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Freeform Lens Design for Scattering Data with General Radiant Fields

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Abstract

We show the existence of a lens, when its lower face is given, such that it refracts radiation emanating from a planar source, with a given field of directions, into the far field that preserves a given distribution of energies. Conditions are shown under which the lens obtained is physically realizable. It is shown that the upper face of the lens satisfies a pde of Monge-Ampère type.

Contents

1. Introduction

In this paper, we solve the following inverse problem in geometric optics concerning the design of a lens: rays are emitted from a planar source Ω with unit direction $e(x)$ and energy density $\mathcal{I}(x)$ for every $x \in \Omega$. The rays first strike a smooth given surface described by the graph of a function u. We are given a target $\Omega^* \subset S^2$, the unit sphere in $\mathbb{R}^3$, equipped with a Radon measure η such that

$$\int_\Omega \mathcal{I}(x)\, dx = \eta(\Omega^*).$$

The problem is then to construct a surface σ so that the lens sandwiched between u and σ refracts all rays with direction $e(x)$ into rays with directions in Ω^* and such that the energy is conserved, i.e.,

$$\int_{\mathcal{T}_\sigma(E)} \mathcal{I}(x)\, dx = \eta(E)$$

for every Borel set $E \subset \Omega^*$ where $\mathcal{T}_\sigma(E)$ is, roughly, the set of points $x \in \Omega$ so that the ray emitted from $(x, 0)$ with direction $e(x)$ is refracted into a direction in E, see Definition 4.1. The material of the lens is assumed to be denser than the media containing the source Ω and the target Ω^*.

The main result concerns the existence of the surface σ and therefore of the desired lens, Theorems 5.1 and 5.5. For this purpose, we use in a crucial way the uniformly refractive surfaces constructed in [13] which are the building blocks of the solution. In [13], Ω^* is a singleton and no energy assumptions were used. That is, given $w \in S^2$, a C^2 function u and a C^1 field $e(x)$, we studied in [13] the existence of parametric surfaces $\sigma_{C,w}$, with C a real parameter chosen properly, so that the lens sandwiched between u and $\sigma_{C,w}$ refracts all rays with direction $e(x)$ into w. Moreover, we showed that for such a surface to exist, the field $e(x)$ must satisfy a curl-free condition [13, Theorem 3.1], that is, $e'(x) = Dh(x)$ for some $h \in C^2(\Omega)$, where $e = (e_1, e_2, e_3)$, and $e' = (e_1, e_2)$. In this case, the surface $\sigma_{C,w}$ is parametrized by

$$f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w)m(x),$$

where $(\varphi(x), u(\varphi(x)))$ is the point of incidence of the ray with direction $e(x)$ on the graph of u, $m(x)$ is the unit direction of the refracted ray at $(\varphi(x), u(\varphi(x)))$, and $d(x, C, w)$ designates the length of the trajectory of the ray with direction $m(x)$ inside the lens. $d(x, C, w)$ is given in (2.3) below, and the constant C is chosen so that d is positive; see Figure 2(b).

Since the surfaces $\sigma_{C,w}$ are given parametrically, they might have self-intersections and also singular points; see Figure 2(a). In such a case, $\sigma_{C,w}$ is not physically realizable. In Sections 2 and 3, we focus on the analysis of solving this difficulty and find sufficient conditions on the surface u, the source Ω, the field e, and the constant C, so that $\sigma_{C,w}$ has no self intersections and is regular at every point.

To place our results in perspective, both from the theoretical and practical points of view, we mention some results from the literature. The problem of finding a convex, analytic, and symmetric lens focusing all rays from a point source into a point image was first solved in [6] in 2d using a fixed point type argument. This result is extended in [17] to 3d to construct freeform lenses that refract rays emitted from a point source into a constant direction or a point image. The general case for an arbitrary incident field and a planar source is solved in [13]; the reflection case is studied in [11].

Illumination problems with one point source are studied in [9,10], and [4] for the refraction case, and in [3,12,21,26], and [20] for reflection. The case of collimated input radiation is considered in [1,14–16], and [19]. Physical limitations, such us as ray obstruction, are discussed in [11] and [12, Section 3.1].

The surfaces constructed in this paper are freeform; in particular, they are not rotationally symmetric. Freeform design is a modern field in Optics. This is a breakthrough in the optical industry due to its applications in illumination, imaging, aerospace and biomedical engineering; see for example the news article [23] and the survey [7] for large set of applications. Due to recent technological advancement in ultra precision cutting, grinding, and polishing machines, manufacturing freeform optical devices with high precision is now possible, see [2]. The systems obtained enhance the performance of traditional designs and provide more flexibility for designers [5]. Moreover, they can achieve imaging tasks that are impossible with symmetric designs. However, the mathematical literature in freeform optics is still limited. In optical engineering, freeform surfaces are designed using the SMS 3D method for various applications but they do not have an analytical expression and are calculated numerically, see [27, Chapter 8] and [22].

In this paper, we develop a mathematical theory to solve an illumination problem involving two refracting surfaces, a planar source, and arbitrary incident field.

A plan and description of the contents of the paper is as follows. In Section 2.1, we prove that if $\sigma_{C,w}$ is a uniformly refractive surface then the function $d(x, C, w)$, given in (2.3), satisfies a Lipschitz estimate which implies by Rademacher's theorem that d is differentiable a.e.. Using this estimate, we prove in Theorem 2.5 that if the norms $||e'||$, $||e' - \kappa_1\kappa_2 w'||$[1] and the Lipschitz constants L_e, L_u, and L_{Du} are small enough, then the constant C can be chosen so that $\sigma_{C,w}$ has no self intersections. Section 3 is devoted to analyze the singular points of $f(x, C, w)$. We say that $f(x, C, w)(:= f(x))$ is regular at x if $f_{x_1}(x) \times f_{x_2}(x) \neq 0$, and is singular otherwise. In Section 3.2, the collimated case $e(x) = (0, 0, 1)$ is considered, and it is shown that for some conditions on the eigenvalues of $D^2 u$, the constant C can be chosen so that $\sigma_{C,w}$ is regular at every point, Theorem 3.2. The case of a general field e is analyzed in Section 3.1. It is shown in Theorem 3.1 that if u is concave, and the derivatives of the components of e' are such that the matrix W given in (3.1) is positive semi-definite, then one can choose C so that $\sigma_{C,w}$ is regular at every point. To summarize, to avoid self intersections, we need to control the size

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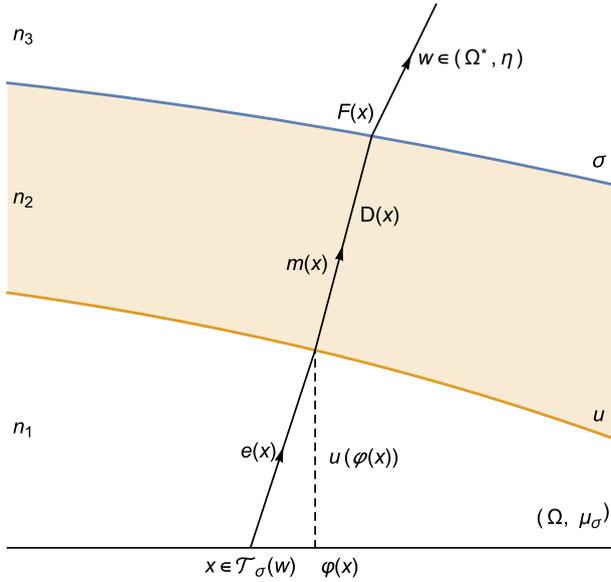[1] Primes here denote the vector of the first two components.

Fig. 1.

of the parameters involved, whereas to avoid singularities one needs to control the curvature of the surface u and that of the potential h, recall $e' = Dh$. In Section 4, we construct refracting surfaces σ so that the lens sandwiched between u and σ refracts incident rays with direction $e(x)$, $x \in \Omega$, into a far field target Ω^*; see Figure 1. In this case, u is assumed to be concave, h convex, and u, Ω, Ω^* and e are so that $\sigma_{C,w}$ satisfies the conditions in Theorem 2.5 for each $w \in \Omega^*$. σ is parametrized the vector $F(x) = (\varphi(x), u(\varphi(x))) + D(x) m(x)$; D is constructed so that the refractor σ is supported at every point by some uniformly refractive surface $\sigma_{C,w}$ with C chosen so that $\sigma_{C,w}$ has no self intersections and no singularities. $D(x)$ represents the length of the trajectory of the ray emanating from x inside the lens (u, σ). We show in Theorem 4.2 that the function D is Lipschitz, σ has no self intersections and is regular a.e. In Section 4.1 , we show that σ induces a Borel measure μ_σ. The energy problem is then reduced to find a collection of uniformly refractive surfaces $\sigma_{C,w}$ with $w \in \Omega^*$ such that the envelope of this collection yields a refractor σ satisfying $\mu_\sigma = \eta$. This is first solved in the discrete case in Section 5.1, that is, when η is a finite linear combination of delta functions. The general case of measure η is then done in Section 5.2 by approximating η by discrete measures. In Section 6 we introduce Aleksandrov solutions to the energy problem and compare them with the notion of solution previously defined. For a connection with generated Jacobian equations see Remark 6.1. Finally, in Section 7, we derive the PDE of the problem and show that D satisfies a Monge-Ampère type differential equation, equation (7.11), that is simplified in the collimated case in Section 7.3.

2. Uniformly Refracting Surfaces for a General Field $e(x)$

Let Ω be a convex bounded region in $\mathbb{R}^2$, and $e(x)$ be a unit field in $\mathbb{R}^3$ defined for every $x \in \Omega$. From each point $(x, 0)$, with $x \in \Omega$, consider the line through $(x, 0)$ with direction $e(x)$. We are given a surface u such that its graph intersects each of these lines at only one point, denoted by $(\varphi(x), u(\varphi(x)))$. Let Ω' be the projection over $\mathbb{R}^2$ of the points $(\varphi(x), u(\varphi(x)))$ with $x \in \Omega$. We assume that the map $\varphi : \Omega \to \Omega'$ is $C^1(\Omega)$, the field $e(x) = (e_1(x), e_2(x), e_3(x)) := (e'(x), e_3(x))$ is $C^1(\Omega)$ with $e_3(x) > 0$, and $u(z)$ is C^2 in an open neighborhood of Ω'.

Given $w \in S^2$, we found in [13] necessary and sufficient conditions between $u, w,$ and e, for the existence of a lens with bottom face u such that all rays emitted from $(x, 0)$ with direction $e(x)$ are refracted uniformly into w. The material the lens is made of has a refractive index n_2, such that $n_2 > n_1, n_3$, where n_1 and n_3 denote, respectively, the refractive indices of the media below and above the lens; n_1 and n_3 are unrelated. We refer to this as a *uniformly refracting lens into the direction* w, and we denote it by Γ_w.

Let $\kappa_1 = n_2/n_1$ and $\kappa_2 = n_3/n_2$. For each $x \in \Omega$, by the Snell law [8, Section 2.1], the ray with direction $e(x)$ is refracted by the first surface at $(\varphi(x), u(\varphi(x)))$ into a unit direction $m(x)$, with

$$m(x) = \frac{1}{\kappa_1}(e(x) - \lambda(x)\nu(x)), \tag{2.1}$$

where $\nu(x) = \dfrac{(-D_z u(\varphi(x)), 1)}{\sqrt{1 + |D_z u(\varphi(x))|^2}}$ denotes the unit normal to u, at $(\varphi(x), u(\varphi(x)))$,

pointing towards medium n_2, and $\lambda(x) = e(x) \cdot \nu(x) - \sqrt{\kappa_1^2 - 1 + (e(x) \cdot \nu(x))^2}$. It is also assumed that $e(x) \cdot \nu(x) > 0$ for all $x \in \Omega$.

The top face of the lens is constructed such that it refracts the rays with direction $m(x)$ uniformly into the direction w. Since $\kappa_2 < 1$, to avoid total internal reflection, we must assume that

$$m(x) \cdot w \geq \kappa_2. \tag{2.2}$$

Under condition (2.2), it is proved in [13, Section 3] that a uniformly refractive lens Γ_w exists if and only if $\text{curl}(e'(x)) = 0$, i.e., e' is generated by some potential function h, $e'(x) = Dh(x)$ with $h \in C^2(\Omega)$. In addition, the top face of the lens, denoted by $\sigma_{C,w}$, is parametrized by the vector

$$f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w)m(x),$$

with

$$d(x, C, w) = \frac{C - h(x) + e(x) \cdot (x, 0) - (e(x) - \kappa_1\kappa_2\, w) \cdot (\varphi(x), u(\varphi(x)))}{\kappa_1 - \kappa_1\kappa_2\, w \cdot m(x)}, \tag{2.3}$$

where C is constant chosen so that $d(x, C, w) > 0$ for all $x \in \Omega$. If we let

$$C^* = \max_{\Omega} h + \max_{z \in \Omega'} |z| + (1 + \kappa_1\kappa_2)\sqrt{\left(\max_{z \in \Omega'} |z|\right)^2 + \left(\max_{\Omega'} u\right)^2}, \tag{2.4}$$

then $d(x, C, w) > 0$ for all $x \in \Omega$ when $C > C^*$.

<div style="text-align:center">

2.1. A Lipschitz Estimate for $d(x, C, w)$

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Notice that for (2.3) to be defined we only need u to be differentiable; in fact, we prove the results in this section only assuming differentiability of u. This yields more precise constants in the inequalities that will be used later. The goal in this section is to prove the following proposition for the distance function d:

Proposition 2.1. *We have*

$$
\begin{aligned}
|d(x, C, w) - d(y, C, w)| \leq \big[& \mathcal{C}_1(\kappa_1, \kappa_2)\,|C| + \mathcal{C}_2(\kappa_1, \kappa_2, u, h, \Omega, \Omega') \big] \\
& \times \big(|e(x) - e(y)| + |Du(x') - Du(y')| \big) \\
& + \mathcal{C}_3(\kappa_1, \kappa_2)\,\|e'\|_{L^\infty(\Omega)}\,|x - y| \\
& + \mathcal{C}_4(\kappa_1, \kappa_2) \left(\max_{z \in \Omega} |e'(z) - \kappa_1\kappa_2\,w'| \right) |y' - x'| \\
& + \mathcal{C}_5(\kappa_1, \kappa_2)\,|u(y') - u(x')|;
\end{aligned}
$$

with $e(x) = (e'(x), e_3(x))$, $x' = \varphi(x)$, $y' = \varphi(y)$, and $w = (w', w_3)$.
If u, Du, φ, e are all Lipschitz, we then obtain

$$
\begin{aligned}
|d(x, C, w) - d(y, C, w)| \leq \big[& \mathcal{C}_1(\kappa_1, \kappa_2)\,|C| + \mathcal{C}_2(\kappa_1, \kappa_2, u, h, \Omega, \Omega') \big] \\
& \times \big(L_e + L_{Du}\,L_\varphi \big)\,|x - y| \\
& + \mathcal{C}_3(\kappa_1, \kappa_2)\,\|e'\|_{L^\infty(\Omega)}\,|x - y| \\
& + \mathcal{C}_4(\kappa_1, \kappa_2) \left(\max_{z \in \Omega} |e'(z) - \kappa_1\kappa_2\,w'| \right) L_\varphi\,|x - y| \\
& + \mathcal{C}_5(\kappa_1, \kappa_2)\,L_u\,L_\varphi\,|x - y|,
\end{aligned}
$$

where the L's are the Lipschitz constants of the corresponding functions.

To prove the proposition we shall prove first two lemmas.

Lemma 2.2. *We have*

$$
|v(x) - v(y)| \leq \sqrt{5}\,|Du(x') - Du(y')| \tag{2.5}
$$

for all $x, y \in \Omega$, where x', y' are defined in Proposition 2.1.

Proof. We write

$$
\begin{aligned}
v(x) - v(y) &= \left(\frac{-Du(x')}{\sqrt{1 + |Du(x')|^2}}, \frac{1}{\sqrt{1 + |Du(x')|^2}} \right) \\
&\quad - \left(\frac{-Du(y')}{\sqrt{1 + |Du(y')|^2}}, \frac{1}{\sqrt{1 + |Du(y')|^2}} \right) \\
&= \left(\frac{Du(y')}{\sqrt{1 + |Du(y')|^2}} - \frac{Du(x')}{\sqrt{1 + |Du(x')|^2}}, \right.
\end{aligned}
$$

$$\frac{1}{\sqrt{1+|Du(x')|^2}} - \frac{1}{\sqrt{1+|Du(y')|^2}}\Bigg)$$
$$= (\alpha, \beta).$$

We have

$$\beta = \frac{\sqrt{1+|Du(y')|^2} - \sqrt{1+|Du(x')|^2}}{\sqrt{1+|Du(x')|^2}\sqrt{1+|Du(y')|^2}}$$

$$= \frac{|Du(y')|^2 - |Du(x')|^2}{\sqrt{1+|Du(x')|^2}\sqrt{1+|Du(y')|^2}\left(\sqrt{1+|Du(y')|^2} + \sqrt{1+|Du(x')|^2}\right)}$$

$$= \frac{\big(|Du(y')| - |Du(x')|\big)\big(|Du(y')| + |Du(x')|\big)}{\sqrt{1+|Du(x')|^2}\sqrt{1+|Du(y')|^2}\left(\sqrt{1+|Du(y')|^2} + \sqrt{1+|Du(x')|^2}\right)}.$$

Then

$$|\beta| \leq \frac{|Du(y') - Du(x')|}{\sqrt{1+|Du(x')|^2}\sqrt{1+|Du(y')|^2}}.$$

Now

$$\alpha = \frac{Du(y')}{\sqrt{1+|Du(y')|^2}} - \frac{Du(x')}{\sqrt{1+|Du(x')|^2}}$$

$$= \frac{Du(y')}{\sqrt{1+|Du(y')|^2}} - \frac{Du(y')}{\sqrt{1+|Du(x')|^2}} + \frac{Du(y')}{\sqrt{1+|Du(x')|^2}} - \frac{Du(x')}{\sqrt{1+|Du(x')|^2}}$$

$$= \left(\frac{1}{\sqrt{1+|Du(y')|^2}} - \frac{1}{\sqrt{1+|Du(x')|^2}}\right) Du(y')$$

$$\quad + \frac{1}{\sqrt{1+|Du(x')|^2}}(Du(y') - Du(x'))$$

$$= -\beta\, Du(y') + \frac{1}{\sqrt{1+|Du(x')|^2}}(Du(y') - Du(x')),$$

and so

$$|\alpha| \leq |\beta|\,|Du(y')| + \frac{1}{\sqrt{1+|Du(x')|^2}}|Du(y') - Du(x')|$$

$$\leq \frac{|Du(y') - Du(x')|}{\sqrt{1+|Du(x')|^2}\sqrt{1+|Du(y')|^2}}|Du(y')|$$

$$\quad + \frac{1}{\sqrt{1+|Du(x')|^2}}|Du(y') - Du(x')|$$

$$\leq \frac{2}{\sqrt{1+|Du(x')|^2}}|Du(y') - Du(x')|.$$

Thus

$$|v(x) - v(y)| = \sqrt{\alpha^2 + \beta^2} \leq \sqrt{5} |Du(x') - Du(y')|.$$

□

Lemma 2.3. *We have*

$$|m(x) - m(y)| \leq C_1(\kappa_1) |e(x) - e(y)| + C_2(\kappa_1) |Du(x') - Du(y')|$$

for all $x, y \in \Omega$ with $C_i(\kappa_1)$, $i = 1, 2$, constants depending only on κ_1.

Proof. From (2.1)

$$m(x) - m(y) = \frac{1}{\kappa_1} (e(x) - e(y)) + \frac{1}{\kappa_1} (\lambda(y) v(y) - \lambda(x) v(x)). \qquad (2.6)$$

Also

$$\lambda(y) v(y) - \lambda(x) v(x)$$

$$= \frac{1}{\sqrt{1 + |Du(y')|^2}} \lambda(y) (-Du(y'), 1) - \frac{1}{\sqrt{1 + |Du(x')|^2}} \lambda(x) (-Du(x'), 1)$$

$$= \left(\frac{\lambda(x)}{\sqrt{1 + |Du(x')|^2}} Du(x') - \frac{\lambda(y)}{\sqrt{1 + |Du(y')|^2}} Du(y'), \right.$$

$$\left. \frac{\lambda(y)}{\sqrt{1 + |Du(y')|^2}} - \frac{\lambda(x)}{\sqrt{1 + |Du(x')|^2}} \right)$$

$$= (A, B).$$

Notice that

$$\frac{\lambda(x)}{\sqrt{1 + |Du(x')|^2}} = \frac{e(x) \cdot v(x) - \sqrt{\kappa_1^2 - 1 + (e(x) \cdot v(x))^2}}{\sqrt{1 + |Du(x')|^2}} \qquad (2.7)$$

$$= \frac{1 - \kappa_1^2}{\sqrt{1 + |Du(x')|^2} \left(e(x) \cdot v(x) + \sqrt{\kappa_1^2 - 1 + (e(x) \cdot v(x))^2} \right)}.$$

We first estimate B. Let

$$\Delta(x, y) = \sqrt{1 + |Du(y')|^2} \left(e(y) \cdot v(y) + \sqrt{\kappa_1^2 - 1 + (e(y) \cdot v(y))^2} \right)$$

$$\sqrt{1 + |Du(x')|^2} \left(e(x) \cdot v(x) + \sqrt{\kappa_1^2 - 1 + (e(x) \cdot v(x))^2} \right).$$

From (2.7) we can write B as follows:

$$\frac{B}{1 - \kappa_1^2}$$

$$= \frac{1}{\sqrt{1 + |Du(y')|^2} \left(e(y) \cdot v(y) + \sqrt{\kappa_1^2 - 1 + (e(y) \cdot v(y))^2} \right)}$$

$$-\frac{1}{\sqrt{1+|Du(x')|^2}\left(e(x)\cdot v(x)+\sqrt{\kappa_1^2-1+(e(x)\cdot v(x))^2}\right)}$$

$$=\frac{\sqrt{1+|Du(x')|^2}\left(e(x)\cdot v(x)+\sqrt{\kappa_1^2-1+(e(x)\cdot v(x))^2}\right)-\sqrt{1+|Du(y')|^2}\left(e(y)\cdot v(y)+\sqrt{\kappa_1^2-1+(e(y)\cdot v(y))^2}\right)}{\sqrt{1+|Du(y')|^2}\left(e(y)\cdot v(y)+\sqrt{\kappa_1^2-1+(e(y)\cdot v(y))^2}\right)\sqrt{1+|Du(x')|^2}\left(e(x)\cdot v(x)+\sqrt{\kappa_1^2-1+(e(x)\cdot v(x))^2}\right)}$$

$$=\frac{\sqrt{1+|Du(x')|^2}\,(e(x)\cdot v(x))-\sqrt{1+|Du(y')|^2}\,(e(y)\cdot v(y))}{\Delta(x,y)}$$

$$+\frac{\sqrt{1+|Du(x')|^2}\sqrt{\kappa_1^2-1+(e(x)\cdot v(x))^2}-\sqrt{1+|Du(y')|^2}\sqrt{\kappa_1^2-1+(e(y)\cdot v(y))^2}}{\Delta(x,y)}$$

$$=\frac{(e(x)\cdot v(x))^2-(e(y)\cdot v(y))^2}{\Delta(x,y)\left(\sqrt{1+|Du(x')|^2}\,(e(x)\cdot v(x))+\sqrt{1+|Du(y')|^2}\,(e(y)\cdot v(y))\right)}$$

$$+\frac{|Du(x')|^2(e(x)\cdot v(x))^2-|Du(y')|^2(e(y)\cdot v(y))^2}{\Delta(x,y)\left(\sqrt{1+|Du(x')|^2}\,(e(x)\cdot v(x))+\sqrt{1+|Du(y')|^2}\,(e(y)\cdot v(y))\right)}$$

$$+\frac{\sqrt{1+|Du(x')|^2}\sqrt{\kappa_1^2-1+(e(x)\cdot v(x))^2}-\sqrt{1+|Du(y')|^2}\sqrt{\kappa_1^2-1+(e(y)\cdot v(y))^2}}{\Delta(x,y)}$$

$$=B_1+B_2+B_3.$$

Since $e\cdot v>0$

$$|B_1|=\frac{|e(x)\cdot v(x)-e(y)\cdot v(y)|\,(e(x)\cdot v(x)+e(y)\cdot v(y))}{\Delta(x,y)\left(\sqrt{1+|Du(x')|^2}\,(e(x)\cdot v(x))+\sqrt{1+|Du(y')|^2}\,(e(y)\cdot v(y))\right)}$$

$$\leq\frac{|e(x)\cdot v(x)-e(y)\cdot v(y)|}{\Delta(x,y)}=\frac{|(e(x)-e(y)+e(y))\cdot v(x)-e(y)\cdot v(y)|}{\Delta(x,y)}$$

$$\leq\frac{|(e(x)-e(y))\cdot v(x)|+|e(y)\cdot(v(x)-v(y))|}{\Delta(x,y)}$$

$$\leq\frac{|e(x)-e(y)|+|v(x)-v(y)|}{\Delta(x,y)}$$

$$\leq\frac{|e(x)-e(y)|+\sqrt{5}\,|Du(x')-Du(y')|}{\Delta(x,y)}\qquad\text{from Lemma 2.2.}$$

Similarly,

$$|B_2|\leq\frac{\big||Du(x')|\,(e(x)\cdot v(x))-|Du(y')|\,(e(y)\cdot v(y))\big|}{\Delta(x,y)}$$

$$=\frac{\big|(|Du(x')|-|Du(y')|)\,(e(x)\cdot v(x))+|Du(y')|\,(e(x)\cdot v(x)-e(y)\cdot v(y))\big|}{\Delta(x,y)}$$

$$\leq\frac{|Du(x')-Du(y')|}{\Delta(x,y)}+\frac{|Du(y')|}{\Delta(x,y)}\,|e(x)\cdot v(x)-e(y)\cdot v(y)|$$

$$\leq\frac{|Du(x')-Du(y')|}{\Delta(x,y)}+\frac{|Du(y')|}{\Delta(x,y)}\left(\sqrt{5}\,|Du(x')-Du(y')|+|e(x)-e(y)|\right)$$

$$=\left(\frac{1+\sqrt{5}\,|Du(y')|}{\Delta(x,y)}\right)|Du(x')-Du(y')|+\frac{|Du(y')|}{\Delta(x,y)}\,|e(x)-e(y)|.$$

It remains to estimate B_3. Let

$$H(x, y) = \sqrt{1 + |Du(x')|^2}\sqrt{\kappa_1^2 - 1 + (e(x) \cdot v(x))^2}$$
$$+ \sqrt{1 + |Du(y')|^2}\sqrt{\kappa_1^2 - 1 + (e(y) \cdot v(y))^2}.$$

Multiplying and dividing by $H(x, y)$ yields

$$B_3 = \frac{\left(1 + |Du(x')|^2\right)\left(\kappa_1^2 - 1 + (e(x) \cdot v(x))^2\right) - \left(1 + |Du(y')|^2\right)\left(\kappa_1^2 - 1 + (e(y) \cdot v(y))^2\right)}{H(x, y)\,\Delta(x, y)}$$

$$= \frac{(\kappa_1^2 - 1)\left(|Du(x')|^2 - |Du(y')|^2\right)}{H(x, y)\,\Delta(x, y)} + \frac{(e(x) \cdot v(x))^2 - (e(y) \cdot v(y))^2}{H(x, y)\,\Delta(x, y)}$$

$$+ \frac{|Du(x')|^2(e(x) \cdot v(x))^2 - |Du(y')|^2(e(y) \cdot v(y))^2}{H(x, y)\,\Delta(x, y)}$$

$$= B_3^1 + B_3^2 + B_3^3.$$

Estimate B_3^1:

$$|B_3^1| = (\kappa_1^2 - 1)\frac{\big||Du(x')| - |Du(y')|\big|\left(|Du(x')| + |Du(y')|\right)}{H(x, y)\,\Delta(x, y)}$$

$$\leq (\kappa_1^2 - 1)\frac{\left|Du(x') - Du(y')\right|\left(|Du(x')| + |Du(y')|\right)}{H(x, y)\,\Delta(x, y)}$$

$$\leq (\kappa_1^2 - 1)\frac{\left|Du(x') - Du(y')\right|\left(|Du(x')| + |Du(y')|\right)}{\sqrt{\kappa_1^2 - 1}\left(|Du(x')| + |Du(y')|\right)\Delta(x, y)}$$

$$= \sqrt{\kappa_1^2 - 1}\frac{\left|Du(x') - Du(y')\right|}{\Delta(x, y)},$$

since $H(x, y) \geq \sqrt{\kappa_1^2 - 1}\left(|Du(x')| + |Du(y')|\right)$.

Estimate B_3^2:

$$|B_3^2| = \frac{|(e(x) \cdot v(x) - e(y) \cdot v(y))\,(e(x) \cdot v(x) + e(y) \cdot v(y))|}{H(x, y)\,\Delta(x, y)}$$

$$\leq \frac{|(e(x) \cdot v(x) - e(y) \cdot v(y))\,(e(x) \cdot v(x) + e(y) \cdot v(y))|}{(e(x) \cdot v(x) + e(y) \cdot v(y))\,\Delta(x, y)}$$

$$= \frac{|e(x) \cdot v(x) - e(y) \cdot v(y)|}{\Delta(x, y)},$$

since $H(x, y) \geq e(x) \cdot v(x) + e(y) \cdot v(y)$. Therefore as in the estimate of B_1 we obtain

$$|B_3^2| \leq \frac{\sqrt{5}}{\Delta(x, y)}\left|Du(x') - Du(y')\right| + \frac{1}{\Delta(x, y)}\left|e(x) - e(y)\right|.$$

Estimate B_3^3:

$$|B_3^3| = \frac{\big||Du(x')| e(x) \cdot v(x) - |Du(y')| e(y) \cdot v(y)\big| \big(|Du(x')| e(x) \cdot v(x) + |Du(y')| e(y) \cdot v(y)\big)}{H(x, y) \, \Delta(x, y)}$$

$$\leq \frac{\big||Du(x')| e(x) \cdot v(x) - |Du(y')| e(y) \cdot v(y)\big|}{\Delta(x, y)}$$

since $H(x, y) \geq |Du(x')| e(x) \cdot v(x) + |Du(y')| e(y) \cdot v(y)$. The estimate then follows as in estimating B_2. So we obtain

$$|B_3^3| \leq \left(\frac{1 + \sqrt{5} |Du(y')|}{\Delta(x, y)} \right) |Du(x') - Du(y')| + \frac{|Du(y')|}{\Delta(x, y)} |e(x) - e(y)|.$$

Collecting estimates we then obtain

$$|B_3| \leq \left(\frac{\sqrt{\kappa_1^2 - 1}}{\Delta(x, y)} + \frac{1 + \sqrt{5} + \sqrt{5} |Du(y')|}{\Delta(x, y)} \right) |Du(x') - Du(y')|$$

$$+ \frac{1 + |Du(y')|}{\Delta(x, y)} |e(x) - e(y)|,$$

and therefore using the estimates of B_1, B_2, and B_3

$$|B| \leq 2 \frac{1 + |Du(y')|}{\Delta(x, y)} (\kappa_1^2 - 1) \, |e(x) - e(y)|$$

$$+ \left(\frac{(\kappa_1^2 - 1)^{3/2}}{\Delta(x, y)} + 2 \frac{1 + \sqrt{5} + \sqrt{5} |Du(y')|}{\Delta(x, y)} (\kappa_1^2 - 1) \right) |Du(x') - Du(y')|.$$

$$(2.8)$$

Next we estimate A:

$$A = \frac{\lambda(x)}{\sqrt{1 + |Du(x')|^2}} Du(x') - \frac{\lambda(y)}{\sqrt{1 + |Du(y')|^2}} Du(y')$$

$$= \frac{\lambda(y)}{\sqrt{1 + |Du(y')|^2}} \left(Du(x') - Du(y') \right)$$

$$+ \left(\frac{\lambda(x)}{\sqrt{1 + |Du(x')|^2}} - \frac{\lambda(y)}{\sqrt{1 + |Du(y')|^2}} \right) Du(x')$$

$$= \frac{\lambda(y)}{\sqrt{1 + |Du(y')|^2}} \left(Du(x') - Du(y') \right) - B \, Du(x').$$

From (2.7) we have

$$\left| \frac{\lambda(y)}{\sqrt{1 + |Du(y')|^2}} \right| \leq \sqrt{\kappa_1^2 - 1}.$$

Then

$$|A| \leq \sqrt{\kappa_1^2 - 1} \, |Du(x') - Du(y')| + |B| \, |Du(x')|.$$

Since $\Delta(x, y) \geq \kappa_1^2 - 1$, $\Delta(x, y) \geq (\kappa_1^2 - 1)|Du(x')|$, $\Delta(x, y) \geq (\kappa_1^2 - 1)|Du(y')|$ and $\Delta(x, y) \geq (\kappa_1^2 - 1)|Du(x')||Du(y')|$, then from (2.8)

$$|B||Du(x')| \leq 4|e(x) - e(y)| + \left(\sqrt{\kappa_1^2 - 1} + 2 + 4\sqrt{5}\right)|Du(x') - Du(y')|,$$

and so

$$|A| \leq 4|e(x) - e(y)| + \left(2\sqrt{\kappa_1^2 - 1} + 2 + 4\sqrt{5}\right)|Du(x') - Du(y')|.$$

Also, from (2.8) and the lower bounds for Δ, we have

$$|B| \leq 4|e(x) - e(y)| + \left(\sqrt{\kappa_1^2 - 1} + 2 + 4\sqrt{5}\right)|Du(x') - Du(y')|.$$

Therefore from (2.6) we obtain

$$\begin{aligned} |m(x) - m(y)| &\leq \frac{1}{\kappa_1}|e(x) - e(y)| + \frac{1}{\kappa_1}(|A| + |B|) \\ &\leq \frac{9}{\kappa_1}|e(x) - e(y)| \\ &\quad + \frac{1}{\kappa_1}\left(3\sqrt{\kappa_1^2 - 1} + 4 + 8\sqrt{5}\right)|Du(x') - Du(y')| \end{aligned}$$

where $x' = \varphi(x)$, $y' = \varphi(y)$ which completes the proof of the lemma. □

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. From (2.3), we write

$$d(x, C, w) = \frac{C + v(x)}{g(x)}, \tag{2.9}$$

with

$$v(x) = -h(x) + e(x) \cdot (x, 0) - (e(x) - \kappa_1\kappa_2 w) \cdot (\varphi(x), u(\varphi(x)), \tag{2.10}$$
$$g(x) = \kappa_1 - \kappa_1\kappa_2 w \cdot m(x). \tag{2.11}$$

Since $g \geq \kappa_1(1 - \kappa_2) > 0$, then

$$\begin{aligned} \left|\frac{C + v(x)}{g(x)} - \frac{C + v(y)}{g(y)}\right| &= \left|\frac{C(g(y) - g(x)) + v(x)g(y) - v(y)g(x)}{g(x)g(y)}\right| \\ &\leq \frac{1}{\kappa_1^2(1 - \kappa_2)^2}(g(y)|v(x) - v(y)| \\ &\quad + (|C| + |v(y)|)|g(y) - g(x)|). \end{aligned}$$

By (2.2), $g(y) \leq \kappa_1(1 - \kappa_2^2)$, and by Lemma 2.3, we have that

$$|g(x) - g(y)| \leq \kappa_1\kappa_2|m(x) - m(y)| \leq C_1|e(x) - e(y)| + C_2|Du(x') - Du(y')|.$$

Note that since Ω is bounded and convex and $|Dh(x)| = |e'(x)| < 1$, then h is bounded in Ω. We also have that u is bounded on Ω', therefore $|v(y)| \leq C(\kappa_1, \kappa_2, u, h, \Omega, \Omega')$. To estimate $|v(y) - v(x)|$ we write

$$
\begin{aligned}
v(x) - v(y) &= h(y) - h(x) + (e(x) \cdot (x, 0) - e(y) \cdot (y, 0)) \\
&\quad + ((e(y) - \kappa_1 \kappa_2 w) \cdot (\varphi(y), u(\varphi(y)))) \\
&\quad - (e(x) - \kappa_1 \kappa_2 w) \cdot (\varphi(x), u(\varphi(x)))) \\
&= A_1 + A_2 + A_3.
\end{aligned}
$$

We have

$$
|A_1| = |h(y) - h(x)| \leq |Dh(\xi)| \, |x - y| \leq \left(\max_{z \in \Omega} |e'(z)| \right) |x - y|.
$$

Also, since $e(x) = (e'(x), e_3(x))$, we have

$$
\begin{aligned}
A_2 = e(x) \cdot (x, 0) - e(y) \cdot (y, 0) &= (e(x) - e(y)) \cdot (x, 0) + e(y) \\
&\quad \cdot ((x, 0) - (y, 0)) \\
&= (e'(x) - e'(y)) \cdot x + e'(y) \cdot (x - y),
\end{aligned}
$$

then

$$
|A_2| \leq \left(\max_{z \in \Omega} |z| \right) |e'(x) - e'(y)| + \left(\max_{z \in \Omega} |e'(z)| \right) |x - y|.
$$

Next

$$
\begin{aligned}
A_3 &= (e(y) - e(x)) \cdot (\varphi(y), u(\varphi(y))) + (e(x) - \kappa_1 \kappa_2 w) \\
&\quad \cdot [(\varphi(y), u(\varphi(y))) - (\varphi(x), u(\varphi(x)))] \\
&= (e(y) - e(x)) \cdot (\varphi(y), u(\varphi(y))) \\
&\quad + (e'(x) - \kappa_1 \kappa_2 w') \cdot (\varphi(y) - \varphi(x)) \\
&\quad + (e_3(x) - \kappa_1 \kappa_2 w_3) (u(\varphi(y)) - u(\varphi(x))),
\end{aligned}
$$

and so

$$
\begin{aligned}
|A_3| &\leq \max_{z \in \Omega'} |(z, u(z))| \, |e(x) - e(y)| \\
&\quad + \left(\max_{z \in \Omega} |e'(z) - \kappa_1 \kappa_2 w'| \right) |\varphi(y) - \varphi(x)| \\
&\quad + (1 + \kappa_1 \kappa_2)|u(\varphi(y)) - u(\varphi(x))|,
\end{aligned}
$$

which concludes the proof of the proposition. ☐

2.2. Analysis of the Self-Intersection of the Surfaces

Since the upper surface $\sigma_{C,w}$ of the lens Γ_w is given parametrically, it might have self intersections, see Figure 2(a). In this case, the lens is not physically realizable.
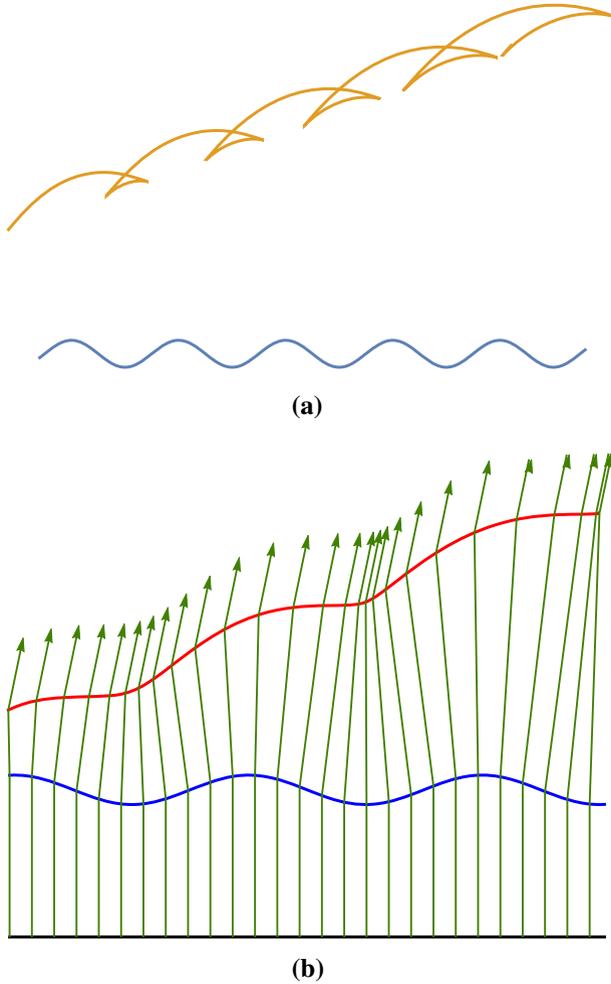
Fig. 2. (**a**) Upper surface with singul ar points and self-intersections (**b**) Lens with ray trajectories

To produce Figures 2(a) and 2(b) we have written a Mathematica code using formulas (2.1) and (2.3) in two dimensions. The lower function used is $u(x) = 2 + A \sin(Mx)$, the incoming field is $(0, 1)$, $h = 0$, $\kappa_1 = 3/2$, $\kappa_2 = 2/3$, $\Omega = (-4, 4)$, and $w = \left(0.2, \sqrt{1 - (0.2)^2}\right)$. For Figure 2(a), we use $C = 2$, $A = -0.2$ and $M = 4$; and for Figure 2(b), $C = 1.31$, $A = -0.2$ and $M = 2$.

In this section, we will use the Lipschitz estimate of d from Proposition 2.1 to show that if the field e, the bottom surface of the lens u, and w, are all suitably chosen, then the constant C can be chosen so that $d(x, C, w) > 0$ and the surface $\sigma_{C,w}$ parametrized by $f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w)m(x)$ does not have self intersections. The special case where $e(x) = w = (0, 0, 1)$ is discussed in [13, Remark 3.4].

Recall that L_F denotes the Lipschitz constant of the map F, i.e., $|F(x) - F(y)| \leq L_F |x - y|$ for all x, y in the corresponding domain. We assume that the incident field e is never horizontal, i.e., $e_3(x) \geq \delta > 0$, for some $\delta > 0$. We first prove that under conditions on the Lipschitz constants of u and e, and the L^∞-norm of e', the map φ is bi-Lipschitz.

Lemma 2.4. *Suppose $e_3(x) \geq \delta > 0$ for $x \in \Omega$. If*

$$\frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} L_u \leq 1/2, \tag{2.12}$$

then φ satisfies the following Lipschitz estimate

$$|\varphi(x) - \varphi(y)| \leq 2 \left\{ 1 + \left(\frac{1}{\delta} + \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} \right) \left(\max_{\Omega'} u \right) L_e \right\} |x - y|. \tag{2.13}$$

If in addition, L_e is small enough, i.e.,

$$\left(\frac{1}{\delta} + \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} \right) \left(\max_{\Omega'} u \right) L_e < 1/2, \tag{2.14}$$

then φ is bi-Lipschitz, and for $x \neq y$,

$$\frac{1}{3} |x - y| < |\varphi(x) - \varphi(y)| < 3 |x - y|. \tag{2.15}$$

Proof. We can write $(\varphi(x), u(\varphi(x))) = (x, 0) + \rho\, e(x)$, for $\rho(x)$ a positive function. Then

$$\varphi(x) = x + \rho\, e'(x)$$
$$u(\varphi(x)) = \rho\, e_3(x),$$

so $\varphi(x)$ satisfies the equation

$$\varphi(x) = x + \frac{u(\varphi(x))}{e_3(x)} e'(x) := x + \psi(x).$$

We write

$$\psi(x) - \psi(y) = \frac{u(\varphi(x))}{e_3(x)} \left(e'(x) - e'(y) \right) + \left(\frac{u(\varphi(x))}{e_3(x)} - \frac{u(\varphi(y))}{e_3(y)} \right) e'(y)$$
$$= A + B.$$

Since $e_3(x) \geq \delta > 0$ for all $x \in \Omega$, it follows that

$$|A| \leq \frac{\max_{\Omega'} u}{\delta} L_e |x - y|.$$

Also

$$|B| = \left| \frac{u\left(\varphi(x)\right) e_3(y) - u\left(\varphi(y)\right) e_3(x)}{e_3(x)\, e_3(y)} \right| |e'(y)|$$

$$\leq \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} \left| u\left(\varphi(x)\right) \left(e_3(y) - e_3(x)\right) + \left(u\left(\varphi(x)\right) - u\left(\varphi(y)\right)\right) e_3(x) \right|$$

$$\leq \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} \left\{ L_e \left(\max_{\Omega'} u \right) |x - y| + L_u \, |\varphi(x) - \varphi(y)| \right\}.$$

Therefore

$$|\psi(x) - \psi(y)| \leq \left(\frac{1}{\delta} + \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} \right) \left(\max_{\Omega'} u \right) L_e \, |x - y|$$

$$+ \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} L_u \, |\varphi(x) - \varphi(y)|.$$

Since

$$\varphi(x) - \varphi(y) = x - y + \psi(x) - \psi(y),$$

we get

$$|\varphi(x) - \varphi(y)|$$
$$\leq \left\{ 1 + \left(\frac{1}{\delta} + \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} \right) \left(\max_{\Omega'} u \right) L_e \right\} |x - y|$$

$$+ \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} L_u \, |\varphi(x) - \varphi(y)|.$$

Then (2.13) follows from (2.12); and the upper estimate in (2.15) follows from (2.14).

On the other hand,

$$|\varphi(x) - \varphi(y)| \geq |x - y| - |\psi(x) - \psi(y)|$$
$$\geq \left\{ 1 - \left(\frac{1}{\delta} + \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} \right) \left(\max_{\Omega'} u \right) L_e \right\} |x - y|$$

$$- \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} L_u \, |\varphi(x) - \varphi(y)|.$$

Hence

$$\left\{ 1 + \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} L_u \right\} |\varphi(x) - \varphi(y)|$$

$$\geq \left\{ 1 - \left(\frac{1}{\delta} + \frac{1}{\delta^2} \|e'\|_{L^\infty(\Omega)} \right) \left(\max_{\Omega'} u \right) L_e \right\} |x - y|,$$

which from (2.14) and (2.12) yields the lower estimate in (2.15). □

With this lemma in hand, we give conditions on the size of the Lipschitz constants of u, Du, and e so that if the constant C is appropriately chosen, then the surface $f(x, C, w)$ does not have self-intersections. The following theorem shows that a small perturbation of the collimated case considered in [13, Remark 3.4] gives also surfaces that are physically realizable:

Theorem 2.5. *Suppose* (2.12) *and* (2.14) *hold. There are positive constants* $C_1 - C_5$ *such that if*

$$\alpha = C_1 \, (L_e + 3 L_{Du}) \tag{2.16}$$

and

$$\beta_w = C_2 \, (L_e + 3 L_{Du}) + C_3 \, \|e'\|_{L^\infty(\Omega)} + 3 C_4 \left(\max_{z \in \Omega} |e'(z) - \kappa_1 \kappa_2 \, w'| \right) + 3 C_5 \, L_u, \tag{2.17}$$

then we have the following: if we choose L_e, L_{Du}, $\|e'\|_{L^\infty(\Omega)}$, $\max_{z \in \Omega} |e'(z) - \kappa_1 \kappa_2 \, w'|$, *and* L_u *all sufficiently small satisfying*

$$\beta_w < 1/3,$$

and

$$C^* < \frac{1/3 - \beta_w}{\alpha},$$

with C^* *given by* (2.4), *then the surface parametrized by*

$$f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w) \, m(x)$$

is physically realizable, i.e., f *is injective and* $d(x, C, w) > 0$, *for* $C > C^*$ *and* $C \in \left[-\dfrac{1/3 - \beta_w}{\alpha}, \dfrac{1/3 - \beta_w}{\alpha} \right]$.

The constants C_1, C_3, C_4, C_5 *depend only on* κ_1 *and* κ_2 *and the constant* C_2 *depends only on* $\kappa_1, \kappa_2, u, h, \Omega$ *and* Ω'.

Proof. Assume f is not injective, then there are two points $x, y \in \Omega$, $x \neq y$, such that $f(y, C, w) = f(x, C, w)$. We first prove that this implies that α in (2.16) is not zero (independently of $C_1 > 0$ to be chosen later). In fact, if $\alpha = 0$, then e is constant and $L_{Du} = 0$. This means the emanating rays are parallel and u is a plane with normal v. Therefore, u refracts all rays into a fixed unit direction m. Since $f(x, C, w) = f(y, C, w)$, we obtain

$$(\varphi(x) - \varphi(y), u(\varphi(x)) - u(\varphi(y))) = (d(x, C, w) - d(y, C, w)) \, m.$$

Since $x \neq y$, from Lemma 2.4 $\varphi(x) \neq \varphi(y)$, so $d(x, C, w) \neq d(y, C, w)$. Since the graph of u is planar, dotting the last identity with v, yields $m \cdot v = 0$, a contradiction with the Snell law since $\kappa_1 > 1$. Therefore, if there are self-intersections, then $\alpha \neq 0$.

On the other hand,

$$
\begin{aligned}
|(\varphi(y), u(\varphi(y))) - (\varphi(x), u(\varphi(x)))| &= |d(y, C, w)m(y) - d(x, C, w)m(x)| \\
&\leq |d(y, C, w) - d(x, C, w)| \\
&\quad + d(x, C, w)|m(y) - m(x)| \\
&:= I + II
\end{aligned}
$$

To estimate I, we use Proposition 2.1. To estimate II, we have from (2.9), (2.10), (2.11), and since $g(x) \geq \kappa_1(1 - \kappa_2)$, that

$$
d(x, C, w) \leq \frac{1}{\kappa_1(1 - \kappa_2)} \left(|C| + C(\kappa_1, \kappa_2, u, h, \Omega, \Omega') \right). \tag{2.18}
$$

From Lemma 2.3, we have

$$
|m(x) - m(y)| \leq C_1(\kappa_1) |e(x) - e(y)| + C_2(\kappa_1) |Du(\varphi(x)) - Du(\varphi(y))|,
$$

then

$$
\begin{aligned}
II &\leq \frac{1}{\kappa_1(1 - \kappa_2)} \left(|C| + C(\kappa_1, \kappa_2, u, h, \Omega, \Omega') \right) (C_1(\kappa_1) |e(x) - e(y)| \\
&\quad + C_2(\kappa_1) |Du(\varphi(x)) - Du(\varphi(y))|).
\end{aligned}
$$

Combining the estimates for I and II we obtain

$$
\begin{aligned}
|(\varphi(y), &u(\varphi(y))) - (\varphi(x), u(\varphi(x)))| \\
&\leq [C_1 |C| + C_2] (|e(x) - e(y)| \\
&\quad + |Du(\varphi(x)) - Du(\varphi(y))|) \\
&\quad + C_3 \|e'\|_{L^\infty(\Omega)} |x - y| \\
&\quad + C_4 \left(\max_{z \in \Omega} |e'(z) - \kappa_1 \kappa_2 w'| \right) |\varphi(y) - \varphi(x)| \\
&\quad + C_5 |u(\varphi(y)) - u(\varphi(x))|.
\end{aligned} \tag{2.19}
$$

Since (2.12) and (2.14) hold, then by Lemma 2.4 we get (2.15), replacing in (2.19) we obtain

$$
\begin{aligned}
1/3 &< [C_1 |C| + C_2] (L_e + 3 L_{Du}) + C_3 \|e'\|_{L^\infty(\Omega)} \\
&\quad + 3 C_4 \left(\max_{z \in \Omega} |e'(z) - \kappa_1 \kappa_2 w'| \right) + 3 C_5 L_u,
\end{aligned} \tag{2.20}
$$

which reads

$$
1/3 < \alpha |C| + \beta_w.
$$

If L_e, L_{Du}, $\|e'\|_{L^\infty(\Omega)}$, $\max_{z \in \Omega} |e'(z) - \kappa_1 \kappa_2 w'|$, and L_u are chosen sufficiently small so that

$$
\beta_w < 1/3,
$$

then from (2.20), $f(x, C, w) = f(y, C, w)$ with $x \neq y$ implies

$$|C| > \frac{1/3 - \beta_w}{\alpha}.$$

Therefore, if $|C| \leq \dfrac{1/3 - \beta_w}{\alpha}$, the surface $\sigma_{C,w}$ cannot have self intersections. Recall that when $C > C^*$, with C^* given in (2.4), $d(x, C, w) > 0$. By choosing, if needed so, $L_e, L_{Du}, \|e'\|_{L^\infty(\Omega)}, \max_{z \in \Omega} |e'(z) - \kappa_1 \kappa_2 \, w'|$, and L_u even smaller than before, we have $C^* < \dfrac{1/3 - \beta_w}{\alpha}$.

In conclusion, if we pick $C > C^*$ and $C \in \left[-\dfrac{1/3 - \beta_w}{\alpha}, \dfrac{1/3 - \beta_w}{\alpha} \right]$, then $d(x, C, w) > 0$ and the surface $f(x, C, w)$ is injective in Ω. □

3. Discussion About the Singular Points of f

We say that a surface parametrized by a function $f(x)$, $x = (x_1, x_2)$, is regular at a point y if $f_{x_1} \times f_{x_2} \neq 0$ at y. That is, at each regular point the surface has a normal vector. Otherwise, y is a singular point.

It is proved in [13] that if a lens sandwiched between the lower surface u and the upper surface f, refracts all rays with direction $e(x)$ into the direction w, and f is a regular surface at each point, then the upper surface is parametrized by $f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w)m(x)$ with $d(x, C, w)$ given by (2.3). In general, such a parametrization might lead to a surface having singular points and therefore at those points there cannot be refraction since the normal is not defined.

The purpose of this section is to show that under appropriate assumptions on u and for a range of values of the constant C, that parametrization indeed leads to a regular surface and therefore the lens sandwiched by u and $f(x, C, w)$ refracts each ray emanating from x with direction $e(x)$ into the direction w.

To simplify the notation in this section we write $f(x)$ instead of $f(x, C, w)$ and $d(x)$ instead of $d(x, C, w)$. Let us first define

$$M(x) = \begin{pmatrix} f_{x_1}(x) \cdot f_{x_1}(x) & f_{x_1}(x) \cdot f_{x_2}(x) \\ f_{x_2}(x) \cdot f_{x_1}(x) & f_{x_2}(x) \cdot f_{x_2}(x) \end{pmatrix},$$

and recall that

$$\left| f_{x_1}(x) \times f_{x_2}(x) \right|^2 = \left| f_{x_1}(x) \right|^2 \left| f_{x_2}(x) \right|^2 - \left(f_{x_1}(x) \cdot f_{x_2}(x) \right)^2 = \det M(x).$$

3.1. Case of a General Field $e(x)$

We consider the unit incident field $e(x) = (e_1(x), e_2(x), e_3(x))$ with $e_3(x) > 0$. The upper face of the lens is parametrized by

$$f(x) = (\varphi(x), u(\varphi(x))) + d(x) \, m(x) = v(x) + d(x) \, m(x)$$

where $\varphi(x) = (\varphi_1(x), \varphi_2(x))$; $x = (x_1, x_2)$; recall $f(x) = f(x, C, w)$ and $d(x) = d(x, C, w)$ given in (2.3). Assume $C > C^*$, where C^* is given by (2.4).

The goal is to find conditions so that a given point y is a regular point of the surface described by f, i.e., $|f_{x_1}(y) \times f_{x_2}(y)| > 0$. This is the contents of the following theorem.

Theorem 3.1. *Suppose* curl $e' = 0$, *u is concave at y, and y is a regular point for the surface $v(x) = (\varphi(x), u(\varphi(x)))$. If the matrix*

$$W := \left(e_{x_i} \cdot v_{x_j} + e_{x_j} \cdot v_{x_i} \right)_{ij} \geq 0 \text{ at } x = y, \tag{3.1}$$

then y is a regular point for the surface $f(x) = v(x) + d(x) m(x)$. In particular, if $e' = Dh$ and h is convex at y, then y is a regular point for f.

Proof. We have $f_{x_i} = v_{x_i} + d_{x_i} m + d\, m_{x_i}$. Since $m(x)$ is a unit vector,

$$f_{x_i} \cdot f_{x_j} = v_{x_i} \cdot v_{x_j} + A_{ij} + B_{ij} + d_{x_i} d_{x_j} + d^2\, m_{x_i} \cdot m_{x_j}, \tag{3.2}$$

where

$$A_{ij} = d_{x_i} m \cdot v_{x_j} + d_{x_j} m \cdot v_{x_i}, \qquad B_{ij} = d\, m_{x_i} \cdot v_{x_j} + d\, m_{x_j} \cdot v_{x_i}.$$

We have $v \cdot v_{x_i} = 0$ and so by (2.1) $m \cdot v_{x_i} = \dfrac{1}{\kappa_1} e \cdot v_{x_i}$. Hence

$$v_{x_i} \cdot v_{x_j} + A_{ij} + d_{x_i} d_{x_j} = v_{x_i} \cdot v_{x_j} + \frac{1}{\kappa_1} \left(d_{x_i} e \cdot v_{x_j} + d_{x_j} e \cdot v_{x_i} \right) + d_{x_i} d_{x_j}$$

$$= \left(1 - \frac{1}{\kappa_1^2} \right) v_{x_i} \cdot v_{x_j} + \left(\frac{v_{x_i}}{\kappa_1} + d_{x_i} e \right) \cdot \left(\frac{v_{x_j}}{\kappa_1} + d_{x_j} e \right).$$

Set $g_i = \dfrac{v_{x_i}}{\kappa_1} + d_{x_i} e$, then the matrix $(g_i \cdot g_j)_{ij} \geq 0$ at each point. Since y is a regular point for v, then the matrix $\left(v_{x_i} \cdot v_{x_j} \right)_{ij} > 0$ at y. It is clear that $\left(d^2\, m_{x_i} \cdot m_{x_j} \right)_{ij}$ is positive semi definite. Therefore, the matrix $\left(v_{x_i} \cdot v_{x_j} + A_{ij} + d_{x_i} d_{x_j} + d^2\, m_{x_i} \cdot m_{x_j} \right)_{ij}$ is positive definite at y.

We shall prove that if u is concave and (3.1) holds, then the matrix (B_{ij}) is positive semi-definite. Since $m_{x_i} = \dfrac{1}{\kappa_1} \left(e_{x_i} - \lambda_{x_i} v - \lambda v_{x_i} \right)$ and $v \cdot v_{x_j} = 0$, it follows that

$$m_{x_i} \cdot v_{x_j} = \frac{1}{\kappa_1} \left(e_{x_i} \cdot v_{x_j} - \lambda\, v_{x_i} \cdot v_{x_j} \right). \tag{3.3}$$

We calculate $v_{x_i} \cdot v_{x_j}$. First notice that

$$v_{x_i} = \left\{ \frac{(-Du(\varphi), 1)}{\sqrt{1 + |Du(\varphi)|^2}} \right\}_{x_i}$$

$$= \left\{ \frac{1}{\sqrt{1 + |Du(\varphi)|^2}} \right\}_{x_i} (-Du(\varphi), 1)$$

$$\quad - \frac{1}{\sqrt{1 + |Du(\varphi)|^2}} (Du_{z_1}(\varphi) \cdot \varphi_{x_i}, Du_{z_2}(\varphi) \cdot \varphi_{x_i}, 0)$$

$$= \left\{ \frac{1}{\sqrt{1 + |Du(\varphi)|^2}} \right\}_{x_i} (-Du(\varphi), 1)$$

$$+ \frac{1}{\sqrt{1 + |Du(\varphi)|^2}} (-D^2 u(\varphi)\varphi_{x_i}, 0). \tag{3.4}$$

Therefore

$$v_{x_j} \cdot v_{x_i} = (\varphi_{x_j}, Du(\varphi) \cdot \varphi_{x_j}) \cdot v_{x_i} = -\frac{1}{\sqrt{1 + |Du(\varphi)|^2}} \varphi_{x_j} \cdot D^2 u(\varphi)\varphi_{x_i}. \tag{3.5}$$

Let us analyze the definiteness of matrix $H := \left(\varphi_{x_j} \cdot D^2 u(\varphi)\varphi_{x_i} \right)_{ij}$. Since u is C^2, $D^2 u$ is symmetric and so H is symmetric. To simplify the writing let $\tau_i = \varphi_{x_i}$, $\bar{\tau}_i = (\tau_i, 0)$, $i = 1, 2$, and

$$\overline{D^2 u} = \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that

$$\varphi_{x_j} \cdot D^2 u(\varphi)\varphi_{x_i} = \tau_j \cdot D^2 u(\varphi)\tau_i = \bar{\tau}_j \cdot \overline{D^2 u}\, \bar{\tau}_i.$$

Then

$$\det H = \det \left(\bar{\tau}_j \cdot \overline{D^2 u}\, \bar{\tau}_i \right) = (\bar{\tau}_1 \times \bar{\tau}_2) \cdot \left(\overline{D^2 u}\, \bar{\tau}_1 \times \overline{D^2 u}\, \bar{\tau}_2 \right)$$

by the Cauchy-Binet formula for the cross product. Next

$$\bar{\tau}_1 \times \bar{\tau}_2 = \det \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \mathbf{k},$$

and since

$$\overline{D^2 u}\, \bar{\tau}_1 = \left((D^2 u)\, \tau_1, 0 \right), \qquad \overline{D^2 u}\, \bar{\tau}_2 = \left((D^2 u)\, \tau_2, 0 \right),$$

we get

$$\overline{D^2 u}\, \bar{\tau}_1 \times \overline{D^2 u}\, \bar{\tau}_2 = \det \begin{pmatrix} (D^2 u)\, \tau_1 \\ (D^2 u)\, \tau_2 \end{pmatrix} \mathbf{k} = \det(D^2 u) \det \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \mathbf{k}.$$

Therefore

$$\det H = \det(D^2 u) \left[\det \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \right]^2.$$

Also trace $H = \tau_1 \cdot D^2 u\, \tau_1 + \tau_2 \cdot D^2 u\, \tau_2$. Since u is concave at y, we obtain $\det H \geq 0$ and trace $H \leq 0$, so $H \leq 0$ at y since H is symmetric. From (3.5) and since $\lambda < 0$, it follows that the symmetric matrix $\left(-\lambda\, v_{x_j} \cdot v_{x_i} \right)_{ij}$ is positive semi-definite at y. From (3.3) and (3.1) we conclude that

$$B_{ij} = d\, m_{x_i} \cdot v_{x_j} + d\, m_{x_j} \cdot v_{x_i}$$

$$= \frac{d}{\kappa_1} \left(e_{x_i} \cdot v_{x_j} + e_{x_j} \cdot v_{x_i} - \lambda \left(v_{x_i} \cdot v_{x_j} + v_{x_j} \cdot v_{x_i} \right) \right)$$

is positive semi-definite at y as desired. Thus, from (3.2) the matrix $(f_{x_i} \cdot f_{x_j})_{ij}$ is positive definite at $x = y$ because it is written as the sum of the positive definite matrix $\left(\left(1 - \dfrac{1}{\kappa_1^2} \right) v_{x_i} \cdot v_{x_j} \right)_{ij}$ plus a positive semi-definite matrix.

Finally, let us analyze condition (3.1). We have

$$v(x) = (\varphi(x), u(\varphi(x))) = (x, 0) + \rho \, e(x),$$

so $v_{x_j} = (x, 0)_{x_j} + \rho_{x_j} e + \rho \, e_{x_j}$ and

$$v_{x_j} \cdot e_{x_i} = (x, 0)_{x_j} \cdot e_{x_i} + \rho_{x_j} e \cdot e_{x_i} + \rho \, e_{x_j} \cdot e_{x_i} = (x, 0)_{x_j} \cdot e_{x_i} + \rho \, e_{x_j} \cdot e_{x_i}. \quad (3.6)$$

Hence

$$\begin{bmatrix} v_{x_1} \cdot e_{x_1} & v_{x_2} \cdot e_{x_1} \\ v_{x_1} \cdot e_{x_2} & v_{x_2} \cdot e_{x_2} \end{bmatrix} = \begin{bmatrix} (e_1)_{x_1} & (e_2)_{x_1} \\ (e_1)_{x_2} & (e_2)_{x_2} \end{bmatrix} + \rho \begin{bmatrix} e_{x_1} \cdot e_{x_1} & e_{x_1} \cdot e_{x_2} \\ e_{x_1} \cdot e_{x_2} & e_{x_2} \cdot e_{x_2} \end{bmatrix}.$$

Since curl $e' = 0$, i.e., $(e_1)_{x_2} = (e_2)_{x_1}$, the matrix $(e_{x_i} \cdot v_{x_j})_{ij}$ is symmetric and so $W = 2 \left(e_{x_i} \cdot v_{x_j} \right)_{ij}$. Thus,

$$\det W = 4 \det \begin{bmatrix} v_{x_1} \cdot e_{x_1} & v_{x_1} \cdot e_{x_2} \\ v_{x_2} \cdot e_{x_1} & v_{x_2} \cdot e_{x_2} \end{bmatrix} = 4 \left(v_{x_1} \times v_{x_2} \right) \cdot \left(e_{x_1} \times e_{x_2} \right),$$

by Cauchy-Binet's formula. Since $e_3 = \sqrt{1 - e_1^2 - e_2^2}$,

$$e_{x_1} \times e_{x_2} = \frac{1}{e_3} \det \begin{bmatrix} (e_1)_{x_1} & (e_2)_{x_1} \\ (e_1)_{x_2} & (e_2)_{x_2} \end{bmatrix} e.$$

Also, since $v_{x_i} = (\varphi_{x_i}, Du(\varphi) \cdot \varphi_{x_i})$, by calculation,

$$v_{x_1} \times v_{x_2} = \sqrt{1 + |Du|^2} \det \begin{bmatrix} (\varphi_1)_{x_1} & (\varphi_2)_{x_1} \\ (\varphi_1)_{x_2} & (\varphi_2)_{x_2} \end{bmatrix} v.$$

Therefore, (3.1) is equivalent to

$$\det W = 4 \frac{\sqrt{1 + |Du|^2}}{e_3} \det \begin{bmatrix} (\varphi_1)_{x_1} & (\varphi_2)_{x_1} \\ (\varphi_1)_{x_2} & (\varphi_2)_{x_2} \end{bmatrix} \det \begin{bmatrix} (e_1)_{x_1} & (e_2)_{x_1} \\ (e_1)_{x_2} & (e_2)_{x_2} \end{bmatrix} (e \cdot v)$$
$$\geq 0 \quad \text{at } x = y, \quad (3.7)$$

and

$$\text{trace } W = 2 \left(v_{x_1} \cdot e_{x_1} + v_{x_2} \cdot e_{x_2} \right) \quad (3.8)$$
$$= 2 \left((e_1)_{x_1} + (e_2)_{x_2} + \rho \left(e_{x_1} \cdot e_{x_1} + e_{x_2} \cdot e_{x_2} \right) \right)$$
$$= 2 \left(h_{x_1 x_1} + h_{x_2 x_2} + \frac{u(\varphi(x))}{e_3(x)} \left(e_{x_1} \cdot e_{x_1} + e_{x_2} \cdot e_{x_2} \right) \right) \geq 0 \quad \text{at } x = y,$$

where $e' = (e_1, e_2) = Dh$; recall that here Du is calculated at $\varphi(y)$, ν is the normal to u at $(\varphi(y), u(\varphi(y)))$, and e_j are calculated at y.

We will simplify (3.7). We first write $\det\left(\dfrac{\partial\varphi}{\partial x}\right)$ (using the notation at the end of paper) in terms of u and Du. Since $(\varphi(x), u(\varphi(x))) = (x, 0) + \rho\, e(x), \varphi(x)$ satisfies the equation

$$x + \frac{u(\varphi(x))}{e_3(x)}\, e'(x) = \varphi(x).$$

Let

$$F(x, z) = x + \frac{u(z)}{e_3(x)}\, e'(x) - z,$$

with $x = (x_1, x_2)$ and $z = (z_1, z_2)$. We have $\dfrac{\partial F}{\partial z} = -Id + Du(z) \otimes \dfrac{e'(x)}{e_3(x)}$ and $\dfrac{\partial F}{\partial x} = Id + u(z)\dfrac{\partial(e'/e_3)}{\partial x}$. By assumption $e \cdot \nu > 0$, and

$$e \cdot \nu = \frac{1}{\sqrt{1 + |Du(\varphi)|^2}}\left(-Du(\varphi) \cdot e'(x) + e_3(x)\right).$$

By Sherman-Morrison's formula [24], $\det\dfrac{\partial F}{\partial z}(x, \varphi(x)) = \det\left(Id - Du(\varphi(x))\right.$ $\left.\otimes \dfrac{e'(x)}{e_3(x)}\right) = 1 - \dfrac{e'(x)}{e_3(x)} \cdot Du(\varphi(x)) = \dfrac{1}{e_3}\sqrt{1 + |Du(\varphi)|^2}\, e \cdot \nu \neq 0$ and therefore $\dfrac{\partial F}{\partial z}(x, \varphi(x))$ is invertible. Since $F(x, \varphi(x)) = 0$, we then get

$$\frac{\partial\varphi}{\partial x} = -\left(\frac{\partial F}{\partial z}(x, \varphi(x))\right)^{-1}\frac{\partial F}{\partial x}(x, \varphi(x))$$

$$= \left(Id - Du(\varphi(x)) \otimes \frac{e'(x)}{e_3(x)}\right)^{-1}\left(Id + u(\varphi(x))\frac{\partial(e'/e_3)}{\partial x}(x)\right)$$

and so

$$\det\frac{\partial\varphi}{\partial x} = \frac{1}{\dfrac{1}{e_3}\sqrt{1 + |Du(\varphi)|^2}\, e \cdot \nu}\det\left(Id + u(\varphi(x))\frac{\partial(e'/e_3)}{\partial x}(x)\right).$$

So (3.7) can be written as

$$\det W = 4\det\left(Id + u(\varphi(x))\frac{\partial(e'/e_3)}{\partial x}(x)\right)\det\left(\frac{\partial e'}{\partial x}(x)\right) \geq 0 \quad \text{at } x = y. \tag{3.9}$$

Notice that since $\dfrac{\partial e'}{\partial x}$ is symmetric,

$$\frac{\partial(e'/e_3)}{\partial x} = \frac{1}{e_3}\frac{\partial e'}{\partial x} + \frac{1}{e_3^3}\left(e' \otimes e'\right)\frac{\partial e'}{\partial x}.$$

In fact, since $e \cdot e_{x_j} = 0$, we have $\partial_{x_j} e_3 = -\dfrac{1}{e_3}\left(\dfrac{\partial e'}{\partial x}e'\right)_j$. Hence $\partial_{x_j}\left(e_i\, e_3^{-1}\right) =$

$\left(\partial_{x_j} e_i\right) e_3^{-1} - e_i\, e_3^{-2}\partial_{x_j} e_3 = \left(\partial_{x_j} e_i\right) e_3^{-1} + e_3^{-3}\, e_i\left(\dfrac{\partial e'}{\partial x}e'\right)_j$ and the formula follows. Therefore (3.9) reads

$$\det W = 4 \det\left(\frac{\partial e'}{\partial x} + \frac{u(\varphi(x))}{e_3}\left(\frac{\partial e'}{\partial x}\right)^2 + \frac{u(\varphi(x))}{e_3^3}\frac{\partial e'}{\partial x}(e' \otimes e')\frac{\partial e'}{\partial x}\right) \geq 0. \tag{3.10}$$

Since $e' = Dh$, (3.10) reads

$$\det W = 4 \det\left(D^2 h + \frac{u(\varphi(x))}{e_3}\left(D^2 h\right)^2 + \frac{u(\varphi(x))}{e_3^3}D^2 h(Dh \otimes Dh)D^2 h\right)$$
$$\geq 0 \quad \text{at } x = y,$$

and if h convex at y this clearly holds and also (3.8). This completes the proof of the theorem. $\quad\square$

3.2. Collimated Case: $e(x) = e_3 = (0,0,1)$

The upper surface of the lens is parametrized by $f(x) = (x, u(x)) + d(x)\, m(x)$, where from (2.3)

$$d(x) = d(x, C, w) = \frac{C - (e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))}{\kappa_1 - \kappa_1\kappa_2 w \cdot m(x)}. \tag{3.11}$$

Since we need $d(x) > 0$ for $x \in \Omega$, C must be such that $C > \max_{x\in\Omega}\{(e_3 - \kappa_1\kappa_2\, w)\cdot (x, u(x))\}$. Since the incident field is now explicit, we obtain more information than in Theorem 3.1 for points where u is not necessarily concave.

Theorem 3.2. *Assume the parametrization* $f(x) = (x, u(x)) + d(x)\, m(x)$ *where* $d(x)$ *is given by* (3.11), $C > \max_{x\in\Omega}\{(e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))\}$, *and* $\mu(y) =$ *maximum eigenvalue of* $D^2 u(y)$. *If*

(1) $\mu(y) \leq 0$, *or*
(2) $\mu(y) > 0$ *and*

$$C < \frac{\kappa_1^2(1 - \kappa_2)\sqrt{1 + |Du(y)|^2}}{\mu(y)\sqrt{\kappa_1^2 - 1}} + \min_{x\in\Omega}\{(e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))\},$$

then y *is a regular point for* f.

Proof. The first part of the theorem follows immediately from Theorem 3.1 since $W = 0$.

As before letting $v(x) = (x, u(x))$, we first find explicit expressions for the terms in (3.2) that will lead to formula (3.24). From (2.1), $m(x) = \dfrac{1}{\kappa_1}(e_3 - \lambda(x)$

$\nu(x))$ with $\nu(x) = \dfrac{(-Du(x), 1)}{\sqrt{1 + |Du(x)|^2}}$, the outer unit normal to the graph of u at $(x, u(x))$, and $\lambda(x) = e_3 \cdot \nu - \sqrt{\kappa_1^2 - 1 + (e_3 \cdot \nu)^2}$. Since v_{x_i} is tangent to the graph of u at $(x, u(x))$, $\nu \cdot (1, 0, u_{x_1}) = 0$,

$$m \cdot v_{x_i} = \frac{1}{\kappa_1} u_{x_i}(x), \qquad i = 1, 2. \tag{3.12}$$

Since $|\nu| = 1$, then $\nu \cdot \nu_{x_i} = 0$ and therefore

$$m_{x_i} \cdot m_{x_j} = \frac{1}{\kappa_1^2}\left(\lambda_{x_i}\lambda_{x_j} + \lambda^2 \, \nu_{x_i} \cdot \nu_{x_j}\right) \tag{3.13}$$

From (3.4)

$$v_{x_i} = \frac{-Du \cdot Du_{x_i}}{\left(1 + |Du|^2\right)^{3/2}}(-Du, 1) + \frac{1}{\sqrt{1 + |Du|^2}}(-Du_{x_i}, 0). \tag{3.14}$$

So from (3.3)

$$v_{x_j} \cdot m_{x_i} = \frac{\lambda \, u_{x_j x_i}}{\kappa_1 \sqrt{1 + |Du|^2}}. \tag{3.15}$$

Replacing (3.12), (3.13), (3.15), in the formulas for $f_{x_i} \cdot f_{x_j}$ the matrix $M = (f_{x_i} \cdot f_{x_j})_{ij}$ can be written as follows:

$$M = Id + Du \otimes Du + \frac{1}{\kappa_1}(Du \otimes Dd + Dd \otimes Du) + \frac{2\,d\lambda}{\kappa_1\sqrt{1 + |Du|^2}}D^2u + Dd \otimes Dd$$

$$+ \frac{d^2}{\kappa_1^2}\left(D\lambda \otimes D\lambda + \lambda^2\begin{pmatrix} \nu_{x_1} \cdot \nu_{x_1} & \nu_{x_1} \cdot \nu_{x_2} \\ \nu_{x_1} \cdot \nu_{x_2} & \nu_{x_2} \cdot \nu_{x_2} \end{pmatrix}\right)$$

$$= Id + \frac{\kappa_1^2 - 1}{\kappa_1^2}Du \otimes Du + \left(\frac{1}{\kappa_1}Du + Dd\right) \otimes \left(\frac{1}{\kappa_1}Du + Dd\right)$$

$$+ \frac{2\,d\lambda}{\kappa_1\sqrt{1 + |Du|^2}}D^2u + \frac{d^2}{\kappa_1^2}\mathcal{L};$$

where

$$\mathcal{L} = D\lambda \otimes D\lambda + \lambda^2\begin{pmatrix} \nu_{x_1} \cdot \nu_{x_1} & \nu_{x_1} \cdot \nu_{x_2} \\ \nu_{x_1} \cdot \nu_{x_2} & \nu_{x_2} \cdot \nu_{x_2} \end{pmatrix}.$$

We next calculate $\mathcal{L}$ by calculating first $D\lambda \otimes D\lambda$. Notice that

$$\lambda = e_3 \cdot \nu - \sqrt{\kappa_1^2 - 1 + (e_3 \cdot \nu)^2} = \frac{1 - \kappa_1^2}{e_3 \cdot \nu + \sqrt{\kappa_1^2 - 1 + (e_3 \cdot \nu)^2}} = \phi_{\kappa_1}(e_3 \cdot \nu),$$

with

$$\phi_\kappa(t) = \frac{1 - \kappa^2}{t + \sqrt{\kappa^2 - 1 + t^2}}; \tag{3.16}$$

and

$$\phi_\kappa'(t) = -\frac{1}{\sqrt{\kappa^2 - 1 + t^2}}\, \phi_\kappa(t). \tag{3.17}$$

Then

$$\lambda_{x_i} = (e_3 \cdot v)_{x_i}\, \phi_{\kappa_1}'(e_3 \cdot v) = -\left\{\frac{1}{\sqrt{1 + |Du|^2}}\right\}_{x_i} \frac{1}{\sqrt{\kappa_1^2 - 1 + \dfrac{1}{1 + |Du|^2}}}\, \lambda$$

$$= \frac{Du \cdot Du_{x_i}}{(1 + |Du|^2)^{3/2}} \frac{\sqrt{1 + |Du|^2}}{\sqrt{(\kappa_1^2 - 1)(1 + |Du|^2) + 1}}\, \lambda = \frac{Du \cdot Du_{x_i}}{(1 + |Du|^2)\sqrt{\Delta}}\, \lambda,$$

with

$$\Delta = \kappa_1^2 + (\kappa_1^2 - 1)|Du|^2. \tag{3.18}$$

Hence

$$D\lambda \otimes D\lambda = \frac{\lambda^2}{\Delta(1 + |Du|^2)^2}\left((Du \cdot Du_{x_1},\, Du \cdot Du_{x_2})\right.$$
$$\left.\otimes (Du \cdot Du_{x_1},\, Du \cdot Du_{x_2})\right)$$

Notice that $(Du \cdot Du_{x_1},\, Du \cdot Du_{x_2}) = D^2u(Du)$, and

$$(D^2u\,Du) \otimes (D^2u\,Du) = (D^2u\,Du)(D^2u\,Du)^t = D^2u\,Du(Du)^t(D^2u)^t$$
$$= D^2u(Du \otimes Du)D^2u.$$

We conclude that

$$D\lambda \otimes D\lambda = \frac{\lambda^2}{\Delta(1 + |Du|^2)^2}\, D^2u(Du \otimes Du)D^2u \tag{3.19}$$

We next calculate the matrix $\bar{v} := \begin{pmatrix} v_{x_1} \cdot v_{x_1} & v_{x_1} \cdot v_{x_2} \\ v_{x_1} \cdot v_{x_2} & v_{x_2} \cdot v_{x_2} \end{pmatrix}$. From (3.14)

$$v_{x_i} \cdot v_{x_j} = \left(\frac{-Du \cdot Du_{x_i}}{(1 + |Du|^2)^{3/2}}(-Du, 1) + \frac{1}{\sqrt{1 + |Du|^2}}(-Du_{x_i}, 0)\right)$$
$$\cdot \left(\frac{-Du \cdot Du_{x_j}}{(1 + |Du|^2)^{3/2}}(-Du, 1) + \frac{1}{\sqrt{1 + |Du|^2}}(-Du_{x_j}, 0)\right)$$

$$= \frac{(Du \cdot Du_{x_i})(Du \cdot Du_{x_j})}{(1 + |Du|^2)^2} - 2\frac{(Du \cdot Du_{x_i})(Du \cdot Du_{x_j})}{(1 + |Du|^2)^2} + \frac{Du_{x_i} \cdot Du_{x_j}}{1 + |Du|^2}$$

$$= \frac{Du_{x_i} \cdot Du_{x_j}}{1 + |Du|^2} - \frac{(Du \cdot Du_{x_i})(Du \cdot Du_{x_j})}{(1 + |Du|^2)^2}.$$

Hence

$$\bar{v} = \frac{(D^2u)^2}{1 + |Du|^2} - \frac{D^2u(Du \otimes Du)D^2u}{(1 + |Du|^2)^2}. \tag{3.20}$$

By (3.19) and (3.20), we obtain

$$
\mathcal{L} = \frac{\lambda^2}{\Delta\left(1+|Du|^2\right)^2} D^2u(Du \otimes Du)D^2u + \lambda^2 \frac{\left(D^2u\right)^2}{1+|Du|^2} - \lambda^2 \frac{D^2u(Du \otimes Du)D^2u}{\left(1+|Du|^2\right)^2}
$$

$$
= \lambda^2 \left(\frac{\left(D^2u\right)^2}{1+|Du|^2} + \left(\frac{1-\Delta}{\Delta}\right) \frac{D^2u(Du \otimes Du)D^2u}{\left(1+|Du|^2\right)^2} \right).
$$

Notice that from (3.18)

$$
1 - \Delta = 1 - \left(\kappa_1^2 + (\kappa_1^2 - 1)|Du|^2\right) = -(\kappa_1^2 - 1)(1 + |Du|^2),
$$

so

$$
\mathcal{L} = \frac{\lambda^2}{1+|Du|^2} D^2u \left(Id - \frac{\kappa_1^2 - 1}{\Delta} Du \otimes Du\right) D^2u. \tag{3.21}
$$

Define

$$
R = Id - \frac{\kappa_1^2 - 1}{\Delta} Du \otimes Du.
$$

Replacing (3.21) in the formula for M yields

$$
M = Id + \frac{\kappa_1^2 - 1}{\kappa_1^2} Du \otimes Du + \left(\frac{1}{\kappa_1}Du + Dd\right) \otimes \left(\frac{1}{\kappa_1}Du + Dd\right)
$$

$$
+ \frac{2d\lambda}{\kappa_1\sqrt{1+|Du|^2}} D^2u + \frac{d^2\lambda^2}{\kappa_1^2\left(1+|Du|^2\right)} D^2u\, R\, D^2u. \tag{3.22}
$$

Notice that R is invertible. In fact, by the Sherman-Morrison formula [24],

$$
\det R = 1 - \frac{(\kappa_1^2 - 1)Du \cdot Du}{\Delta} = \frac{\Delta - (\kappa_1^2 - 1)|Du|^2}{\Delta} = \frac{\kappa_1^2}{\Delta} > 0,
$$

and

$$
R^{-1} = Id + \frac{\dfrac{\kappa_1^2 - 1}{\Delta} Du \otimes Du}{\dfrac{\kappa_1^2}{\Delta}} = Id + \frac{\kappa_1^2 - 1}{\kappa_1^2} Du \otimes Du. \tag{3.23}
$$

Hence M becomes

$$
M = R^{-1} + \left(\frac{1}{\kappa_1}Du + Dd\right) \otimes \left(\frac{1}{\kappa_1}Du + Dd\right) + \frac{2d\lambda}{\kappa_1\sqrt{1+|Du|^2}} R^{-1}\left(RD^2u\right)
$$

$$
+ \frac{d^2\lambda^2}{\kappa_1^2\left(1+|Du|^2\right)} R^{-1}\left(RD^2u\right)^2
$$

We conclude that

$$M = \left(\frac{1}{\kappa_1}Du + Dd\right) \otimes \left(\frac{1}{\kappa_1}Du + Dd\right) + R^{-1}\left(Id + \frac{d\lambda}{\kappa_1\sqrt{1+|Du|^2}}RD^2u\right)^2$$
(3.24)

with $R = Id - \dfrac{\kappa_1^2 - 1}{\Delta}Du \otimes Du$, and R^{-1} given by (3.23).

The point y is a regular point for the surface parametrized by the vector f if and only if $\det M(y) \neq 0$. We find sufficient conditions so that $\det M \neq 0$.

It is easy to check that if v is a function of two variables, then $Dv \otimes Dv$ is symmetric positive semi-definite with eigenvalues 0 and $|Dv|^2$, and so $\det Dv \otimes Dv = 0$. Hence $\left(\dfrac{1}{\kappa_1}Du + Dd\right) \otimes \left(\dfrac{1}{\kappa_1}Du + Dd\right)$ is symmetric positive semi-definite with determinant 0. Also R and R^{-1} are symmetric positive definite. Let

$$H = R^{-1}\left(Id + \frac{d\lambda}{\kappa_1\sqrt{1+|Du|^2}}RD^2u\right)^2 = R\left(R^{-1} + \frac{d\lambda}{\kappa_1\sqrt{1+|Du|^2}}D^2u\right)^2.$$

Since M is symmetric, from (3.24) H is symmetric. Since $B = R^{-1} + \dfrac{d\lambda}{\kappa_1\sqrt{1+|Du|^2}}D^2u$ is symmetric, B^2 is symmetric and positive semi-definite. Since R is symmetric and positive definite we get that H is positive semi-definite.[2] Therefore by (3.24) and the concavity of the det function on positive semi-definite matrices we deduce

$$\det M \geq \det\left[\left(\frac{1}{\kappa_1}Du + Dd\right) \otimes \left(\frac{1}{\kappa_1}Du + Dd\right)\right]$$

$$+ \det\left[R^{-1}\left(Id + \frac{d\lambda}{\kappa_1\sqrt{1+|Du|^2}}RD^2u\right)^2\right]$$

$$= \det\left(R^{-1}\right)\left(\det\left(Id + \frac{d\lambda}{\kappa_1\sqrt{1+|Du|^2}}RD^2u\right)\right)^2$$

$$= \det\left(R^{-1}\right)(\det R)^2\left(\det\left(R^{-1} + \frac{d\lambda}{\kappa_1\sqrt{1+|Du|^2}}D^2u\right)\right)^2$$

$$= \frac{\kappa_1^2}{\Delta}\left(\det\left(R^{-1} + \frac{d\lambda}{\kappa_1\sqrt{1+|Du|^2}}D^2u\right)\right)^2.$$

---

[2] We use here that if A, B are symmetric, positive semi-definite and AB is symmetric, then AB is positive semi-definite.

To prove the second part of the theorem, suppose y is a singular point for f. So $\det M(y) = 0$, and then $\det \left(R^{-1} + \dfrac{d\,\lambda}{\kappa_1\sqrt{1+|Du|^2}} D^2 u \right) = 0$ at y. Hence there exists a unit vector v such that $\langle R^{-1} v + \dfrac{d\,\lambda}{\kappa_1\sqrt{1+|Du|^2}} (D^2 u) v, v \rangle = 0$ at y, so

$$-\frac{d\,\lambda}{\kappa_1\sqrt{1+|Du|^2}} \langle (D^2 u) v, v \rangle = \langle R^{-1} v, v \rangle \quad \text{at } y.$$

By (3.23), the eigenvalues of R^{-1} are 1 and $1 + \dfrac{\kappa_1^2 - 1}{\kappa_1^2} |Du(y)|^2$. Hence $\langle R^{-1} v, v \rangle \geq \|v\| = 1$. On the other hand, since $\langle D^2 u(y)\, v, v \rangle \leq \mu(y)$, it follows that

$$-\frac{d(y)\,\lambda(y)}{\kappa_1\sqrt{1+|Du(y)|^2}} \mu(y) \geq 1.$$

Recall that $\lambda = \phi_{\kappa_1}(e_3 \cdot v)$, and $\phi_{\kappa_1}(t)$ is increasing for $t \geq 0$, where ϕ_{κ_1} given in (3.16). Since $0 \leq e_3 \cdot v \leq 1$, $\phi_{\kappa_1}(0) \leq \lambda \leq \phi_{\kappa_1}(1)$, i.e.,

$$-\sqrt{\kappa_1^2 - 1} \leq \lambda \leq 1 - \kappa_1.$$

Hence

$$\frac{\sqrt{\kappa_1^2 - 1}}{\kappa_1\sqrt{1+|Du(y)|^2}} d(y)\,\mu(y) \geq 1,$$

and therefore

$$d(y) \geq \frac{\kappa_1\sqrt{1+|Du(y)|^2}}{\mu(y)\sqrt{\kappa_1^2 - 1}} \tag{3.25}$$

when f has a singular point at y. By (3.11) we then have

$$d(x, C, w) \leq \frac{C - \min_{x \in \Omega}\{(e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))\}}{\kappa_1 - \kappa_1\kappa_2}.$$

So if f has a singular point at y, then from (3.25) we obtain

$$C \geq \frac{\kappa_1^2(1 - \kappa_2)\sqrt{1+|Du(y)|^2}}{\mu(y)\sqrt{\kappa_1^2 - 1}} + \min_{x \in \Omega}\{(e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))\}. \tag{3.26}$$

This completes the proof of the theorem.   □

Corollary 3.3. *Let*

$$\gamma = \min_{\{y : \mu(y) > 0\}} \frac{\kappa_1^2(1 - \kappa_2)\sqrt{1+|Du(y)|^2}}{\mu(y)\sqrt{\kappa_1^2 - 1}},$$

(if $\{y : \mu(y) > 0\} = \emptyset$, then $\gamma = +\infty$) and suppose that

$$\max_{x \in \Omega}\{(e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))\} < \min_{x \in \Omega}\{(e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))\} + \gamma. \quad (3.27)$$

If

$$\max_{x \in \Omega}\{(e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))\} < C < \min_{x \in \Omega}\{(e_3 - \kappa_1\kappa_2\, w) \cdot (x, u(x))\} + \gamma,$$

then the upper surface of the lens $f(x) = (x, u(x)) + d(x)\, m(x)$ is regular at each x.

4. Lenses Refracting a Field e into a Target Ω^*

We are given the source Ω a bounded convex region in $\mathbb{R}^2$, and the far field target Ω^*, a closed subset of S^2. The incident unit field $e(x) = (e'(x), e_3(x))$ is given so that $e_3(x) \geq \delta > 0$, for every $x \in \bar{\Omega}$, and $e' = Dh$ where h is a C^2 convex function in Ω. Ω^* and e are such that condition (2.2) is satisfied for every $x \in \bar{\Omega}$ and $w \in \Omega^*$. The lower face of the lens is given by the graph of a C^2 concave function u in Ω' as at the beginning of Section 2. Further, we assume that L_e, L_{Du}, L_u, $\|e'\|_{L^\infty}$, $\max_{x \in \bar{\Omega}, w=(w', w_3) \in \Omega^*} |e' - \kappa_1\kappa_2\, w'|$ are small enough so that (2.12), and (2.14) are satisfied, $\beta_w < 1/3$ and $C^* < \dfrac{1/3 - \beta_w}{\alpha}$ for every $w \in \Omega^*$, where C^*, α, and β_w are defined respectively in (2.4), (2.16), and (2.17). We set $\beta = \max_{w \in \Omega^*} \beta_w$, by compactness of Ω^*, we have $\beta < 1/3$ and $C^* < \dfrac{1/3 - \beta}{\alpha}$. Theorems 2.5 and 3.1 imply that for every $w \in \Omega^*$, and $C > C^*$ with $|C| \leq \dfrac{1/3 - \beta}{\alpha}$, the surface $\sigma_{C,w}$ parametrized by the vector $f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w)\, m(x)$, with m and d given respectively in (2.1) and (2.3), has no self-intersections and is regular at every point. Moreover, the lens enclosed between u and $\sigma_{C,w}$ refracts uniformly the field e into w. We use these uniformly refractive surfaces to construct a lens with lower face u and upper face σ that refracts all rays emitted from $(x, 0)$ with direction $e(x)$ into the far field target Ω^*. σ is parametrized by the vector $F(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}(x)\, m(x)$ where $\mathcal{D}$ is constructed so that for every point $x \in \Omega$, σ is supported from above at $F(x)$ by a uniformly refractive surface $\sigma_{C,w}$ with some $w \in \Omega^*$. More precisely, we have the following definition:

Definition 4.1. Let $0 < \varepsilon < \dfrac{1/3 - \beta}{\alpha} - C^*$. We say that the surface σ, parametrized by $F(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}(x)\, m(x)$, yields a lens refracting Ω into Ω^* if for each $x_0 \in \bar{\Omega}$ there exists $w \in \Omega^*$ and C, with $C \geq C^* + \varepsilon$ and $|C| \leq \dfrac{1/3 - \beta}{\alpha}$, such that the surface $\sigma_{C,w}$ supports σ at $F(x_0)$, i.e., $\mathcal{D}(x) \leq d(x, C, w)$ for every x, with equality at $x = x_0$. We also define the corresponding normal map of σ

$$\mathcal{N}_\sigma(x_0) = \left\{ w \in \Omega^* : \text{there exists } C \geq C^* + \varepsilon, |C| \right.$$
$$\left. \leq \frac{1/3 - \beta}{\alpha} \text{ such that } \sigma_{C,w} \text{ supports } \sigma \text{ at } x_0 \right\},$$

and the tracing map $\mathcal{T}_\sigma(w) = \{ x \in \bar{\Omega} : w \in \mathcal{N}_\sigma(x) \}$.

We show in the following theorem that the surfaces σ given parametrically by Definition 4.1 have no self-intersections and have a normal vector at almost every point. This will follow from the conditions on the constant C and that u is concave and $e' = Dh$ with h convex.

Theorem 4.2. *Let σ be a surface given by Definition 4.1 and let $N = \{ x \in \bar{\Omega} : \mathcal{D}$ is not differentiable at $x \}$. Then:*

(1) *σ has no self-intersections;*
(2) *$|N| = 0$;*
(3) *If $y \in \Omega \backslash N$, then F is regular at y, i.e., σ has a normal at y;*
(4) *If $y \in \Omega \backslash N$, then $\mathcal{N}_\sigma(y)$ is a singleton and the ray emitted from y with direction $e(y)$ is refracted by the lens enclosed by u and σ into $\mathcal{N}_\sigma(y)$.*

To show the theorem, we first prove the following lemma:

Lemma 4.3. *Suppose σ, parametrized by $F(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}(x) m(x)$, yields a lens in the sense of Definition 4.1 that refracts Ω into Ω^*. Then:*

(1) *$\mathcal{D}$ is a Lipschitz continuous function,*
(2) *F is Lipschitz;*

where the Lipschitz constants are bounded uniformly by a constant depending only on e, h, u, Ω, Ω', and Ω^*.

Proof. Let $x, y \in \Omega$, and $w_1 \in \mathcal{N}_\sigma(x)$, then there exists $C_1 \geq C^* + \varepsilon$ and $|C_1| \leq \dfrac{1/3 - \beta}{\alpha}$ such that σ_{C_1, w_1} supports σ from above at $F(x)$. Therefore $\mathcal{D}(x) = d(x, C_1, w_1)$, and $\mathcal{D}(y) \leq d(y, C_1, w_1)$. By the second part of Proposition 2.1

$$\mathcal{D}(y) - \mathcal{D}(x) \leq d(y, C_1, w_1) - d(x, C_1, w_1) \leq (A_1 |C_1| + A_2)|x - y|$$
$$\leq \left(A_1 \frac{1/3 - \beta}{\alpha} + A_2 \right) |x - y| := A|x - y|,$$

where A, A_1, A_2 are constants independent of x, y, and depending only on e, h, u, Ω, Ω', and Ω^*. Switching the roles of x and y we conclude that $\mathcal{D}$ is Lipschitz.

To prove the second part of the lemma, we use the above estimate for $\mathcal{D}$, Lemma 2.3, and inequalities (2.13), (2.18), and obtain the following:

$$|F(x) - F(y)| = |(\varphi(x), u(\varphi(x))) - (\varphi(y), u(\varphi(y)))|$$
$$+ |\mathcal{D}(x)m(x) - \mathcal{D}(y)m(y)|$$
$$\leq |\varphi(x) - \varphi(y)| + |u(\varphi(x)) - u(\varphi(y))|$$

$$+ \mathcal{D}(x)|m(x) - m(y)| + |\mathcal{D}(x) - \mathcal{D}(y)||m(y)|$$

$$\leq (B_1|C_1| + B_2)|x - y| \leq \left(B_1 \frac{1/3 - \beta}{\alpha} + B_2 \right) |x - y|$$

$$:= B|x - y|,$$

where B, B_1, B_2 are constants independent of x and y. □

Proof of Theorem 4.2. The proof of (2) follows from Lemma 4.3, and Rademacher theorem.

To prove (1) we proceed by contradiction. Assume that F is not injective, then there exist $x \neq y$ such that $F(x) = F(y)$. Without loss of generality, suppose $\mathcal{D}(y) \geq \mathcal{D}(x)$ and let σ_{C_1, w_1} be a uniformly refractive surface supporting σ at $F(x)$ with $w_1 \in \bar{\Omega}^*$, $C_1 \geq C^* + \varepsilon$, and $|C_1| \leq \dfrac{1/3 - \beta}{\alpha}$. Then

$$\begin{aligned}
|(\varphi(y), u(\varphi(y))) - (\varphi(x), u(\varphi(x)))| &= |\mathcal{D}(x)m(x) - \mathcal{D}(y)m(y)| \\
&= \mathcal{D}(x)|m(x) - m(y)| \\
&\quad + |\mathcal{D}(x) - \mathcal{D}(y)||m(y)| \\
&= d(x, C_1, w_1)|m(x) - m(y)| \\
&\quad + \mathcal{D}(y) - \mathcal{D}(x).
\end{aligned}$$

Using the fact that $\mathcal{D}(y) - \mathcal{D}(x) \leq d(y, C_1, w_1) - d(x, C_1, w_1)$, and the estimates of I and II in the proof of Theorem 2.5, we get that

$$|(\varphi(y) - \varphi(x), u(\varphi(y)) - u(\varphi(x)))| \leq (\alpha|C_1| + \beta_{w_1})|x - y|,$$

with α, and β_{w_1} defined in (2.16), and (2.17). Therefore, by (2.15), $1/3 < \alpha|C_1| + \beta_{w_1}$, and hence

$$|C_1| > \frac{1/3 - \beta_{w_1}}{\alpha} \geq \frac{1/3 - \beta}{\alpha}, \quad \text{which is a contradiction.}$$

We next prove (3). Recall that F regular at y means that $F_{x_1}(y) \times F_{x_2}(y) \neq 0$. Let $\sigma_{C, w}$ be a supporting surface to σ at $F(y)$. We claim that if $y \in \Omega \setminus N$ then $\nabla \mathcal{D}(y) = \nabla d(y, C, w)$ (here to avoid confusion we use ∇ to denote the gradient). In fact, since $\mathcal{D}$ and $d(\cdot, C, w)$ are differentiable at y, and $\mathcal{D}(x) \leq d(x, C, w)$ for every $x \in \Omega$, then by Taylor's theorem,

$$\nabla \mathcal{D}(y) \cdot (x - y) + o(|x - y|) \leq \nabla d(y, C, w) \cdot (x - y) + o(|x - y|),$$

For $\tau > 0$ small enough, we have $x = y + \tau v \in \Omega$ for every v with $|v| = 1$. Then

$$\tau \nabla \mathcal{D}(y) \cdot v + o(\tau) \leq \tau \nabla d(y, C, w) \cdot v + o(\tau).$$

Dividing by τ and letting $\tau \to 0^+$ we get $\nabla \mathcal{D}(y) \cdot v \leq \nabla d(y, C, w) \cdot v$ for every $v \in S^2$, and the claim follows. Therefore

$$F_{x_1}(y) \times F_{x_2}(y) = f_{x_1}(y, C, w) \times f_{x_2}(y, C, w),$$

where $f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w) \, m(x)$. Since h is convex, and u is concave at y then Theorem 3.1 implies that

$$f_{x_1}(y, C, w) \times f_{x_2}(y, C, w) \neq 0,$$

and so y is a regular point for F.

Proof of (4). We have $y \in \Omega \backslash N$, and by part (3) of the theorem, y is a regular point for F. Assume there exist $w_1, w_2 \in \mathcal{N}_\sigma(y)$, and let $\sigma_{C_1, w_1}, \sigma_{C_2, w_2}$ be two supporting surfaces to σ at $F(y)$. Let $\nu_\sigma, \nu_{C_1, w_1}, \nu_{C_2, w_2}$ be the unit normals to $\sigma, \sigma_{C_1, w_1}, \sigma_{C_2, w_2}$ at y, respectively, towards medium n_3. By Snell's law at $f(y, C_1, w_1)$ and $f(y, C_2, w_2)$, we have

$$m(y) - \kappa_2 w_1 = \lambda_{C_1, w_1} \nu_{C_1, w_1}, \qquad m(y) - \kappa_2 w_2 = \lambda_{C_2, w_2} \nu_{C_2, w_2},$$

with $\lambda_{C_i, w_i} = \phi_{\kappa_2}(m(y) \cdot \nu_{C_i, w_i}(y))$, $i = 1, 2$, where ϕ_{κ_2} as defined in (3.16), and the incident direction $m(y)$ is given by (2.1). From the proof of Part (3) we have

$$\begin{aligned} F_{x_1}(y) \times F_{x_2}(y) &= f_{x_1}(y, C_1, w_1) \times f_{x_2}(y, C_1, w_1) \\ &= f_{x_1}(y, C_2, w_2) \times f_{x_2}(y, C_2, w_2) \neq 0. \end{aligned}$$

Then $\nu_\sigma(y) = \nu_{C_1, w_1}(y) = \nu_{C_2, w_2}(y)$, and therefore $\lambda_{C_1, w_1} = \lambda_{C_2, w_2}$, and hence $w_1 = w_2$, which ends the proof of Part (4). $\square$

Remark 4.4. If $y \in \Omega \backslash N$, then from part (4) of the theorem, there exists a unique $w \in \mathcal{N}_\sigma(y)$. Will show also that there is a unique $C \geq C^* + \varepsilon$ and $|C| \leq \dfrac{1/3 - \beta}{\alpha}$ such that $\sigma_{C, w}$ support σ at $F(y)$. In fact, assume there exist C_1, C_2 such that $\sigma_{C_1, w}$ and $\sigma_{C_2, w}$ supports σ at $F(y)$. Then

$$\mathcal{D}(y) = d(y, C_1, w_1) = d(y, C_2, w_1),$$

and from (2.3) we get $C_1 = C_2$.

4.1. The Refractor Measure

Let $\mathcal{I} \in L^1(\Omega)$ with $\mathcal{I} \geq 0$. The energy received on a set $E \subset \Omega^*$ is given by

$$\int_{\mathcal{T}_\sigma(E)} \mathcal{I}(x) \, dx \tag{4.1}$$

where $\mathcal{T}_\sigma$ is the tracing map from Definition 4.1. We prove in this section that (4.1) is well defined for each E Borel subset of Ω^* and is a finite measure on Ω^* which will be called *the refractor measure and denoted by* μ_σ.

Proposition 4.5. *If $A, B \subset \Omega^*$ with $A \cap B = \emptyset$, then $|\mathcal{T}_\sigma(A) \cap \mathcal{T}_\sigma(B)| = 0$.*

Proof. Let N be the set from Theorem 4.2. Since A and B are disjoints, by Theorem 4.2(4)

$$\mathcal{T}_\sigma(A) \cap \mathcal{T}_\sigma(B) \subseteq \{x \in \bar{\Omega} : \mathcal{N}_\sigma(x) \text{ is not single valued}\} \subseteq N \cup \partial\Omega.$$

The conclusion then follows since $|N| = 0$ by Theorem 4.2(2), and that $|\partial\Omega| = 0$ because Ω is convex. $\square$

We define the set $\mathcal{S}_\sigma = \{E \subseteq \Omega^* : \mathcal{T}_\sigma(E)$ is Lebesgue measurable$\}$.

Proposition 4.6. $\mathcal{S}_\sigma$ *contains all closed subsets of* Ω^*.

Proof. We show that $\mathcal{T}_\sigma(E)$ is compact for each E closed subset of Ω^*. Let x_n be a sequence in $\mathcal{T}_\sigma(E)$ converging to x_0, i.e., there exists σ_{C_n, w_n} with $w_n \in E$, $C_n \geq C^* + \varepsilon$ and $|C_n| \leq (1/3 - \beta)/\alpha$ supporting σ at $F(x_n)$. Then there exist a subsequence $\{n_k\}$ such that w_{n_k} and C_{n_k} converges to w_0 and C_0, respectively. Since E is closed, $w_0 \in E$ and we also have $C_0 \geq C^* + \varepsilon$ and $|C_0| \leq (1/3 - \beta)/\alpha$. We prove that $x_0 \in \mathcal{T}_\sigma(w_0)$. In fact $\mathcal{D}(x) \leq d(x, C_{n_k}, w_{n_k})$ with equality at $x = x_{n_k}$. Letting $k \to \infty$, we get $\mathcal{D}(x) \leq d(x, C_0, w_0)$ with equality for $x = x_0$. Therefore $\mathcal{T}_\sigma(E)$ is compact and hence $E \in \mathcal{S}_\sigma$. □

Lemma 4.7. $\mathcal{S}_\sigma$ *is closed under complements.*

Proof. If $E \in \mathcal{S}_\sigma$, then the set $\mathcal{T}_\sigma(E^c) = (\mathcal{T}_\sigma(E))^c \cup (\mathcal{T}_\sigma(E) \cap \mathcal{T}_\sigma(E^c))$ is measurable from Proposition 4.5. □

We then conclude the following:

Theorem 4.8. *For each refractor* σ *in the sense of Definition 4.1, the class* $\mathcal{S}_\sigma$ *is a Borel sigma-algebra of* Ω^*. *If* $\mathcal{I}$ *is nonnegative and* $\mathcal{I} \in L^1(\Omega)$, *then* $\mu_\sigma(E) = \int_{\mathcal{T}_\sigma(E)} \mathcal{I}(x)\, dx$ *is a finite Borel measure on* Ω^*.

We end this section by showing the following stability result:

Proposition 4.9. *Let* σ_n *be a sequence of refractors from* $\bar{\Omega}$ *to* Ω^* *parametrized by the vector*

$$F_n(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}_n(x)\, m(x),$$

such that $\mathcal{D}_n(x) \to \mathcal{D}(x)$ *point-wise in* $\bar{\Omega}$. *Let* σ *be the surface parametrized by* $F(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}(x)\, m(x)$, *then*

(1) σ *is a refractor from* $\bar{\Omega}$ *to* Ω^* *in the sense of Definition 4.1;*
(2) $\mu_n \to \mu$ *weakly, where* μ_n *and* μ *are the refractor measures associated to* F_n *and* F.

Proof. Let $x_0 \in \bar{\Omega}$ and $w_n \in \mathcal{N}_{\sigma_n}(x_0)$. There exists $C_n \geq C^* + \varepsilon$ with $|C_n| \leq (1/3 - \beta)/\alpha$ such that

$$D_n(x) \leq d(x, C_n, w_n)$$

with equality at $x = x_0$. There exists a subsequence C_{n_k} and w_{n_k} converging to C_0 and w_0, respectively, with $w_0 \in \Omega^*$. Since $d(x, C_{n_k}, w_{n_k}) \to d(x, C_0, w_0)$, $\mathcal{D}(x) \leq d(x, C_0, w_0)$ with equality at $x = x_0$, and $C_0 \geq C^* + \varepsilon$, $|C_0| \leq (1/3 - \beta)/\alpha$. This shows that σ_{C_0, w_0} supports σ at $F(x_0)$, and part (1) is then proved.

We now prove (2). Let $\hat{N}$ be the set of all points where $\mathcal{D}$, and $\{\mathcal{D}_n\}$ with $n = 1, \ldots$ are not differentiable. By Theorem 4.2(4), $\mathcal{N}_{\sigma_n}, \mathcal{N}_\sigma$ are single valued for $x \in \bar{\Omega} \setminus \hat{N}$, and by Theorem 4.2(2) $|\hat{N}| = 0$. Then for every $h \in C(\Omega^*)$,

$$\int_{\bar{\Omega}^*} h\, d\mu_n = \int_{\bar{\Omega} \setminus \hat{N}} h(\mathcal{N}_{\sigma_n}(x))\, \mathcal{I}(x)\, dx.$$

It remains to show that $\mathcal{N}_{\sigma_n}(x) \to \mathcal{N}_\sigma(x)$ for $x \in \bar{\Omega} \backslash \hat{N}$. In fact, let $w_0 = \mathcal{N}_\sigma(x)$ and $w_n = \mathcal{N}_{\sigma_n}(x)$. From the proof of (1), every subsequence w_{n_k} of w_n has a sub-subsequence converging to an element of $\mathcal{N}_\sigma(x)$, and hence to w_0. $\square$

5. The Energy Problem

In this section, we are given a non-negative function $\mathcal{I}$ in $L^1(\Omega)$, and a Radon measure η in Ω^*, that satisfy the following conservation of energy condition:

$$\int_\Omega \mathcal{I}(x)\,\mathrm{d}x = \eta(\Omega^*). \tag{5.1}$$

As in Section 4, we assume that $e_3(x) \geq \delta > 0$ for every $x \in \bar{\Omega}$, and $e' = Dh$ where h is a C^2 convex function. Also Ω^* and e are such that (2.2) is satisfied. The lower face of the lens is given by the graph of $u \in C^2$ concave. The constants L_e, L_u, L_{Du}, $\|e'\|_{L^\infty}$, $\max_{x \in \bar{\Omega}, w=(w', w_3) \in \Omega^*} |e'(x) - \kappa_1 \kappa_2 \, w'|$, are chosen small enough so that (2.12), and (2.14) are satisfied, $\beta < 1/3$ and $C^* < \dfrac{1/3 - \beta}{\alpha}$, where C^*, α are given in (2.4), (2.16) respectively, and $\beta = \max_{w \in \Omega^*} \beta_w$ with β_w defined in (2.17). We recall once again that all these choices are to avoid surfaces with self intersections and singular points. The goal of this section is to construct a refractor σ from $\bar{\Omega}$ to Ω^*, in the sense of Definition 4.1, such that

$$\mu_\sigma(E) = \eta(E), \qquad \text{for each Borel set } E \subset \Omega^*,$$

where μ_σ is the measure defined in Theorem 4.8.

Let $\varepsilon > 0$ be fixed, and define the following constants:

$$C_1^* = C^* + (1 + \kappa_2)\left(2\,C^* - \left(\max_\Omega h + \min_\Omega h\right) + \varepsilon\right), \tag{5.2}$$

$$C_2^* = C^* + (1 + \kappa_2)\left(C_1^* + C^* - \left(\max_\Omega h + \min_\Omega h\right)\right). \tag{5.3}$$

We have

$$C_2^* = C^* + C_1^* + \kappa_2 C_1^* + (1 + \kappa_2)\left(C^* - \left(\max_\Omega h + \min_\Omega h\right)\right)$$

$$= C_1^* + C^* + \kappa_2\left(C^* + (1 + \kappa_2)\left(2\,C^* - \left(\max_\Omega h + \min_\Omega h\right) + \varepsilon\right)\right)$$

$$\quad + (1 + \kappa_2)\left(C^* - \left(\max_\Omega h + \min_\Omega h\right)\right)$$

$$= C_1^* + (1 + \kappa_2)\left(2C^* - \left(\max_\Omega h + \min_\Omega h\right)\right)$$

$$\quad + \kappa_2(1 + \kappa_2)\left(2C^* - \left(\max_\Omega h + \min_\Omega h\right) + \varepsilon\right).$$

From (2.4), $2C^* - (\max_\Omega h + \min_\Omega h) > 0$, and we obtain

$$C_2^* > C_1^* > C^* + \varepsilon. \tag{5.4}$$

To show the existence of σ, we require, in addition to the choices above, that α is even smaller so that the interval $\left[-\dfrac{1/3 - \beta}{\alpha}, \dfrac{1/3 - \beta}{\alpha} \right]$ contains $C^* + \varepsilon, C_1^*, C_2^*$, i.e.,

$$|C^* + \varepsilon| \le (1/3 - \beta)/\alpha \tag{5.5}$$
$$|C_1^*| \le (1/3 - \beta)/\alpha \tag{5.6}$$
$$\left| C_2^* \right| \le (1/3 - \beta)/\alpha. \tag{5.7}$$

5.1. Existence in the Discrete Case

We are given Ω^* compact in S^2 equipped with a discrete measure $\eta = \sum_{i=1}^{K} g_i \, \delta_{w_i}$ with $g_1, \ldots, g_K > 0$, and

$$\{w_1, \ldots, w_K\} \subset \Omega^*, \qquad w_i \ne w_j \text{ for } i \ne j$$

and satisfying the conservation of energy condition

$$\int_\Omega \mathcal{I}(x) \, dx = \sum_{i=1}^{K} g_i. \tag{5.8}$$

We define a discrete refractor σ as follows. Let C_i be constants such that $C_i \ge C^* + \varepsilon$ and $|C_i| \le (1/3 - \beta)/\alpha$, for $i = 1, \ldots, K$. Consider σ_{C_i, w_i} the surfaces parametrized by the vectors

$$f(x, C_i, w_i) = (\varphi(x), u(\varphi(x))) + d(x, C_i, w_i) \, m(x)$$

with $d(x, C_i, w_i)$ given by (2.3) and m given in (2.1). We let σ be the surface parametrized by

$$F(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}(x) \, m(x),$$

with

$$\mathcal{D}(x) = \min_{1 \le i \le K} d(x, C_i, w_i).$$

σ is clearly a refractor from $\bar{\Omega}$ to Ω^* in the sense of Definition 4.1, and we identify σ with the vector $(C_1, \ldots, C_K)$.

We shall prove the following theorem:

Theorem 5.1. *There exist constants $C_1, \ldots, C_K$ with $C_i \ge C^* + \varepsilon$ and $|C_i| \le (1/3 - \beta)/\alpha$, for $i = 1, \ldots, K$, such that the refractor σ corresponding to $(C_1, \ldots, C_K)$ satisfies*

$$\int_{\mathcal{T}_\sigma(w_i)} \mathcal{I}(x) \, dx = g_i, \qquad \forall 1 \le i \le K$$

and therefore $\mu_\sigma = \eta = \sum_{i=1}^{K} g_i \delta_{w_i}$.

Proof. We establish the theorem by proving a sequence of claims.

Let

$$W = \{(C_1^*, C_2, \ldots, C_K) : C_i \geq C^* + \varepsilon, |C_i|$$
$$\leq (1/3 - \beta)/\alpha, \mu_\sigma(w_i) \leq g_i \text{ for } i = 2, \ldots, K\},$$

where C_1^* given in (5.2).

Claim 1. $W \neq \emptyset$. We prove that $(C_1^*, C_2, \ldots, C_K) \in W$, with $C_i = C_2^*$ for $2 \leq i \leq K$ where C_2^* given in (5.3). We have from (5.4) that $C_i > C_1^* > C^* + \varepsilon$, and by (5.7), $|C_i| \leq \dfrac{1/3 - \beta}{\alpha}$. It remains to show that $\mu_\sigma(w_i) \leq g_i$ for $i = 2, \ldots, K$.

By (2.9), (2.10) and the definition of C^* in (2.4), we have

$$-C^* \leq v(x) \leq C^* - \left(\max_\Omega h + \min_\Omega h\right) \qquad (5.9)$$

so

$$d(x, C_1^*, w_1) \leq \frac{C_1^* + C^* - (\max_\Omega h + \min_\Omega h)}{\kappa_1(1 - \kappa_2)} \quad \text{and}$$
$$d(x, C_2^*, w_i) \geq \frac{C_2^* - C^*}{\kappa_1(1 - \kappa_2^2)},$$

$i = 2, \ldots, K$. Therefore, by the definition of C_2^* in (5.3) it follows that

$$d(x, C_1^*, w_1) \leq d(x, C_2^*, w_i), \qquad i = 2, \ldots, K.$$

Hence $\mathcal{D}(x) = d(x, C_1^*, w_1) \wedge \min_{2 \leq k \leq K} d(x, C_2^*, w_i) = d(x, C_1^*, w_1)$ for every $x \in \bar{\Omega}$. Thus $\mathcal{T}_\sigma(w_i) = \mathcal{T}_\sigma(w_i) \cap \mathcal{T}_\sigma(w_1)$, so by Proposition 4.5, $|\mathcal{T}_\sigma(w_i)| = 0$ and we get $\mu_\sigma(w_i) = 0 < g_i$ for every $2 \leq i \leq K$.

Claim 2. W is compact. We first prove the following lemma:

Lemma 5.2. *Let* $(C_1^n, C_2^n, \ldots, C_K^n)$ *with* $C_i^n \geq C^* + \varepsilon$ *and* $|C_i^n| \leq (1/3 - \beta)/\alpha$, *for* $i = 1, \ldots, K$, *and suppose* $(C_1^n, C_2^n, \ldots, C_K^n) \to (C_1, C_2, \ldots, C_K)$ *as* $n \to \infty$. *Let* σ_n *and* σ *be the corresponding refractors with* $\mathcal{D}_n(x) = \min_{1 \leq i \leq K} d(x, C_i^n, w_i)$, *and* $\mathcal{D}(x) = \min_{1 \leq i \leq K} d(x, C_i, w_i)$. μ_n *and* μ *are the corresponding refractor measures to* σ_n *and* σ. *Then* $\mu(w_i) = \lim_{n \to \infty} \mu_n(w_i), \forall 1 \leq i \leq K$.

Proof. Since $d(x, C, w)$ is continuous in the variable C, we get that $\mathcal{D}_n(x) \to \mathcal{D}(x)$ point-wise in Ω. Then by Proposition 4.9, $\mu_n \to \mu$ weakly. By the weak convergence $\mu(w_i) \geq \limsup_{n \to \infty} \mu_n(w_i)$. We claim that $\mu(w_i) \leq \liminf_{n \to \infty} \mu_n(w_i)$. Fix $1 \leq i \leq K$, and let G be an open set containing w_i such that $G \cap \{w_1, \ldots, w_K\} = \{w_i\}$. If $y \in \bar{\Omega}$, then $y \in \cup_{j=1}^K \mathcal{T}_{\sigma_n}(w_j)$ for all n and $y \in \cup_{j=1}^K \mathcal{T}_\sigma(w_j)$. Hence $\mathcal{T}_{\sigma_n}(G \setminus w_i) = \left(\cup_{j=1}^K \mathcal{T}_{\sigma_n}(w_j)\right) \cap \mathcal{T}_{\sigma_n}(G \setminus w_i)$, and so $|\mathcal{T}_{\sigma_n}(G \setminus w_i)| = 0$ by Proposition 4.5. Similarly $|\mathcal{T}_\sigma(G \setminus w_i)| = 0$. Therefore $\mu_n(G) = \mu_n(w_i)$ and $\mu(G) = \mu(w_i)$ for all n and $1 \leq i \leq K$. By the weak convergence $\mu(w_i) = \mu(G) \leq \liminf_{n \to \infty} \mu_n(G) = \liminf_{n \to \infty} \mu_n(w_i)$ for $1 \leq i \leq K$ which completes the proof of the lemma. $\square$

In particular, if $(C_1^*, C_2^n, \ldots, C_K^n) \in W$ and $(C_1^*, C_2^n, \ldots, C_K^n) \to (C_1^*, C_2, \ldots, C_K)$ as $n \to \infty$ we obtain $\mu(w_i) \leq g_i$ for $2 \leq i \leq K$, that is, $(C_1^*, C_2, \ldots, C_K) \in W$, obtaining Claim 2.

Claim 3. For each $\sigma \in W$ the corresponding refractor measure satisfies $\mu_\sigma(w_1) \geq g_1$. In fact, since $\mathcal{T}_\sigma(\cup_1^K w_i) = \bar{\Omega}$, by (5.8) and Proposition 4.5

$$g_1 + \cdots + g_K = \int_\Omega \mathcal{I}(x)\,dx = \int_{\mathcal{T}_\sigma(\cup_{i=1}^K w_i)} \mathcal{I}(x)\,dx$$

$$= \sum_{i=1}^K \int_{\mathcal{T}_\sigma(w_i)} \mathcal{I}(x)\,dx = \sum_{i=1}^K \mu_\sigma(w_i)$$

$$\leq \mu_\sigma(w_1) + g_2 + \cdots + g_K,$$

and the claim follows.

For each $2 \leq i \leq K$, we define

$$\bar{C}_i = \inf\{C_i : (C_1^*, \ldots, C_i, \ldots, C_K) \in W\}.$$

Let $\bar{\sigma}$ be the refractor parametrized by the vector $\bar{F}(x) = (\varphi(x), u(\varphi(x))) + \bar{\mathcal{D}}(x)\,m(x)$, with $\bar{\mathcal{D}}(x) = d(x, C_1^*, w_1) \wedge \min_{2 \leq i \leq K} d(x, \bar{C}_i, w_i)$; and let $\bar{\mu}$ be its corresponding refractor measure. We will show that $\bar{\sigma}$ is the desired solution.

Claim 4. $(C_1^*, \bar{C}_2, \ldots, \bar{C}_i, \ldots, \bar{C}_K) \in W$. We first need the following lemma:

Lemma 5.3. *We are given* $(C_1, \ldots, C_\ell, \ldots, C_K)$ *and* $(\tilde{C}_1, \ldots, \tilde{C}_\ell, \ldots, \tilde{C}_K)$ *such that* $C_i = \tilde{C}_i$ *for every* $i \neq \ell$ *and* $C_\ell \leq \tilde{C}_\ell$. *Let* σ *and* $\tilde{\sigma}$ *be the corresponding refractors. Then* $\mathcal{T}_\sigma(w_i) \subseteq \mathcal{T}_{\tilde{\sigma}}(w_i)$ *for each* $i \neq \ell$, *where the inclusion is up to a set of measure zero, and so*

$$\mu_\sigma(w_i) \leq \mu_{\tilde{\sigma}}(w_i), \quad i \neq \ell.$$

Proof. We set $\mathcal{D}(x) = \min_{1 \leq i \leq K} d(x, C_i, w_i)$, and $\tilde{\mathcal{D}}(x) = \min_{1 \leq i \leq K} d(x, \tilde{C}_i, w_i)$. Let N be the set where $\mathcal{D}$ is not differentiable. Fix $i \neq \ell$, and $y \in \mathcal{T}_\sigma(w_i) \backslash N$. By Theorem 4.2(4), and Remark 4.4, the surface σ_{C_i, w_i} supports σ at y, i.e.

$$\mathcal{D}(y) = d(y, C_i, w_i) \leq d(y, C_j, w_j) \quad \text{for all } j.$$

For $j \neq \ell$, $d(y, C_j, w_j) = d(y, \tilde{C}_j, w_j)$. Since $d(x, C, w)$ is increasing in C, we have $d(y, C_\ell, w_\ell) \leq d(y, \tilde{C}_\ell, w_\ell)$, then, since $C_i = \tilde{C}_i$,

$$d(y, \tilde{C}_i, w_i) \leq d(y, \tilde{C}_j, w_j) \quad \text{for all } j.$$

Therefore $\tilde{\mathcal{D}}(y) = d(y, \tilde{C}_i, w_i)$, and $y \in \mathcal{T}_{\tilde{\sigma}}(w_i)$. Hence $\mathcal{T}_\sigma(w_i) \backslash N \subseteq \mathcal{T}_{\tilde{\sigma}}(w_i)$ and by Theorem 4.2(2) the lemma follows. □

From this monotonicity result, we obtain, as in [12, Corollary 4.4], the following corollary:

Corollary 5.4. *We are given the vectors* $(C_1^1, C_2^1, \ldots, C_K^1)$, $(C_1^2, C_2^2, \ldots, C_K^2)$, $(C_1, C_2, \ldots, C_K)$, *with* $C_i = \min(C_i^1, C_i^2)$ *for* $i = 1, \ldots, K$. *Let* μ_1, μ_2, μ *be their corresponding refractor measures, then*

$$\mu(w_i) \le \max\{\mu_1(w_i), \mu_2(w_i)\} \quad \text{for all } 1 \le i \le K.$$

Now we can complete the proof of Claim 4. Since W is compact then for each $2 \le i \le K$, the infimum in the definition of $\bar{C}_i$, is attained at some vector

$$v^i = (C_1^*, C_2^i, \ldots, C_{i-1}^i, \bar{C}_i, C_{i+1}^i, \ldots, C_K^i) \in W.$$

Let μ_i be the refractor measure associated with the refractor given by v^i, then by Corollary 5.4

$$\bar{\mu}(w_i) \le \max\{\mu_2(w_i), \ldots, \mu_K(w_i)\} \le g_i \quad \text{for } i = 2, \ldots, K.$$

Claim 5. $\bar{C}_i > C^* + \varepsilon$ *for all* $2 \le i \le K$. Without loss of generality, we assume that $\bar{C}_2 = C^* + \varepsilon$, then by (2.9) and (5.9)

$$d(x, \bar{C}_2, w_2) = \frac{\bar{C}_2 + v(x)}{g(x)} \le \frac{\bar{C}_2 + C^* - (\max_\Omega h + \min_\Omega h)}{\kappa_1 (1 - \kappa_2)},$$

$$d(x, C_1^*, w_1) \ge \frac{C_1^* - C^*}{\kappa_1 (1 - \kappa_2^2)},$$

and hence, by the choice of C_1^* in (5.2),

$$d(x, C_1^*, w_1) \ge d(x, \bar{C}_2, w_2) \quad \forall x.$$

Since $\bar{\mathcal{D}}(x) = d(x, C_1^*, w_1) \wedge \min_{2 \le i \le K} d(x, \bar{C}_i, w_i)$, it follows that $\mathcal{T}_{\bar{\sigma}}(w_1) \subseteq \mathcal{T}_{\bar{\sigma}}(w_2)$ up to a set of measure zero and therefore $|\mathcal{T}_{\bar{\sigma}}(w_1)| = 0$ by Proposition 4.5. Thus $\bar{\mu}(w_1) = 0$, which contradicts Claim 3.

Claim 6. $\bar{\mu}(w_i) = g_i$ *for* $i = 2, \ldots, K$. Without loss of generality, assume that $\bar{\mu}(w_2) < g_2$. Define $v_\lambda = (C_1^*, \bar{C}_2 - \lambda, \bar{C}_3, \ldots, \bar{C}_K)$, with $\lambda > 0$ small and let σ_λ be its corresponding refractor. By Claim 5 and (5.5), for $\lambda > 0$ small enough $\bar{C}_2 - \lambda \ge C^* + \varepsilon$ and $|\bar{C}_2 - \lambda| \le (1/3 - \beta)/\alpha$. From Lemma 5.2, we also have $\mu_{\sigma_\lambda}(w_2) < g_2$ for λ small. Moreover by Lemma 5.3 and Claim 4, $\mu_{\sigma_\lambda}(w_i) \le \bar{\mu}(w_i) \le g_i$ for $i \ge 3$. Hence $v_\lambda \in W$ contradicting the definition of $\bar{C}_2$.

Claim 7. $\bar{\mu}(w_1) = g_1$. By (5.8), and Claim 6

$$0 = \bar{\mu}(\Omega^*) - \eta(\Omega^*) = (\bar{\mu}(w_1) - g_1) + (\bar{\mu}(w_2) - g_2) + \cdots + (\bar{\mu}(w_N) - g_N)$$
$$= \bar{\mu}(w_1) - g_1,$$

and the claim follows.

Therefore the proof of Theorem 5.1 is complete. $\quad \square$

5.2. *Existence for General Radon Measures η*

In this section, we show existence of Brenier type solutions to the energy problem when the measure η is not necessarily discrete.

Theorem 5.5. *Assume a compact target Ω^* is equipped with a Radon measure η satisfying* (5.1). *Then there exists a refractor σ from $\bar{\Omega}$ to Ω^*, in the sense of Definition 4.1, such that*

$$\mu_\sigma(E) = \eta(E), \quad \text{for every Borel set } E \subset \Omega^*.$$

Proof. We subdivide the set Ω^* into finite union of disjoint Borel sets with diameter less than $1/2$. We discard the sets of η-measure zero, and label the remaining sets $\Omega_1^1, \Omega_2^1, \ldots, \Omega_{N_1}^1$. Fix $w_1^1, \ldots, w_{N_1}^1$ so that $w_i^1 \in \Omega_i^1$ for each $i = 1, \ldots, N_1$, and define the measure

$$\eta_1 = \sum_{i=1}^{N_1} \eta\left(\Omega_i^1\right) \delta_{w_i^1}.$$

Notice that

$$\eta_1\left(\Omega^*\right) = \int_\Omega \mathcal{I}(x)\,\mathrm{d}x,$$

then by Theorem 5.1, there exists a vector $\left(C_1^*, C_2^1, \ldots, C_{N_1}^1\right)$ associated to a refractor σ_1 with C_1^* given in (5.2), $C_i^1 \geq C^* + \varepsilon$, and $|C_i^1| \leq \dfrac{1/3 - \beta}{\alpha}$, and such that

$$\mu_{\sigma_1} = \eta_1.$$

σ_1 is parametrized by the vector $F_1(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}_1(x)\,m(x)$, with

$$\mathcal{D}_1(x) = d(x, C_1^*, w_1^1) \wedge \min_{2 \leq i \leq N_1} d(x, C_i^1, w_i^1).$$

Next, we subdivide each set Ω_i^1 into disjoint Borel sets of diameter less than $1/4$. We discard the sets of η-measure zero, and relabel the remaining sets $\Omega_1^2, \Omega_2^2, \ldots, \Omega_{N_2}^2$. We fix $w_1^2, \ldots, w_{N_2}^2$ so that $w_i^2 \in \Omega_i^2$ for each $i = 1, \ldots, N_2$, and define the measure

$$\eta_2 = \sum_{i=1}^{N_2} \eta\left(\Omega_i^2\right) \delta_{w_i^2}.$$

Again, notice that

$$\eta_2\left(\Omega^*\right) = \int_\Omega \mathcal{I}(x)\,\mathrm{d}x,$$

then by Theorem 5.1, there exists a vector $\left(C_1^*, C_2^2, \ldots, C_{N_2}^2\right)$ associated to a refractor σ_2 with C_1^* given in (5.2), $C_i^2 \geq C^* + \varepsilon$, and $|C_i^2| \leq \dfrac{1/3 - \beta}{\alpha}$, and such that

$$\mu_{\sigma_2} = \eta_2.$$

σ_2 is parametrized by the vector $F_2(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}_2(x) m(x)$, with

$$\mathcal{D}_2(x) = d(x, C_1^*, w_1^2) \wedge \min_{2 \leq i \leq N_2} d(x, C_i^2, w_i^2).$$

Recursively, we subdivide the sets $\Omega_1^{n-1}, \ldots, \Omega_{N_{n-1}}^{n-1}$ into disjoint Borel sets with diameter less than $1/2^n$. We discard the sets of η-measure zero and relabel the remaining $\Omega_1^n, \Omega_2^n, \ldots, \Omega_{N_n}^n$. Pick $w_1^n, \ldots, w_{N_n}^n$ so that $w_i^n \in \Omega_i^n$ for each $i = 1, \ldots, N_n$, and define the measure

$$\eta_n = \sum_{i=1}^{N_n} \eta\left(\Omega_i^n\right) \delta_{w_i^n}.$$

Once again

$$\eta_n\left(\Omega^*\right) = \int_\Omega \mathcal{I}(x) \, dx,$$

and by Theorem 5.1, there exists a vector $\left(C_1^*, C_2^n, \ldots, C_{N_n}^n\right)$ associated to a refractor σ_n with C_1^* given in (5.2), $C_i^n \geq C^* + \varepsilon$, and $|C_i^n| \leq \dfrac{1/3 - \beta}{\alpha}$, and such that

$$\mu_{\sigma_n} = \eta_n.$$

σ_n is parametrized by the vector $F_n(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}_n(x) m(x)$, with

$$\mathcal{D}_n(x) = d(x, C_1^*, w_1^n) \wedge \min_{2 \leq i \leq N_n} d(x, C_i^n, w_i^n).$$

By Lemma 4.3, the sequence $\mathcal{D}_n$ is equicontinuous. Moreover, by (2.9) and (5.9), for every $x \in \Omega$, $C \geq C^* + \varepsilon$, and $w \in \Omega^*$,

$$\frac{\varepsilon}{\kappa_1(1 - \kappa_2^2)} \leq d(x, C, w) \leq \frac{C + C^* - (\max_\Omega h + \min_\Omega h)}{\kappa_1(1 - \kappa_2)}.$$

Therefore, since $|C_i^j| \leq (1/3 - \beta)/\alpha$ for $i = 2, \ldots, N_j$; $j = 1, 2, \ldots$, then the sequence $\mathcal{D}_n$ is uniformly bounded. Therefore by Arzelà-Ascoli there exists a subsequence $\mathcal{D}_{n_k}$ converging uniformly to some strictly positive function $\mathcal{D}$. Hence, by Proposition 4.9, the surface σ parametrized by the vector $F(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}(x) m(x)$, is a refractor from Ω to Ω^* in the sense of Definition 4.1, and $\mu_{\sigma_{n_k}} \to \mu_\sigma$ weakly. We also have that $\eta_n \to \eta$ weakly and $\eta_n = \mu_{\sigma_n}$, hence $\mu_\sigma(E) = \eta(E)$ for every Borel set $E \subset \Omega^*$. $\square$

6. Aleksandrov Type Solutions

Let $\mathcal{G} \in L^1(\Omega^*)$ with $\mathcal{G} \geq 0$. The purpose of this section is to construct Aleksandrov type solutions to the energy problem described in Section 5. Given a set $E \subset \Omega^*$ measurable we shall first show that the set function given by

$$\int_{\mathcal{N}_\sigma(E)} \mathcal{G}(x)\, dx \tag{6.1}$$

is a Borel measure in Ω^*, where $\mathcal{N}_\sigma$ is the normal mapping from Definition 4.1; and next compare this notion with the Brenier definition (4.1).

6.1. Legendre Type Transform

Suppose the upper surface σ of the lens we are seeking is parametrized by

$$F(x) = (\varphi(x), u(\varphi(x))) + \mathcal{D}(x)\, m(x), \qquad \text{for } x \in \Omega,$$

where at each $x_0 \in \Omega$ there is a support surface $\sigma_{C,w}$ as in Definition 4.1, for some $w \in \Omega^*$ and $\sigma_{C,w}$ is parametrized by $f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w)\, m(x)$; and where $d(x, C, w)$ is given by (2.3). Set $\Phi(x) = (\varphi(x), u(\varphi(x)))$.

Let $w_0 \in \mathcal{N}_\sigma(x_0)$. Then there exists C such that $f(x_0, C, w_0) = F(x_0)$, so $F(x_0) = \Phi(x_0) + d(x_0, C, w_0)\, m(x_0)$, and therefore $d(x_0, C, w_0) = |F(x_0) - \Phi(x_0)|$. Also for $x \neq x_0$

$$d(x, C, w_0) \geq \mathcal{D}(x) = |F(x) - \Phi(x)|.$$

Hence solving C in (2.3) yields

$$C = d(x, C, w_0)\, (\kappa_1 - \kappa_1\, \kappa_2 w_0 \cdot m(x)) + h(x) - e'(x) \cdot x + (e(x) - \kappa_1\, \kappa_2\, w_0) \cdot \Phi(x),$$

and therefore

$$\begin{aligned}
C &\geq |F(x) - \Phi(x)|\, (\kappa_1 - \kappa_1\, \kappa_2 w_0 \cdot m(x)) + h(x) - e'(x) \cdot x \\
&\quad + (e(x) - \kappa_1\, \kappa_2\, w_0) \cdot \Phi(x) \\
&:= \ell(x, w_0)
\end{aligned} \tag{6.2}$$

for all $x \in \Omega$ with equality at $x = x_0$.

Therefore for each $w \in \Omega^*$ we introduce the Legendre type transform given by

$$F^*(w) = \sup_{x \in \Omega} \ell(x, w), \tag{6.3}$$

and so if $\sigma_{C,w}$ supports σ then $C = F^*(w)$.

Remark 6.1. We can translate these definitions in terms of the generated Jacobian equations introduced in [25]. We set $G(x, w, z)$ for $x \in \Omega$, $w \in \Omega^*$, $z \in I$, with I the interval for the admissible values of C in Definition 4.1, by

$$G(x, w, z) = d(x, z, w),$$

where d is defined in (2.3). And also set

$$H(x, w, z) = z \, (\kappa_1 - \kappa_1 \kappa_2 w_0 \cdot m(x)) + h(x) - e'(x) \cdot x$$
$$+ (e(x) - \kappa_1 \kappa_2 \, w_0) \cdot \Phi(x).$$

We then have [25, Formula (1.17)]:

$$G \, (x, w, H(x, w, z)) = z.$$

Also [25, Formula (1.21)] translates to

$$\sup_{x \in \Omega} H(x, w, \mathcal{D}(x)) = F^*(w).$$

For each $x_0 \in \Omega$, there exist $w_0 \in \Omega^*$ and $C_0 \in I$, such that $\mathcal{D}(x) \leq G(x, w_0, C_0)$ for all $x \in \Omega$ with equality at $x = x_0$, analogously as in [25, Formula (2.1)].

We then have

Lemma 6.2. F^* *is differentiable in* Ω^* *a.e.*

Proof. Let us write

$$\alpha(x) = (-\kappa_1 \kappa_2) \, (|F(x) - \Phi(x)| \, m(x) + \Phi(x)) \qquad (6.4)$$
$$\beta(x) = \kappa_1 \, |F(x) - \Phi(x)| \, m(x) + h(x) - e'(x) \cdot x + e(x) \cdot \Phi(x).$$

So

$$\ell(x, w) = \alpha(x) \cdot w + \beta(x).$$

Writing $w = (w', w_3)$ with $w' = (w_1, w_2)$ and $w_3 > 0$, we have $\bar{\ell}(x, w') = \ell(x, w)$ with $\bar{\ell}(x, w') = \alpha'(x) \cdot w' + \beta(x) + \alpha_3(x) \sqrt{1 - |w'|^2}$. Since $m_3(x) > 0$ and $u > 0$,

$$\alpha_3(x) = (-\kappa_1 \kappa_2) \, (|F(x) - \Phi(x)| \, m_3(x) + u(\varphi(x))) < 0.$$

Since $\sqrt{1 - |w'|^2}$ is concave, $\bar{\ell}(x, w')$ is convex as a function of w' and therefore F^* is convex as a function of w'. ☐

We then have the following lemma, similar to the Aleksandrov lemma for the subdifferential [18, Lemma 1.1.12]:

Lemma 6.3. *The set*

$$\mathcal{S} = \{w \in \Omega^* : w \in \mathcal{N}_\sigma(x_1) \cap \mathcal{N}_\sigma(x_2) \text{ for some } x_1, x_2 \in \Omega, x_1 \neq x_2\}$$

has surface measure zero.

Proof. Recall σ is parametrized by $F(x) = \Phi(x) + \mathcal{D}(x) \, m(x)$. We shall prove that

$$\mathcal{S} \subset \{w \in \Omega^* : F^* \text{ is not differentiable at } w\}.$$

If $w \in \mathcal{S}$, then there exist $x_1 \neq x_2$ in Ω and C_1, C_2 such that $d(x, C_1, w)$ and $d(x, C_2, w)$ support $\mathcal{D}(x)$ at $x = x_1$ and $x = x_2$ respectively. Then $C_1 = C_2 = F^*(w) := C$. Let us write $w = (w', w_3)$ with $w_3 > 0$; $w_3 = \sqrt{1 - |w'|^2}$. We can think of F^* as a function of w'. Suppose that F^* were differentiable at w'. Since $w \in \mathcal{N}_\sigma(x_i)$, $D_{w'} F^*(w) = D_{w'} \ell(x_i, w)$ for $i = 1, 2$, and therefore $D_{w'} \ell(x_1, w) = D_{w'} \ell(x_2, w)$. By definition of ℓ, $D_{w'} \ell(x, w) = (\alpha_1(x), \alpha_2(x)) - \alpha_3(x) \dfrac{w'}{w_3} = \alpha'(x) - \alpha_3(x) \dfrac{w'}{w_3}$; where $\alpha(x)$ is from (6.4). Hence

$$\alpha'(x_1) - \alpha'(x_2) = (\alpha_3(x_1) - \alpha_3(x_2)) \frac{w'}{w_3}.$$

Writing $f(x, C, w) = \Phi(x) + d(x, C, w) m(x) = (f_1(x, C, w), f_2(x, C, w), f_3(x, C, w)) = (f'(x, C, w), f_3(x, C, w))$, it follows that

$$f'(x_1, C, w) - f'(x_2, C, w) = (f_3(x_1, C, w) - f_3(x_2, C, w)) \frac{w'}{w_3}, \qquad i = 1, 2. \tag{6.5}$$

We will prove in Remark 6.4 below that by choosing the Lipschitz constants in Theorem 2.5 sufficiently small, if $|C|$ is sufficiently small, then (6.5) implies that $x_1 = x_2$; obtaining in this way a contradiction. Consequently, w' cannot be a point of differentiability of F^*. □

Remark 6.4. Suppose that $w_3 \geq \Delta > 0$ for all $w \in \Omega^*$.[3] We show that by choosing the Lipschitz constants in Theorem 2.5 sufficiently small, (6.5) implies that $x_1 = x_2$. Suppose by contradiction that $x_1 \neq x_2$ and from (6.5) proceed as in the proof of Theorem 2.5 to obtain the inequality

$$1/3 < \left(1 + \frac{1}{\Delta}\right)(\alpha |C| + \beta_w) + \frac{3}{\Delta} L_u,$$

where A is the right hand side of (2.20), with α from (2.16) and β_w from (2.17). Then

$$|C| > \left(\frac{1}{3} - \frac{3}{\Delta} L_u - \left(1 + \frac{1}{\Delta}\right) \beta_w\right) / \left(\left(1 + \frac{1}{\Delta}\right) \alpha\right) := \bar{C}.$$

If $\left(1 + \dfrac{1}{\Delta}\right) \beta_w + \dfrac{3}{\Delta} L_u > 1/3$, then $\bar{C} > 0$. Therefore if $|C| \leq \bar{C}$, then it follows $x_1 = x_2$. Notice that $\bar{C} < \dfrac{1/3 - \beta_w}{\alpha}$ which is the size of the interval in Theorem 2.5. In other words by choosing the constants in Theorem 2.5 sufficiently small and choosing $|C| \leq \bar{C}$ we obtain the desired result.

---

[3] Notice that this is saying that all vectors in the target Ω^* are pointing up, i.e., the lens refracts all rays upwards.

Notice that in the definition of the normal mapping $\mathcal{N}_\sigma$, this requires possibly to take C in a smaller interval, however the size of this interval depends only on the initial configuration.

From arguments similar to the ones in Section 4.1 we obtain

Corollary 6.5. *If $\mathcal{G} \in L^1(\Omega^*)$ with $\mathcal{G} \geq 0$, then the set function*

$$\int_{\mathcal{N}_\sigma(E)} \mathcal{G}(x)\, dx \tag{6.6}$$

is a finite Borel measure in Ω.

6.2. Comparison Between Brenier and Aleksandrov Type Solutions

Let $\mathcal{I} \in L^1(\Omega)$ and $\mathcal{G} \in L^1(\Omega^*)$ be such that

$$\int_\Omega \mathcal{I}(x)\mathrm{d}x = \int_{\Omega^*} \mathcal{G}(y)\, dy, \tag{6.7}$$

and let Ω^* be contained in the upper unit sphere. We showed in Section 5 the existence of lens (u, σ) such that $\mathcal{T}_\sigma(\Omega^*) = \bar{\Omega}$ and

$$\int_{\mathcal{T}_\sigma(E)} \mathcal{I}(x)\, \mathrm{d}x = \int_E \mathcal{G}(y)\, dy, \tag{6.8}$$

for each Borel set $E \subseteq \Omega^*$, that is, σ is a Brenier solution.

As in [20, Section 4] and [19, Section 10] we define Aleksandrov solution.

Definition 6.6. We say that σ is an Aleksandrov solution if $\Omega^* = \mathcal{N}_\sigma(\bar{\Omega})$ and for each Borel set $E \subseteq \Omega$

$$\int_{\mathcal{N}_\sigma(E)} \mathcal{G}(y)\, dy = \int_E \mathcal{I}(x)\, \mathrm{d}x. \tag{6.9}$$

As in Lemma [8, Lemma 3.8], (6.7) implies that if σ is a Brenier solution, then for every φ continuous in Ω^*,

$$\int_{\Omega^*} \varphi(y)\mathcal{G}(y)\, dy = \int_\Omega \varphi\left(\mathcal{N}_\sigma(x)\right) \mathcal{I}(x)\, \mathrm{d}x. \tag{6.10}$$

Theorem 6.7. *If $\mathcal{I} > 0$ a.e., then each Brenier solution is an Aleksandrov solution.*

Proof. Let σ be a Brenier solution. We claim that $|\mathcal{T}_\sigma(\mathcal{N}_\sigma(K))\backslash K| = 0$, for each compact set K. We first prove that $\mathcal{T}_\sigma(\mathcal{N}_\sigma(K))\backslash K \subset \mathcal{T}_\sigma(\mathcal{N}_\sigma(K)) \cap \mathcal{T}_\sigma(\mathcal{N}_\sigma(K^c))$. If $x \in \mathcal{T}_\sigma(\mathcal{N}_\sigma(K))\backslash K$, then there is $y \in K$ such that $x \in \mathcal{T}_\sigma(\mathcal{N}_\sigma(y))$ and $x \notin K$. We always have $x \in \mathcal{T}_\sigma(\mathcal{N}_\sigma(x))$. Therefore $x \in \mathcal{T}_\sigma(\mathcal{N}_\sigma(K)) \cap \mathcal{T}_\sigma(\mathcal{N}_\sigma(K^c))$. Second, let $A = \mathcal{N}_\sigma(K)$ and $B = \mathcal{N}_\sigma(K^c)$ and notice that by Lemma 6.3, $|A \cap B| = 0$. Since σ is a Brenier solution, $\int_{\mathcal{T}_\sigma(A \cap B)} \mathcal{I}(x)\, \mathrm{d}x = \int_{A \cap B} \mathcal{G}(y)\, dy = 0$. If $\mathcal{I} > 0$ a.e., we conclude that $|\mathcal{T}_\sigma(A \cap B)| = 0$. Third, we show that $\mathcal{T}_\sigma(A) \cap \mathcal{T}_\sigma(B) \subset \mathcal{T}_\sigma(A \cap B) \cup \mathcal{D}$; where $\mathcal{D} = \{x \in \Omega : \mathcal{N}_\sigma \text{ is not single valued at } x\}$ has measure

zero from Theorem 4.2(4). In fact, if $x \in \mathcal{T}_\sigma(A) \cap \mathcal{T}_\sigma(B)$, then there exist $y_1 \in A$ and $y_2 \in B$ such that $x \in \mathcal{T}_\sigma(y_1) \cap \mathcal{T}_\sigma(y_2)$ which implies that $y_1, y_2 \in \mathcal{N}_\sigma(x)$. So if $x \notin \mathcal{D}$, then $y_1 = y_2 \in A \cap B$ and the inclusion follows. Therefore the claim is proved.

Now write

$$
\int_{\mathcal{N}_\sigma(K)} \mathcal{G}(y)\, dy = \int_{\mathcal{T}_\sigma(\mathcal{N}_\sigma(K))} \mathcal{I}(x)\, dx, \qquad \text{since } \sigma \text{ is a Brenier solution}
$$

$$
= \int_{\mathcal{T}_\sigma(\mathcal{N}_\sigma(K))\setminus K} \mathcal{I}(x)\, dx + \int_K \mathcal{I}(x)\, dx = \int_K \mathcal{I}(x)\, dx
$$

for all compact sets $K \subset \Omega$. Since the measures are regular we obtain by approximation (6.9) for all Borel subsets of Ω.

If remains to show that $\Omega^* = \mathcal{N}_\sigma(\bar{\Omega})$. Since σ is a Brenier solution $\mathcal{T}_\sigma(\Omega^*) = \bar{\Omega}$ and so $\Omega^* \subset \mathcal{N}_\sigma(\mathcal{T}_\sigma(\Omega^*)) = \mathcal{N}_\sigma(\bar{\Omega}) \subset \Omega^*$. □

Remark 6.8. If $\mathcal{G} > 0$ a.e., then each Aleksandrov solution is a Brenier solution. We have $\mathcal{T}_\sigma(y) \neq \emptyset$ for each $y \in \Omega^*$, and it is enough to show that $|\mathcal{N}_\sigma(\mathcal{T}_\sigma(K))\setminus K| = 0$, for each compact set K. This follows regarding writing the argument in the first part of the proof of Theorem 6.7 now using Proposition 4.5 and then Lemma 6.3.

7. Differential Equation of the Energy Problem

Rays are emitted from $(x, 0)$, $x \in \Omega$, with unit direction $e(x) = (e_1(x), e_2(x), e_3(x)) := (e'(x), e_3(x))$. The bottom of the lens is given parametrically by

$$
v(x) = (x, 0) + \rho(x)\, e(x), \tag{7.1}
$$

with ρ a positive smooth function. The top surface of the lens is parametrized by the vector

$$
f(x) = v(x) + d(x)\, m(x),
$$

where the vector m is given by (2.1) and d is a scalar function calculated so that the lens sandwiched by v and f refracts the source Ω into the target $\Omega^* \subseteq S^2$ and solves the energy problem in Section 5. To avoid confusion with the notation for the gradient, we let d denote the distance function $\mathcal{D}$ introduced in Definition 4.1, and also f denotes F in the same definition.

The purpose of this section is to show that the distance function d satisfies the Monge-Ampère type equation (7.11). We assume that v has a normal $\nu(x)$ at each point satisfying $e(x) \cdot \nu(x) > 0$ and $\nu_3(x) > 0$. In Section 7.3 we analyze the collimated case and find a sufficient condition on the refractive indices of the media so that this assumption holds.

We begin with a lemma giving a formula for the normal vector to a general parametric surface.

Lemma 7.1. *Suppose $f(x) = (f_1(x), f_2(x), f_3(x))$ is any C^1 surface given parametrically, $x = (x_1, x_2)$, that is regular at x, i.e., $f_{x_1} \times f_{x_2}(x) \neq 0$. If $v(x) = (v^1(x), v^2(x), v^3(x))$ is the unit normal vector with $v^3(x) > 0$, then we have*

$$v(x) = \frac{(-A^{-1}Df_3(x), 1)}{\sqrt{1 + |A^{-1}Df_3(x)|^2}},$$

where

$$A = \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(x_1, x_2) & \dfrac{\partial f_2}{\partial x_1}(x_1, x_2) \\ \dfrac{\partial f_1}{\partial x_2}(x_1, x_2) & \dfrac{\partial f_2}{\partial x_2}(x_1, x_2) \end{bmatrix}.$$

Proof. From the assumption on the normal $v^3(x) = \pm \dfrac{\det A}{|f_{x_1}(x) \times f_{x_2}(x)|} > 0$, so A is invertible. We have for $i = 1, 2$ that

$$0 = v(x) \cdot f_{x_i}(x) = (v^1(x), v^2(x)) \cdot ((f_1)_{x_i}(x), (f_2)_{x_i}(x)) + v^3(x)(f_3)_{x_i}(x),$$

so

$$A \begin{bmatrix} v^1(x) \\ v^2(x) \end{bmatrix} = -v^3(x)Df_3(x).$$

Therefore $v(x) = v^3(x)\left(-A^{-1}Df_3(x), 1\right)$. Since $|v| = 1$ and $v^3 > 0$, we get $v^3(x) = \dfrac{1}{\sqrt{1 + |A^{-1}Df_3(x)|^2}}$. $\quad\square$

We next calculate the normal to v in (7.1). Recall that v is regular at every point and the normal v satisfies $v_3 > 0$. Therefore from Lemma 7.1

$$v(x) = \frac{(-B^{-1}Dv_3(x), 1)}{\sqrt{1 + |B^{-1}Dv_3(x)|^2}}, \tag{7.2}$$

with $B = \dfrac{\partial(v_1, v_2)}{\partial(x_1, x_2)} = \begin{bmatrix} \dfrac{\partial v_1}{\partial x_1}(x_1, x_2) & \dfrac{\partial v_2}{\partial x_1}(x_1, x_2) \\ \dfrac{\partial v_1}{\partial x_2}(x_1, x_2) & \dfrac{\partial v_2}{\partial x_2}(x_1, x_2) \end{bmatrix}$ invertible. From (7.1)

$$\frac{\partial v_j}{\partial x_i}(x) = \delta_i^j + \rho_{x_i}(x) e_j(x) + \rho(x)(e_j)_{x_i}(x),$$

and recalling $\dfrac{\partial e'}{\partial x} = ((e_j)_{x_i})_{ij}$, $e'(x) = (e_1(x), e_2(x))$, then

$$B = Id + D\rho \otimes e'(x) + \rho(x)\frac{\partial e'}{\partial x}(x). \tag{7.3}$$

If

$$C = Id + \rho(x)\frac{\partial e'}{\partial x}(x)$$

is invertible, then

$$C^{-1} = \frac{1}{\det C} \left(Id + \rho(x) \frac{\partial e'}{\partial x}(x) \right)^* = \frac{1}{\det C} \left(Id + \rho(x) \left(\frac{\partial e'}{\partial x}(x) \right)^* \right),$$

where $*$ denotes the adjoint. Notice that, for the energy problem considered in Section 5, the field e' is the gradient of a convex function, and since $\rho > 0$, in this case the corresponding matrix C is invertible. From (7.3), [24], and since $H(a \otimes b)H = (Ha) \otimes (H^t b)$, for any matrix H, we obtain

$$B^{-1} = C^{-1} - \frac{C^{-1} \left(D\rho(x) \otimes e'(x) \right) C^{-1}}{1 + e'(x) \cdot C^{-1} D\rho(x)}$$

$$= C^{-1} - \frac{\left(C^{-1} D\rho(x) \right) \otimes \left((C^{-1})^t e'(x) \right)}{1 + e'(x) \cdot C^{-1} D\rho(x)}.$$

Letting $t \rightsquigarrow \rho(x)$, $p \rightsquigarrow D\rho(x)$, $q \rightsquigarrow e'(x)$, and $M \rightsquigarrow \frac{\partial e'}{\partial x}(x)$, gives

$$C^{-1} = \frac{1}{\det(Id + t M)} \left(Id + t M^* \right),$$

and

$$\frac{\left(C^{-1} D\rho(x) \right) \otimes \left((C^{-1})^t e'(x) \right)}{1 + e'(x) \cdot C^{-1} D\rho(x)} = \frac{1}{\det(Id + t M)}$$

$$\times \frac{\left((Id + t M^*) p \right) \otimes \left((Id + t M^*)^t q \right)}{\det(Id + t M) + q \cdot (Id + t M^*) p}.$$

Therefore

$$B^{-1} = L(t, p, q, M) := \frac{1}{\det(Id + t M)}$$

$$\times \left(Id + t M^* - \frac{\left((Id + t M^*) p \right) \otimes \left((Id + t M^*)^t q \right)}{\det(Id + t M) + q \cdot (Id + t M^*) p} \right).$$

(7.4)

On the other hand, from (7.1) $v_3(x) = \rho(x) e_3(x) = \rho(x) \sqrt{1 - |e'(x)|^2}$. Then

$$Dv_3(x) = \sqrt{1 - |e'(x)|^2} D\rho(x) - \rho(x) \frac{\frac{\partial e'}{\partial x}(x)(e'(x))}{\sqrt{1 - |e'(x)|^2}}$$

(7.5)

$$= \sqrt{1 - |q|^2} \, p - t \frac{Mq}{\sqrt{1 - |q|^2}} := R(t, p, q, M),$$

using the notation before. From (7.4), (7.5),

$$B^{-1} Dv_3(x) = L(t, p, q, M) \, R(t, p, q, M),$$

and so from (7.2) we obtain the following formula for the normal to v:

$$v(x) = \frac{(-L(t, p, q, M) \, R(t, p, q, M), 1)}{\sqrt{1 + |L(t, p, q, M) \, R(t, p, q, M)|^2}}, \tag{7.6}$$

where $t \rightsquigarrow \rho(x)$, $p \rightsquigarrow D\rho(x)$, $q \rightsquigarrow e'(x)$, and $M \rightsquigarrow \dfrac{\partial e'}{\partial x}(x)$.

We calculate now the refracted vector $m(x)$. Snell's law applied at the point $v(x)$ and (7.6) yields

$$m(x) = \frac{1}{\kappa_1}(e(x) - \lambda(x) \, v(x)) := W(t, p, q, M), \tag{7.7}$$

with the notation $t \rightsquigarrow \rho(x)$, $p \rightsquigarrow D\rho(x)$, $q \rightsquigarrow e'(x)$, and $M \rightsquigarrow \dfrac{\partial e'}{\partial x}(x)$.

7.1. Calculation of the Refractor Map T for the Lens with Upper Surface f

The lens sandwiched by v and f, refracts incoming rays at the point $f(x)$ into the unit direction Tx, where T is a map from the source Ω into the target $\Omega^* \subseteq S^2$. We are going to calculate an expression for T. By Snell's law at $f(x)$,

$$Tx = \frac{1}{\kappa_2}(m(x) - \lambda_2(x)v_2(x)),$$

where $v_2(x)$ is the unit normal at the striking point on the surface $f(x)$; and $\lambda_2(x) = \phi_{\kappa_2}(m(x) \cdot v_2(x))$.

We assume that f satisfies the conditions of Lemma 7.1, i.e., f is regular at x, and $v_2^3(x) := v_2(x) \cdot e_3 > 0$. In this case $v_2(x) = \dfrac{(-A^{-1}Df_3(x), 1)}{\sqrt{1 + |A^{-1}Df_3(x)|^2}}$, with $A = \dfrac{\partial(f_1, f_2)}{\partial(x_1 x_2)}$. We shall calculate an expression for A^{-1}. We have for $i, j = 1, 2$

$$\frac{\partial f_j}{\partial x_i}(x) = \frac{\partial v_j}{\partial x_i}(x) + d(x)\frac{\partial m_j}{\partial x_i}(x) + d_{x_i}(x)m_j(x).$$

Let $m' = (m_1, m_2)$, and $\dfrac{\partial m'}{\partial x}(x) = \left(\dfrac{\partial m_j}{\partial x_i}(x)\right)_{ij}$, then we get, as in (7.3), that

$$\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \frac{\partial(v_1, v_2)}{\partial(x_1, x_2)} + d(x)\frac{\partial m'}{\partial x}(x) + Dd(x) \otimes m'(x)$$

$$= Id + D\rho(x) \otimes e'(x) + \rho(x)\frac{\partial e'}{\partial x}(x) + d(x)\frac{\partial m'}{\partial x}(x)$$

$$+ Dd(x) \otimes m'(x).$$

Let p_1, p_2, q_1, q_2 be n-column vectors, and define the matrices

$$P = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \qquad Q = \begin{bmatrix} q_1^t \\ q_2^t \end{bmatrix};$$

clearly, P is $n \times 2$ and Q is $2 \times n$. Then

$$p_1 \otimes q_1 + p_2 \otimes q_2 = p_1 q_1^t + p_2 q_2^t = P\,Q.$$

Now use the Woodbury matrix identity: if H, P, Q are $n \times n$ matrices, then

$$(H + P\,Q)^{-1} = H^{-1} - H^{-1} P \left(Id + Q H^{-1} P \right)^{-1} Q H^{-1}.$$

We first write

$$
\begin{aligned}
\frac{\partial(f_1,\,f_2)}{\partial(x_1,\,x_2)} &= Id + \rho(x)\frac{\partial e'}{\partial x}(x) + d(x)\frac{\partial m'}{\partial x}(x) \\
&\quad + D\rho(x) \otimes e'(x) + Dd(x) \otimes m'(x) \\
&= Id + \rho(x)\frac{\partial e'}{\partial x}(x) + d(x)\frac{\partial m'}{\partial x}(x) \\
&\quad + \left[D\rho(x)\ Dd(x) \right]\begin{bmatrix} e'(x)^t \\ m'(x)^t \end{bmatrix} \\
&= H + \left[D\rho(x)\ Dd(x) \right]\begin{bmatrix} e'(x)^t \\ m'(x)^t \end{bmatrix}, \\
H &= Id + \rho(x)\frac{\partial e'}{\partial x}(x) + d(x)\frac{\partial m'}{\partial x}(x).
\end{aligned}
$$

Notice that by (7.7) $\dfrac{\partial m'}{\partial x}(x)$ depends on $\rho(x), D\rho(x), D^2\rho(x), e'(x), \dfrac{\partial e'}{\partial x}(x)$ and on the second derivatives of the components of e'. Since $e' = (e_1, e_2)$,

$$H = \mathcal{H}\left(\rho(x), D\rho(x), D^2\rho(x), e'(x), \frac{\partial e'}{\partial x}(x), D^2 e_1(x), D^2 e_2(x), d(x) \right).$$

Assuming the invertibility of the matrices involved, from Woodbury's identity we get

$$
\begin{aligned}
\left(\frac{\partial(f_1,\,f_2)}{\partial(x_1,\,x_2)} \right)^{-1} &= H^{-1} - H^{-1}\left[D\rho(x)\ Dd(x) \right] \\
&\quad \times \left(Id + \begin{bmatrix} e'(x)^t \\ m'(x)^t \end{bmatrix} H^{-1}\left[D\rho(x)\ Dd(x) \right] \right)^{-1}\begin{bmatrix} e'(x)^t \\ m'(x)^t \end{bmatrix} H^{-1} \\
&= \overline{\mathcal{H}}\left(\rho(x), D\rho(x), D^2\rho(x), e'(x), \frac{\partial e'}{\partial x}(x), \right. \\
&\qquad\qquad \left. D^2 e_1(x), D^2 e_2(x), d(x), Dd(x) \right).
\end{aligned}
$$

We also have from the form of f, $f_3(x) = v_3(x) + d(x)m_3(x)$, and so $Df_3(x) = Dv_3(x) + Dd(x)m_3(x) + d(x)Dm_3(x)$. Hence from (7.5), (7.7) and Lemma 7.1, we get the normal to $f(x)$ in terms of the variables involved:

$$v_2(x) = \frac{(-A^{-1}Df_3(x), 1)}{\sqrt{1 + |A^{-1}Df_3(x)|^2}}$$

$$:= V\left(\rho(x), D\rho(x), D^2\rho(x), e'(x), \frac{\partial e'}{\partial x}(x), D^2e_1(x),\right.$$

$$\left. D^2e_2(x), d(x), Dd(x)\right). \tag{7.8}$$

Therefore, from Snell's law at $f(x)$, we obtain

$$Tx = (T_1x, T_2x, T_3x) = \frac{1}{\kappa_2}\left(m(x) - \phi_{\kappa_2}(m(x) \cdot v_2(x))\, v_2(x)\right)$$

$$= F\left(\rho(x), D\rho(x), D^2\rho(x), e'(x), \frac{\partial e'}{\partial x}(x), D^2e_1(x), D^2e_2(x), d(x), Dd(x)\right)$$

$$= (F_1, F_2, F_3). \tag{7.9}$$

7.2. Derivation of the PDE for d

The energy densities at the source Ω and the target Ω^* are given by positive integrable functions $\mathcal{I}$ and $\mathcal{G}$ respectively, such that conservation of energy condition (5.1) is satisfied ($\eta = \mathcal{G}\, dy$). If $E \subseteq \Omega$, then T maps E into $T(E)$. We require the energy to be conserved on each patch E, that is,

$$\int_{T(E)} \mathcal{G}(y)\, dy = \int_E \mathcal{I}(x)\, dx.$$

Writing $y = Tx$, and formally applying the formula of change of variables yields

$$\int_E \mathcal{G}(Tx)|J_T(x)|dx = \int_E \mathcal{I}(x)\, dx.$$

Therefore, we obtain the following PDE:

$$|J_Tx| = \frac{\mathcal{I}(x)}{\mathcal{G}(Tx)},$$

with $|J_Tx| = \left|(Tx)_{x_1}(x) \times (Tx)_{x_2}(x)\right|$. Since $|Tx| = 1$, then $Tx \cdot (Tx)_{x_i} = 0$, $i = 1, 2$. Hence, assuming $T_3 \neq 0$, we get

$$(T_3)_{x_i} = -\frac{T_1(T_1)_{x_i} + T_2(T_2)_{x_i}}{T_3}.$$

Therefore

$$(Tx)_{x_1}(x) \times (Tx)_{x_2}(x) = \frac{1}{T_3x} \det \begin{pmatrix} (T_1x)_{x_1} & (T_1x)_{x_2} \\ (T_2x)_{x_1} & (T_2x)_{x_2} \end{pmatrix} Tx. \tag{7.10}$$

Let us calculate

$$\frac{\partial T_i x}{\partial x_j}, \qquad 1 \leq i, j \leq 2$$

using (7.9). We group the variables of T_i that are independent of d and Dd and write

$$T_i x = F_i \left(\rho(x), D\rho(x), D^2\rho(x), e'(x), \frac{\partial e'}{\partial x}(x), D^2 e_1(x), D^2 e_2(x), d(x), Dd(x) \right)$$
$$= F_i\left(X, d, Dd \right),$$

where

$$X = \left(\rho, D\rho, D^2\rho, e', \frac{\partial e'}{\partial x}, D^2 e_1, D^2 e_2 \right).$$

Thus, $F_i = F_i(X, t, p)$, with $t \in R$ and $p = (p_1, p_2)$; $i = 1, 2, 3$. We now differentiate T_i with respect to x_j:

$$\frac{\partial T_i}{\partial x_j} = \mathcal{F}_{ij} + \frac{\partial F_i}{\partial t}\frac{\partial d}{\partial x_j} + \frac{\partial F_i}{\partial p_1}\frac{\partial^2 d}{\partial x_1 \partial x_j} + \frac{\partial F_i}{\partial p_2}\frac{\partial^2 d}{\partial x_2 \partial x_j},$$

where

$$\mathcal{F}_{ij} = \sum_\ell \frac{\partial F_i}{\partial X_\ell}\frac{\partial X_\ell}{\partial x_j}.$$

Notice that $\mathcal{F}_{ij}$ depends on ρ and its derivatives up to order three, it depends on e' and its derivatives up order three, and it depends on d and Dd. If write $F' = (F_1, F_2)$, and set

$$D_p F'(X, d, Dd) = \left(\frac{\partial F_i}{\partial p_j}(X, d, Dd) \right)_{i,j=1,2},$$

then

$$\left(\frac{\partial T_i}{\partial x_j} \right)_{i,j=1,2} = (\mathcal{F}_{ij}) + \frac{\partial F'}{\partial t}(X, d, Dd) \otimes Dd + D_p F'(X, d, Dd) D^2 d(x)$$
$$= \mathcal{B} + \mathcal{A}(X, d, Dd) D^2 d,$$

where

$$\mathcal{B} = (\mathcal{F}_{ij}) + \frac{\partial F'}{\partial t}(X, d, Dd) \otimes Dd \quad \text{and} \quad \mathcal{A}(X, d, Dd) = D_p F'(X, d, Dd)$$

are both independent of $D^2 d$. Therefore, from (7.10),

$$|J_T(x)| = \frac{1}{F_3(X, d, Dd)} \det\left(\mathcal{A}(X, d, Dd) D^2 d + \mathcal{B} \right),$$

and so d satisfies the following Monge-Ampère type equation:

$$\det\left(\mathcal{A}(X(x), d(x), Dd(x)) D^2 d(x) + \mathcal{B} \right) = \frac{F_3(X(x), d(x), Dd(x))\mathcal{I}(x)}{\mathcal{G}(F(X(x), d(x), Dd(x)))}.$$
$$\text{(7.11)}$$

Notice that $\mathcal{B}$ depends on ρ and its derivatives up to order three, it depends on e' and its derivatives up to order three, and it depends on d and Dd.

7.3. The Collimated Case

Assuming that the refractor σ constructed in Theorem 5.5 is smooth, i.e., $d \in C^2$, we will show that in the collimated case the pde (7.11) satisfied by d has a simpler form and will give sufficient conditions for the invertibility of the matrices involved in the derivation of the pde. We assume that the field $e(x)$ is vertical, i.e., $e(x) = e_3 = (0, 0, 1)$, and the lower surface of the lens u is $C^3(\Omega)$ and is concave. The surface σ is parametrized by the vector $f(x) = (x, u(x)) + d(x) m(x)$, with $d \in C^2(\Omega)$.

From (7.6), we have $t = u$, $p = Du$, $q = 0$, $M = 0$; so $L(t, p, 0, 0) = Id$, $R(t, p, 0, 0) = p$. Then the normal $\nu(x) = \dfrac{(-Du(x), 1)}{\sqrt{1 + |Du(x)|^2}}$. Notice that ν does not depend on u.

Next, from (7.7), we get

$$
m(x) = \frac{1}{\kappa_1} \left((0, 0, 1) - \phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta(x)}} \right) \frac{(-Du(x), 1)}{\sqrt{\Delta(x)}} \right) := (m'(x), m_3(x)),
\tag{7.12}
$$

where $\Delta(x) = 1 + |Du(x)|^2$.

We next calculate $\partial m' / \partial x$. By (7.12), $m'(x) = \dfrac{\phi_{\kappa_1}\left(\dfrac{1}{\sqrt{\Delta}}\right)}{\kappa_1 \sqrt{\Delta}} Du(x)$, then

$$
\frac{\partial m_j}{\partial x_i} = \frac{1}{\kappa_1 \sqrt{\Delta}} \left(\frac{\partial \left(\frac{1}{\sqrt{\Delta}} \right)}{\partial x_i} \phi'_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) u_{x_j} + \phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) u_{x_j x_i} \right)
$$

$$
+ \frac{\partial \left(\frac{1}{\sqrt{\Delta}} \right)}{\partial x_i} \times \frac{\phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) u_{x_j}}{\kappa_1}
$$

$$
= \frac{1}{\kappa_1 \sqrt{\Delta}} \left(-\frac{1}{2 \Delta \sqrt{\Delta}} \phi'_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) \frac{\partial \Delta}{\partial x_i} u_{x_j} + \phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) u_{x_j x_i} \right)
$$

$$
- \frac{1}{2 \Delta \sqrt{\Delta}} \frac{\partial \Delta}{\partial x_i} \frac{\phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) u_{x_j}}{\kappa_1}
$$

$$
= \frac{1}{\kappa_1 \sqrt{\Delta}} \left[\left(-\frac{1}{2 \Delta \sqrt{\Delta}} \phi'_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) - \frac{1}{2 \Delta} \phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) \right) \frac{\partial \Delta}{\partial x_i} u_{x_j} \right.
$$

$$
\left. + \phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) u_{x_j x_i} \right].
$$

Therefore

$$
\frac{\partial m'}{\partial x} = \frac{1}{\kappa_1 \sqrt{\Delta}} \left[\left(-\frac{1}{2 \Delta \sqrt{\Delta}} \phi'_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) - \frac{1}{2 \Delta} \phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) \right) D\Delta \otimes Du \right.
$$

$$
\left. + \phi_{\kappa_1} \left(\frac{1}{\sqrt{\Delta}} \right) D^2 u \right].
\tag{7.13}
$$

Since $\Delta = 1 + |Du(x)|^2$, $\dfrac{\partial \Delta}{\partial x_i} = 2\,u_{x_1 x_i} u_{x_1} + 2\,u_{x_2 x_i} u_{x_2}$. Therefore

$$DA(x) = 2\,D^2 u(x)\,(Du(x)).$$

Replacing this in (7.13), we get

$$
\begin{aligned}
\frac{\partial m'}{\partial x} &= \frac{1}{\kappa_1 \sqrt{\Delta}} \left[\left(-\frac{1}{\Delta \sqrt{\Delta}} \phi'_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) - \frac{1}{\Delta} \phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) \right) \left(D^2 u (Du) \right) \otimes Du \right. \\
&\qquad \left. + \phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) D^2 u \right] \\
&= \frac{1}{\kappa_1 \sqrt{\Delta}} \left[\left(-\frac{1}{\Delta \sqrt{\Delta}} \phi'_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) - \frac{1}{\Delta} \phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) \right) D^2 u \, (Du \otimes Du) \right. \\
&\qquad \left. + \phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) D^2 u \right] \\
&= \frac{1}{\kappa_1 \sqrt{\Delta}} D^2 u \left[\phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) Id \right. \\
&\qquad \left. - \left(\frac{1}{\Delta \sqrt{\Delta}} \phi'_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) + \frac{1}{\Delta} \phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) \right) Du \otimes Du \right].
\end{aligned}
$$

From (3.17), we have

$$
\phi'_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) = \frac{-1}{\sqrt{\kappa_1^2 - 1 + \dfrac{1}{\Delta}}} \phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right) = \frac{-\sqrt{\Delta}}{\sqrt{1 + (\kappa_1^2 - 1)\Delta}} \phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}} \right).
$$

Therefore

$$
\frac{\partial m'}{\partial x} = \frac{\phi_{\kappa_1}\left(\dfrac{1}{\sqrt{\Delta}} \right)}{\kappa_1 \sqrt{\Delta}} D^2 u \left(Id + \left(\frac{1}{\Delta \sqrt{1 + (\kappa_1^2 - 1)\Delta}} - \frac{1}{\Delta} \right) Du \otimes Du \right).
$$
(7.14)

We prove that $\det\left(\dfrac{\partial m'}{\partial x} \right) \geq 0$. Define

$$
\mathcal{M} = Id + \frac{1}{\Delta} \left(\frac{1}{\sqrt{1 + (\kappa_1^2 - 1)\Delta}} - 1 \right) Du \otimes Du.
$$

For (7.14), we write

$$
\frac{\partial m'}{\partial x} = \frac{\phi_{\kappa_1}\left(\dfrac{1}{\sqrt{\Delta}} \right)}{\kappa_1 \sqrt{\Delta}} D^2 u \, \mathcal{M}.
$$

Notice that the matrix $Du \otimes Du$ has eigenvalues 0 and $|Du|^2 = \Delta - 1$, then $\mathcal{M}$ has eigenvalues 1 and $1 + \dfrac{\Delta - 1}{\Delta}\left(\dfrac{1}{\sqrt{1 + (\kappa_1^2 - 1)\Delta}} - 1\right)$, which are both positive and therefore $\mathcal{M}$ is positive definite. Since u is concave, we conclude that $\det\left(\dfrac{\partial m'}{\partial x}\right) \geq 0$.

We next calculate the normal ν_2 to f towards medium n_3. First notice that the existence of ν_2 follows from Theorem 4.2(3) and the assumption that $d \in C^2$. To calculate the normal we use Lemma 7.1, for which we need to show that $e_3 \cdot \nu_2 > 0$.

Lemma 7.2. *Assume the medium containing the source Ω is denser than or equal to the medium containing Ω^*, that is, $n_1 \geq n_3$, then $e_3 \cdot \nu_2(x) > 0$.*

Proof. By (2.1),

$$e_3 \cdot \nu_2 = (\kappa_1 m + \lambda \nu) \cdot \nu_2 = \kappa_1 m \cdot \nu_2 + \lambda \nu \cdot \nu_2 := I + II.$$

From (2.2) we have $m(x) \cdot Tx \geq \kappa_2$. Then by Snell's law [8, Subsection 2.1] we have $m(x) \cdot \nu_2(x) \geq \sqrt{1 - \kappa_2^2}$. Since $n_1 \geq n_3$, we get

$$I \geq \kappa_1 \sqrt{1 - \kappa_2^2} \geq \sqrt{\kappa_1^2 - 1}.$$

Since $\phi_{\kappa_1}(t)$ is increasing in $[0, 1]$, and ν, ν_2 are unit vectors, it follows that

$$II \geq \lambda = \phi_{\kappa_1}\left(1/\sqrt{\Delta}\right) > \phi_{\kappa_1}(0) = -\sqrt{\kappa_1^2 - 1}.$$

Combining both estimates, we conclude that $e_3 \cdot \nu_2 > 0$. $\square$

By Lemma 7.1, when $n_1 \geq n_3$, we get that $\nu_2 = \dfrac{\left(-A^{-1}Df_3(x), 1\right)}{\sqrt{1 + |A^{-1}Df_3(x)|^2}}$, where $A = \left(\dfrac{\partial f_j(x)}{\partial x_i}\right)_{i,j}$ is invertible, with $f(x) = (x, u(x)) + d(x)m(x)$.

Therefore it will be assumed in the rest of this section that $n_1 \geq n_3$. To calculate the matrix A^{-1}, we write for every $1 \leq i, j \leq 2$,

$$\frac{\partial f_j}{\partial x_i} = \delta_i^j + d\frac{\partial m_j}{\partial x_i} + \frac{\partial d}{\partial x_i}m_j,$$

so

$$A = Id + d(x)\frac{\partial m'}{\partial x}(x) + Dd(x) \otimes m'(x) := H + Dd(x) \otimes m'(x). \quad (7.15)$$

Since $\det\left(\dfrac{\partial m'}{\partial x}\right) \geq 0$ and $d > 0$ then $\det H > 0$ and H is invertible. From [24], it follows that

$$\det A = 1 + \left(H^{-1}Dd\right) \cdot m' \neq 0$$

$$A^{-1} = H^{-1} + \frac{H^{-1}\left(Dd \otimes m'\right)H^{-1}}{1 + \left(H^{-1}Dd\right) \cdot m'}. \tag{7.16}$$

It remains to calculate Df_3. We have from (7.12) that

$$m_3 = \frac{1}{\kappa_1}\left(1 - \frac{\phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}}\right)}{\sqrt{\Delta}}\right), \tag{7.17}$$

then, as in the calculation of $\partial m'/\partial x$, we obtain

$$Dm_3 = \frac{1}{\kappa_1\sqrt{\Delta}}\left(\frac{1}{2\,\Delta\sqrt{\Delta}}\phi'_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}}\right) + \frac{1}{2\,\Delta}\phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}}\right)\right)D\Delta$$

$$= \frac{1}{\kappa_1\sqrt{\Delta}}\left(\frac{1}{\Delta\sqrt{\Delta}}\phi'_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}}\right) + \frac{1}{\Delta}\phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}}\right)\right)D^2u(Du)$$

$$= \left(1 - \frac{1}{\sqrt{1 + (\kappa_1^2 - 1)\Delta}}\right)\frac{\phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}}\right)}{\kappa_1\Delta\sqrt{\Delta}}D^2u(Du).$$

Since $f_3(x) = u(x) + d(x)m_3(x)$,

$$Df_3(x) = Du(x) + \frac{1}{\kappa_1}\left(1 - \frac{\phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}}\right)}{\sqrt{\Delta}}\right)Dd(x)$$

$$+ d(x)\left(1 - \frac{1}{\sqrt{1 + (\kappa_1^2 - 1)\Delta}}\right)\frac{\phi_{\kappa_1}\left(\frac{1}{\sqrt{\Delta}}\right)}{\kappa_1\Delta\sqrt{\Delta}}D^2u(Du). \tag{7.18}$$

We conclude that $v_2(x) = \dfrac{\left(-A^{-1}Df_3(x), 1\right)}{\sqrt{1 + |A^{-1}Df_3(x)|^2}}$, with A^{-1} and Df_3 given by (7.16) and (7.18) respectively. In contrast with the case of a general field e, in the collimated case v_2 depends only on Du, D^2u, d and Dd, see (7.8).

Observe that from (7.17), $m_3(x) > 0$. Then by Snell's law $T_3x = \kappa_2\,m_3(x) + \lambda_2(x)\,v_2^3(x)$. By Lemma 7.2, $v_2^3(x) > 0$, and since $\kappa_2 < 1$, $\lambda_2(x) > 0$. Therefore $T_3x > 0$, and (7.10) is well defined.

Proceeding as in the previous section, we obtain that that d satisfies the Monge-Ampère type equation (7.11), where in this case $X(x) = (Du(x), D^2u(x))$, and the matrix $\mathcal{B}$ depends on d, Dd, and on the derivatives up to order three of u but does not depend on u.

Summary of Notation

- If u is a scalar function, Du or ∇u denote its gradient and $D^2 u$ denotes its Hessian.
- For a field $F(x) = (F_1(x), F_2(x))$ with $x = (x_1, x_2)$, we write $\dfrac{\partial F}{\partial x} =$

$$\frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_2}{\partial x_1} \\ \dfrac{\partial F_1}{\partial x_2} & \dfrac{\partial F_2}{\partial x_2} \end{pmatrix}.$$

- n denotes an homogenous medium and at the same time its refractive index.
- We consider homogeneous media n_1, n_2, n_3 with n_2 bigger than both n_1 and n_3, but n_1 and n_3 are unrelated.
- $\kappa_1 = n_2/n_1$ and $\kappa_2 = n_3/n_2$.
- $e(x) = (e_1(x), e_2(x), e_3(x))$ denotes a unit vector, $e'(x) = (e_1(x), e_2(x))$, $e_3(x) \geq \delta > 0$.
- If a, b are column vector of the same dimension, $a \otimes b = ab^t$.
- Given a map F, L_F denotes its Lipschitz constant.
- Ω denotes the source and Ω^* the target.
- S^2 denotes the unit sphere in $\mathbb{R}^3$.

Conflict of interest The authors declare that they have no conflict of interest.

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