HOMOGENIZATION AND CONVERGENCE OF CORRECTORS IN CARNOT GROUPS

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1. Introduction

In this paper we consider periodic homogenization of differential operators in the setting of a Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ)$. In fact, Carnot groups, i.e., connected and simply connected stratified nilpotent Lie groups, provide a natural setting for periodic homogenization, being endowed with intrinsic dilations and translations. All these notions will be clearly stated in Section 2. If $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}$, that is, the class of left invariant vector fields with respect to the multiplication $\circ$, let $X_1, \cdots, X_m$ be a family of linearly independent vector fields in $\mathbb{R}^N$ that by commutation generate $\mathfrak{g}$, see Subsection 2.1. We shall refer to $X_1, \cdots, X_m$ as the horizontal derivatives. Then the homogenization problem we consider here has the form

$$-\sum_{i,j=1}^{m} X_i(a_{ij}(\delta_{1/\epsilon}(\xi))X_j) = f,$$

where the coefficients a_{ij} are $\mathbf{Y}$-periodic functions with respect to a reference pavage or tiling of the group $\mathbb{G}$, and $\delta_{1/\epsilon}$ are the dilations in the group, see Subsections 2.3 and 2.5. It is known that, when $\epsilon \to 0$, we obtain the homogenized equation (3.3) and the main result in the paper is the strong convergence of the horizontal gradients of correctors in $L^{2+\theta}$ for some $\theta > 0$, see Theorem 6.1 and Remark 6.2. To this end, we prove in Section 5 Meyers type estimates, having independent interest, for solutions to equations defined with general vector fields satisfying Hörmander's condition on regular domains using reverse Hölder estimates from [10, Theorem 3.1]. These combined with the convergence of correctors in L^2 proved in Theorem 4.4 yield the desired convergence.

When $\mathbb{G}$ is the Heisenberg group, homogenization results are proved in [2] and [8]. The convergence of gradients of higher order correctors to uniformly elliptic linear equations has been considered in [13] using a different method.

The organization of the paper is as follows. Section 2 contains the basic definitions about Carnot groups, examples, the construction of the pavage, the spaces considered, and the notion of periodicity in this setting. In Section 3 we describe the equations considered and the homogenized equation. Section 4 concerns with gradients of correctors and its convergence in L^2. Section 5 has the proofs of the Meyers type estimates needed later, the main result is Theorem 5.4. Finally, Section 6 contains the proof of the convergence of gradients of correctors in $L^{2+\theta}$, Theorem 6.1.

2. Preliminaries

2.1. **Carnot groups.** Let $\circ$ be an operation in $\mathbb{R}^N$ (we call it multiplication) that makes $\mathbb{G} = (\mathbb{R}^N, \circ)$ be a Lie group, that is, a group where the multiplication $\circ : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and the inversion $x \to x^{-1}$ are smooth maps. From now on, we assume the origin is the identity element. Given $x \in \mathbb{G}$, the *left–translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined by $\tau_x z = x \circ z$.

We write $N = N_1 + N_2 + \cdots + N_k$ and given $x \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k}$ we put $x = (x'_1, x'_2, \cdots, x'_k)$ with $x'_j \in \mathbb{R}^{N_j}$ for $j = 1, \cdots, k$. We assume that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k$ are real numbers and for $\lambda > 0$ the family of dilations

$$\delta_\lambda x = (\lambda^{\alpha_1} x'_1, \cdots, \lambda^{\alpha_k} x'_k), \tag{2.1}$$

forms a group of automorphisms of $\mathbb{G}$, i.e. $\delta_\lambda(x \circ y) = \delta_\lambda x \circ \delta_\lambda y$. Let $\mathfrak{g}$ denote the Lie algebra of $\mathbb{G}$, that is, the linear space of all $\circ$–left invariant vector fields on $\mathbb{G}$, and let $X_1, \cdots, X_N$ be a basis of $\mathfrak{g}$ such that $X_j(0) = D_j$, $j = 1, \cdots, N$ (the elements of $\mathfrak{g}$ are uniquely determined by their value at the origin). If $V_1 = \text{span}\{X_1, \cdots, X_{N_1}\}$, then we assume that $\mathfrak{g}$ admits a *stratification* of the form

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k, \quad [V_1, V_i] = V_{i+1} \text{ for } i = 1, ..., k-1 \text{ and } [V_1, V_k] = \{0\} \tag{2.2}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. This is expressed by saying that $\mathbb{G} = (\mathbb{R}^N, \circ)$ is a *Carnot group of step k with N_1 generators*. The integer $Q = N_1\alpha_1 + \cdots + N_k\alpha_k$ is the homogeneous dimension of $\mathbb{G}$, which is also the Hausdorff dimension of $\mathbb{R}^N$ with respect to the Carnot-Carathéodory distance defined by means of subunit curves. The N-dimensional Lebesgue measure (from now on denoted by $|E|$ for any measurable set $E \subset \mathbb{R}^N$) is the natural measure in $\mathbb{G}$, since it is the Haar measure of the group, i.e. $|\tau_x E| = |E|$ for all $x \in \mathbb{G}$ and for any measurable set $E \subset \mathbb{R}^N$. Moreover, if $\lambda > 0$ then $|\delta_\lambda(E)| = \lambda^Q |E|$. Note also that thanks to the stratification assumption, Hörmander's finite rank condition is satisfied by the vector fields $X_1, \cdots, X_{N_1}$.

Under these assumptions we have that the multiplication $\circ$ and the vector fields X_j's have the form given in Propositions 2.1 and 2.2 below, see [16, pp. 618–622] and [7, Proposition 2.1]. The following proposition is important in Lemma 2.4 for the construction of a pavage of $\mathbb{G}$. We include its proof for the sake of completeness.

Proposition 2.1. *There exists a unique polynomial vector-valued function $\mathcal{H} = (\mathcal{H}_1, \cdots, \mathcal{H}_k) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ with $\mathcal{H}_i = (H_i^1, \cdots, H_i^{N_i})$ satisfying $\mathcal{H}(\delta_\lambda x, \delta_\lambda y) = \delta_\lambda \mathcal{H}(x, y)$, for all $x, y \in \mathbb{R}^N$ and $\lambda > 0$, such that the multiplication $\circ$ is given by*

$$x \circ y = x + y + \mathcal{H}(x, y), \qquad \textit{for all } x, y \in \mathbb{R}^N. \tag{2.3}$$

In addition, for each $x, y \in \mathbb{G}$ we have $\mathcal{H}_1(x, y) = 0$ and

$$\mathcal{H}_j(x, y) = \mathcal{R}_j(x'_1, \cdots, x'_{j-1}, y'_1, \cdots, y'_{j-1}), \quad \textit{for} \quad 1 < j \le k, \tag{2.4}$$

where $\mathcal{R}_j$ are polynomials.

Proof. Given $\beta \in \mathbb{Z}^N_{\ge 0}$ a multi–index, we write $\beta = (\beta'_1, \beta'_2, \cdots, \beta'_k)$ with $\beta'_j \in \mathbb{Z}^{N_j}_{\ge 0}$. We write

$$x \circ y = ((x \circ y)'_1, (x \circ y)'_2, \cdots, (x \circ y)'_k),$$

and set

$$P_j(x, y) = (x \circ y)'_j = (P_j^1(x, y), \cdots, P_j^{N_j}(x, y)).$$

We have $P_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^{N_j}$, and

$$P_j(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_j} P_j(x, y).$$

If $\beta = (\beta'_1, \beta'_2, \cdots, \beta'_k), \gamma = (\gamma'_1, \gamma'_2, \cdots, \gamma'_k) \in \mathbb{Z}^N_{\ge 0}$, then

$$D_x^\beta D_y^\gamma (P_j(\delta_\lambda x, \delta_\lambda y)) = \lambda^{\sum_{i=1}^k (|\beta'_i| + |\gamma'_i|)\alpha_i} (D_x^\beta D_y^\gamma P_j)(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_j} (D_x^\beta D_y^\gamma P_j)(x, y);$$

that is,

$$(D_x^\beta D_y^\gamma P_j)(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_j - \sum_{i=1}^k (|\beta_i'|+|\gamma_i'|)\alpha_i} (D_x^\beta D_y^\gamma P_j)(x,y)$$

so $(D_x^\beta D_y^\gamma P_j)(x,y)$ is homogeneous of degree $\alpha_j - \sum_{i=1}^k (|\beta_i'| + |\gamma_i'|)\alpha_i$ with respect to δ_λ. Since $P_j \in C^\infty$, it follows that $(D_x^\beta D_y^\gamma P_j)(x,y) \equiv 0$ for all $\beta, \gamma \in \mathbb{Z}_{\geq 0}^N$ such that

$$\alpha_j - \sum_{i=1}^k (|\beta_i'| + |\gamma_i'|)\alpha_i < 0, \tag{2.5}$$

and so P_j is a polynomial. Since $x \circ 0 = x$ and $0 \circ y = y$, we have that $P_j(0,0) = 0$, $P_j(x,0) = x_j'$, and $P_j(0,y) = y_j'$, for $j = 1, \cdots, k$. Let us define

$$\mathcal{H}_j(x,y) = P_j(x,y) - x_j' - y_j',$$

for $j = 1, \cdots, k$.

Let us first analyze $P_1(x,y)$. Recall that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k$. We claim that the only derivatives $(D_x^\beta D_y^\gamma P_1)(x,y)$ that might not be identically zero are those for which

$$|\beta_1'| + |\gamma_1'| \leq 1, \qquad \text{and} \qquad |\beta_i'| + |\gamma_i'| = 0 \quad \text{for} \quad i > 1.$$

From (2.5), we have that the only derivatives $(D_x^\beta D_y^\gamma P_1)(x,y)$ that might not be identically zero are those for which

$$\alpha_1 \geq \sum_{i=1}^k (|\beta_i'| + |\gamma_i'|)\alpha_i$$

which implies that $\alpha_1 \geq (|\beta_1'| + |\gamma_1'|)\alpha_1$, that is, $|\beta_1'| + |\gamma_1'| \leq 1$. We prove that $|\beta_i'| + |\gamma_i'| = 0$ for $i > 1$. Because if on the other hand we would have $|\beta_i'| + |\gamma_i'| > 0$ for some $i > 1$, since $\alpha_i > \alpha_1$, we would get that $\alpha_1 \geq (|\beta_i'| + |\gamma_i'|)\alpha_i > (|\beta_i'| + |\gamma_i'|)\alpha_1$, that is, $|\beta_i'| + |\gamma_i'| < 1$, a contradiction. Therefore, $P_1^\ell(x,y)$ is an affine function of x_1' and y_1' for each $1 \leq \ell \leq N_1$, that is,

$$P_1^\ell(x,y) = a_\ell + b_1^\ell x_1 + b_2^\ell x_2 + \cdots + b_{N_1}^\ell x_{N_1} + c_1^\ell y_1 + c_2^\ell y_2 + \cdots + c_{N_1}^\ell y_{N_1}.$$

Since $P_1(0,0) = 0$ we get that $a_\ell = 0$, and since $P_1(x,0) = x_1'$ and $P_1(0,y) = y_1'$ we get that $P_1^\ell(x,y) = x_\ell + y_\ell$.

Similarly we analyze $P_2(x,y)$. We claim that the only derivatives $(D_x^\beta D_y^\gamma P_2)(x,y)$ that might not be identically zero are those for which

$$|\beta_1'| + |\gamma_1'| \leq \alpha_2/\alpha_1, \qquad |\beta_2'| + |\gamma_2'| \leq 1, \qquad \text{and} \qquad |\beta_i'| + |\gamma_i'| = 0 \quad \text{for} \quad i > 2.$$

From (2.5), we have that the only derivatives $(D_x^\beta D_y^\gamma P_2)(x,y)$ that might not be identically zero are those for which

$$\alpha_2 \geq \sum_{i=1}^k (|\beta_i'| + |\gamma_i'|)\alpha_i \tag{2.6}$$

which implies that $\alpha_2 \geq (|\beta_2'| + |\gamma_2'|)\alpha_2$, that is, $|\beta_2'| + |\gamma_2'| \leq 1$. Also from (2.6), we get that $|\beta_1'| + |\gamma_1'| \leq \alpha_2/\alpha_1$. We prove that $|\beta_i'| + |\gamma_i'| = 0$ for $i > 2$. Because if on the other hand we would have $|\beta_i'| + |\gamma_i'| > 0$ for some $i > 2$, since $\alpha_i > \alpha_2$, we would get that $\alpha_2 \geq (|\beta_i'| + |\gamma_i'|)\alpha_i > (|\beta_i'| + |\gamma_i'|)\alpha_2$, that is, $|\beta_i'| + |\gamma_i'| < 1$, a contradiction. Therefore, $P_2^\ell(x, y)$ is an affine function of x_2' and y_2' plus a polynomial of x_1', y_1' for each $1 \leq \ell \leq N_2$, that is,

$$\begin{aligned} &P_2^\ell(x, y) \\ &= a_\ell + b_1^\ell x_{N_1+1} + b_2^\ell x_{N_1+2} + \cdots + b_{N_2} x_{N_2} + c_1^\ell y_{N_1+1} + c_2^\ell y_{N_1+2} + \cdots + c_{N_2}^\ell y_{N_2} \\ &\qquad + h_2^\ell(x_1', y_1'). \end{aligned}$$

Since $P_2(0, 0) = 0$ we get that $a_\ell = 0$, and since $P_2(x, 0) = x_2'$ and $P_2(0, y) = y_2'$ we get that $P_2^\ell(x, y) = x_{N_1+\ell} + y_{N_1+\ell} + h_2^\ell(x_1', y_1')$, for $1 \leq \ell \leq N_2$. Continuing in this way we get that $P_j(x, y) = x_j' + y_j' + h_j(x_1', \cdots, x_{j-1}', y_1', \cdots, y_{j-1}')$ for $j = 1, \cdots, k$. □

As a consequence of Proposition 2.1 we get that the vector fields X_j's have the form given in the following proposition (see [7, Proposition 2.2]). We relabel the polynomials H_r^ℓ in the statement of Proposition 2.1 by writing $H_r^\ell = H_{N_1+N_2+\cdots+N_{r-1}+\ell}$ for $1 \leq \ell \leq N_r$, $1 \leq r \leq k$, with the convention that $N_0 = 0$.

Proposition 2.2. *The vector fields X_j have polynomial coefficients and*

$$X_j = D_j + \sum_{\substack{1 \leq r \leq k \\ j < N_1+N_2+\cdots+N_{r-1}+1}} \sum_{i=N_1+N_2+\cdots+N_{r-1}+1}^{N_1+N_2+\cdots+N_{r-1}+N_r} q_{ij}(x)\, D_i, \tag{2.7}$$

for $j = 1, \cdots, N$, where $q_{ij}(x) = \left.\dfrac{\partial H_i}{\partial y_j}(x, y)\right|_{y=0}$.

Proof. Since X_j is a left–invariant vector field

$$(X_j f)(x) = \left.\frac{\partial f(x \circ y)}{\partial y_j}\right|_{y=0}.$$

From Proposition 2.1, $\dfrac{\partial H_i^\ell}{\partial y_j} = 0$ for $j \geq i > 1$ and $1 \leq \ell \leq N_i$, and $H_1^\ell = 0$ for $1 \leq \ell \leq N_1$. Thus,

$$\begin{aligned} \frac{\partial f(x \circ y)}{\partial y_j} &= \sum_{r=1}^{k} \sum_{i=N_1+N_2+\cdots+N_{r-1}+1}^{N_1+N_2+\cdots+N_{r-1}+N_r} \frac{\partial f}{\partial x_i}\left(\delta_{ij} + \frac{\partial H_i}{\partial y_j}(x, y)\right) \\ &= \frac{\partial f}{\partial x_j} + \sum_{\substack{1 \leq r \leq k \\ j < N_1+N_2+\cdots+N_{r-1}+1}} \sum_{i=N_1+N_2+\cdots+N_{r-1}+1}^{N_1+N_2+\cdots+N_{r-1}+N_r} \frac{\partial f}{\partial x_i} \frac{\partial H_i}{\partial y_j}(x, y). \end{aligned}$$

□

Notice that $X_i^* = -X_i$ for $i = 1, \cdots, N$.

2.2. **Examples.**

2.2.1. *Example 1.* The simplest example is given by the Euclidean space $\mathbb{G} = (\mathbb{R}^N, +)$, that is the commutative Carnot group. Here $k = 1$, $N_1 = N$, $\alpha_1 = 1$, and $X_1 = D_1, \cdots, X_N = D_N$.

2.2.2. *Example 2: The Heisenberg group.* Here $N = 2n + 1$ and $\circ$ is given by

$$[z_1, t_1] \circ [z_2, t_2] = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2 \cdot y_1 - x_1 \cdot y_2)),$$

where $[z_1, t_1] = (x_1, y_1, t_1), [z_2, t_2] = (x_2, y_2, t_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. We have $k = 2$, $N_1 = 2n$, $N_2 = 1$, the dilations are $\delta_\lambda[z, t] = (\lambda(x, y), \lambda^2 t)$, and $X_1 = D_{x_1} + 2\, y_1\, D_t, \cdots, X_n = D_{x_n} + 2\, y_n\, D_t, Y_1 = D_{y_1} - 2\, x_1\, D_t, \cdots, Y_n = D_{y_n} - 2\, x_n\, D_t$.

2.2.3. *Example 3.* In the analysis of 2nd order pde's, a typical situation is to have a family of smooth vector fields $X_1, \cdots, X_m$ on $\mathbb{R}^N$, that are suggested by the form of the 2nd order differential operator under investigation, and the corresponding Carnot group structure with respect to which the X_j's are left–invariant is a priori not known. Bonfiglioli [3, Theorem 3.8] gives necessary and sufficient conditions on the X_j's so that such a compatible Lie structure can be explicitly constructed. We refer to [3, Section 4] for several interesting examples arising in the applications.

2.3. **Pavage.** The existence in $\mathbb{G}$ of intrinsic dilations and translations yields the existence of an intrinsic pavage (see Definition 2.3 below) of $\mathbb{G}$, as well as of an intrinsic notion of periodicity (see e.g. [2], [8], [7]).

Definition 2.3. *The family of sets $\{Q_k\}_{k\in\mathbb{Z}^N}$ is a pavage of $\mathbb{G}$ if $Q_k \cap Q_m = \emptyset$ for $k \neq m$, and $\cup_{k\in\mathbb{Z}^N} Q_k = \mathbb{G}$.*

Let $\mathbf{Y} := \{x \in \mathbb{G} : -1/2 < x_i \leq 1/2 \text{ for } i = 1, \cdots, N\}$ denote the unit cube in $\mathbb{G}$ centered at the origin. For $k \in \mathbb{Z}^N$, denote $Q_k = k \circ \mathbf{Y}$. Let $\epsilon > 0$ and consider group dilations of Q_k with respect to ϵ:

$$Q_k^\epsilon = \delta_\epsilon Q_k = \delta_\epsilon(k \circ \mathbf{Y}) = \delta_\epsilon k \circ \delta_\epsilon \mathbf{Y}, \tag{2.8}$$

and we set $Q^\epsilon = \delta_\epsilon \mathbf{Y}$.

Lemma 2.4. *For each $\epsilon > 0$ the family $\{Q_k^\epsilon\}_{k\in\mathbb{Z}^N}$ defined by* (2.8) *is a pavage of $\mathbb{G}$.*

Proof. It is enough to prove the lemma for $\epsilon = 1$, that is, for each $y \in \mathbb{R}^N$ there exist unique $k \in \mathbb{Z}^N$ and $x \in \mathbf{Y}$ such that $y = k \circ x$; the general case immediately follows from (2.8). From Proposition 2.1, the first N_1 coordinates of k and x are *uniquely* determined by the equations

$$k_i + x_i = y_i \qquad i = 1, \cdots, N_1,\ k_i \in \mathbb{Z},\ -1/2 < x_i \leq 1/2. \tag{2.9}$$

If $N_1 + 1 \leq i \leq N_2$, then, again from Proposition 2.1, the equations determining k_i and x_i are

$$k_i + x_i + H_2^{i-N_1}(k, x) = y_i \qquad k_i \in \mathbb{Z},\ -1/2 < x_i \leq 1/2 \tag{2.10}$$

where $H_2^{i-N_1}(k, x)$ depends only on k_j, x_j with $1 \leq j \leq N_1$ which are already determined by (2.9). Using Proposition 2.1 and continuing in this way we obtain the lemma. □

Remark 2.5. Let $\Omega \subset \mathbb{G}$ be a bounded open set, then arguing as in the proof of [8, Lemma 2.21] we get $|\Omega| \approx \epsilon^Q \#\{Q_k^\epsilon \subset \Omega\}$, for ϵ small, since $|Q_k^\epsilon| = \epsilon^Q$, where Q is defined in Subsection 2.1.

2.4. **Function spaces.** Let $\Omega \subset \mathbb{R}^N$ be open. If $f : \Omega \to \mathbb{R}$, we define the horizontal gradient by $\nabla_{\mathbb{G}} f = (X_1 f, \cdots, X_m f)$ where $m = N_1$. Let $1 \leq p < \infty$, we define

$$D^{1,p}(\Omega) = \{u \in L^p(\Omega) : X_i u \in L^p(\Omega), \forall i = 1, \cdots, m\},$$

endowed with the norm

$$\|u\|_{D^{1,p}(\Omega)} = \left(\int_\Omega |u|^p \, dx + \int_\Omega |\nabla_{\mathbb{G}} u|^p \, dx \right)^{1/p}.$$

$D_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $D^{1,p}(\Omega)$, and by Sobolev inequality the norm $\|\cdot\|_{D^{1,p}(\Omega)}$ in $D_0^{1,p}(\Omega)$ is equivalent to

$$\left(\int_\Omega |\nabla_{\mathbb{G}} u|^p \, dx \right)^{1/p},$$

and from now on we set

$$\|u\|_{D_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla_{\mathbb{G}} u|^p \, dx \right)^{1/p}.$$

As usual, $D^{-1,p'}(\Omega)$ denotes the dual space of $D_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

2.5. **Periodicity in $\mathbb{G}$.**

Definition 2.6. *The function $g : \mathbb{G} \to \mathbb{R}$ is said to be $\mathbf{Y}$–periodic with respect to the reference pavage $\{Q_k\}$ if $g(k \circ \xi) = g(\xi)$ for all $\xi \in \mathbf{Y}$, and for all $k \in \mathbb{Z}^N$.*

If g is $\mathbf{Y}$-periodic with respect to the reference pavage $\{Q_k\}$, then $g^\epsilon(\xi) = g(\delta_{1/\epsilon}\xi)$ is Q^ϵ-periodic with respect to the pavage $\{Q_k^\epsilon\}$. Given a function v defined in Q^ϵ, the Q^ϵ-periodic extension of v to $\mathbb{G}$ with respect to the pavage $\{Q_k^\epsilon\}$ is defined by setting $\bar{v}(\delta_\epsilon k \circ \xi) = v(\xi)$, for $\xi \in Q^\epsilon$ and $k \in \mathbb{Z}^N$.

Definition 2.7. *The function $g : \mathbf{Y} \to \mathbb{R}$ is regular periodic in $\mathbf{Y}$ if there exists a sequence $g_m \in C^1(\mathbb{G})$ of $\mathbf{Y}$-periodic functions with respect to the reference pavage $\{Q_k\}$ such that $\|g_m - g\|_{D^{1,2}(\mathbf{Y})} \to 0$ as $m \to \infty$.*

By $D^{1,2}_{\text{per}}(\mathbf{Y})$ we denote the closed subspace of $D^{1,2}(\mathbf{Y})$ of regular periodic functions in $\mathbf{Y}$; and

$$D^{1,2}_{\#}(\mathbf{Y}) = \{u \in D^{1,2}_{\text{per}}(\mathbf{Y}) : \int_{\mathbf{Y}} u\, dx = 0\}.$$

We also define

$$L^p_{\#,\mathbb{G}}(\mathbf{Y}) = \{\, f \text{ is } \mathbf{Y}\text{–periodic} : f|_{\mathbf{Y}} \in L^p(\mathbf{Y})\},$$

with the norm $\|\cdot\|_{L^p(\mathbf{Y})}$.

Lemma 2.8 (Lemma 2.21, [8])**.** *Let $1 < p < \infty$ and $f \in L^p_{\#,\mathbb{G}}(\mathbf{Y})$. Then*

$$f^\epsilon(\xi) = f(\delta_{1/\epsilon}\xi) \rightharpoonup \frac{1}{|\mathbf{Y}|}\int_{\mathbf{Y}} f(\xi)\, d\xi$$

weakly in $L^p(\Omega)$ as $\epsilon \to 0$ for each $\Omega \subset \mathbb{G}$ open and bounded.

3. The equations and homogenization

Let $m = N_1$ be the number of generators of $\mathbb{G}$ as in Subsection 2.1, and $A(\xi) = (a_{ij}(\xi))^m_{i,j=1}$ an $m \times m$ matrix whose entries are bounded $\mathbf{Y}$-periodic functions with respect to the reference pavage Q_k in $\mathbb{G}$ and such that there exists a constant $\alpha > 0$ satisfying

$$\sum_{i,j=1}^m a_{ij}(\xi)\, \lambda_i\lambda_j \geq \alpha \sum_{i=1}^m \lambda_i^2, \tag{3.1}$$

for a.e. $\xi \in \mathbf{Y}$ and for each $\lambda \in \mathbb{R}^m$. Let $A^\epsilon(\xi) = A(\delta_{1/\epsilon}\xi)$, and

$$L^\epsilon = -\sum_{i,j=1}^m X_i(a^\epsilon_{ij}(\xi)\, X_j).$$

Given $\{f_\epsilon\}_{\epsilon>0} \subset D^{-1,2}(\Omega)$ with $\Omega \subset \mathbb{G}$ open bounded, for each $\epsilon > 0$ consider the problem

$$\begin{cases} L^\epsilon u_\epsilon &= f_\epsilon, \text{ in } \Omega \\ u_\epsilon &= 0, \text{ on } \partial\Omega. \end{cases} \tag{3.2}$$

Using variational techniques one obtains that (3.2) has a unique weak solution $u_\epsilon \in D^{1,2}_0(\Omega)$ and u_ϵ converges weakly in $D^{1,2}_0(\Omega)$ and strongly in $L^2(\Omega)$ as $\epsilon \to 0$. The following result is a simple extension of [2, Theorem 6] proved for the case of the Heisenberg group.

Theorem 3.1. *If $f_\epsilon \to f$ strongly in $D^{-1,2}(\Omega)$, then the weak solutions $u_\epsilon \in D^{1,2}_0(\Omega)$ to* (3.2) *converge weakly in $D^{1,2}_0(\Omega)$ to a function $u \in D^{1,2}_0(\Omega)$ as $\epsilon \to 0$ where u is a weak solution to*

$$\begin{cases} -\sum_{i,j=1}^m X_i(a^0_{ij}\, X_j u) &= f, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega, \end{cases} \tag{3.3}$$

with

$$a^0_{ij} = \frac{1}{|\mathbf{Y}|}\int_{\mathbf{Y}} [a_{ij} + \sum_{k=1}^m a_{ik}\, X_k(\chi^j)]\, d\xi$$

and where the correctors χ^j, $j = 1, \cdots, m$, are regular periodic functions on $\mathbf{Y}$ satisfying in the weak sense

$$\begin{cases} -\sum_{i,k=1}^m X_i(a_{ik} X_k(\chi^j)) &= \sum_{i=1}^m X_i(a_{ij}), \text{ in } \mathbf{Y} \\ \chi^j &\in D^{1,2}_{\#}(\mathbf{Y}). \end{cases} \tag{3.4}$$

Moreover, we have

$$A^\epsilon(\xi)\nabla_{\mathbb{G}} u_\epsilon \rightharpoonup A^0 \nabla_{\mathbb{G}} u$$

weakly in $L^2(\Omega, \mathbb{R}^m)$, $A^0 = (a^0_{ij})$.

4. Correctors

Let $\chi = (\chi^1, \cdots, \chi^m)$ be given in Theorem 3.1, and

$$\nabla_{\mathbb{G}}\chi = \begin{pmatrix} X_1\chi^1 & \cdots & X_m\chi^1 \\ \vdots & \ddots & \vdots \\ X_1\chi^m & \cdots & X_m\chi^m \end{pmatrix}.$$

Then

$$\nabla_{\mathbb{G}} u \, \nabla_{\mathbb{G}}\chi = \sum_{k=1}^m X_k u \, \nabla_{\mathbb{G}}\chi^k.$$

Given $p \in (1, +\infty]$, p' denotes the conjugate of p, and for $p \in (2, +\infty]$ we denote

$$p^* = \begin{cases} \dfrac{2p}{p-2} & \text{if } 2 < p < \infty \\ 2 & \text{if } p = \infty. \end{cases}$$

4.1. **Compensated compactness.**

Lemma 4.1. *Let $1 < p < +\infty$. Suppose that $\{u_j\}$ is a sequence converging weakly to u in $D^{1,p}(\Omega)$, $\{g_j\}$ with $g_j = (g^i_j)_{i=1}^m$ is a sequence in $L^{p'}(\Omega, \mathbb{R}^m)$ converging weakly to $g = (g^i)_{i=1}^m$ in $L^{p'}(\Omega, \mathbb{R}^m)$, and $\{-\sum_{i=1}^m X_i(g^i_j)\}_{j=1}^\infty$ converges strongly to $-\sum_{i=1}^m X_i(g^i)$ in $D^{-1,p'}(\Omega)$. Then*

$$\lim_{j\to\infty} \int_\Omega g_j \cdot \nabla_{\mathbb{G}} u_j \, \phi \, dx = \int_\Omega g \cdot \nabla_{\mathbb{G}} u \, \phi \, dx \tag{4.1}$$

for each $\phi \in C_0^\infty(\Omega)$. If in addition $\{g_j \cdot \nabla_{\mathbb{G}} u_j\}_j$ is a bounded sequence in $L^r(\Omega)$ and $g \cdot \nabla_{\mathbb{G}} u \in L^r(\Omega)$, for some $r \in (1, +\infty]$, then (4.1) holds for each $\phi \in L^{r'}(\Omega)$.

Proof. Let $\langle \cdot, \cdot \rangle$ denote the duality pair between $D^{-1,p'}(\Omega)$ and $D^{1,p}_0(\Omega)$. Given $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} \langle -\sum_{i=1}^m X_i(g^i_j), u_j\phi \rangle &= \int_\Omega g_j \cdot \nabla_{\mathbb{G}}(u_j\phi) \, dx \\ &= \int_\Omega g_j \cdot \nabla_{\mathbb{G}} u_j \, \phi \, dx + \int_\Omega g_j \cdot \nabla_{\mathbb{G}}\phi \, u_j \, dx. \end{aligned}$$

Hence

$$\lim_{j\to\infty}\int_\Omega g_j\cdot\nabla_{\mathbb{G}}u_j\ \phi\,dx=\langle-\sum_{i=1}^m X_i(g^i),u\phi\rangle-\int_\Omega g\cdot\nabla_{\mathbb{G}}\phi\ u\,dx=\int_\Omega g\cdot\nabla_{\mathbb{G}}u\ \phi\,dx,$$

which is (4.1).

To prove the last assertion in the Lemma, let $\phi\in L^{r'}(\Omega)$, $\delta>0$, and $\psi\in C_0^\infty(\Omega)$ such that $\|\phi-\psi\|_{L^{r'}(\Omega)}<\delta$. We write

$$\begin{aligned}&\int_\Omega g_j\cdot\nabla_{\mathbb{G}}u_j\ \phi\,dx-\int_\Omega g\cdot\nabla_{\mathbb{G}}u\ \phi\,dx\\&\quad=\int_\Omega g_j\cdot\nabla_{\mathbb{G}}u_j\ \psi\,dx-\int_\Omega g\cdot\nabla_{\mathbb{G}}u\ \psi\,dx\\&\quad+\int_\Omega g_j\cdot\nabla_{\mathbb{G}}u_j\ (\phi-\psi)\,dx-\int_\Omega g\cdot\nabla_{\mathbb{G}}u\ (\phi-\psi)\,dx=I+II.\end{aligned}$$

From (4.1), $I\to 0$ as $j\to\infty$; and by Hölder's inequality $|II|\le C\delta$. □

Lemma 4.2. *Suppose $\chi\in D^{1,s}(\Omega,\mathbb{R}^m)$ for some $s\in(2,+\infty]$, and $w_\epsilon\rightharpoonup w$ weakly in $D^{1,2}(\Omega)$ with $\nabla_{\mathbb{G}}w=0$. Then*

$$\lim_{\epsilon\to0}\int_\Omega(A^\epsilon(x)[\eta+(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\eta]\cdot\nabla_{\mathbb{G}}w_\epsilon)\,h(x)\,dx=0$$

for each $h\in L^{s^*}(\Omega)$ and for all $\eta\in\mathbb{R}^m$.

Proof. We have

(1) $A^\epsilon(\cdot)[\eta+(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}\cdot)\eta]\rightharpoonup A^0\eta$ weakly in $L^2(\Omega,\mathbb{R}^m)$. This follows from Lemma 2.8.

(2) $-\sum_{i=1}^m X_i(a_{ij}^\epsilon(x)[\eta_j+\sum_{k=1}^m X_j\chi^k(\delta_{1/\epsilon}x)\eta_k])=0$ in $D^{-1,2}(\Omega)$ for $j=1,\cdots,m$; which follows using an argument similar to the proof of [2, Lemma 5].

(3) Since $w_\epsilon\rightharpoonup w$ weakly in $D^{1,2}(\Omega)$, by Hölder's inequality we get that $\|A^\epsilon(\cdot)[\eta+(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}\cdot)\eta]\cdot\nabla_{\mathbb{G}}w_\epsilon\|_{L^{(s^*)'}(\Omega)}\le C$, and $A^0\eta\cdot\nabla_{\mathbb{G}}w=0$.

The lemma then follows from Lemma 4.1. □

Lemma 4.3. *Let u_ϵ and u as in Theorem 3.1. Suppose $\chi\in D^{1,s}(\mathbf{Y},\mathbb{R}^m)$ and $u\in D_0^{1,s^*}(\Omega)$ for some $s\in(2,+\infty]$. Then*

$$\lim_{\epsilon\to0}\int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}}u(x)+(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\nabla_{\mathbb{G}}u(x)]\cdot\nabla_{\mathbb{G}}(u_\epsilon-u)(x)\ dx=0;\ \textit{and}\tag{4.2}$$

$$\lim_{\epsilon\to0}\int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}}u(x)+(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\nabla_{\mathbb{G}}u(x)]\cdot(\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x)\ h(x)dx=0,\tag{4.3}$$

for all $h\in L^{s^*}(\Omega)$ and $k=1,2,\cdots,m$. In particular,

$$\lim_{\epsilon\to0}\int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}}u(x)+(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\nabla_{\mathbb{G}}u(x)]\cdot(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\nabla_{\mathbb{G}}u(x)\ dx=0.$$

Proof. We begin proving (4.2). Let $\delta > 0$. Since $\nabla_{\mathbb{G}} u \in L^{s^*}(\Omega, \mathbb{R}^m)$, there exists a simple function $\Psi = \sum_{\ell=1}^r \eta_\ell 1_{\Omega_\ell}$ with $\eta_\ell \in \mathbb{R}^m$, $\Omega = \cup_{\ell=1}^r \Omega_\ell$, $\Omega_\ell \cap \Omega_k = \emptyset$ if $\ell \neq k$, and $\|\nabla_{\mathbb{G}} u - \Psi\|_{L^{s^*}(\Omega)} \leq \delta$. We write

$$\begin{aligned}
&\int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}} u(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\nabla_{\mathbb{G}} u(x)] \cdot \nabla_{\mathbb{G}}(u_\epsilon - u)(x)\ dx \\
&\quad = \int_\Omega A^\epsilon(x)[\Psi(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\Psi(x)] \cdot \nabla_{\mathbb{G}}(u_\epsilon - u)(x)\ dx \\
&\quad + \int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}} u(x) - \Psi(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,(\nabla_{\mathbb{G}} u(x) - \Psi(x))] \cdot \nabla_{\mathbb{G}}(u_\epsilon - u)(x)\ dx.
\end{aligned}$$

From Lemma 4.2 we get

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_\Omega A^\epsilon(x)[\Psi(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\Psi(x)] \cdot \nabla_{\mathbb{G}}(u_\epsilon - u)(x)\ dx \\
&\quad = \lim_{\epsilon\to 0} \sum_{\ell=1}^r \int_{\Omega_\ell} A^\epsilon(x)[\eta_\ell + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\eta_\ell] \cdot \nabla_{\mathbb{G}}(u_\epsilon - u)(x)\ dx \\
&\quad = \lim_{\epsilon\to 0} \sum_{\ell=1}^r \int_{\Omega} A^\epsilon(x)[\eta_\ell + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\eta_\ell] \cdot \nabla_{\mathbb{G}}(u_\epsilon - u)(x)\, 1_{\Omega_\ell}(x)\, dx = 0.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\left|\int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}} u(x) - \Psi(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,(\nabla_{\mathbb{G}} u(x) - \Psi(x))] \cdot \nabla_{\mathbb{G}}(u_\epsilon - u)(x)\ dx\right| \\
&\quad \leq C\,\|\nabla_{\mathbb{G}}(u_\epsilon - u)\|_{L^2(\Omega)}\,\|\nabla_{\mathbb{G}} u - \Psi + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}\cdot)\,(\nabla_{\mathbb{G}} u - \Psi)\|_{L^2(\Omega)} \\
&\quad \leq C\,\left(\|\nabla_{\mathbb{G}} u - \Psi\|_{L^2(\Omega)} + \|(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}\cdot)\,(\nabla_{\mathbb{G}} u - \Psi)\|_{L^2(\Omega)}\right) \\
&\quad \leq C\,\left(\|\nabla_{\mathbb{G}} u - \Psi\|_{L^2(\Omega)} + \|(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}\cdot)\|_{L^s(\Omega)}\,\|\nabla_{\mathbb{G}} u - \Psi\|_{L^{s^*}(\Omega)}\right) \\
&\quad \leq C\,\|\nabla_{\mathbb{G}} u - \Psi\|_{L^{s^*}(\Omega)} \leq C\,\delta,
\end{aligned}$$

uniformly in ϵ. This completes the proof of (4.2).

To prove (4.3), let $h \in L^{s^*}(\Omega)$, $\delta > 0$ and Ψ as in the proof of (4.2). We write

$$\begin{aligned}
&\int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}} u(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\nabla_{\mathbb{G}} u(x)] \cdot (\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x)\ h(x)dx \\
&\quad = \int_\Omega A^\epsilon(x)[\Psi(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\Psi(x)] \cdot (\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x)\ h(x)dx \\
&\quad + \int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}} u(x) - \Psi(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,(\nabla_{\mathbb{G}} u(x) - \Psi(x))] \cdot (\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x)\ h(x)dx.
\end{aligned}$$

By Lemma 2.8,

$$(\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x) \rightharpoonup \frac{1}{|\mathbf{Y}|}\int_{\mathbf{Y}} \nabla_{\mathbb{G}}\chi^k(\xi)\, d\xi = 0,$$

weakly in $L^2(\Omega)$. Moreover, $\|\chi^k(\delta_{1/\epsilon}\cdot)\|_{D^{1,2}(\Omega)} \leq C$. Therefore, up to a subsequence, $\chi^k(\delta_{1/\epsilon}\cdot) \rightharpoonup w^k$ weakly in $D^{1,2}(\Omega)$ for some $w^k \in D^{1,2}(\Omega)$ with $\nabla_{\mathbb{G}} w^k = 0$. Thus by Lemma

4.2 we get

$$A^\epsilon(x)[\eta_\ell + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\eta_\ell]\cdot(\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x) \rightharpoonup 0$$

weakly in $L^{(s^*)'}(\Omega)$ for $\ell = 1, \cdots, r$. Therefore

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_\Omega A^\epsilon(x)[\Psi(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\Psi(x)]\cdot(\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x)\,h(x)\,dx\\
&\quad = \lim_{\epsilon\to 0}\sum_{\ell=1}^{r}\int_\Omega A^\epsilon(x)[\eta_\ell + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,\eta_\ell]\cdot(\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x)\,h(x)\,1_{\Omega_\ell}(x)\,dx = 0.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\left|\int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}}u(x) - \Psi(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\,(\nabla_{\mathbb{G}}u(x) - \Psi(x))]\cdot(\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x)\;\,h(x)dx\right|\\
&\quad\le C\|\nabla_{\mathbb{G}}u - \Psi + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}\cdot)\,(\nabla_{\mathbb{G}}u - \Psi)\|_{L^2(\Omega)}\|(\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}\cdot)\|_{L^s(\Omega)}\|h\|_{L^{s^*}(\Omega)}\\
&\quad\le C\left(\|\nabla_{\mathbb{G}}u - \Psi\|_{L^2(\Omega)} + \|(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}\cdot)\,(\nabla_{\mathbb{G}}u - \Psi)\|_{L^2(\Omega)}\right)\\
&\quad\le C\left(\|\nabla_{\mathbb{G}}u - \Psi\|_{L^2(\Omega)} + \|(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}\cdot)\|_{L^s(\Omega)}\|\nabla_{\mathbb{G}}u - \Psi\|_{L^{s^*}(\Omega)}\right)\\
&\quad\le C\|\nabla_{\mathbb{G}}u - \Psi\|_{L^{s^*}(\Omega)} \le C\delta,
\end{aligned}$$

uniformly in ϵ. This completes the proof of (4.3). □

4.2. **Convergence of correctors in L^2.** Let us now define the gradient of the corrector

$$w_\epsilon(x) = \nabla_{\mathbb{G}}u_\epsilon(x) - \nabla_{\mathbb{G}}u(x) - (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\nabla_{\mathbb{G}}u(x), \tag{4.4}$$

and we shall prove the main result of this section.

Theorem 4.4. *Suppose $f_\epsilon \to f$ strongly in $D^{-1,2}(\Omega)$ as $\epsilon \to 0$, $\chi \in D^{1,s}(\mathbf{Y};\mathbb{R}^m)$[1] and $u \in D^{1,s^*}(\Omega)$ for some $s \in (2,\infty]$. Then $w_\epsilon \to 0$ strongly in $L^2(\Omega;\mathbb{R}^m)$ as $\epsilon \to 0$.*

Proof. We have

$$\begin{aligned}
\|w_\epsilon\|^2_{L^2(\Omega;\mathbb{R}^m)} &= \int_\Omega |w_\epsilon|^2\,dx \le C\int_\Omega A^\epsilon(x)w_\epsilon(x)\cdot w_\epsilon(x)\,dx\\
&= C\Bigg(\int_\Omega A^\epsilon(x)\nabla_{\mathbb{G}}u_\epsilon(x)\cdot\nabla_{\mathbb{G}}(u_\epsilon(x) - u(x))\,dx\\
&\quad - \int_\Omega A^\epsilon(x)\nabla_{\mathbb{G}}u_\epsilon(x)\cdot\nabla_{\mathbb{G}}(\chi)(\delta_{1/\epsilon}x)\nabla_{\mathbb{G}}u(x)\,dx\\
&\quad - \int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}}u(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\nabla_{\mathbb{G}}u(x)]\cdot\nabla_{\mathbb{G}}(u_\epsilon(x) - u(x))\,dx\\
&\quad + \int_\Omega A^\epsilon(x)[\nabla_{\mathbb{G}}u(x) + (\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\nabla_{\mathbb{G}}u(x)]\cdot(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon}x)\nabla_{\mathbb{G}}u(x)\,dx\Bigg)\\
&= C(I_1^\epsilon - I_2^\epsilon - I_3^\epsilon + I_4^\epsilon).
\end{aligned}$$

[1]By Theorem 5.4 we can always find such s.

Since $f_\epsilon \to f$ strongly in $D^{-1,2}(\Omega)$ and $u_\epsilon \rightharpoonup u$ weakly in $D_0^{1,2}(\Omega)$ we get $I_1^\epsilon = \langle f_\epsilon, u_\epsilon - u\rangle \to 0$ as $\epsilon \to 0$. As in the proof of (4.3), we have that up to a subsequence in ϵ, $\chi^k(\delta_{1/\epsilon}\cdot) \rightharpoonup w^k$ weakly in $D^{1,2}(\Omega)$ for some $w^k \in D^{1,2}(\Omega)$ with $\nabla_{\mathbb{G}} w^k = 0$ for $k = 1, 2, \cdots, m$. Moreover, we have

$$-\sum_{i=1}^m X_i(a_{ij}^\epsilon(x) X_j u_\epsilon(x)) = f_\epsilon \to f = -\sum_{i=1}^m X_i(a_{ij}^0 X_j u)$$

strongly in $D^{-1,2}(\Omega)$; $A^\epsilon(x)\nabla_{\mathbb{G}} u(x) \rightharpoonup A^0 \nabla_{\mathbb{G}} u(x)$ weakly in $L^2(\Omega, \mathbb{R}^m)$; and

$$\|A^\epsilon(\cdot)\nabla_{\mathbb{G}} u_\epsilon \cdot (\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}\cdot)\|_{L^{(s^*)'}(\Omega)} \le C,$$

for all $\epsilon > 0$.

Therefore from Lemma 4.1 we get

$$\lim_{\epsilon\to 0} \int_\Omega A^\epsilon(x)\nabla_{\mathbb{G}} u_\epsilon(x) \cdot (\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x) X_k u \; dx = 0$$

for $k = 1, 2, \cdots, m$. Hence

$$\lim_{\epsilon\to 0} I_2^\epsilon = \lim_{\epsilon\to 0} \sum_{k=1}^m \int_\Omega A^\epsilon(x)\nabla_{\mathbb{G}} u_\epsilon(x) \cdot (\nabla_{\mathbb{G}}\chi^k)(\delta_{1/\epsilon}x) X_k u \; dx = 0.$$

From Lemma 4.3 we get $\lim_{\epsilon\to 0} I_3^\epsilon = \lim_{\epsilon\to 0} I_4^\epsilon = 0$, and the proof of the theorem is complete. □

5. Meyers type estimates for vector fields satisfying Hörmander's condition

Let $X_1, ..., X_m$ be smooth real vector fields in $\mathbb{R}^N$ satisfying Hörmander's finite rank condition, i.e., $X_1, ..., X_m$, together with their commutators of length up to a fixed order k span $\mathbb{R}^N$ at each point $x \in \mathbb{R}^N$. We recall some basic facts about the metric generated by the vector fields $X_1, ..., X_m$, see for example [4], [15]. We say that an absolutely continuous curve $\gamma : [0, T] \longrightarrow \mathbb{R}^N$ is a sub-unit curve with respect to X if for any $\xi \in \mathbb{R}^N$

$$\langle \dot\gamma(t), \xi\rangle^2 \le \sum_{j=1}^m \langle X_j(\gamma(t)), \xi\rangle^2$$

for a.e. $t \in [0, T]$. If $x_1, x_2 \in \mathbb{R}^N$, we define

$$d(x_1, x_2) = \inf\{T > 0 : \exists \text{ a sub-unit curve } \gamma : [0, T] \to \mathbb{R}^n, \gamma(0) = x_1,\ \gamma(T) = x_2\}.$$

Notice that, by Hörmander's finite rank condition, $d(x, y) < \infty$ for all $x, y \in \mathbb{R}^N$ (Chow's Theorem). Using d we can define metric balls $B(x, \delta)$ on $\mathbb{R}^N$ by $B(x, \delta) = \{y \in \mathbb{R}^N | d(y, x) < \delta\}$. An important property for the balls $B(x, \delta)$ is the following doubling condition: Given a bounded set $U \subset \mathbb{R}^N$, there exist $C = C_U > 0$ and $\delta_U > 0$ such that for every $x \in U$ and $\delta \in (0, \delta_U)$

$$|B(x, 2\delta)| \le C|B(x, \delta)|.$$

From now on, we restrict ourselves in a fixed open set U, and thus we can assume the above constant are uniform for all x. Throughout this Section $B(x, r)$ always denotes the

metric ball associated with d, and we shall use the following Poincaré type inequality (see [14]): there exist $p \in (1,2)$ and $C > 0$ such that

$$\left(\fint_{B_R} |u - u_{B_R}|^2\, dx\right)^{1/2} \leq C\, R \left(\fint_{B_R} |Xu|^p\, dx\right)^{1/p} \tag{5.1}$$

for every $B_R = B(x,R) \subset \mathbb{R}^N$ and for every $u \in D^{1,p}(B_R)$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $a = (a_{ij}) : \Omega \longmapsto \mathbb{R}^{m\times m}$ be measurable satisfying the following conditions

$$|a_{ij}| \leq c_0 \quad \text{a.e.} \quad x \in \Omega \quad \text{and} \quad \forall i, j = 1, .., m, \tag{5.2}$$

$$c_1|\xi|^2 \leq (a\xi, \xi) \quad \text{a.e.} \quad x \in \Omega \quad \text{and} \quad \forall \xi \in \mathbb{R}^m. \tag{5.3}$$

Let $\gamma : \Omega \longmapsto \mathbb{R}$ be a measurable function satisfying

$$0 < \alpha_0 \leq \gamma(x) \leq \beta_0 \quad \text{a.e.} \quad x \in \Omega. \tag{5.4}$$

5.1. **Linear case.** We first consider the following problem

$$\sum_{i,j=1}^{m} X_i^*(a_{ij}X_j u) + \gamma\, u = f^0 + \sum_{i=1}^{m} X_i^*(f^i),\ f^i \in L^2(\Omega)\ \forall i = 0, ..., m. \tag{5.5}$$

The function $u \in D^{1,2}(\Omega)$ is a weak solution to (5.5) if

$$\int_\Omega a_{ij}X_j u X_i\phi\, dx + \int_\Omega \gamma u\phi\, dx = \int_\Omega f^0\phi\, dx + \int_\Omega \sum_{i=1}^{m} f^i X_i\phi\, dx$$

for all $\phi \in D_0^{1,2}(\Omega)$. By Sobolev type inequality and using Lax-Milgram's Lemma we see that (5.5) has a unique weak solution in $D_0^{1,2}(\Omega)$. We shall use the following notation: $f = (f^1, ..., f^m)$ and $F = (f^0, f^1, ..., f^m)$.

Lemma 5.1 (Caccioppoli Estimate)**.** *Suppose that* (5.2)*,* (5.3)*, and* (5.4) *hold. Let* $u \in D^{1,2}(\Omega)$ *be a weak solution to* (5.5)*. Let* $0 < \alpha < \beta \leq 1$*. Then there exists a positive constant* C *depending only on the structural constants,* α, β *and* Ω *such that for any ball* $B_{\beta R} = B(x, \beta R) \Subset \Omega$ *we have*

$$\begin{aligned}&\left(\fint_{B_{\alpha R}} \left(|Xu|^2 + |u|^2\right) dx\right)^{1/2}\\ &\leq C\left\{\left(\fint_{B_{\beta R}} (|Xu|^p + |u|^p)\, dx\right)^{1/p} + \left(\fint_{B_{\beta R}} |F|^2\, dx\right)^{1/2}\right\},\end{aligned} \tag{5.6}$$

where p *is the exponent in* (5.1)*.*

Proof. By [6] there exists a cut–off function $\varphi \in D_0^{1,2}(\Omega)$ with $0 \le \varphi \le 1, \varphi = 1$ on $B_{\alpha R}$, $\operatorname{supp}\varphi \subset B_{\beta R}$ and $|X_i\varphi| \le C/R\ \forall i = 1, ..., m$ where the constant C is independent of R. Choosing $v = (u - u_{B_{\beta R}})\varphi^2 \in D_0^{1,2}(\Omega)$ as a test function we get

$$\int_{B_{\beta R}} a_{ij}X_j u X_i u\ \varphi^2\, dx + 2\int_{B_{\beta R}} a_{ij}X_j u X_i\varphi\ (u - u_{B_{\beta R}})\varphi\, dx + \int_{B_{\beta R}} \gamma u(u - u_{B_{\beta R}})\varphi^2\, dx$$
$$= \int_{B_{\beta R}} f^0(u - u_{B_{\beta R}})\varphi^2 dx + \sum_{i=1}^m \int_{B_{\beta R}} f^i X_i u\ \varphi^2\, dx + 2\sum_{i=1}^m \int_{B_{\beta R}} f^i X_i\varphi\ (u - u_{B_{\beta R}})\varphi\, dx.$$

Hence by (5.2), (5.3), and (5.4) we get

$$c_1 \int_{B_{\beta R}} |Xu|^2\varphi^2\, dx$$
$$\le 2c_0 \int_{B_{\beta R}} |Xu|\,|X\varphi|\,|u - u_{B_{\beta R}}|\varphi\, dx + \beta_0 \int_{B_{\beta R}} |u|\,|u - u_{B_{\beta R}}|\varphi^2\, dx$$
$$+ \int_{B_{\beta R}} |f^0|\,|u - u_{B_{\beta R}}|\varphi^2\, dx + \int_{B_{\beta R}} |f|\,|Xu|\varphi^2\, dx + 2\int_{B_{\beta R}} |f|\,|X\varphi|\,|u - u_{B_{\beta R}}|\varphi\, dx.$$

From Young's inequality, and since $|X\varphi| \le C/R$, it follows moving some terms that

$$\int_{B_{\beta R}} \left(|Xu|^2 + |u|^2\right)\varphi^2\, dx$$
$$\le \frac{C}{R^2}\int_{B_{\beta R}} |u - u_{B_{\beta R}}|^2\, dx + C\int_{B_{\beta R}} |u|^2\, dx + C\int_{B_{\beta R}} |u - u_{B_{\beta R}}|^2\, dx + C\int_{B_{\beta R}} |F|^2\, dx$$
$$\le \frac{C}{R^2}\int_{B_{\beta R}} |u - u_{B_{\beta R}}|^2\, dx + C\int_{B_{\beta R}} |u - u_{B_{\beta R}}|^2\, dx + C|B_{\beta R}||u_{B_{\beta R}}|^2 + C\int_{B_{\beta R}} |F|^2\, dx.$$

Therefore, using the properties of φ, Poincaré's and Hölder's inequalities and the fact that the Lebesgue measure is doubling with respect to the metric balls we derive that

$$\fint_{B_{\alpha R}} \left(|Xu|^2 + |u|^2\right)\, dx \le C\left(\fint_{B_{\beta R}} (|Xu|^p + |u|^p)\, dx\right)^{2/p} + C\fint_{B_{\beta R}} |F|^2\, dx,$$

since $R \le C(\Omega)$ by the assumption. This completes the proof. □

In the following theorem we will prove higher integrability of weak solutions to (5.5). To prove the boundary estimate of Theorem 5.4(2) we need the following condition for the domain Ω.

Definition 5.2 (Condition (S))**.** *We say that the domain Ω satisfies condition (S) if there exist $0 < R^* < 1$ and $\lambda > 0$ such that for any ball $B(x, R)$ with $x \in \partial\Omega$ and $R \le R^*$ we have*

$$|B(x, R) \setminus \Omega| \ge \lambda|B(x, R)|.$$

Condition (S) holds for instance if the following uniform exterior ball property holds: there exists $a > 0$ such that for each $x \in \partial\Omega$ there exists a ball $B(y, ar) \subset B(x, r) \setminus \bar{\Omega}$. For

example, this is true when $\partial\Omega$ is continuously differentiable, provided x is not a characteristic point of $\partial\Omega$, i.e., if at least one of the vector fields is transversal to $\partial\Omega$ at x (see e.g., [9, Theorem 7]). Nevertheless, condition (S) can be satisfied also at characteristic points. For example, in the Heisenberg group $\mathbb{H}^1$ identified with $\mathbb{R}^3$, the origin is a characteristic point of the boundary of the set $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_3 > (x_1^2 + x_2^2)^{\alpha/2}\}$, $\alpha > 1$, and clearly condition (S) is always satisfied at that point.

We recall the following result proved in [10] (see also [5]) that will be used in the proof of Theorem 5.4.

Theorem 5.3 (Theorem 3.1, [10])**.** *Let f and g be nonnegative functions such that $g \in L^q_{loc}(\mathbb{R}^N)$ for some $q > 1$, and $f \in L^r_{loc}(\mathbb{R}^N)$ for some $r > q$. Fix a ball $B_1 = B(x_0, R_0)$. Suppose that for each ball $B_R = B(x, R) \subset B_1$ we have*

$$\fint_{B_{3R/4}(x)} g^q\,dx \le b\left\{\left(\fint_{B_R(x)} g\,dx\right)^q + \fint_{B_R(x)} f^q\,dx\right\} + \theta \fint_{B_R(x)} g^q\,dx,$$

where b, θ are positive constants. If θ is sufficiently small, then $g \in L^p_{\mathrm{loc}}(B_1)$ for $q \le p < q + \epsilon$ and

$$\left(\fint_{B_R(x)} g^p\,dx\right)^{1/p} \le C\left\{\left(\fint_{B_{2R}(x)} g^q\,dx\right)^{1/q} + \left(\fint_{B_{2R}(x)} f^p\,dx\right)^{1/p}\right\}$$

for each ball $B_{2R}(x) \subset B_1$, where C and ϵ are positive constants depending only on the structural constants, b, θ, q and r.

Theorem 5.4. *Suppose that* (5.2)*,* (5.3)*, and* (5.4) *hold and assume further that $F \in L^\sigma(\Omega; \mathbb{R}^{m+1})$ for some $\sigma > 2$. There exists $p_0 > 2$ such that if $u \in D^{1,2}(\Omega)$ is a weak solution to* (5.5)*, then*

(1) *$u \in D^{1,q}_{\mathrm{loc}}(\Omega)$ for all $q \in [2, p_0)$, and for any ball $B_R = B(x, R)$ with $B(x, 2R) \Subset \Omega$ we have*

$$\left(\fint_{B_R} (|Xu|^q + |u|^q)\,dx\right)^{1/q} \le C\left\{\left(\fint_{B_{2R}} \left(|Xu|^2 + |u|^2\right)dx\right)^{1/2} + \left(\fint_{B_{2R}} |F|^q\,dx\right)^{1/q}\right\}, \tag{5.7}$$

where p is the constant in (5.1) *and C depends only on p, σ and the constant in* (5.6)*.*

(2) *If in addition Ω satisfies condition (S) and $u \in D^{1,2}_0(\Omega)$, then $u \in D^{1,q}_0(\Omega)$ for all $q \in [2, p_0)$, and*

$$\int_\Omega |Xu|^q\,dx \le C\int_\Omega |F|^q\,dx, \tag{5.8}$$

where C depends only on the structural constants, p, σ, λ, Ω and q.

Proof. To prove part 1 it is enough to show (5.7). Let $B(\bar{x}, \bar{R})$ be a ball such that $B(\bar{x}, 2\bar{R}) \Subset \Omega$ and set $B_1 = B(\bar{x}, 2\bar{R})$. By Lemma 5.1 with $\alpha = 3/4, \beta = 1$ we then get for any ball $B_R \subset B_1$ that

$$\begin{aligned}
&\fint_{B_{3R/4}} (|Xu|^p + |u|^p)^{2/p}\, dx \le C(p) \fint_{B_{3R/4}} \left(|Xu|^2 + |u|^2\right) dx \\
&\le b \left\{ \left(\fint_{B_R} (|Xu|^p + |u|^p)\, dx \right)^{2/p} + \fint_{B_R} |F|^2\, dx \right\}.
\end{aligned}$$

Therefore by the reverse Hölder inequality from Theorem 5.3 applied with $q = 2/p$, there exists $p_1 > 2/p$ depending only on b, p, σ such that for any $r \in [\frac{2}{p}, p_1)$

$$\begin{aligned}
&\left(\fint_{B_R} (|Xu|^p + |u|^p)^r\, dx \right)^{1/rp} \\
&\le C \left\{ \left(\fint_{B_{2R}} \left(|Xu|^2 + |u|^2\right) dx \right)^{1/2} + \left(\fint_{B_{2R}} |F|^{rp}\, dx \right)^{1/rp} \right\}
\end{aligned}$$

for any ball B_R satisfying $B_{2R} \subset B_1$. In particular, this inequality holds for $B_R = B(\bar{x}, \bar{R})$. So part 1 is proved with $p_0 = p_1 p$.

To prove part 2, first let $V \subset \mathbb{R}^N$ be an open set such that $\Omega \Subset V$ and extend $u, f^0, ..., f^m$ as zero outside Ω. Notice that if $\tilde{u}$ is such an extension of u, then $X\tilde{u} = \widetilde{Xu}$, where $\widetilde{Xu}$ is the extension of Xu to V as zero outside Ω. However, to simplify the notation, we will write u and Xu instead of $\tilde{u}$ and $\widetilde{Xu}$ respectively.

Let us fix a ball $B_R(x) \subset V$ with $R \le R^*$. We shall prove that

$$\fint_{B_{3s/4}(z)} (|Xu|^2 + |u|^2)\, dx \le C \left(\left(\fint_{B_s(z)} (|Xu|^p + |u|^p)\, dx \right)^{2/p} + \fint_{B_s(z)} |F|^2\, dx \right), \tag{5.9}$$

for each ball $B_s(z) \subset B_R(x)$. Recall that $1 < p < 2$ is the exponent in (5.1). We have $B_{3s/4}(z) \subset B_{7s/8}(z)$.

Case 1: $B_{7s/8}(z) \subset \Omega$.

In this case (5.9) follows applying (5.6) with $\alpha = 3/4$ and $\beta = 7/9$.

Case 2: $B_{7s/8}(z) \subset V \setminus \Omega$.

This case is trivial since u is extended as zero outside Ω.

Case 3: $B_{7s/8}(z) \cap \partial\Omega \neq \emptyset$.

Let φ be a cut-off function like in the proof of Lemma 5.1, $\varphi = 1$ on $B_{3s/4}(z)$, supp $\varphi \subset B_{7s/8}(z)$, and choose $v = u\varphi^2 \in D_0^{1,2}(\Omega)$ as a test function. We then have

$$\int_{B_{7s/8}(z)} a_{ij}X_j u X_i u\, \varphi^2\, dx + 2\int_{B_{7s/8}(z)} a_{ij}X_j u X_i\varphi\, u\varphi\, dx + \int_{B_{7s/8}(z)} \gamma u^2\varphi^2\, dx$$
$$= \int_{B_{7s/8}(z)} f^0 u\varphi^2 dx + \sum_{i=1}^m \int_{B_{7s/8}(z)} f^i X_i u\, \varphi^2\, dx + 2\sum_{i=1}^m \int_{B_{7s/8}(z)} f^i X_i\varphi\, u\varphi\, dx.$$

Arguing as in the proof of Lemma 5.1 we can derive that

$$\int_{B_{3s/4}(z)} |Xu|^2\, dx \le \frac{C}{s^2}\int_{B_{7s/8}(z)} u^2\, dx + C\int_{B_{7s/8}(z)} |u|^2\, dx + C\int_{B_{7s/8}(z)} |F|^2\, dx$$
$$\le \frac{C}{s^2}\int_{B_{7s/8}(z)} u^2\, dx + C\int_{B_{7s/8}(z)} |F|^2\, dx. \tag{5.10}$$

Let $y \in B_{7s/8}(z) \cap \partial\Omega$, then $B_{s/8}(y) \subset B_s(z) \subset B_{15s/8}(y)$. Moreover we have $s \le 2R \le 2R^*$ because $B_s(z) \subset B_R(x)$. Hence since Ω satisfies condition (S), it follows from the doubling property that $|B_s(z) \setminus \Omega| \ge C(\lambda)\, |B_s(z)|$. But $u = 0$ on $B_s(z) \setminus \Omega$ and we then get

$$\int_{B_s(z)} u^2\, dx \le C(\lambda)\int_{B_s(z)} |u - u_{B_s(z)}|^2\, dx. \tag{5.11}$$

From (5.10), (5.11) and Poincaré inequality we obtain

$$\fint_{B_{3s/4}(z)} (|Xu|^2 + |u|^2)\, dx \le C(\lambda)\left(\fint_{B_s(z)} |Xu|^p\, dx\right)^{2/p} + C\fint_{B_s(z)} |F|^2\, dx,$$

and (5.9) also holds in Case 3.

From (5.9) and the Reverse Hölder Inequality from Theorem 5.3, there exists $p_0 > 2$, now also depending on λ, such that for any $q \in [2, p_0)$ we obtain

$$\left(\fint_{B_t(z)} |Xu|^q\, dx\right)^{1/q}$$
$$\le C\left(\left(\fint_{B_{2t}(z)} |Xu|^2\, dx\right)^{1/2} + \left(\fint_{B_{2t}(z)} |u|^2\, dx\right)^{1/2} + \left(\fint_{B_{2t}(z)} |F|^q\, dx\right)^{1/q}\right)$$

for any ball $B_t(z)$ such that $B_{2t}(z) \subset B_R(x)$, in particular, for $B_{R/2}(x)$. Covering the set Ω with a finite family of balls $B_{R/2}(x)$ with $B_R(x) \subset V$ and $R \le R^*$ and then adding up on this family of balls yields

$$\left(\int_\Omega |Xu|^q\, dx\right)^{1/q} \le C(q,\Omega)\left(\left(\int_\Omega |Xu|^2\, dx\right)^{1/2} + \left(\int_\Omega |u|^2\, dx\right)^{1/2} + \left(\int_\Omega |F|^q\, dx\right)^{1/q}\right).$$

Since $u \in D_0^{1,2}(\Omega)$, it follows by Sobolev type inequality that on the right hand side of the last inequality, the second term can be estimated by the first. Also, choosing $v = u$ as a

test function, it is easy to see that

$$\int_\Omega |Xu|^2\,dx \le \int_\Omega |F|^2\,dx.$$

Therefore we obtain (5.8) and the proof is complete. □

5.2. **Nonlinear case.** In this subsection we consider the following nonlinear problem

$$\begin{cases} \sum_{i,j=1}^m X_i^*(a_{ij}X_j u) + \gamma(x)\,u = H(x,u,Xu) \\ u \in D_0^{1,2}(\Omega)\cap L^\infty(\Omega), \end{cases} \tag{5.12}$$

where we assume that conditions (5.2), (5.3), (5.4) hold, and $H(x,s,\xi)$ is a Carathéodory function satisfying

$$|H(x,s,\xi)| \le c_2 + b(|s|)|\xi|^2 \quad \text{a.e.} \quad x\in\Omega, \forall s\in\mathbb{R} \quad \text{and} \quad \forall \xi\in\mathbb{R}^m \tag{5.13}$$

with $b:\mathbb{R}^+ \longmapsto \mathbb{R}^+$ a continuous nondecreasing function. We shall prove Meyer's type estimates for the equation (5.12). We mention that when the equation (5.12) is uniformly elliptic, homogenization results are proved in [1] using the standard Meyers estimates and the convergence in L^2 of the gradients of correctors of the corresponding linear equation. Following their method the Meyers estimates from this subsection together with the convergence of correctors given in Theorem 4.4 would lead to similar results of those in [1] but this time for the equation (5.12).

We begin with a Caccioppoli type estimate.

Lemma 5.5. *Suppose that* (5.2), (5.3), (5.4) *and* (5.13) *hold. Let* $u\in D_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ *be any weak solution to the problem* (5.12). *Let* $0<\alpha<\beta\le 1$. *Then there exists a constant* C *depending only on the structural constants,* α *and* β *such that for any ball* $B(x,\beta R)\Subset\Omega$ *we have*

$$\fint_{B_{\alpha R}} \left(|Xu|^2+|u|^2\right)dx \le C\,\exp\left\{C\left(\|u\|_\infty b(\|u\|_\infty)\right)^2\right\}\times\left(\left(\fint_{B_{\beta R}}|Xu|^p\,dx\right)^{2/p} + (\|u\|_\infty+1)^2\right). \tag{5.14}$$

Proof. Let $B_R = B(x,R)$ be a ball such that $B(x,\beta R)\Subset\Omega$. Constructing the cut-off function φ the same as in the proof of Lemma 5.1 and choose $v=(u-u_{\beta R})\,E_t(x)\,\varphi^2\in D_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ as a test function with $E_t(x)=e^{t|u-u_{\beta R}|^2}$, $u_{\beta R}=u_{B_{\beta R}}$, and where the constant $t>0$ will be fixed in the moment. We then have

$$\begin{aligned}
&\int_{B_{\beta R}} a_{ij}X_j u X_i u\; E_t(x)\,\varphi^2\,dx + 2\,t\int_{B_{\beta R}} a_{ij}X_j u X_i u\,(u-u_{\beta R})^2 E_t(x)\,\varphi^2\,dx \\
&+2\int_{B_{\beta R}} a_{ij}X_j u X_i\varphi\,(u-u_{\beta R})E_t(x)\,\varphi\,dx + \int_{B_{\beta R}}\gamma\,u\,(u-u_{\beta R})\,E_t(x)\,\varphi^2\,dx \\
&= \int_{B_{\beta R}} H(x,u,Xu)(u-u_{\beta R})\,E_t(x)\,\varphi^2 dx.
\end{aligned}$$

Hence from (5.2), (5.3), (5.4), (5.13), Young's inequality and since $|X\varphi| \le C/R$ we get

$$
\begin{aligned}
&c_1 \int_{B_{\beta R}} |Xu|^2 E_t\, \varphi^2\, dx + 2c_1 t \int_{B_{\beta R}} |Xu|^2 |u-u_{\beta R}|^2 E_t \varphi^2\, dx + \alpha_0 \int_{B_{\beta R}} |u-u_{\beta R}|^2 E_t \varphi^2\, dx \\
&\le \beta_0 |u_{\beta R}| \int_{B_{\beta R}} |u-u_{\beta R}| E_t \varphi^2\, dx + 2c_0 \int_{B_{\beta R}} |Xu||X\varphi||u-u_{\beta R}| E_t \varphi\, dx \\
&\qquad + \int_{B_{\beta R}} \left(c_2 + b(\|u\|_\infty)|Xu|^2\right) |u-u_{\beta R}| E_t \varphi^2 dx \\
&= \left(\beta_0 |u_{\beta R}| + c_2\right) \int_{B_{\beta R}} |u-u_{\beta R}| E_t \varphi^2\, dx + 2c_0 \int_{B_{\beta R}} |Xu||X\varphi||u-u_{\beta R}| E_t \varphi\, dx \\
&\qquad + \int_{B_{\beta R}} b(\|u\|_\infty)|Xu|^2 |u-u_{\beta R}| E_t \varphi^2 dx \\
&\le \frac{\alpha_0}{2} \int_{B_{\beta R}} |u-u_{\beta R}|^2 E_t \varphi^2\, dx + C\left(|u_{\beta R}|+1\right)^2 \int_{B_{\beta R}} E_t \varphi^2\, dx + \frac{c_1}{2} \int_{B_{\beta R}} |Xu|^2 E_t \varphi^2\, dx \\
&\qquad + \frac{C}{R^2} \int_{B_{\beta R}} |u-u_{\beta R}|^2 E_t\, dx + C\, b(\|u\|_\infty)^2 \int_{B_{\beta R}} |Xu|^2 |u-u_{\beta R}|^2 E_t \varphi^2 dx.
\end{aligned}
$$

Choosing $t = \dfrac{C\, b(\|u\|_\infty)^2}{2\, c_1}$ and since $E_t(x) \ge 1$, we obtain

$$
\int_{B_{\beta R}} \left(|Xu|^2 + |u-u_{\beta R}|^2\right) \varphi^2\, dx \le C\left(|u_{\beta R}|+1\right)^2 \int_{B_{\beta R}} E_t \varphi^2\, dx + \frac{C}{R^2} \int_{B_{\beta R}} |u-u_{\beta R}|^2 E_t\, dx.
$$

Now taking into account that $|u-u_{\beta R}|^2 \ge u^2 - \dfrac{|u_{\beta R}|^2}{2}$ we find

$$
\begin{aligned}
&\int_{B_{\beta R}} \left(|Xu|^2 + |u|^2\right) \varphi^2\, dx \\
&\le C\left(|u_{\beta R}|+1\right)^2 \int_{B_{\beta R}} E_t \varphi^2\, dx + C|u_{\beta R}|^2 \int_{B_{\beta R}} \varphi^2\, dx + \frac{C}{R^2} \int_{B_{\beta R}} |u-u_{\beta R}|^2 E_t\, dx \\
&\le C e^{4t\|u\|_\infty^2} \left(\|u\|_\infty + 1\right)^2 |B_{\beta R}| + \frac{C e^{4t\|u\|_\infty^2}}{R^2} \int_{B_{\beta R}} |u-u_{\beta R}|^2\, dx.
\end{aligned}
$$

Hence by Poincaré's inequality (5.1) and taking into account the choice of t we get (5.14). □

We next recall the following result that can be used for the homogenization results concerning the equation (5.12); see inequalities (4.22) and (4.26) in [10].

Theorem 5.6. *There exists a weak solution v to the problem* (5.12) *such that*

$$
\|v\|_{L^\infty(\Omega)} \le \frac{c_2}{\alpha_0}, \tag{5.15}
$$

and

$$\|v\|_{D_0^{1,p}(\Omega)} \le \sqrt{\frac{2|\Omega|}{c_1\alpha_0}}\,\exp\left(\frac{[c_2 b(c_2/\alpha_0)]^2}{8c_1\alpha_0^2}\right). \tag{5.16}$$

We now prove the higher integrability of weak solutions to the nonlinear problem (5.12).

Theorem 5.7. *Suppose that* (5.2), (5.3), (5.4), (5.13) *hold. Then there exists an exponent* $p_0 > 2$ *such that if* $u \in D_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *is any weak solution to the problem* (5.12) *satisfying* (5.15), *we have*

(1) $u \in D^{1,q}_{\mathrm{loc}}(\Omega)$ *for all* $q \in [2, p_0)$, *and for any ball* $B_R = B(x,R)$ *satisfying* $B(x,2R) \Subset \Omega$ *we have*

$$\left(\frac{1}{|B_R|}\int_{B_R} |Xu|^q\,dx\right)^{1/q} \le C\left(\left(\fint_{B_{2R}} |Xu|^2\,dx\right)^{1/2} + 1\right),$$

where C depends only on the structural constants and p.

(2) *If in addition* Ω *satisfies condition* (S), *then* $u \in D_0^{1,q}(\Omega)$ *for all* $q \in [2, p_0)$, *and we have*

$$\left(\int_\Omega |Xu|^q\,dx\right)^{1/q} \le C\left(\left(\int_\Omega |Xu|^2\,dx\right)^{1/2} + 1\right), \tag{5.17}$$

where C depends only on the structural constants, p, λ, *and* Ω.

Proof. The proof runs parallel to the proof of Theorem 5.4, and so we omit most details. By Lemma 5.5 and inequality (5.15), there exists a positive constant b depending only on the structural constants such that

$$\fint_{B_{3R/4}} |Xu|^2\,dx \le b\left(\left(\fint_{B_R} |Xu|^p\,dx\right)^{2/p} + 1\right) \tag{5.18}$$

for any ball $B_R = B(x,R)$ satisfying $B(x,2R) \Subset \Omega$. Therefore by the reverse Hölder inequality, Theorem 5.3, we obtain part 1. For part 2, let V be an open set in $\mathbb{R}^N$ such that $\Omega \Subset V$ and extending u as zero outside Ω. Again by the reverse Hölder inequality, Theorem 5.3, we only need to show that (5.18) holds for any ball $B_R = B(x,R) \subset V$ satisfying $R \le R^*$. The argument is similar to the proof of part 2 of Theorem 5.4 with $v = u\,e^{tu^2}\,\varphi^2 \in D_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ taken as a test function in Case 3. In this way we derive the desired result. □

6. Convergence of correctors in $L^{2+\theta}$

In this section we prove that the correctors w_ε defined by (4.4) converge strongly in $L^{2+\theta}(\Omega)$ as $\varepsilon \to 0$ for some $\theta > 0$ provided that

(H_1) There exists $\sigma > 2$ such that $F_\epsilon := (f^0_\epsilon, f^1_\epsilon, ..., f^m_\epsilon)$ is bounded in $L^\sigma(\Omega; \mathbb{R}^{m+1})$ for all sufficiently small ϵ;

(H_2) $\nabla_{\mathbb{G}} u \in L^p(\Omega, \mathbb{R}^m)$ for p sufficiently large.

Let p_0 be the constant in Theorem 5.4 corresponding to the number σ in (H_1). We first show that for any $\beta \geq 0$, $2+\beta < p_0$, there exist positive constants ε_1 and C_1 depending on β such that

$$\|w_\varepsilon\|_{L^{2+\beta}(\Omega;\mathbb{R}^m)} \leq C_1, \tag{6.1}$$

for all $\varepsilon \leq \varepsilon_1$. We have

$$\|w_\varepsilon\|_{L^{2+\beta}(\Omega;\mathbb{R}^m)} \leq \|\nabla_{\mathbb{G}} u_\epsilon\|_{2+\beta} + \|\nabla_{\mathbb{G}} u\|_{2+\beta} + \|(\nabla_{\mathbb{G}}\chi)(\delta_{1/\epsilon})\nabla_{\mathbb{G}} u\|_{2+\beta}, \tag{6.2}$$

From (5.8) and (H_1), $\|\nabla_{\mathbb{G}} u_\epsilon\|_{2+\beta} \leq C\,\|F_\varepsilon\|_{2+\beta} \leq M_1$ uniformly for all ε sufficiently small. From (H_2), $\|\nabla_{\mathbb{G}} u\|_{2+\beta}$ is finite. To estimate the last summand in (6.2) it is enough to estimate

$$J = \int_\Omega \left|\left(X_j\chi^k\right)(\delta_{1/\varepsilon}x) X_j u(x)\right|^{2+\beta} dx,$$

for all $1 \leq j, k \leq m$. Applying Hölder's inequality with exponents $p = \dfrac{2+\alpha}{2+\beta}$ and $p' = \dfrac{2+\alpha}{\alpha-\beta}$, where α is chosen so that $2+\beta < 2+\alpha < p_0$, we get

$$\begin{aligned} J &\leq \left(\int_\Omega \left|\left(X_j\chi^k\right)(\delta_{1/\varepsilon}x)\right|^{2+\alpha} dx\right)^{(2+\beta)/(2+\alpha)} \left(\int_\Omega \left|X_j u(x)\right|^{(2+\alpha)(2+\beta)/(\alpha-\beta)} dx\right)^{(\alpha-\beta)/(2+\alpha)} \\ &\leq C\left(\int_\Omega \left|\left(X_j\chi^k\right)(\delta_{1/\varepsilon}x)\right|^{2+\alpha} dx\right)^{(2+\beta)/(2+\alpha)} \\ &= C\, I^{(2+\beta)/(2+\alpha)}, \end{aligned}$$

where we have used (H_2) in the second inequality. Let Ω_0 be an open bounded set such that $\bar{\Omega} \subset \Omega_0$, and consider $\mathcal{J}_\varepsilon = \{i \in \mathbb{Z}^n : Q^\varepsilon_i \subset \Omega_0\}$. We have that $\Omega \subset \cup_{i\in\mathcal{J}_\varepsilon} Q^\varepsilon_i$ for all ε sufficiently small. Therefore, letting $x = \delta_\varepsilon y$ yields

$$\begin{aligned} I &\leq \varepsilon^Q \sum_{i\in\mathcal{J}_\varepsilon} \int_{i\circ\mathbf{Y}} |X_j\chi^k(y)|^{2+\alpha}\, dy \\ &= \varepsilon^Q \sum_{i\in\mathcal{J}_\varepsilon} \int_{\mathbf{Y}} |X_j\chi^k(y)|^{2+\alpha}\, dy, \text{ by periodicity} \\ &= \varepsilon^Q \,\#\{\mathcal{J}_\varepsilon\} \int_{\mathbf{Y}} |X_j\chi^k(y)|^{2+\alpha}\, dy \\ &\approx |\Omega_0| \int_{\mathbf{Y}} |X_j\chi^k(y)|^{2+\alpha}\, dy, \text{ by Remark 2.5.} \end{aligned}$$

We want to show that the last integral is bounded. To do this, fix a bounded open set $U \subset \mathbb{G}$ such that $\mathbf{Y} \Subset U$. Since $\chi^k \in D^{1,2}_{\#}(\mathbf{Y})$ is a weak solution of equation (3.4) in $\mathbf{Y}$ and a_{ij} are bounded $\mathbf{Y}$-periodic functions, by arguing as in the proof of [2, Lemma 5] we see that

$$-\sum_{i,j=1}^{m} X_i(a_{ij}\, X_j(\chi^k)) = \sum_{i=1}^{m} X_i(a_{ik}), \text{ in } D^{-1,2}(U).$$

Therefore, we can apply the first part of Theorem 5.4 noticing that in our case $X_i^* = -X_i$ to get

$$\left(\int_{\mathbf{Y}} |X_j\chi^k(y)|^{2+\alpha}\, dy\right)^{\frac{1}{2+\alpha}} \leq C\left[\left(\int_U |X_j\chi^k|^2\, dx\right)^{\frac{1}{2}} + \left(\int_U |\chi^k|^2\, dx\right)^{\frac{1}{2}} + 1\right] < \infty.$$

This completes the proof of (6.1).

Now for any nonnegative θ satisfying $2+\theta < p_0$, pick a number β such that $2+\theta < 2+\beta < p_0$. We then have by interpolation inequality that

$$\|w_\varepsilon\|_{2+\theta} \leq \|w_\varepsilon\|_2^t\, \|w_\varepsilon\|_{2+\beta}^{1-t} \leq \|w_\varepsilon\|_2^t\, C_1^{1-t},$$

where $\dfrac{1}{2+\theta} = \dfrac{t}{2} + \dfrac{1-t}{2+\beta}$. On the other hand, from Theorem 4.4 $\|w_\varepsilon\|_{L^2(\Omega;\mathbb{R}^m)} \to 0$ as $\varepsilon \to 0$. Therefore letting $\varepsilon \to 0$ we obtain the following theorem.

Theorem 6.1. *Suppose Ω is a bounded domain satisfying condition (S), and conditions (H_1) and (H_2) above hold. Then $\|w_\varepsilon\|_{L^{2+\theta}(\Omega)} \to 0$ as $\varepsilon \to 0$ for all nonnegative θ satisfying $2+\theta < p_0$.*

Remark 6.2. If we relax assumption (H_2) above replacing it by

$$(H_2)_{\text{loc}} \qquad \nabla_{\mathbb{G}} u \in L^p_{\text{loc}}(\Omega, \mathbb{R}^m) \text{ for } p \text{ sufficiently large,}$$

then the conclusion of Theorem 6.1 still holds without assuming condition (S) and provided we replace the convergence of the correctors in $L^{2+\theta}(\Omega)$ by the convergence in $L^{2+\theta}_{\text{loc}}(\Omega)$.

In fact, in the standard elliptic case, the higher integrability of the gradient in (H_2) follows easily by Schauder estimates, from regularity assumptions on the data. On the other hand, when dealing with elliptic operators on Carnot groups, things can be dramatically different, because of the possible presence of characteristic points at the boundary (a point $x \in \partial\Omega$ is said a characteristic point for $X_1, \ldots, X_m$ if $X_1(x), \ldots, X_m(x)$ belong to the tangent space to $\partial\Omega$ at x). In general, even smooth open sets have characteristic points, that – roughly speaking – can affect the regularity of solutions as cusps do when dealing with elliptic operators. Notice that, for instance, every continuously differentiable closed surface in the Heisenberg group (identified with $\mathbb{R}^3$) that is diffeomorphic to a sphere contains

characteristic points. Notice also that condition (S) may hold even at the characteristic points of the boundary.

On the other hand, if in (3.3) f belongs to $L^p(\Omega)$ for $p > Q$, then by Hölder inequality condition (H_2) is satisfied if the following uniform estimate for the Green function of Ω holds:

$$|\nabla_{\mathbb{G}} G(x,y)| \leq C d(x,y)^{1-Q}, \quad x, y \in \Omega. \tag{6.3}$$

For instance, if $\mathbb{G} = \mathbb{H}^n$, then (6.3) is proved in [17, Theorem 3.6], provided Ω satisfies the uniform exterior ball property (see [17, Definition 3.3]). The same result holds for general Carnot groups, and it can be derived from the proof of estimate (54) in [18]. In turn, the uniform exterior ball property – which may hold for domains having characteristic points – implies by doubling our assumption (S) given in Definition 5.2.

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