then we construct A_{i+1} and verify (1), (2) and (5) for $k = i+1$. It follows from the hypothesis the claim holds up to $k = i$ that

$$B_{(1-\delta_i)\sqrt{2}}(0) \subset \mu^{\frac{-i}{2}}A_iS_{\mu^i}(\phi) \subset B_{(1+\delta_i)\sqrt{2}}(0),$$

where A_i is a positive definite matrix with $\det A_i = 1$ and

$$|A_ix|^2 \leq c_2(1-C\delta_1)\cdots(1-C\delta_{i-1})|x|^2. \tag{4.39}$$

+ Constructing $l_i(x)$: Let

$$\phi^*(y) := \frac{1}{\mu^i}[\phi(\mu^{\frac{i}{2}}A_i^{-1}y) - \phi(0) - \mu^i] \quad \text{and} \quad v(y) := \frac{(u-l_{i-1})(\mu^{\frac{i}{2}}A_i^{-1}y)}{\mu^{\frac{i-1}{2}(1+\alpha)}}, \quad \text{for} \quad y \in \Omega_i^*$$

where $\Omega_i^* := \mu^{\frac{-i}{2}} A_i S_{\mu^i}(\phi)$. Then as $\det A_i = 1$ and since

$$D^2\phi^*(y) = (A_i^{-1})^t D^2\phi(\mu^{\frac{i}{2}} A_i^{-1} y) A_i^{-1} \quad \text{and} \quad D^2 v(y) = \mu\mu^{\frac{i-1}{2}(1-\alpha)} (A_i^{-1})^t D^2 u(\mu^{\frac{i}{2}} A_i^{-1} y) A_i^{-1},$$

we get

$$\begin{cases} 1-\theta \le \det D^2\phi^* \le 1+\theta & \text{in} \quad \Omega_i^* \\ \phi^* = 0 & \text{on} \quad \partial\Omega_i^* \end{cases}$$

and $\Phi^*(y) := \det D^2\phi^*(y)\,(D^2\phi^*(y))^{-1} = A_i \Phi(\mu^{\frac{i}{2}} A_i^{-1} y) A_i^t$. Consequently,

$$\begin{aligned} \mathcal{L}_{\phi^*} v(y) &= \operatorname{trace}(\Phi^*(y) D^2 v(y)) \\ &= \mu\mu^{\frac{i-1}{2}(1-\alpha)} \operatorname{trace}\Big(A_i \Phi(\mu^{\frac{i}{2}} A_i^{-1} y) A_i^t (A_i^{-1})^t D^2 u(\mu^{\frac{i}{2}} A_i^{-1} y) A_i^{-1}\Big) \\ &= \mu\mu^{\frac{i-1}{2}(1-\alpha)} \operatorname{trace}\Big(\Phi(\mu^{\frac{i}{2}} A_i^{-1} y) D^2 u(\mu^{\frac{i}{2}} A_i^{-1} y)\Big) \\ &= \mu\mu^{\frac{i-1}{2}(1-\alpha)} \mathcal{L}_\phi u(\mu^{\frac{i}{2}} A_i^{-1} y) = \mu\mu^{\frac{i-1}{2}(1-\alpha)} f(\mu^{\frac{i}{2}} A_i^{-1} y) =: \tilde{f}(y) \qquad \text{in} \quad \Omega_i^*. \end{aligned}$$

Notice that

$$\begin{aligned} \Big(\frac{1}{|\Omega_i^*|} \int_{\Omega_i^*} |\tilde{f}(y)|^n dy\Big)^{\frac{1}{n}} &= \mu\mu^{\frac{i-1}{2}(1-\alpha)} \Big(\frac{1}{|\mu^{\frac{-i}{2}} A_i S_{\mu^i}(\phi)|} \int_{\mu^{\frac{-i}{2}} A_i S_{\mu^i}(\phi)} |f(\mu^{\frac{i}{2}} A_i^{-1} y)|^n dy\Big)^{\frac{1}{n}} \\ &= \mu\mu^{\frac{i-1}{2}(1-\alpha)} \Big(\frac{1}{|S_{\mu^i}(\phi)|} \int_{S_{\mu^i}(\phi)} |f(x)|^n dx\Big)^{\frac{1}{n}} \\ &\le \mu\mu^{\frac{i-1}{2}(1-\alpha)} \theta \mu^{\frac{i(\alpha-1)}{2}} = \theta\mu^{\frac{1+\alpha}{2}} \le \theta. \end{aligned}$$

We apply Lemma 4.2 with $\phi \rightsquigarrow \phi^*$, $f \rightsquigarrow \tilde{f}$, $\Omega \rightsquigarrow \Omega_i^*$ and $\varphi := v|_{\partial B_1}$. Note that by (3) for $k = i$ we have $\|\varphi\|_{L^\infty(\partial B_1)} \le 1$, and also φ has modulus of continuity $\rho(s) = 2C^*(\sqrt{c_1\mu})^{-\beta} s^\beta$ by (5) for $k = i$. Hence if h is the solution of

$$\begin{cases} \mathcal{L}_w h = 0 & \text{in} \quad B_1 \\ h = \varphi = v & \text{on} \quad \partial B_1 \end{cases} \quad \text{where} \quad \begin{cases} \det D^2 w = 1 & \text{in} \quad \Omega_i^* \\ w = 0 & \text{on} \quad \partial\Omega_i^*, \end{cases}$$

then

$$\|v - h\|_{L^\infty(B_1)} \le 3c_e C_2^2 \mu.$$

We have $\|h\|_{L^\infty(B_1)} \le 1$ by the maximum principle and it follows from the formulas (3.13) and (3.15) that $S_{2\mu}(\phi^*) \subset B_{C_2\sqrt{2\mu + C_3\theta^{1/2}}}(0) \subset B_{C_2\sqrt{3\mu}}(0)$. Thus by applying Theorem 2.7, we get

$$\|h - \bar{l}\|_{L^\infty(S_{2\mu}(\phi^*))} \le \|h - \bar{l}\|_{L^\infty(B_{C_2\sqrt{3\mu}}(0))} \le 3c_e C_2^2 \mu,$$

where $\bar{l}(y) := h(0) + Dh(0) \cdot y$. Therefore,

$$\|v - \bar{l}\|_{L^\infty(S_{2\mu}(\phi^*))} \le \|v - h\|_{L^\infty(S_{2\mu}(\phi^*))} + \|h - \bar{l}\|_{L^\infty(S_{2\mu}(\phi^*))} \le 6c_e C_2^2 \mu \le \mu^{\frac{1}{2}(1+\alpha)}. \tag{4.40}$$

Define

$$l_i(x) := l_{i-1}(x) + \mu^{\frac{i-1}{2}(1+\alpha)}\bar{l}(\mu^{\frac{-i}{2}}A_i x). \tag{4.41}$$

Then since $S_\mu(\phi^*) = \mu^{\frac{-i}{2}}A_i S_{\mu^{i+1}}(\phi)$, we obtain from (4.40) for $x \in S_{\mu^{i+1}}(\phi)$ that

$$\begin{aligned}|u(x) - l_i(x)| &= |(u - l_{i-1})(x) - \mu^{\frac{i-1}{2}(1+\alpha)}\bar{l}(\mu^{\frac{-i}{2}}A_i x)| = \mu^{\frac{i-1}{2}(1+\alpha)}|v(\mu^{\frac{-i}{2}}A_i x) - \bar{l}(\mu^{\frac{-i}{2}}A_i x)| \\ &\le \mu^{\frac{i-1}{2}(1+\alpha)}\|v - \bar{l}\|_{L^\infty(S_\mu(\phi^*))} \le \mu^{\frac{i-1}{2}(1+\alpha)}\mu^{\frac{1}{2}(1+\alpha)} = \mu^{\frac{i}{2}(1+\alpha)}.\end{aligned}$$

Thus (3) for $k = i + 1$ is verified. Also (4) for $k = i$ holds because it follows from the definition (4.41) and the definition of $\bar{l}$ that $a_i = a_{i-1} + \mu^{\frac{i-1}{2}(1+\alpha)}h(0)$ and $b_i = b_{i-1} + \mu^{\frac{i-1}{2}(1+\alpha)}\mu^{\frac{-i}{2}}A_i^t Dh(0)$. Hence by using Theorem 2.7, we get

$$|a_i - a_{i-1}| + \mu^{\frac{i}{2}}\|(A_i^{-1})^t \cdot (b_i - b_{i-1})\| = \mu^{\frac{i-1}{2}(1+\alpha)}\big[|h(0)| + \|Dh(0)\|\big] \le 2c_e\mu^{\frac{i-1}{2}(1+\alpha)}$$

giving (4) for $k = i$.

+ Constructing A_{i+1}: Applying Lemma 3.3 for ϕ^* and Ω_i^* we obtain a positive definite matrix $M = A^tA$ with $\det M = 1$, $(1 - C\delta_i)I \le M \le (1 + C\delta_i)I$ such that

$$B_{(1-\delta_{i+1})\sqrt{2}}(0) \subset \mu^{\frac{-1}{2}}AS_\mu(\phi^*) \subset B_{(1+\delta_{i+1})\sqrt{2}}(0), \quad \text{with} \quad \delta_{i+1} := C(\delta_i\mu^{1/2} + \mu^{-1}\theta^{1/2}).$$

Define $A_{i+1} := AA_i$ which implies in particular that A_{i+1} is a positive definite matrix with $\det A_{i+1} = 1$. Then

$$B_{(1-\delta_{i+1})\sqrt{2}}(0) \subset \mu^{\frac{-(i+1)}{2}}A_{i+1}S_{\mu^{i+1}}(\phi) \subset B_{(1+\delta_{i+1})\sqrt{2}}(0).$$

Thus (2) and the first part of (1) for $k = i + 1$ hold obviously since $A_iA_{i+1}^{-1} = A^{-1}$ and $\|A^{-1}\| \le \frac{1}{\sqrt{1-C\delta_i}} \le \frac{1}{\sqrt{c_1}}$. Next observe from the definition of A that

$$(1 - C\delta_i)|A_i x|^2 \le |A_{i+1}x|^2 = |AA_i x|^2 \le (1 + C\delta_i)|A_i x|^2.$$

Consequently by combining with (4.39), we get $|A_{i+1}x|^2 \le c_2(1 + C\delta_1)\cdots(1 + C\delta_i)|x|^2$ yielding the second part of (1), i.e., $\|A_{i+1}\| \le \sqrt{c_2(1 + C\delta_1)\cdots(1 + C\delta_i)}$. It remains to verify (5) for $k = i + 1$. Using the definitions of v and l_i we have

$$\begin{aligned}(u - l_i)(\mu^{\frac{i+1}{2}}A_{i+1}^{-1}x) &= (u - l_{i-1})(\mu^{\frac{i+1}{2}}A_{i+1}^{-1}x) - \mu^{\frac{i-1}{2}(1+\alpha)}\bar{l}(\mu^{\frac{1}{2}}A_iA_{i+1}^{-1}x) \\ &= \mu^{\frac{i-1}{2}(1+\alpha)}\big[v(\mu^{\frac{1}{2}}A_iA_{i+1}^{-1}x) - \bar{l}(\mu^{\frac{1}{2}}A_iA_{i+1}^{-1}x)\big] \\ &= \mu^{\frac{i-1}{2}(1+\alpha)}(v - \bar{l})(\mu^{\frac{1}{2}}A^{-1}x) \qquad \text{for all} \quad x \in \mu^{\frac{-(i+1)}{2}}A_{i+1}S_{\mu^{i+1}}(\phi).\end{aligned}$$

Moreover since $\mathcal{L}_{\phi^*}(v - \bar{l})(y) = \mathcal{L}_{\phi^*}v(y) = \tilde{f}(y) = \mu\mu^{\frac{i-1}{2}(1-\alpha)}f(\mu^{\frac{i}{2}}A_i^{-1}y)$ for $y \in \Omega_i^*$ and

$$\begin{aligned}\|\tilde{f}\|_{L^n(S_{2\mu}(\phi^*))} &= \mu\mu^{\frac{i-1}{2}(1-\alpha)}\Big(\int_{\mu^{\frac{-i}{2}}A_iS_{2\mu^{i+1}}(\phi)} |f(\mu^{\frac{i}{2}}A_i^{-1}y)|^n\,dy\Big)^{\frac{1}{n}} \\ &= \mu\mu^{\frac{i-1}{2}(1-\alpha)}\mu^{\frac{-i}{2}}\Big(\int_{S_{2\mu^{i+1}}(\phi)} |f(x)|^n\,dx\Big)^{\frac{1}{n}} \le C_n\theta\mu^{\frac{1}{2}(1+\alpha)}(2\mu)^{\frac{\alpha}{2}},\end{aligned}$$

we get from the Hölder estimate (2.2) and (4.40) that

$$\begin{aligned}|(v-\bar{l})(x_1)-(v-\bar{l})(y_1)| &\le C^*\mu^{-\beta}|x_1-y_1|^\beta\Big\{\|v-\bar{l}\|_{L^\infty(S_{2\mu}(\phi^*))}+(2\mu)^{\frac{1}{2}}\|\tilde{f}\|_{L^n(S_{2\mu}(\phi^*))}\Big\}\\ &\le C^*\mu^{-\beta}|x_1-y_1|^\beta\mu^{\frac{1}{2}(1+\alpha)}\Big\{1+C_n\theta(2\mu)^{\frac{1+\alpha}{2}}\Big\}\\ &\le 2C^*\mu^{-\beta}|x_1-y_1|^\beta\mu^{\frac{1}{2}(1+\alpha)} \qquad \forall x_1,y_1\in S_\mu(\phi^*).\end{aligned}$$

Therefore for any $x,y\in\mu^{\frac{-(i+1)}{2}}A_{i+1}S_{\mu^{i+1}}(\phi)$, by taking $x_1:=\mu^{\frac{1}{2}}A^{-1}x=\mu^{\frac{1}{2}}A_iA_{i+1}^{-1}x\in\mu^{\frac{-i}{2}}A_iS_{\mu^{i+1}}(\phi)=S_\mu(\phi^*)$ and $y_1:=\mu^{\frac{1}{2}}A^{-1}y\in S_\mu(\phi^*)$ we obtain

$$\begin{aligned}\frac{\Big|(u-l_i)(\mu^{\frac{i+1}{2}}A_{i+1}^{-1}x)-(u-l_i)(\mu^{\frac{i+1}{2}}A_{i+1}^{-1}y)\Big|}{\mu^{\frac{i}{2}(1+\alpha)}} &= \frac{\Big|(v-\bar{l})(\mu^{\frac{1}{2}}A^{-1}x)-(v-\bar{l})(\mu^{\frac{1}{2}}A^{-1}y)\Big|}{\mu^{\frac{1}{2}(1+\alpha)}}\\ \le 2C^*\mu^{-\beta}|\mu^{\frac{1}{2}}A^{-1}(x-y)|^\beta &\le 2C^*\mu^{-\beta}(\mu^{\frac{1}{2}}\|A^{-1}\|)^\beta|x-y|^\beta\le 2C^*(\sqrt{c_1\mu})^{-\beta}|x-y|^\beta\end{aligned}$$

giving (5) for $k=i+1$ as desired.

4. Proof of (4.35)**:** Take $0<\alpha^*<1$ such that $\frac{(1+\alpha^*)(1+\alpha)}{2}=1+\alpha'$. This is possible since $\alpha'<\alpha$. In particular, we have $\frac{1-\alpha^*}{1+\alpha^*}<\alpha$. Next observe that by taking θ even smaller if needed (now also depends on α'), we can assume that

$$\theta^{1/2}\le\frac{1-\alpha^*}{1+\alpha^*}\,\frac{\mu\ln\mu^{-1}}{2C}. \tag{4.42}$$

Then we have the following growth estimate for the norm of the matrix A_k

$$\|A_k\|\le C\mu^{\frac{-k(1-\alpha^*)}{2(1+\alpha^*)}}. \tag{4.43}$$

To see this we note first that since $\delta_1<1$, the sequence $\{\delta_k\}_{k=1}^\infty$ is decreasing. By induction we obtain for all $k\ge1$

$$\begin{aligned}\delta_k &= (C\sqrt{\mu})^k+\frac{C\sqrt{\theta}}{\mu}(C\sqrt{\mu})^{k-1}\sum_{i=0}^{k-1}(C\sqrt{\mu})^{-i}\\ &= (C\sqrt{\mu})^k+\frac{C\sqrt{\theta}}{\mu}(C\sqrt{\mu})^{k-1}\frac{[C^{-1}\mu^{-\frac{1}{2}}]^k-1}{[C^{-1}\mu^{-\frac{1}{2}}]-1}\\ &\le (C\sqrt{\mu})^k+\frac{C\sqrt{\theta}}{\mu}(C\sqrt{\mu})^{k-1}\frac{C^{-k}\mu^{-\frac{k}{2}}}{\frac{C^{-1}\mu^{-\frac{1}{2}}}{2}}=(C\sqrt{\mu})^k+\frac{2C\sqrt{\theta}}{\mu}.\end{aligned}$$

It follows that for all $k \geq 2$,

$$\begin{aligned}\prod_{i=1}^{k-1}(1+C\delta_i) &= \exp\left(\sum_{i=1}^{k-1}\log(1+C\delta_i)\right) \leq \exp\left(C\sum_{i=1}^{k-1}\delta_i\right)\\ &\leq \exp\left(C\sum_{i=1}^{k-1}(C\sqrt{\mu})^i + \frac{(k-1)2C\sqrt{\theta}}{\mu}\right)\\ &= \exp\left(C\sum_{i=1}^{k-1}(C\sqrt{\mu})^i\right)\exp\left(\frac{(k-1)2C\sqrt{\theta}}{\mu}\right) \leq C\exp\left(\frac{k2C\sqrt{\theta}}{\mu}\right).\end{aligned}$$

Hence as $\sqrt{\theta} \leq \dfrac{1-\alpha^*}{1+\alpha^*}\dfrac{\mu\ln\mu^{-1}}{2C}$ by (4.42), we get

$$\prod_{i=1}^{k-1}(1+C\delta_i) \leq C\mu^{\frac{(\alpha^*-1)k}{1+\alpha^*}},$$

which yields $\|A_k\|^2 \leq C\mu^{\frac{(\alpha^*-1)k}{1+\alpha^*}}$ due to (1) of the claim. Therefore (4.43) is proved.

By using (4.43) we can now conclude that

$$\|b_k - b_{k-1}\| \leq \|(A_k^{-1})^t\cdot(b_k-b_{k-1})\|\,\|A_k\| \leq \frac{2c_eC}{\mu^{(1+\alpha)/2}}\mu^{\frac{k}{2}\left(\alpha-\frac{1-\alpha^*}{1+\alpha^*}\right)} \tag{4.44}$$

where the second inequality is due to (4) of the claim. Since $\alpha - \frac{1-\alpha^*}{1+\alpha^*} > 0$, it follows from (4.44) and (4) of the claim that $\{a_k\} \subset \mathbb{R}$ and $\{b_k\} \subset \mathbb{R}^n$ are Cauchy sequences, and hence $a_k \to a$ and $b_k \to b$ for some $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$. Let $l(x) := a + b\cdot x$. As $b_0 = 0$ and by using (4.44), we have

$$\|b_k\| \leq \sum_{i=0}^{k-1}\|b_{i+1}-b_i\| \leq \frac{2c_eC}{\mu^{(1+\alpha)/2}}\sum_{i=0}^{k-1}\mu^{\frac{i}{2}\left(\alpha-\frac{1-\alpha^*}{1+\alpha^*}\right)} = \frac{2c_eC}{\mu^{(1+\alpha)/2}}\frac{1-\mu^{\frac{k}{2}\left(\alpha-\frac{1-\alpha^*}{1+\alpha^*}\right)}}{1-\mu^{\frac{1}{2}\left(\alpha-\frac{1-\alpha^*}{1+\alpha^*}\right)}}$$

giving

$$\|Dl(0)\| = \|b\| = \lim_{k\to\infty}\|b_k\| \leq \frac{2c_eC}{\mu^{(1+\alpha)/2}\left(1-\mu^{\frac{1}{2}\left(\alpha-\frac{1-\alpha^*}{1+\alpha^*}\right)}\right)}. \tag{4.45}$$

Similarly, (4) of the claim also yields

$$|l(0)| = |a| = \lim_{k\to\infty}|a_k| \leq \frac{2c_e}{1-\mu^{\frac{1+\alpha}{2}}}. \tag{4.46}$$

Next observe that $B_{(\sqrt{2}C)^{-1}\mu^{\frac{k}{1+\alpha^*}}}(0) \subset S_{\mu^k}(\phi)$ for all $k = 1, 2, \ldots$. Indeed if $x \in B_{(\sqrt{2}C)^{-1}\mu^{\frac{k}{1+\alpha^*}}}(0)$, then we have by (4.43)

$$\left|\mu^{\frac{-k}{2}}A_kx\right| \leq \mu^{\frac{-k}{2}}\|A_k\|\|x| \leq C\mu^{\frac{-k}{2}}\mu^{\frac{-k(1-\alpha^*)}{2(1+\alpha^*)}}(\sqrt{2}C)^{-1}\mu^{\frac{k}{1+\alpha^*}} = \frac{1}{\sqrt{2}} < (1-\delta_k)\sqrt{2}$$

since $\delta_k < \delta_1 < 1 - \frac{6}{5\sqrt{2}}$. Hence $\mu^{\frac{-k}{2}}A_kx \in \mu^{\frac{-k}{2}}A_kS_{\mu^k}(\phi)$ by Claim (2) and so $x \in S_{\mu^k}(\phi)$.

Therefore for any integer number $k_0 \geq 1$ and any $x \in B_{(\sqrt{2}C)^{-1}\mu^{\frac{k_0}{1+\alpha^*}}}(0)$, we obtain

$$x \in S_{\mu^{k_0}}(\phi) \quad \text{and} \quad \|A_k x\| \leq \|A_k\|\,|x| \leq (\sqrt{2})^{-1}\mu^{\frac{-k(1-\alpha^*)}{2(1+\alpha^*)}}\mu^{\frac{k_0}{1+\alpha^*}}.$$

Consequently,

$$\begin{aligned}
|u(x)-l(x)| &\leq |u(x)-l_{k_0-1}(x)| + |l(x)-l_{k_0-1}(x)| = |u(x)-l_{k_0-1}(x)| + \lim_{m\to\infty}|l_m(x)-l_{k_0-1}(x)| \\
&\leq \mu^{\frac{k_0-1}{2}(1+\alpha)} + \sum_{k=k_0}^{\infty}|l_k(x)-l_{k-1}(x)| \\
&= \mu^{\frac{k_0-1}{2}(1+\alpha)} + \sum_{k=k_0}^{\infty}|(a_k-a_{k-1}) + \langle (A_k^{-1})^t\cdot(b_k-b_{k-1}), A_k x\rangle| \\
&\leq \mu^{\frac{k_0-1}{2}(1+\alpha)} + \sum_{k=k_0}^{\infty}\left[|a_k-a_{k-1}| + \|(A_k^{-1})^t\cdot(b_k-b_{k-1})\|\,\|A_k x\|\,\right] \\
&\leq \mu^{\frac{k_0-1}{2}(1+\alpha)} + 2c_e\sum_{k=k_0}^{\infty}\left[\mu^{\frac{k-1}{2}(1+\alpha)} + \mu^{\frac{(k-1)\alpha-1}{2}}(\sqrt{2})^{-1}\mu^{\frac{-k(1-\alpha^*)}{2(1+\alpha^*)}}\mu^{\frac{k_0}{1+\alpha^*}}\right] \\
&= \mu^{(k_0-1)(\alpha+1)/2} + 2c_e\sum_{k=k_0}^{\infty}\mu^{(k-1)(\alpha+1)/2} + \sqrt{2}\,c_e\,\mu^{k_0/(1+\alpha^*)}\,\mu^{-(\alpha+1)/2}\sum_{k=k_0}^{\infty}\mu^{\frac{k}{2}\left(\alpha-\frac{1-\alpha^*}{1+\alpha^*}\right)} \\
&= \left[1 + \frac{2c_e}{1-\mu^{\frac{1+\alpha}{2}}} + \frac{\sqrt{2}\,c_e}{1-\mu^{\frac{1}{2}\left(\alpha-\frac{1-\alpha^*}{1+\alpha^*}\right)}}\right]\mu^{\frac{k_0-1}{2}(1+\alpha)} \\
&= C'\left((\sqrt{2}C)^{-1}\mu^{\frac{k_0}{1+\alpha^*}}\right)^{\frac{(1+\alpha^*)(1+\alpha)}{2}} = C'\left((\sqrt{2}C)^{-1}\mu^{\frac{k_0}{1+\alpha^*}}\right)^{1+\alpha'}.
\end{aligned}$$

This together with (4.45) and (4.46) gives (4.35) as desired with $\mu^* := (\sqrt{2}C)^{-1}\mu^{\frac{1}{1+\alpha^*}}$. □

By a perturbation argument, we obtain the next theorem as a consequence of Theorem 4.5. We use the following notation:

Definition 4.6. *For $f \in L^n_{loc}(\Omega)$, let*

$$[f]^n_{\alpha,\Omega} := \sup_{S_r(\phi,x)\Subset\Omega} r^{\frac{1-\alpha}{2}}\left(\frac{1}{|S_r(\phi,x)|}\int_{S_r(\phi,x)}|f|^n\,dx\right)^{1/n}.$$

It is clear from Lemma 2.3 and Hölder's inequality that if $f \in L^p(\Omega)$ for $p > n$, then $[f]^n_{\alpha,\Omega} < \infty$ for some $\alpha > 0$.

Theorem 4.7. *Let Ω be a normalized convex domain and $\phi \in C(\bar{\Omega})$ be a convex solution to* $\det D^2\phi = g$ *in Ω and $\phi = 0$ on $\partial\Omega$, where $g \in C(\Omega)$ satisfying $\lambda \leq g(x) \leq \Lambda$ in Ω. Then if $u \in W^{2,n}_{loc}(\Omega)$ is a solution of $\mathcal{L}_\phi u = f$ in Ω with $[f]^n_{\alpha,\Omega} < \infty$ for some $0 < \alpha < 1$, we have*

$u \in C^{1,\alpha'}_{loc}(\Omega)$ for any $\alpha' \in (0,\alpha)$. Moreover, for $\Omega' \Subset \Omega$ there holds

$$\|u\|_{C^{1,\alpha'}(\Omega')} \le C\Big\{\|u\|_{L^\infty(\Omega)} + [f]^n_{\alpha,\Omega}\Big\},$$

where C depends on n, α, α', λ, Λ, $dist(\Omega', \partial\Omega)$ and the modulus of continuity of g.

Proof. Let $0 < \alpha' < \alpha$ and $\Omega' \Subset \Omega$. Given $\epsilon_0 > 0$, since $g \in C(\Omega)$ and by Lemma 2.1 there exists $h_0 > 0$ such that for any $x_0 \in \Omega'$,

$$B_{C_1 h_0}(x_0) \subset S_{h_0}(\phi, x_0) \subset B_{C_2 h_0^b}(x_0) \quad \text{and} \quad |g(y) - g(x_0)| \le \epsilon_0 \quad \forall y \in S_{h_0}(\phi, x_0).$$

Let $Tx = A(x - x_0) + y_0$ be the affine transformation such that $B_1(0) \subset TS_{h_0}(\phi, x_0) \subset B_n(0)$. In particular by combining with Lemma 2.3 we get $C_1 \le |\det A|^{\frac{2}{n}} h_0 \le C_2$ for some positive constants C_1 and C_2 depending only on n, λ and Λ.

Define $\Omega^* := TS_{h_0}(\phi, x_0)$ and consider the functions

$$\phi^*(y) := \kappa_0[\phi(T^{-1}y) - l_{x_0}(T^{-1}y) - h_0] \quad \text{and} \quad v(y) := g(x_0)\kappa_0^{\frac{\alpha-3}{2}} u(T^{-1}y), \quad \text{for} \quad y \in \Omega^*$$

where $\kappa_0 := \frac{|\det A|^{\frac{2}{n}}}{g(x_0)^{\frac{1}{n}}}$ and $l_{x_0}(x)$ is the supporting function of ϕ at x_0. Then

$$D^2\phi^*(y) = \kappa_0\,(A^{-1})^t D^2\phi(T^{-1}y)A^{-1}, \qquad \det D^2\phi^*(y) = \frac{\det D^2\phi(T^{-1}y)}{g(x_0)} = \frac{g(T^{-1}y)}{g(x_0)},$$

$$\Phi^*(y) = \frac{\kappa_0}{g(x_0)} A\Phi(T^{-1}y)A^t, \qquad D^2 v(y) = g(x_0)\kappa_0^{\frac{\alpha-3}{2}}(A^{-1})^t D^2 u(T^{-1}y)A^{-1}.$$

As $g(x_0) - \epsilon_0 \le g(T^{-1}y) \le g(x_0) + \epsilon_0$ in Ω^*, these imply that

$$1 - \frac{\epsilon_0}{\lambda} \le \det D^2\phi^*(y) \le 1 + \frac{\epsilon_0}{\lambda} \quad \text{and} \quad \mathcal{L}_{\phi^*} v(y) = \kappa_0^{\frac{\alpha-1}{2}} f(T^{-1}y) =: \tilde{f}(y) \quad \text{in} \quad \Omega^*.$$

Note that y_0 is the minimum point of ϕ^* in Ω^* with $\phi^*(y_0) = -\kappa_0 h_0$. Since

$$S_r(\phi^*) := S_r(\phi^*, y_0) = TS_{\kappa_0^{-1} r}(\phi, x_0) \qquad \text{for} \quad r \le \kappa_0 h_0,$$

we have

$$\begin{aligned}
\left(\frac{1}{|S_r(\phi^*)|}\int_{S_r(\phi^*)} |\tilde{f}(y)|^n dy\right)^{\frac{1}{n}} &= \kappa_0^{\frac{\alpha-1}{2}} \left(\frac{1}{|TS_{\kappa_0^{-1}r}(\phi, x_0)|}\int_{TS_{\kappa_0^{-1}r}(\phi,x_0)} |f(T^{-1}y)|^n dy\right)^{\frac{1}{n}} \\
&= \kappa_0^{\frac{\alpha-1}{2}} \left(\frac{1}{|S_{\kappa_0^{-1}r}(\phi, x_0)|}\int_{S_{\kappa_0^{-1}r}(\phi,x_0)} |f(x)|^n dx\right)^{\frac{1}{n}} \\
&\le \kappa_0^{\frac{\alpha-1}{2}} [f]^n_{\alpha,\Omega}(\kappa_0^{-1} r)^{\frac{\alpha-1}{2}} = [f]^n_{\alpha,\Omega} r^{\frac{\alpha-1}{2}} \qquad \text{for all} \quad r \le \kappa_0 h_0.
\end{aligned}$$

Moreover $\kappa_0 h_0 = \frac{|\det A|^{\frac{2}{n}} h_0}{g(x_0)^{\frac{1}{n}}} \ge C(\lambda, \Lambda) > 0$ as $|\det A|^{\frac{2}{n}} h_0 \approx 1$. Therefore if we choose $\epsilon_0 := \lambda\theta$, where $\theta > 0$ is the constant given in Theorem 4.5 corresponding to $r_0 := C(\lambda, \Lambda)$,

then by Theorem 4.5 there exist constants $\mu^*, C > 0$ depending only on n, α, α', λ and Λ, and an affine function $\bar{l}$ such that

$$|v(y) - \bar{l}(y)| \le C|y - y_0|^{1+\alpha'}\Big\{\|v\|_{L^\infty(\Omega^*)} + [f]^n_{\alpha,\Omega}\Big\} \quad \text{for all} \quad y \in B_{\mu^*}(y_0) \Subset \Omega^*. \tag{4.47}$$

Observe that as $B_{C_1 h_0}(x_0) \subset S_{h_0}(\phi, x_0)$, we have $TB_{C_1 h_0}(x_0) \subset B_n(0)$, i.e., $AB_{C_1 h_0}(0) + y_0 \subset B_n(0)$. This yields $\|A\| \le Ch_0^{-1}$. Thus $TB_{\frac{\mu^* h_0}{C}}(x_0) \subset B_{\mu^*}(y_0)$ and we obtain from (4.47) and by rescaling back that

$$\begin{aligned}
&|u(x) - \ell(x, x_0)| = g(x_0)^{-1}\kappa_0^{\frac{3-\alpha}{2}}|v(Tx) - \bar{l}(Tx)| \\
&\le C\|A\|^{1+\alpha'}|x - x_0|^{1+\alpha'} g(x_0)^{-1}\kappa_0^{\frac{3-\alpha}{2}}\Big\{\|v\|_{L^\infty(\Omega^*)} + [f]^n_{\alpha,\Omega}\Big\} \\
&= C\|A\|^{1+\alpha'}|x - x_0|^{1+\alpha'}\left\{\|u\|_{L^\infty(S_{h_0}(\phi,x_0))} + g(x_0)^{-1}\Big(\frac{|\det A|^{\frac{2}{n}}}{g(x_0)^{\frac{1}{n}}}\Big)^{\frac{3-\alpha}{2}}[f]^n_{\alpha,\Omega}\right\} \\
&\le Ch_0^{-(1+\alpha')}h_0^{\frac{\alpha-3}{2}}|x - x_0|^{1+\alpha'}\Big\{\|u\|_{L^\infty(S_{h_0}(\phi,x_0))} + [f]^n_{\alpha,\Omega}\Big\} \qquad \text{for all} \quad x \in B_{\frac{\mu^* h_0}{C}}(x_0),
\end{aligned}$$

where $\ell(\cdot, x_0)$ is the function given by $\ell(x, x_0) := g(x_0)^{-1}\kappa_0^{\frac{3-\alpha}{2}}\bar{l}(Tx)$. That is u is $C^{1,\alpha'}$ at x_0. In other words, we proved that for any $x_0 \in \Omega'$ there exists a linear function $\ell(x, x_0)$ such that

$$|u(x) - \ell(x, x_0)| \le Ch_0^{\frac{\alpha-2\alpha'-5}{2}}\Big\{\|u\|_{L^\infty(\Omega)} + [f]^n_{\alpha,\Omega}\Big\}|x - x_0|^{1+\alpha'} \qquad \text{for all} \quad x \in B_{\frac{\mu^* h_0}{C}}(x_0). \tag{4.48}$$

We claim that this implies

$$|Du(x_1) - Du(x_2)| \le C\Big\{\|u\|_{L^\infty(\Omega)} + [f]^n_{\alpha,\Omega}\Big\}|x_1 - x_2|^{\alpha'} \qquad \text{for all} \quad x_1, x_2 \in \Omega', \tag{4.49}$$

where C depends also on h_0 and hence on the modulus of continuity of g. In order to prove the claim we use the following lemma of Calderón-Zygmund, [CZ61, Lemma 2.6].

Lemma 4.8. *Given an integer $m \ge 0$, there exists a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball such that $\varphi_\epsilon * P = P$ for each $\epsilon > 0$ and every polynomial P of degree $\le m$. As usual, $\varphi_\epsilon(x) := \epsilon^{-n}\varphi(x/\epsilon)$.*

Let $C_1 := Ch_0^{\frac{\alpha-2\alpha'-5}{2}}\Big\{\|u\|_{L^\infty(\Omega)} + [f]^n_{\alpha,\Omega}\Big\}$. From (4.48) it follows immediately that $\ell(x_0, x_0) = u(x_0)$ and

$$\left|\frac{u(h\,e_j + x_0) - u(x_0)}{h} - \frac{\ell(h\,e_j + x_0, x_0) - \ell(x_0, x_0)}{h}\right| \le \frac{C_1\,|h\,e_j|^{1+\alpha'}}{|h|} \to 0,$$

as $h \to 0$. So $D_x\ell(x_0, x_0) = Du(x_0)$ and hence $D_x\ell(x, x_0) \equiv Du(x_0)$ for each $x_0 \in \Omega'$.

By dividing the segment connecting x_1 and x_2 into a finite number of segments if necessary, we only need to verify (4.49) for $x_1, x_2 \in \Omega'$ satisfying $|x_1 - x_2| \le \frac{\mu^* h_0}{C}$. For such

x_1 and x_2, write

$$u(x) = u(x) - \ell(x, x_1) + \ell(x, x_1)$$
$$u(x) = u(x) - \ell(x, x_2) + \ell(x, x_2),$$

and convolving these expressions with φ_ϵ and using Lemma 4.8 with $m = 1$ we get

$$u_\epsilon(x) = [u - \ell(\cdot, x_1)] * \varphi_\epsilon(x) + \ell(x, x_1)$$
$$u_\epsilon(x) = [u - \ell(\cdot, x_2)] * \varphi_\epsilon(x) + \ell(x, x_2),$$

for $\mathrm{dist}(x, \partial\Omega) > \epsilon$, and taking derivatives

$$D_j u_\epsilon(x) = [u - \ell(\cdot, x_1)] * D_j\varphi_\epsilon(x) + D_j u(x_1)$$
$$D_j u_\epsilon(x) = [u - \ell(\cdot, x_2)] * D_j\varphi_\epsilon(x) + D_j u(x_2).$$

Hence $D_j u(x_1) - D_j u(x_2) = [u - \ell(\cdot, x_2)] * D_j\varphi_\epsilon(x) - [u - \ell(\cdot, x_1)] * D_j\varphi_\epsilon(x) = I_1 - I_2$, where

$$I_i := \epsilon^{-n-1} \int_{|y-x|<\epsilon} [u(y) - \ell(y, x_i)] D_j\varphi((x-y)/\epsilon)\, dy.$$

If we let $x := (x_1 + x_2)/2$, and $\epsilon := |x_1 - x_2|/2$, then we get that $B_\epsilon(x) \subset B_{2\epsilon}(x_i) \subset B_{\frac{\mu^* h_0}{C}}(x_i)$ for $i = 1, 2$, and so from (4.48) we obtain

$$\begin{aligned} |I_i| &\le \epsilon^{-n-1} \int_{|y-x_i|<2\epsilon} |u(y) - \ell(y, x_i)|\, |D_j\varphi((x-y)/\epsilon)|\, dy \\ &\le \epsilon^{-n-1} C_1 \, \|D_j\varphi\|_\infty \int_{|y-x_i|<2\epsilon} |y - x_i|^{1+\alpha'}\, dy \le C_n\, C_1\, (2\epsilon)^{\alpha'} = C_n\, C_1\, |x_1 - x_2|^{\alpha'}, \end{aligned}$$

and (4.49) follows as claimed. The proof of the theorem is completed. □

We remark that Theorem 4.7 still holds if one replaces the condition $\phi = 0$ on $\partial\Omega$ by the condition $\phi = \psi$ on $\partial\Omega$ where $\psi \in C^{1,\beta}(\partial\Omega)$ for some $\beta > 1 - \frac{2}{n}$. This new condition is necessary as shown by Pogorelov's examples to ensure that the graph of the function ϕ in Ω does not contain any line segment (see for example [Gut01, Theorem 5.4.7]).

We end the paper by a comment on Hölder estimates for second derivatives of solutions to the linearized Monge-Ampère equation. Assume that $\det D^2\phi = g$ in Ω and $\phi = 0$ on $\partial\Omega$, where $g \in C^\alpha_{loc}(\Omega)$ and $\lambda \le g \le \Lambda$ in Ω. Then as mentioned in the introduction if u is a solution to the equation $\mathcal{L}_\phi u = f$ in Ω with $f \in C^\alpha_{loc}(\Omega)$, we have $C^{2,\alpha}$ interior estimate for u. This follows from Caffarelli $C^{2,\alpha}$ estimate for the function ϕ and the classical Schauder's estimate for linear uniformly elliptic equation. However a direct proof of this $C^{2,\alpha}$ estimate for u can be derived from the method used in this paper.

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