Learning Improper Fractions with the Number Line and the Area Model

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Acknowledgement

This research was supported by NSF CAREER DRL-1452000, NSF ECR-1760144, and James S. McDonnell Foundation Award #220020546 to Elizabeth A. Gunderson. We thank the children and teachers who gave their time to this research and the research assistants who helped carry it out including Yiqiao Wang, Jorge Carvalho Pereira, Grace Bennett-Pierre, Nadia Tavassolie, Khushi Sibal, Paula Ueki, Dianna Wambach, Madyson Kolbes, Su Dam, Nadhia Marie Engle, and Brianna Stein.

In Press, *Journal of Cognition and Development.*

Abstract

Number lines and area models are both used pervasively in teaching fractions. Prior studies found that second and third graders demonstrated better magnitude knowledge of proper fractions after a 15-minute training using the number line as compared to using the area model. The current study aimed to extend these findings to improper fractions. We randomly assigned fourth and fifth graders to a number line training, an area model training, or a non-numerical control condition. The number line and area model trainings involved both proper and improper fractions and were closely modeled on the training procedures in prior studies. Fraction training with the area model produced improvements in children's area model estimation of proper and improper fractions. However, contrary to our expectations, training with the number line did not improve number line estimation, and neither training led to improvements in transfer tasks assessing fraction magnitude knowledge. These findings suggest that children can develop the skill to represent improper fractions on area models with brief training. Nevertheless, it is unclear whether this skill enhances a comprehensive understanding of fraction magnitudes.

Keywords: number line; area model; fractions; intervention

Introduction

Many children and even adults struggle with mastering fractions. On the 2013 National Assessment of Educational Progress (NAEP), only 55% of 4th graders chose the correct answer to 2/5 + 3/5 + 4/5, and 39% chose 9/15 (an answer that can be obtained by separately adding the numerators and the denominators of the three addends; U.S. Department of Education, 2013). Children's poor performance is not restricted to arithmetic: on the 2007 NAEP, only 49% of 8th graders correctly ordered three fractions, 2/7, 1/2, and 5/9, from the least to the greatest (Martin, Strutchens, & Elliott, 2007). This lack of fraction knowledge often persists into adulthood: in a sample of more than 1,600 community college students, only 33% correctly identified the smallest among four fractions (Stigler, Givvin, & Thompson, 2010).

Children's poor knowledge of fractions is especially unfortunate given the importance of mastering fractions for academic achievement, career development, and life functioning. Fractions are essential for learning more advanced math, such as algebra (Booth & Newton, 2012). In nationally-representative US and UK samples, knowledge of fractions in 5th grade predicted general math achievement in high school, over and above IQ, working memory, family background, and whole number knowledge (Siegler et al., 2012). The importance of fractions extends beyond school. In a representative sample of US workers, 68% reported using fractions at work (Handel, 2016). Fractions are also ubiquitous in adults' daily life, such as in adjusting recipes, making medical decisions, and managing personal finances (e.g., Reyna, Nelson, Han, & Dieckmann, 2009).

The importance of mastering fractions, and many children's failure to do so, underscores the need for improving fraction instruction. Much intervention research has been done to improve children's fraction understanding (Fazio, Kennedy, & Siegler, 2016; Jordan et al., 2013; Moss & Case, 1999; Saxe et al., 2007). However, many fraction interventions have relatively small effects, suggesting that it is hard to mitigate children's difficulty with fractions and that more efforts are needed to develop effective fraction instruction (Hwang, Riccomini, Hwang, & Morano, 2019; Misquitta, 2011; Roesslein & Codding, 2019). The current intervention study targeted one aspect of fraction instruction, the use of visual representations. In particular, we investigated whether learning fractions with the area model or the number line would lead to better understanding of fraction magnitude.

Area models and number lines are frequently used in teaching fractions (National Governors Association Center for Best Practices, 2010). Representing fractions on area models involves shading parts of whole shapes. For instance, to represent a fraction, x/y, using an area model, a 2D shape is divided into y equal segments with x segments shaded (Figure 1A). Such representations capture the part-whole meaning of fractions, which might be familiar to children given that fractions are frequently used in part-whole contexts in children's daily life (e.g., sharing pizza cut into pieces). However, representing fractions with area models may reinforce children's misconceptions about fractions. Many children tend to view fractions as two separate whole numbers (i.e., whole number bias; Ni & Zhou, 2005). Area models may strengthen this bias by emphasizing the part (the numerator) and the whole (the denominator) rather than the relation between the two (Hamdan & Gunderson, 2017).

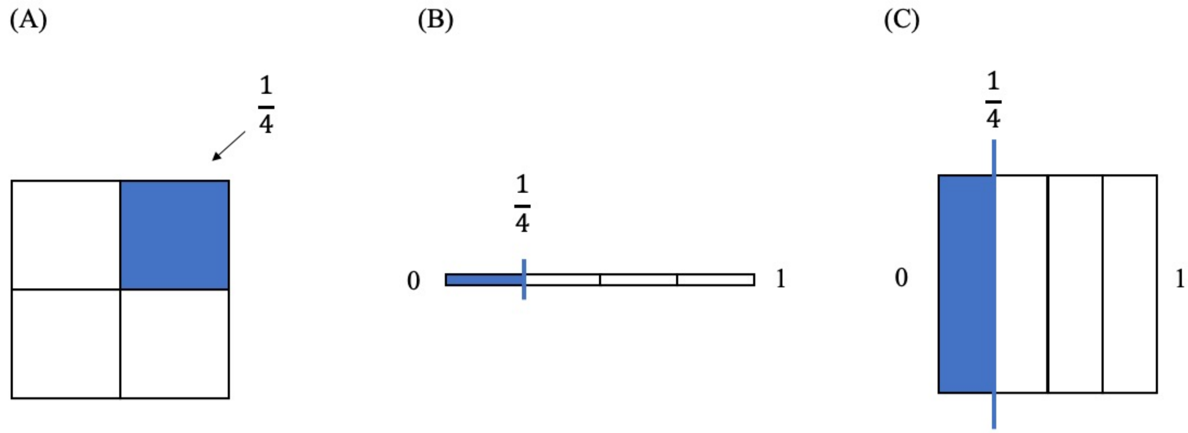

Figure 1. Example of representation of the fraction 1/4 with (A) an area model, (B) a number line, and (C) a two-dimensional square number line (adapted from “Number Line Unidimensionality Is a Critical Feature for Promoting Fraction Magnitude Concepts,” By E. Gunderson et al., 2019, *Journal of Experimental Child Psychology*, 187, p. 4. Copyright 2019, Elsevier Inc.)

On number lines, in contrast, fractions are represented as integrated magnitudes. In other words, instead of representing fractions using multiple parts, a fraction can be represented on the number line as a single mark, with the distance between the mark and the zero point corresponding to the magnitude of the fraction (Figure 1B). To determine where a fraction goes on the number line, both the numerator and the denominator of the fraction must be considered simultaneously, in relation to one another. This process may help mitigate children’s whole number bias, which occurs when students instead consider the numerator and denominator as separate whole number magnitudes (Ni & Zhou, 2005). Moreover, number lines take advantage of unidimensional mental representations of numbers: extensive behavioral and neuroimaging evidence suggests that numbers are represented in a manner similar to a number line, with smaller numbers on the left and larger numbers on the right (Ansari, 2008; Dehaene, Bossini, &

Giraux, 1993; Toomarian & Hubbard, 2018). The number line allows children to integrate fractions with whole numbers, which conforms with the developmental trajectory of numerical magnitudes set forth by the integrated theory of numerical development (Siegler, Thompson, & Schneider, 2011). This theory posits that numerical development is a process of enhancing the magnitude representations of a broadening range of numbers and that learning fractions involves integrating fractions with existing whole number knowledge. Therefore, compared to the area model, using the number line seems to be more beneficial for children's fraction learning.

Consistent with this view, several interventions that focused on building fraction magnitude knowledge using number lines yielded better learning than regular school curricula (Dyson, Jordan, Rodrigues, Barbieri, & Rinne, 2018; Fuchs et al., 2016, 2013, 2014; Saxe et al., 2007). For example, Fuchs et al. (2013) randomly assigned 4th graders to a 12-week intervention focused on learning fraction magnitudes on the number line or to a control curriculum that emphasized the part-whole interpretation of fractions with area models. Children who received the intervention using the number line improved more on fraction magnitude knowledge and arithmetic than those who were in the control condition.

The extensive intervention curricula in these studies involved multiple components (Dyson et al., 2018; Fuchs et al., 2016, 2013, 2014; Saxe et al., 2007), making it impossible to distinguish the effects of the number line on children's fraction learning from other intervention components. Two recent experiments provided direct evidence for the advantages of using number lines, particularly over area models, in learning about fraction magnitude (Gunderson, Hamdan, Hildebrand, & Bartek, 2019; Hamdan & Gunderson, 2017). Hamdan and Gunderson (2017) randomly assigned 2nd and 3rd graders to a number line training, an area model training, or a non-numerical control condition. The number line and the area model trainings involved similar procedures and only differed in the visual representation used. After a 15-minute training, children in the number line condition were more accurate at estimating fractions on number lines than children in the other two conditions. Critically, compared to children in the area model training and the control condition, children in the number line training condition were also more accurate at comparing fractions, a task which none of the children were directly taught. These findings suggest that using number lines is more beneficial than area models for children to develop fraction magnitude understanding.

Gunderson et al. (2019) replicated these findings and demonstrated that the unidimensionality of the number line is essential for it to produce superior fraction learning than the area model. This study involved four training conditions: pure unidimensional number line training (i.e., number lines being pure lines with no width), hybrid unidimensional number line training (i.e., number lines being long thin rectangles, same as the number line training condition in Hamdan and Gunderson, 2017; Figure 1B), square number line training (i.e., squares that were partitioned from left to right and had “0” at the left end and “1” at the right end; Figure 1C), and square area model training (Figure 1A). Similar to Hamdan and Gunderson (2017), compared to children who were taught fractions with area models, those who were taught fractions with the unidimensional number lines (either pure or hybrid) were more accurate on number line estimation and magnitude comparison at posttest, controlling for pretest performance (Gunderson et al., 2019). Moreover, children who were taught fractions with two-dimensional square number lines showed similar performance at posttest to those who were taught with area models. Thus, being unidimensional is an essential feature of number lines to be more beneficial than area models for learning fraction magnitudes (Gunderson et al., 2019).

The main purpose of the current study was to extend prior findings on the advantage of the number line over the area model to the learning of improper fractions. Many children experience similar, if not more, difficulty understanding improper fractions as compared to proper fractions (Resnick et al., 2016; Siegler et al., 2011; D. Zhang, Stecker, & Beqiri, 2017). In math class, improper fractions are introduced later than proper fractions (Grade 4 versus Grade 3; National Governors Association Center for Best Practices, 2010). Before learning improper fractions, children's exposure to fractions has been limited to proper fractions, leading to the belief that fractions are always smaller than one (Stafylidou & Vosniadou, 2004). Therefore, understanding the magnitudes of improper fractions, which are always greater than one, imposes great challenges for many children. Resnick et al. (2016) tracked the development of children's estimation of proper and improper fractions from Grade 4 through 6. They found that most fourth graders estimated both proper and improper fractions as being smaller than one. While some children's estimates of improper fractions gradually became reasonably accurate, more than 40% of children still estimated improper fractions to be smaller than one in 6th grade. This failure in understanding magnitudes of improper fractions is unfortunate as such understanding may be essential for a comprehensive understanding of fraction magnitudes to emerge (Rinne, Ye, & Jordan, 2017).

In the current study, we aimed to test whether the number line or the area model better facilitates children's learning of improper fractions. The number line, which naturally extends beyond one, may better facilitate children's transition from learning proper to improper fractions (Tian & Siegler, 2017). In contrast, representing improper fractions on area models may be awkward because more than one identical shape, with each shape representing one whole unit, must be involved (Behr, Wachsmuth, & Post, 1988; Wu, 2009). Therefore, we expected that teaching improper fractions with the number line would lead to greater learning than with the area model.

To test this expectation, we randomly assigned 4th and 5th graders to a number line training, an area model training, or a non-numerical control condition. We implemented a training design involving a pretest, training, immediate posttest, and delayed posttest. The number line and the area model trainings involved both proper and improper fractions and were closely modeled on the training procedures in prior studies (Gunderson et al., 2019; Hamdan & Gunderson, 2017). The current study employed the hybrid rather than the pure unidimensional number line in the training because both Hamdan and Gunderson (2017) and Gunderson et al. (2019) used the hybrid number line in the intervention. Moreover, in Gunderson et al. (2019), training with both types of unidimensional number lines yielded greater improvement in fraction magntiude comparison than training with the square area model; however, only the hybrid number line training led to greater improvement than the square number line training. In the pretest and posttests of the present study, children completed a number line estimation task, an area model estimation task, a magnitude comparison task, and a comparison to one task (on which children judged whether a given fraction was smaller than, equal to, or greater than one). To be consistent with prior studies using the number line estimation task (e.g., Gunderson et al., 2019; Hamdan & Gunderson, 2017; Siegler, Thompson, & Schneider, 2011), pure unidimensional number lines, rather than hybrid unidimensional number lines, were employed in the pretest and posttests so that our findings would be comparable to prior work. We chose an older age group (4th and 5th graders) than prior studies (which focused on 2nd and 3rd graders) because improper fractions are taught later than proper fractions in school and because pilot testing suggested that this older age group was not at ceiling in improper fraction knowledge.

Because our study was modeled on prior studies training children on proper fractions, we expected to conceptually replicate the results of those studies with improper fraction training (Gunderson et al., 2019; Hamdan & Gunderson, 2017). Therefore, we had four preregistered hypotheses:

Hypothesis 1. **At the immediate posttest,** children in either the number line or the area model training condition will be more accurate at estimating fractions on the model they have received training on than children in the other two conditions.

Hypothesis 2. The effects described in Hypothesis 1 will hold on both fractions that appear in training and fractions that do not appear in training.

Hypothesis 3. On the magnitude comparison task at immediate posttest, children in the number line condition will have higher accuracy (across all magnitude comparison items) than children in the area model condition, and children in the area model condition will have higher accuracy than those in the control condition.

Hypothesis 4. The effects of condition on magnitude comparison in Hypothesis 3 will be present among ambiguous fraction pairs (in which the fraction with the smaller numerator has a larger denominator than the other fraction, such as 2/8 vs. 4/6).

Hypothesis 4 was based on the finding that compared to the area model training, the number line training led to greater improvements in comparing ambiguous fraction pairs both in Hamdan and Gunderson (2017) and in Gunderson et al. (2019). However, the effect of training condition was not consistent across the two studies on whole-number consistent pairs (in which the larger fraction also has a larger numerator and a larger denominator than the other fraction, such as 8/4 vs. 3/2) or whole-number inconsistent pairs (in which the larger fraction has a smaller numerator and a smaller denominator than the other fraction, such as 2/6 vs. ½).

Besides these four hypotheses, we also explored whether training effects would transfer to better performance on an additional task assessing fraction magnitude understanding (i.e., comparing fractions to one) and would persist two weeks after training. The study was pre-registered on the Open Science Framework (OSF; https://osf.io/9wp5r). All study materials, deidentified data, and analysis scripts have been made publicly available on OSF (https://osf.io/c7a5q).

Method

Participants

Participants were fourth and fifth grade students recruited from six schools (18 classrooms) in a large city in the northeastern US (n = 133; 66 4th graders; 73 girls; M_{age} = 10.25, SD_{age} = 0.66). One hundred and twenty-nine participants completed all three sessions of the study – two participants were absent on one or more testing days, and another two participants withdrew during the study. Another 10 participants were excluded because they were tested outside the pre-determined time window (see the Procedure section below for more details). Our analytic sample included 119 participants.

Participants came from diverse backgrounds. Participants' parents reported their children's race/ethnicity (demographic information reported here and below were of the 119 participants in the analytic sample; $n_{\text{race/ethnicity}}$ = 105; 40.0% Caucasian, 24.8% Black/African American, 19.0% Hispanic, 6.7% Asian/Asian American, and 9.5% Multi-race), annual family income ($n_{\text{family income}}$ = 90; M = \$56,869, SD = 32,290, range = < \$15,000 to > \$100,000), and parental education level ($n_{\text{parental education}}$ = 103; years of education: M = 14.54, SD = 2.54, range = 10 [less than high school] to 18 [graduate degree]). On average, participants came from middle-income families, and their parents completed 2 years of college.

Procedure

Each participant worked with a trained experimenter for three 20- to 30-minute sessions in a quiet space at their school. In Session 1, participants completed the pretest. In Session 2, participants were randomly assigned, within each classroom, to the number line training condition, the area model training condition, or the non-numerical control condition. After about 15 - 20 minutes of training (or control activities), participants completed the immediate posttest. In Session 3, participants completed the delayed posttest. Session 2 was planned to be administered within seven days after Session 1, and Session 3 was planned to be administered between 14-16 days after Session 2. Due to scheduling errors, 10 participants were tested outside of the 14-16-day time window between Sessions 2 and 3 and were thus excluded from the sample based on our pre-registered Sampling Plan. The remaining sample included 119 participants, with 40 in the number line training condition, 41 in the area model training condition, and 38 in the non-numerical control condition. The average time between Sessions 1 and 2 was 2.97 days (SD = 1.19) and between Sessions 2 and 3 was 14.26 days (SD = 0.48).

The pretest and both posttests (immediate and delayed) consisted of the same four tasks: number line estimation, area model estimation, magnitude comparison, and comparison to one (see Figure 2 for an example problem from each task). The two estimation tasks were administered using PDF Expert (Readdle Inc., 2019) on an iPad, and the other two tasks were administered using E-Prime 2.0 (Schneider, Eschman, & Zuccolotto, 2002) on a laptop computer. The four tasks were presented in one of two orders, the following order or the reverse of this order: number line estimation, comparison to one, magnitude comparison, and area model estimation. Participants completed the four tasks in one order in the pretest and delayed posttest, and in the other order in the immediate posttest. The order of tasks was randomly assigned to each participant.

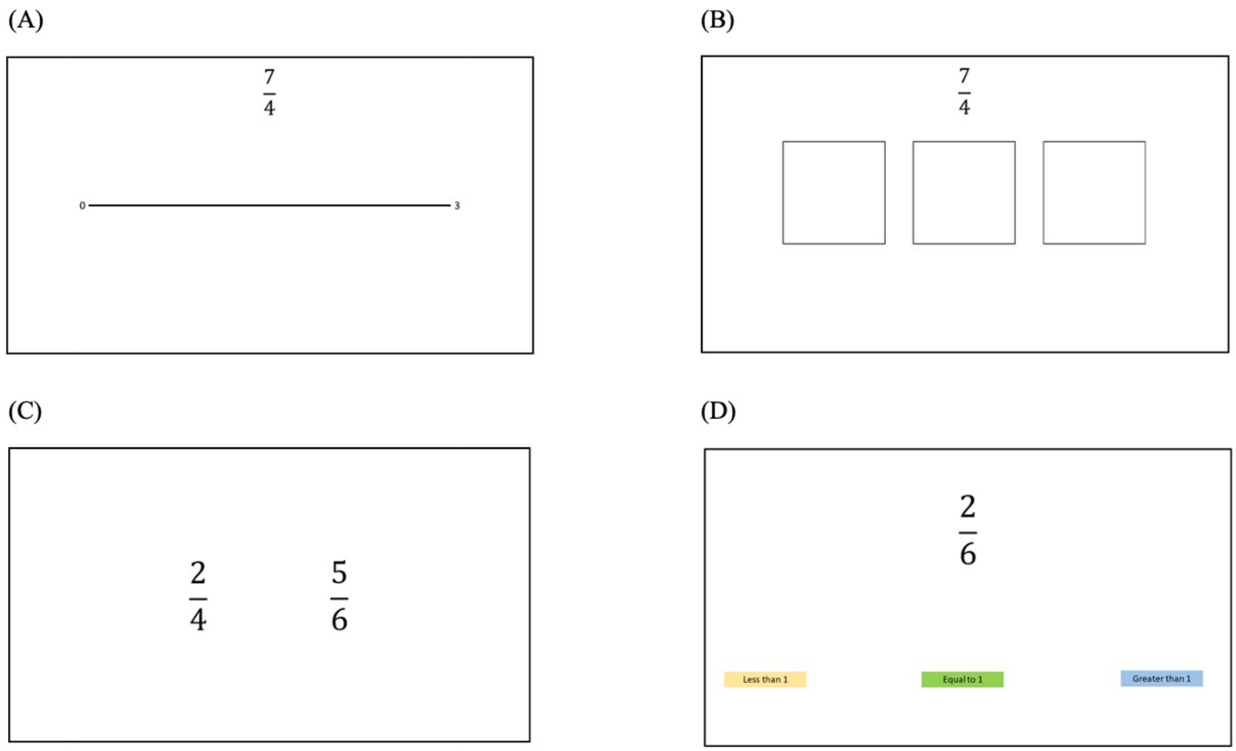

Figure 2. Example items of (A) the number line estimation task, (B) the area model estimation task, (C) the magnitude comparison task, and (D) the comparison to one task.

Training conditions

In the two fraction training conditions, the experimenter showed participants how to represent fractions on number lines or area models, and participants practiced representing fractions with feedback. The training procedures were developed based on prior studies (Gunderson et al., 2019; Hamdan & Gunderson, 2017) and were parallel between the number line and the area model conditions (see Figure 3 for an overview of the training procedures and Supplementary Materials Sections A - C for the scripts used in the training and the control condition).

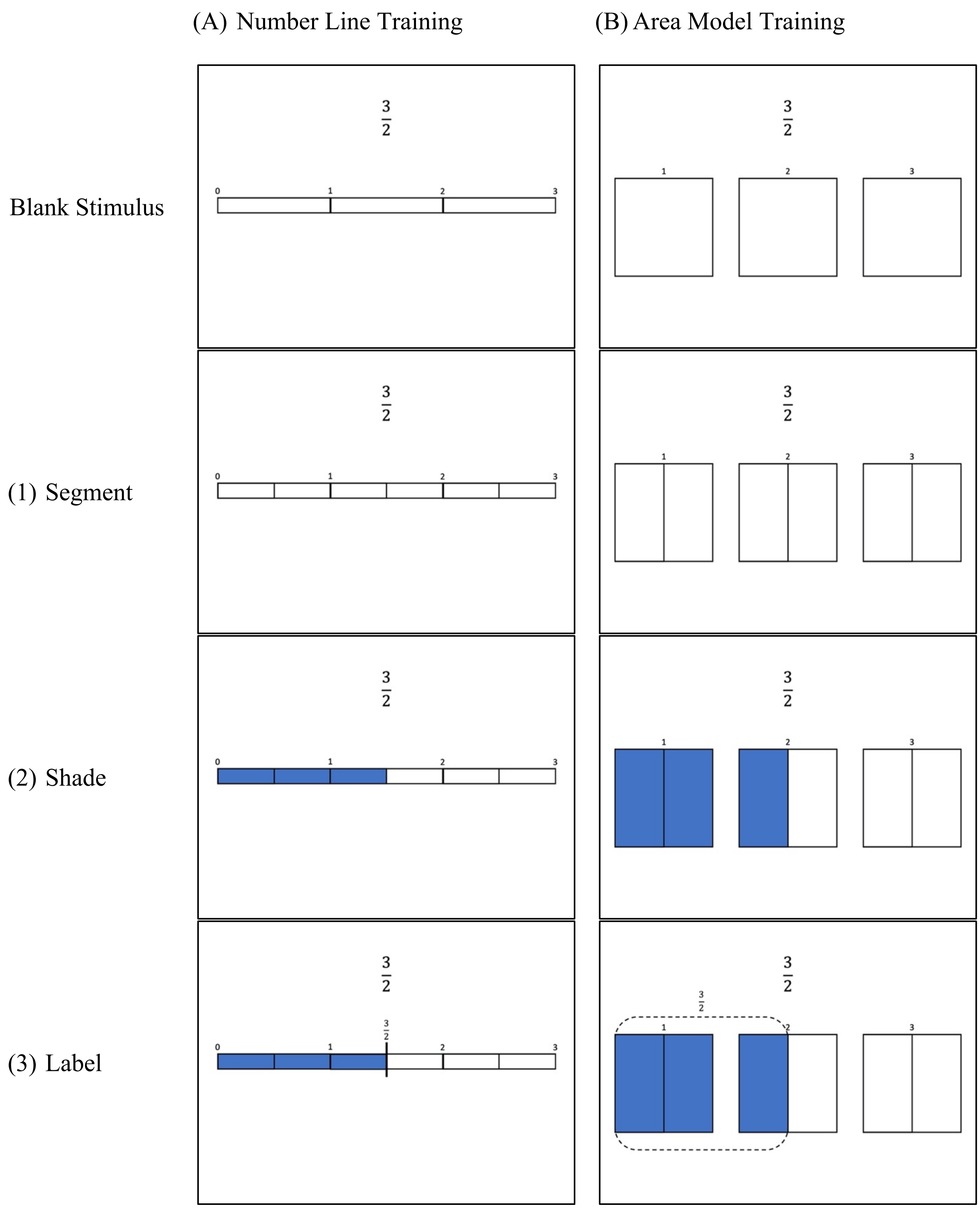

Figure 3. Procedures of (A) number line training and (B) area model training.

At the beginning of the training, fractions were introduced as numbers that have a number on the top (i.e., the numerator) and a number on the bottom (i.e., the denominator). Then, participants were taught to represent fractions in three steps: 1) segment each unit of the visual representations (each unit of a visual representation represents one) into the number of equal segments corresponding to the denominator, 2) shade the number of segments corresponding to the numerator, and 3) label the visual representation by drawing a hash mark at the end of the shaded segments (number line training) or circling the shaded segments (area model training) and by writing the fraction next to the hash mark or the circle.

Participants were taught to represent eight fractions during training (3/2, 1/2, 5/2, 3/4, 7/4, 12/5, 2/5, and 4/5, in that order). For each fraction, participants practiced each of the three steps following the experimenter's explanation of that step. If a participant performed a step incorrectly, the experimenter would demonstrate how to correctly perform that step and ask the participant to practice that step along with the steps leading up to it again on a blank number line or area model. For example, if a participant incorrectly labeled a fraction on the number line, the participant would be asked to segment, shade, and label a blank number line for that fraction. For fractions 3/2, 3/4, and 12/5 (the first fraction in each group of fractions with the same denominator), the experimenter demonstrated each step before the participant's practice. Each training session took approximately 15 to 20 minutes and was administered with PDF Expert (Readdle Inc., 2019) on an iPad.

Number line training

Participants in the number line training condition (n = 40) were taught to show fractions on 0-3 number lines with labeled hash marks at 1 and 2 (Figure 3A). The number line (8 mm high × 180 mm wide) was similar to the "hybrid" number line in Gunderson et al. (2019). In the training, the experimenter demonstrated representing fractions by segmenting each unit of the number line with vertical hash marks, shading segments from left to right, and labeling a longer hash mark at the end of the shaded portion with the represented fraction.

Area model training

Participants in the area model training condition (n = 41) were taught to show fractions on three squares (each square was 50 mm × 50 mm) with the numbers 1, 2, or 3 above the middle of each square. In the training, the experimenter demonstrated representing fractions by segmenting each square area model with vertical lines; shading the segments from left to right; and finally, circling the shaded portion and writing the represented fraction next to the shaded portion. In the area model training (but not the number line training), it was considered correct if participants segmented using horizontal or diagonal lines or shaded in a different order than from left to right.

Non-numerical control

Participants in the non-numeric control condition (n = 38) worked on crossword puzzles in collaboration with the experimenter. The activity lasted for 18 minutes, which was similar in length to the number line or area model training session based on pilot testing. Participants referred to crossword puzzle clues and had the option to use word banks for each puzzle printed on paper. There were six puzzles available for the participant to complete, and no participant completed all six puzzles.

Pretest and posttest measures

Number line estimation

In this task, participants were asked to show fractions on 0-3 number lines (Figure 2A). At the beginning of the task, the experimenter illustrated where 1/2 goes on the number line.

Then, participants were asked to represent 12 fractions (i.e. 3/4, 4/5, 2/5, 2/6, 3/5, 2/3, 7/4, 12/5, 5/2, 6/5, 7/6, and 8/3). Among the 12 fractions, six were proper fractions, and six were improper fractions; and six fractions were included in the number line and area model training, and six were not. Participants received the 12 fractions in a predetermined random order or the reverse of that order.

On each trial, participants' response value was calculated by dividing the length between the zero endpoint and the hash mark drawn by the participants by the total length of the line (i.e., 180 mm) and then multiplying the quotient by the number line range (i.e., 3). For example, if the length between the hash mark and the zero endpoint was 90 mm, the response value = 90/180 × 3 = 3/2. Trained researchers scored participants' responses (see Supplementary Materials, Section D for detailed coding scheme adapted from Gunderson et al., 2019, Appendix D) and calculated percent absolute error (PAE) for each item: (|response value - correct value|)/(number line range). For example, if the participant was asked to represent 4/5 and responded at a point equivalent to 3/2, PAE = |3/2 – 4/5|/3 = 0.23. Individual PAEs were averaged for the analyses. Internal consistency of the task was adequate ($\alpha_{pretest} = .69, \alpha_{immediate\ posttest} = .73, \alpha_{delayed\ posttest} = .67$).

Area model estimation

In this task, participants were asked to show fractions on area models. The area model consisted of three squares (50 mm high × 50 mm wide) presented side by side (Figure 2B). At the beginning of the task, the experimenter demonstrated 1/2 on the area model by shading 1/2 of the leftmost square. Then, participants were asked to show each of the same 12 fractions as in the number line estimation task on area models. On each trial, participants' response value was calculated by dividing the shaded pixels by the total pixels of the three squares and then multiplying the quotient by the numerical range of the area models (i.e., 3). Trained researchers scored participants' responses by calculating the pixels of shaded area using Adobe Photoshop 2017 (Faulkner & Chavez, 2017; see Supplementary Materials, Section D for detailed coding scheme adapted from Gunderson et al., 2019, Appendix D) and calculated the PAE for each trial: (|response value - correct value|)/(total area). Individual PAEs were averaged for the analyses. Reliability was good ($\alpha_{pretest} = .76, \alpha_{immediate\ posttest} = .88, \alpha_{delayed\ posttest} = .86$).

Magnitude comparison

In this task, participants were asked to choose the larger fraction in each pair of fractions (Figure 2C). On each trial, participants were asked to press the yellow button (the "A" key covered with a yellow sticker) if the fraction on the left was larger or the blue button (the "L" key covered with a blue sticker) if the fraction on the right was larger. The fraction pair remained on the screen until a valid response was detected. Each trial was preceded by a blank screen of 500ms. Participants were asked to respond as quickly and accurately as possible.

Each participant completed 24 trials, with a unique pair of fractions on each trial (see Supplementary Materials Table E1 for the fraction pairs). Among the 24 fraction pairs, eight included a proper and an improper fraction, eight included two improper fractions, and eight included two proper fractions. As in past studies (Gunderson et al., 2019; Hamdan and Gunderson, 2017), the fraction pairs included "consistent" pairs (6 items), "inconsistent" pairs (6 items), and "ambiguous" pairs (12 items). In the consistent pairs, the larger fraction also had a larger numerator and larger denominator than the other fraction (e.g., 8/4 vs. 3/2). Conversely, in the inconsistent pairs, the larger fraction had a smaller numerator and denominator than the other fraction (e.g., 2/6 vs. 1/2). In the ambiguous pairs, the fraction with the smaller numerator had a larger denominator than the other fraction (e.g., 2/8 vs. 4/6). We deliberately included more ambiguous pairs to increase our power of detecting the effect of interest in Hypothesis 4. Order of fraction pairs was randomized for each participant. Children's accuracy was scored by calculating the percentage of items answered correctly. Reliability was fair overall ($\alpha_{pretest} = .71, \alpha_{immediate\ posttest} = .71, \alpha_{delayed\ posttest} = .76$) and excellent for each item type ($\alpha_{consistent} = .92, \alpha_{inconsistent} = .93, \alpha_{ambiguous} = .91$).

Comparison to one

In this task, participants were asked to judge whether a fraction was less than 1, equal to 1, or greater than 1. On each trial, a fraction was presented in the center of the screen with the three options (i.e., "less than one", "equal to one", and "greater than one") below it (Figure 2D). Participants were asked to press the yellow button (the "A" key covered with a yellow sticker) if the fraction was less than one, the green button (the "G" key covered with a green sticker) if the fraction was equal to one, or the blue button (the "L" key covered with a blue sticker) if the fraction was greater than one. The problem remained on the screen until a valid response was detected. Each trial was preceded by a blank screen of 500ms. Participants were asked to respond as quickly and accurately as possible.

Each participant completed 16 trials. Among the 16 fractions, 12 were the same fractions as in the number line and area model estimation tasks (6 proper fractions and 6 improper fractions), and four were fractions equal to one (2/2, 3/3, 4/4, and 6/6). Order of the trials was randomized for each participant. Children's accuracy was scored by calculating the percentage of items answered correctly. Reliability was good overall ($\alpha_{pretest} = .83, \alpha_{immediate\ posttest} = .82, \alpha_{delayed\ posttest} = .84$).

Results

Data exclusion

According to our preregistered data analysis plan, participants' data were excluded from analyses of a task (or a subset of items in a task, e.g., ambiguous magnitude comparison items) if they did not finish at least half of the relevant items. Among the participants included in each analysis, we used the mean scores of each participant's available trials relevant to that analysis. We also preregistered that we would exclude a participant if the experimenter made an error during training. No participants needed to be excluded for this reason.

Descriptive statistics

Table 1 shows descriptive statistics of children's demographic characteristics as well as performance on each task at pretest, immediate posttest, and delayed posttest by condition (see Supplementary Materials, Section F for descriptive statistics of children's performance on different types of problems on each measure at pretest and posttests). Analyses comparing demographic characteristics and performance on each task at pretest revealed no differences among children assigned to each condition.

Table 2 shows the correlations among all measures collapsed across conditions. Notably, accuracy of concurrent fraction measures did not correlate or only weakly correlated with each other ($.02 < |r| < .32$). For each fraction measure, accuracy across the three testing sessions correlated moderately to strongly ($.38 < |r| < .64$). Among all fraction measures at pretest and posttests, grade level only correlated with accuracy on the comparison to one task at pretest.

Table 1

Means (and Standard Deviations) of Children's Performance on All Measures by Condition.

	NL Training (n = 40)	AM Training (n = 41)	CW Control (n = 38)	Condition Difference
Demographics				
Child Gender	22 female, 18 male	19 female, 22 male	24 female, 14 male	$\chi^2(2, N = 119) = 2.25, p = .324$
Grade Level	18 in G4, 22 in G5	20 in G4, 21 in G5	19 in G4, 19 in G5	$\chi^2(2, N = 119) = 0.21, p = .898$
Child Age (years)	10.37 (0.76)	10.24 (0.63)	10.20 (0.58)	$F\ (2, 116) = 0.74, p = .480,\ \eta_p^2 = .01$
Pretest Measures				
NL Est. (PAE)	0.28 (0.09)	0.27 (0.10)	0.27 (0.11)	$F\ (2, 111) = 0.05, p = .956, \eta_p^2 < .01$
AM Est. (PAE)	0.26 (0.08)	0.25 (0.11)	0.26 (0.13)	$F\ (2, 116) = 0.08, p = .926, \eta_p^2 < .01$
MC (accuracy)	0.57 (0.15)	0.61 (0.16)	0.59 (0.21)	$F\ (2, 116) = 0.55, p = .579, \eta_p^2 < .01$
CO (accuracy)	0.59 (0.24)	0.67 (0.27)	0.70 (0.22)	$F\ (2, 112) = 2.04, p = .134, \eta_p^2 = .04$
Immediate-Posttest Measures				
NL Est. (PAE)	0.24 (0.10)	0.24 (0.10)	0.25 (0.10)	$F\ (2, 114) = 0.05, p = .952, \eta_p^2 < .01$
AM Est. (PAE)	0.22 (0.11)[a]	0.08 (0.09)[b]	0.22 (0.13)[a]	$F\ (2, 116) = 23.14, p < .001, \eta_p^2 = .29$
MC (accuracy)	0.56 (0.15)	0.59 (0.17)	0.62 (0.19)	$F\ (2, 115) = 1.36, p = .261, \eta_p^2 = .02$
CO (accuracy)	0.65 (0.24)	0.70 (0.26)	0.73 (0.23)	$F\ (2, 114) = 0.95, p = .389, \eta_p^2 = .02$
Delayed-Posttest Measures				
NL Est. (PAE)	0.26 (0.08)	0.23 (0.10)	0.26 (0.10)	$F\ (2, 113) = 1.66, p = .195, \eta_p^2 = .03$
AM Est. (PAE)	0.23 (0.11)[a]	0.16 (0.12)[b]	0.21 (0.12)[a, b]	$F\ (2, 114) = 4.09, p = .019, \eta_p^2 = .07$

MC (accuracy)	0.56 (0.13)[a]	0.66 (0.17)[b]	0.60 (0.22)[a, b]	$F\ (2, 115) = 3.22, p = .044, \eta_p^2 = .05$
CO (accuracy)	0.69 (0.25)	0.70 (0.25)	0.75 (0.22)	$F\ (2, 115) = 0.85, p = .429, \eta_p^2 = .01$

Note. In rows where conditions differ, post-hoc comparisons have been conducted with the Bonferroni correction for multiple comparison, and the conditions that significantly differ ($p < .05$) are labeled with different superscript letters. For example, in the immediate-posttest, PAEs of area model estimation of the number line training and crossword control group did not differ (indicated by the superscript a), and both are different than the PAE of area model estimation of the area model training group (indicated by the superscript b). G, grade; NL, number line; AM, area model; CW, cross-word puzzle; Est., estimation; MC, magnitude comparison; CO, comparison to one; PAE, percent absolute error.

Table 2

Correlations Among All Measures Across Conditions.

	N	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Demographics															
1. Child Gender (female = 0, male = 1)	119	-													
2. Grade Level (G4 = 0, G5 = 1)	119	-.21 **	-												
3. Child Age (year)	119	-.10	.75 ***	-											
Pretest Measures															
4. NL Est. (PAE)	114	-.16	.04	.02	-										
5. AM Est. (PAE)	119	-.01	.10	.01	.02	-									
6. MC (accuracy)	119	.07	.07	.04	-.14	-.17	-								
7. CO (accuracy)	115	.05	.22 **	.27 **	.02	-.26 **	.22 **	-							
Immediate-Posttest Measures															
8. NL Est. (PAE)	117	-.06	-.03	.05	.45 ***	.07	-.28 **	-.18	-						
9. AM Est. (PAE)	119	-.11	.03	.07	.09	.39 ***	-.27 **	-.19 **	.16	-					
10. MC (accuracy)	118	.16	.04	.05	-.07	-.09	.56 ***	.16	-.20 **	-.17	-				
11. CO (accuracy)	117	.03	.04	.10	-.03	-.29 **	.20 **	.57 ***	-.15	-.23 **	.24 **	-			
Delayed-Posttest Measures															
12. NL Est. (PAE)	116	-.15	.03	-.05	.52 ***	.05	-.27 **	-.20 **	.54 ***	.14	-.21 **	-.23 **	-		
13. AM Est. (PAE)	117	-.06	-.02	.02	.07	.43 ***	-.27 **	-.20 **	.13	.64 ***	-.28 **	-.26 **	.13	-	
14. MC (accuracy)	118	.20 **	.02	.04	-.19 **	-.16	.38 ***	.16	-.20 **	-.34 ***	.49 ***	.07	-.27 **	-.32 ***	-
15. CO (accuracy)	118	.08	.13	.21 **	.01	-.33 ***	.17	.58 ***	-.03	-.17	.13	.62 ***	-.16	-.19 **	.09

Note. G, grade; NL, number line; AM, area model; Est., estimation; MC, magnitude comparison; CO, comparison to one; PAE, percent absolute error. * $p < .05$, ** $p < .01$, *** $p < .001$

Preregistered analyses

Based on our preregistered analysis plan, we conducted analyses of covariance (ANCOVA) to test each of the four hypotheses. In each ANCOVA, the dependent variable was children's mean score on a measure at immediate posttest. Condition (i.e., number line training, area model training, or cross-word puzzle control) was entered as the independent variable, and children's age and mean score on the same measure at pretest were entered as covariates. In cases where a significant effect of condition was found, pairwise comparisons between conditions based on the estimated marginal (EM) means from each ANCOVA were conducted. The Bonferroni correction was applied to adjust for multiple comparisons.

In all the ANCOVA analyses reported below, pretest scores significantly predicted posttest scores, but age did not. Only the effects of condition are reported below (see Supplementary Materials, Section G for detailed statistics on the effects of pretest scores and age).

Hypothesis 1

We expected children in either of the two training conditions to be more accurate at estimating fractions on the model that they received training on than children in the other two conditions at immediate posttest (Hypothesis 1). As expected, PAE on area model estimation at immediate posttest significantly differed across conditions, after adjusting for age and PAE on area model estimation at pretest, F (2, 114) = 27.25, p < .001, η_p^2= .32 (Figure 4A).

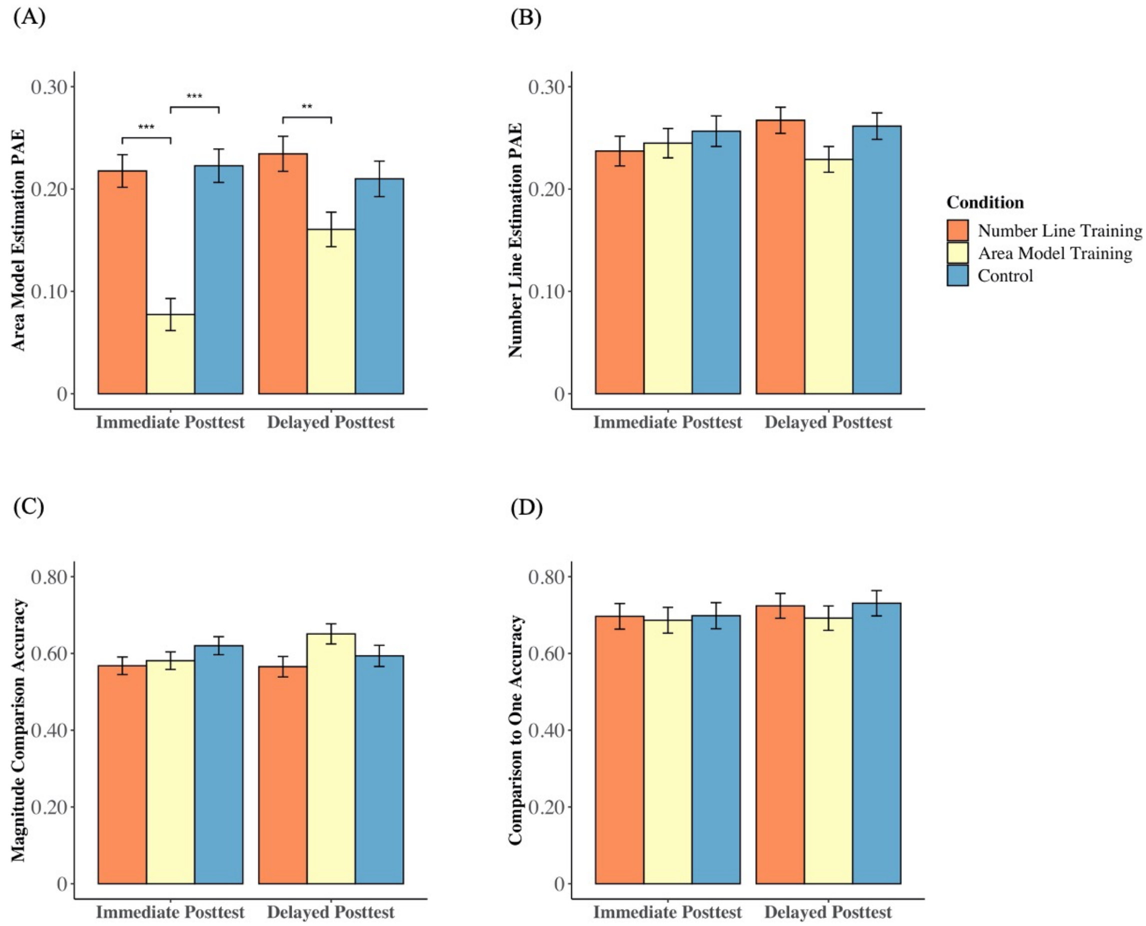

Figure 4. Performance on (A) area model estimation, (B) number line estimation, (C) magnitude comparison, and (D) comparison to one at immediate posttest and delayed posttest. Values are adjusted means, controlling for age and pretest performance on the same measure. Error bars represent one standard error. ** $p < .01$; *** $p < .001$.

Post-hoc analyses revealed that PAE on area model estimation at immediate posttest was significantly smaller among children in the area model training condition (EM mean = 0.08, *SE* = 0.02) than those in the number line training condition (EM mean = 0.22, *SE* = 0.02), *p.adj* < .001, and those in the cross-word puzzle control condition (EM mean = 0.22, *SE* = 0.02), *p.adj* < .001. However, controlling for age and PAE on number line estimation at pretest, PAE on number line estimation at immediate posttest did not differ across conditions, F (2, 108) = 0.44, p = .644, η_p^2= .01 (Figure 4B).

Hypothesis 2

Children in the area model condition yielded more accurate estimates of fractions on area models than children in the other two conditions at immediate posttest, and we expected this effect to hold for both the trained and untrained fractions (Hypothesis 2). As expected, on area model estimation, a significant effect of condition emerged for PAE of both trained and untrained fractions at immediate posttest, adjusting for pretest PAE and age (trained fractions, F (2, 114) = 25.76, p < .001, η_p^2= .31; untrained fractions, F (2, 114) = 21.77, p < .001, η_p^2 = .28). Post-hoc analyses showed that children in the area model condition yielded significantly smaller area model PAEs on both trained (EM mean = 0.09, *SE* = 0.02) and untrained fractions (EM mean = 0.07, *SE* = 0.02) at immediate posttest than children in the number line condition (trained, EM mean = 0.22, *SE* = 0.02; untrained, EM mean = 0.21, *SE* = 0.02) and than children in the cross-word puzzle condition (trained, EM mean = 0.23, *SE* = 0.02; untrained, EM mean = 0.22, *SE* = 0.02), *p.adjs* < .001.

Hypothesis 3

For the magnitude comparison task at immediate posttest, we hypothesized that children in the number line condition would have higher accuracy than children in the area model condition, and children in the area model condition would have higher accuracy than those in the crossword puzzle control condition. However, controlling for accuracy on magnitude comparison at pretest and children's age, no difference in accuracy at immediate posttest was seen across conditions, F (2, 113) = 1.36, p = .261, η_p^2= .02 (Figure 4C).

Hypothesis 4

We hypothesized that the effects of condition on magnitude comparison in Hypothesis 3 would be present among ambiguous items. However, similar to the lack of effect of condition on overall magnitude comparison accuracy, we found no significant effect of condition on magnitude comparison of ambiguous items, F (2, 113) = 0.17, p = .847, $\eta_p^2 < .01$.

In summary, only children who received area model training, but not those who received number line training, estimated fractions more accurately on the models they were trained on in the immediate posttest. However, the area model training did not lead to significantly higher accuracy on the magnitude comparison task.

Exploratory analyses

We conducted exploratory analyses to understand the scope of improvement among children who received area model training and the lack of learning among children who received number line training. Similar to the pre-registered analyses, when comparing performance across conditions on posttests, we conducted ANCOVAs with pretest performance and age as covariates and condition as the independent variable. In cases where there was a significant effect of condition, we conducted post-hoc pairwise comparisons of estimated marginal means with the Bonferroni correction.

Scope of improvement among children who received area model training

We first explored whether area model training led to more accurate estimates of both proper and improper fractions on the area model. Separate ANCOVAs on PAE of area model estimation, adjusting for pretest accuracy and children's age, were conducted for proper and improper fractions. Both ANCOVAs yielded a significant effect of condition (proper fractions, F (2, 114) = 10.08, $p < .001$, $\eta_p^2 = .15$; improper fractions, F (2, 113) = 29.06, $p < .001$, $\eta_p^2 = .34$).

Post-hoc analyses revealed that children in the area model condition had significantly smaller PAEs on both proper (EM mean = 0.05, *SE* = 0.02) and improper fractions (EM mean = 0.10, *SE* = 0.02) at immediate posttest than children in the number line condition (proper fractions, EM mean = 0.17, *SE* = 0.02; improper fractions, EM mean = 0.26, *SE* = 0.02) and children in the cross-word puzzle condition (proper fractions, EM mean = 0.15, *SE* = 0.02; improper fractions, EM mean = 0.31, *SE* = 0.02), *p.adjs* < .01.

Next, we examined whether children's improvement in area model estimation transferred to higher accuracy on the comparison to one task. Adjusting for pretest accuracy of the same task and children's age, there was no significant effect of condition on accuracy at immediate posttest, F (2, 108) = 0.04, p = .964, $\eta_p^2 < .01$ (Figure 4D).

Additionally, we explored whether children's improvement in area model estimation after area model training remained on the delayed posttest. Adjusting for pretest PAE on area model estimation and age, PAE on area model estimation at delayed posttest significantly differed by condition, F (2, 112) = 4.91, p = .009, η_p^2= .08 (Figure 4A). Post-hoc analyses showed that children in the area model condition yielded significantly smaller PAEs (EM mean = 0.16, *SE* = 0.02) than those in the number line condition (EM mean = 0.23, *SE* = 0.02), *p.adj* = .008, but not than those in the cross-word puzzle condition (EM mean = 0.21, *SE* = 0.02), *p.adjs* = .130.

Therefore, after learning how to estimate fractions on area models, children's improvement in area model estimation persisted to some extent on the delayed posttest, which was administered around two weeks after training. Yet the improvement on area model estimation did not transfer to any of the other tasks in the current study (number line estimation, magnitude comparison, and comparison to one).

Lack of improvement among children who received number line training

We first examined whether children in the number line training condition at least improved on estimating the same fractions that they received training on. Results of the ANCOVA on PAE of number line estimation of the trained fractions suggested that they did not; no effect of condition arose, F (2, 108) = 0.53, p = .593, η_p^2 = .01.

Then, we examined whether children in the number line training condition at least improved on estimating proper fractions, on which improvements have been seen in prior studies with similar training (Gunderson et al., 2019; Hamdan & Gunderson, 2017). We did not find evidence for improvement even on proper fractions: results of the ANCOVA on PAE of number line estimation of proper fractions suggested that children in the number line training condition performed similarly at posttest as those in the other two conditions, F (2, 108) = 1.45, p = .240, η_p^2 = .03.

Finally, we explored the possibility that the number line training procedure was more confusing for children than the area model training procedure. To do so, we compared the number of training trials on which children received corrective feedback between the two training conditions. During training, corrective feedback was provided when a child performed a step incorrectly following the experimenter's instructions on that step. Receiving a large number of corrective feedback trials suggests difficulty in understanding the experimenter's instructions. For children in the two training conditions, an analysis of variance (ANOVA) with training condition as the between-subject variable showed that children in the number line condition (M = 2.42, SD = 3.02) received corrective feedback on a marginally greater number of training trials than those in the area model condition (M = 1.46, SD = 1.55), F (1, 79) = 3.27, p = .074, η_p^2 = .04.

Discussion

In prior studies, using the number line, as compared to the area model, in teaching fractions led to better fraction magnitude knowledge among children (Dyson et al., 2018; Fuchs et al., 2016, 2013, 2014; Gunderson et al., 2019; Hamdan & Gunderson, 2017; Moss & Case, 1999; Saxe et al., 2007). The current study extended this work by testing the effects of a brief training on improper fractions with the number line versus the area model. Unexpectedly, fraction training with the area model produced improvements in children's area model estimation of proper and improper fractions but training with the number line did not improve number line estimation. Further, neither training led to improvements in transfer tasks assessing fraction magnitude knowledge. We discuss potential reasons for these unexpected results, implications of these findings for learning improper fractions, and implications for educational practice.

We expected the number line to better support children's learning of improper fractions than the area model. However, contrary to this expectation, compared to children in the number line or the control condition, children who were taught fractions with the area models estimated both proper and improper fractions more accurately on the area model at immediate posttest. The improvement brought by the area model training was not only greater than the number line training or the control activities but also impressive in absolute terms: among children who received the area model training, PAE of area model estimation decreased from 0.25 to 0.08 from pretest to posttest. Two weeks after training, children in the area model condition persistently yielded more accurate estimates on area model estimation than those in the number line condition (but not more accurate than those in the control condition). In contrast, children who received the number line training were no better than children in the area model or the control condition at estimating fractions on the number line after training – not even on estimating the fractions on which they received training. In prior work training proper fractions

(Gunderson et al., 2019), number line training also appeared to transfer to area model estimation, in that both trainings led to equivalent area model estimation performance. This was not the case in our study, providing further evidence that the number line training in the present study was not effective at improving fraction concepts.

One reason for the ineffectiveness of the number line training might be that the training procedures were hard to follow. We developed the procedures for both the number line and the area model training based on prior interventions that enhanced 2nd and 3rd graders' magnitude knowledge of proper fractions (Gunderson et al., 2019; Hamdan & Gunderson, 2017). The training included the same steps (i.e., segment, shade, and label) and lasted a similar amount of time as in prior studies (i.e., 15 minutes). However, because the current training involved both proper and improper fractions, children may have found the training more difficult to understand than in prior studies, which only involved proper fractions. Consistent with this idea, whereas prior number line training with only proper fractions led to more accurate estimates of proper fractions on the number line than the area model training and the control activity (Gunderson et al., 2019; Hamdan & Gunderson, 2017), children who completed number line training in the current study did not improve at estimating proper fractions. Training being hard to follow might be more evident in the number line than in the area model condition because children are more familiar with the area model than the number line for representing fractions (Ni, 2001; X. Zhang, Clements, & Ellerton, 2015). Consistent with this view, during training, children in the number line condition tended to execute procedures incorrectly more often than children in the area model condition. Future studies should explore whether a more extensive number line training (e.g., training children on more fractions and providing more trials of experimenter demonstrations) would yield greater learning outcomes.

The effectiveness of the area model training and the ineffectiveness of the number line training might also be due to the limitations of the stimuli used in training and at posttest. We expected the continuous number line to support learning of improper fractions because number lines naturally extend beyond one. To convert a 0-1 number line to represent improper fractions, one only needs to extend the number line beyond the endpoint of 1. Such a representation is analogous with the mental number line, on which proper fractions, the number one, and improper fractions are ordered in a continuous manner (Dehaene, 1992). In contrast, converting a one-unit area model to incorporate improper fractions requires adding discrete shapes identical to the one-unit model (Behr et al., 1988; Wu, 2009). The area models employed in the current study (i.e., three blank squares; Figure 2B), however, eliminated the potential difficulty of adding identical shapes – because three shapes were already provided, children did not have to actively extend a one-unit area model to a multi-unit area model to represent improper fractions.

Moreover, the perceptual dissimilarity between the number lines used in training and at posttest might be another reason why no improvement was seen on number line estimation. The number line used at posttest had labeled endpoints at 0 and 3 but did not have hash marks or numerical labels at 1 and 2, whereas the number line used in the training did. In the training, we included the hash marks and numerical labels with the goal of helping students connect fraction magnitudes to their existing whole-number knowledge. However, we did not include them in the number line estimation task at posttests to be consistent with prior studies using this task (e.g., Gunderson et al., 2019; Hamdan & Gunderson, 2017; Siegler, Thompson, & Schneider, 2011). Eliminating the hash marks and labels at 1 and 2 may have substantially increased the difficulty of representing fractions on the number line: Most of children's successful strategies of estimating fractions on number lines beyond 1 involve an initial step of segmenting the number line into whole number units, and some children have trouble with this step (D. Zhang et al., 2017). Although the area models in the training also had numerical labels (i.e., 1, 2, and 3), eliminating these labels from the area models at posttest may not impose much difficulty as the three squares had clear boundaries for them to be considered as three whole number units. Future research should explore whether these perceptual features of the number line and the area model influence children's performance of estimating fractions on them.

Implications for Learning Improper Fractions

The effectiveness of number line training for improving proper fraction knowledge in prior studies, and the ineffectiveness of similar number line training for improving improper fraction knowledge in the current study, suggest that improper fraction magnitudes might be harder to learn than proper fraction magnitudes. In two prior studies, a 15-minute number line training effectively improved knowledge of proper fractions among second and third graders who had limited formal instruction on fractions (Gunderson et al., 2019; Hamdan & Gunderson, 2017). Although the current study involved older students (i.e., fourth and fifth graders), a number line training similar to that in the prior studies failed to yield any learning of fraction magnitudes. The lack of learning was not due to ceiling performance at pretest: the PAE of number line estimation among children in the number line condition was 0.28, similar to that on area model estimation, which decreased substantially after area model training.

One reason for improper fractions to be harder than proper fractions might arise from the process of integrating improper fractions with existing whole number knowledge. The integrated theory of numerical development posits that learning new types of numbers involves extending existing number knowledge, and understanding fractions requires integrating fractions with existing whole number knowledge (Siegler et al., 2011). This integration process can be challenging. Much of the difficulty in learning fractions is due to overgeneralization of whole number knowledge (Ni & Zhou, 2005). As compared to proper fractions, learning improper fractions may impose an additional challenge: the magnitudes of improper fractions may fall between any two whole numbers whereas proper fractions are always between 0 and 1. Learning magnitudes of improper fractions may thus require fluency in whole number arithmetic, particularly division.

Another reason that improper fractions may be challenging is that understanding improper fraction magnitudes requires inhibiting the tendency to treat all fractions as smaller than one. The tendency to treat all fractions as smaller than one is seen when children estimate fractions on the number line (Resnick et al., 2016) and when children reason about whether there is a smallest or biggest fraction (Stafylidou & Vosniadou, 2004). In the current study, most children did not seem to explicitly hold such a belief given that their performance on the comparison to one task was well above chance. However, even college students have this tendency to treat all fractions as being smaller than one on tasks involving automatic processing of fraction magnitudes (Kallai & Tzelgov, 2009). It is likely that for children to learn and process the magnitudes of improper fractions, they need to inhibit an implicit tendency to treat all fractions as smaller than one.

Although the present training on improper fractions was closely matched to prior ones on proper fractions (Gunderson et al., 2019; Hamdan & Gunderson, 2017), there are still important differences between the trainings. For example, the prior and current trainings both lasted about 15 minutes, but the prior training only involved proper fractions whereas the current involved both proper and improper fractions. This and other differences preclude direct comparison between the prior and current findings to reveal whether learning improper fractions is more challenging than learning proper fractions. To better understand whether and, if so, why learning improper fractions is harder than proper fractions, future studies need to closely match the proper and improper fraction trainings and include measures of factors that might contribute to differences in learning proper and improper fractions, such as whole number arithmetic fluency and inhibition.

Educational implications

Our findings indicate that children can learn to represent improper fractions on the area model with brief training, and this learning can last at least two weeks after training. Representing improper fractions with the area model can be challenging (Behr et al., 1988; Wu, 2009). Nevertheless, children in the current study quickly learned how to do so after a 15-minute training, during which they saw the experimenter show three fractions on the area model and practiced and received feedback on eight fractions.

However, it is less clear whether progress in estimating fractions on the area model promoted a comprehensive understanding of fraction magnitudes. Despite the considerable improvement in area model estimation, children who received the area model training did no better than other children who received the number line training or control activities on any other fraction tasks in the current study (i.e., number line estimation, fraction magnitude comparison, and comparison to one). These findings are in sharp contrast with prior interventions, in which the number line training not only led to more accurate number line estimation than the other conditions, but also transferred to fraction magnitude comparison (Gunderson et al., 2019; Hamdan & Gunderson, 2017) and led to similar improvement in area model estimation as the area model training (Gunderson et al., 2019). However, they are in line with other findings that many children who could successfully express fractions on the area model did not develop conceptual understanding of fractions in non-area-model contexts (X. Zhang et al., 2015). It is possible that, given appropriate dosage and scaffolding, estimating fractions on the number line would better facilitate the development of a comprehensive understanding of fraction magnitudes than on the area model. It is also possible that children need more scaffolding to transfer the improvement in the area model estimation task to solving other fraction tasks, for example, by triggering children's awareness of the relations between the tasks (Cooper & Sweller, 1987). These possibilities should be examined in future research.

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Learning Improper Fractions with the Number Line and the Area Model

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Supplementary Materials

Section A. Script of the number line training.

Section B. Script of the area model training.

Section C. Script of the cross-word puzzle control condition.

Section D. Coding scheme of the number line estimation and the area model estimation tasks.

Section E. Table E1. Fraction pairs used in the magnitude comparison task.

Section F. Descriptive statistics of children's performance on different types of problems on each measure at pretest and posttests.

Section G. Detailed results of ANCOVA analyses.

Section A. Script of the number line training.

Experimenter Shows Number Line

SHOW: The whole-number segmented number line to the child.

Say only the first time: Today we are going to be playing a game where we are going to learn about fractions. This is a number line (*point to number line*). This number line goes from 0 at one end to 3 at one end. We can use this number line to show fractions.

We're going to use this pen and this iPad to show the fractions.

Point to Fraction

Say: This is a fraction It has a number on top and a number on the bottom. We can call this fraction x over y, because the x is on top and the y is on the bottom (*say while pointing to appropriate part of the fraction*). We can also call it "x/y".

Segment

Say: Now I'm going to show x/y on the number line. First, we need to look at the bottom number (*point to denominator*). The number on the bottom tells us how many equal parts we need to make on each unit on the number line.

Say: We are going to divide each unit on the number line into y equal parts (*segment from left to right)*.

Shade

Say: Now, we need to look at the top number (*say while pointing to the numerator*). This number tells us how many equal parts we need to shade, starting from zero. So, we need to shade x equal part(s), like this (*shade appropriate part of number line, and afterwards, count part(s) shaded, while pointing to parts shaded*).

Label

Say: If someone asks where x/y is on the number line, we should say here (*point to the right endpoint of the shaded part*).We can show it by drawing a hash-mark at the end of the part we shaded, like this (*draw hash-mark*). We should write x/y above the hash-mark, like this (*write fraction above has- mark location*). Remember, we can call this x over y.

Student Practice

Move to Unsegmented Number Line

Segment

Say: Now, you're going to show x/y on this number line. First, we need to know how many equal parts we should make in each unit on the number line. Where do we look to figure this out? (Let child answer).

> ***If correct ("number on bottom" or indicating that they mean the denominator):** That's right!
>
> ***If no response or wrong response:** We need to look at the number on the bottom (*say while pointing to denominator*) to know how many equal parts we need to make in each unit on the number line.

Can you please make the right number of equal parts in each unit?

> ***Answers correctly:** That's right! It looks like this (*show appropriate SEGMENT remed.*).
>
> ***Answers incorrectly:** Actually, to show x/y you make y equal parts in each unit on the number line. Remember, you should look at the number on the bottom (*say while pointing to denominator*) to tell how many equal parts you need to make in each unit on the number line. See, like this (*show appropriate SEGMENT remed.).* (*Point to space between hash marks while counting. Count in each unit*) One, two,…, one, two, See how each unit on this number line has y equal parts? (*Present child with new worksheet for that fraction SEGMENT task*) Can you make this number line, look like this? (*Say while pointing to SEGMENT remed.*) (*Keep remediation page out while child completes task)* Great! (*Continue even if child is still unable to replicate correct answer*).

Shade

Say: Now we need to know how many equal parts to shade starting from zero. Where do we look to know how many equal parts (let child answer)? (*Say while pointing to numerator*).

> ***If correct ("number on top" or indicating that they mean the numerator):** That's right!
>
> ***If no response or wrong response:** We need to look at the number on the top (*say while pointing to numerator*) to know how many equal parts we need to shade starting from zero.

Can you please shade the right number of parts?

> ***Answers correctly:** That's right! It looks like this (*show child appropriate SHADE remed.).*
>
> ***Answers incorrectly:** Actually, to show x/y, you need to shade x equal parts on the number line, because the x is the number on top (*say while pointing to numerator*) and it tells us how many equal parts to shade, starting from zero. See, like this (*show child*

appropriate SHADE remed and count shaded parts out loud). (*Present child with new worksheet for that fraction SHADE task*) Can you make this number line look like this (*say while pointing to SHADE remed.)*? (*Keep remediation page out while child completes task)* Great! (*Continue even if child is still unable to replicate correct answer*)

<u>Label</u>

Say: Now, can you show where x/y goes on the number line?

***Answers correctly:** That's right! It looks like this (*show child appropriate PLACE remed.)*.

***If draws hash mark without labeling:** Can you put the fraction where it goes?

***If labels without drawing hash mark:** Can you put the hash mark where it goes?

***Answers incorrectly:** Actually, you show where x/y is by drawing a hash-mark at the end of the part you shaded and writing the fraction above the hash-mark. See, like this (*show appropriate PLACE remed.*) (*Present child with new worksheet for that fraction PLACE task*) Now, can you make this number line, look like this? (*Show PLACE remed.*) (*Keep Remediation page out while child completes task)* Great! (*Continue even if child is still unable to replicate correct answer*).

<u>*-Repeat -*</u>

Section B. Script of the area model training.

Experimenter Shows Squares

SHOW: The unsegmented squares to the child.

Say only the first time: Today we are going to be playing a game where we are going to learn about fractions. These are squares (*point to squares*). There are three squares. We can use these squares to show fractions.

We're going to use this pen and this iPad to show the fractions.

Point to Fraction

This is a fraction. It has a number on top and a number on the bottom. We can call this fraction x over y, because the x is on top and the y is on the bottom (*say while pointing to appropriate part of the fraction*). We can also call it "x/y".

<u>Segment</u>

Say: Now I'm going to show x/y on the squares. First, we need to look at the bottom number (*point to denominator*). The number on the bottom tells us how many equal parts we need to make in each square.

Say: We are going to divide each square into y equal parts (*segment from left to right).*

<u>Shade</u>

Say: Now, we need to look at the top number (*say while pointing to the numerator*). This number tells us how many equal parts we need to shade. So, we need to shade x equal part(s), like this (*shade appropriate part of squares, starting from left, and afterwards, count part(s) shaded, while pointing to parts shaded*).

<u>Label</u>

Say: If someone asks where x/y is on the squares, we should say here *(point to the middle of the shaded part*). We can show it by drawing a circle around the part we shaded, like this (*draw circle around shaded part of squares).* We should write x/y above the circle, like this. Remember, we can call this x over y.

Student Practice

Move to Unsegmented Squares

Segment

Say: Now, you're going to show x/y on these squares. First, we need to know how many equal parts we should make in each square. Where do we look to figure this out? (Let child answer).

***If correct ("number on bottom" or indicating that they mean the denominator):** That's right!

***If no response or wrong response:** We need to look at the number on the bottom (*say while pointing to denominator*) to know how many equal parts we need to make in each square.

Can you please make the right number of equal parts in each square?

***Answers correctly:** That's right! It looks like this *(show appropriate SEGMENT remed.*).

***Answers incorrectly:** Actually, to show x/y you make y equal parts in each square. Remember, you should look at the number on the bottom (*say while pointing to denominator*) to tell how many equal parts you need to make in each square. See, like this (*show appropriate SEGMENT remed.*). (*Point to space between lines while counting. Count in each square*) One, two,…, one, two, See how each of these squares has y equal parts? (*Present child with new worksheet for that fraction SEGMENT task*) Can you make these squares, look like this? (*Say while pointing to SEGMENT remed.*) (*Keep remediation page out while child completes task)* Great! (*Continue even if child is still unable to replicate correct answer*).

Shade

Say: Now we need to know how many equal parts to shade. Where do we look to know how many equal parts (let child answer)? (*Say while pointing to numerator*).

***If correct ("number on top" or indicating that they mean the numerator):** That's right!

***If no response or wrong response:** We need to look at the number on the top (*say while pointing to numerator*) to know how many equal parts we need to shade.

Can you please shade the right number of parts?

***Answers correctly:** That's right! It looks like this (*show child appropriate SHADE remed.).*

***Answers incorrectly:** Actually, to show x/y, you need to shade x equal parts on the squares, because the x is the number on top (*say while pointing to numerator*) and it tells us how many equal parts to shade. See, like this (show child appropriate SHADE remed and count shaded parts out loud). (*Present child with new worksheet for that fraction SHADE task*) Can you make these squares look like this (say while pointing to SHADE remed.)? (*Keep remediation page out while child completes task)* Great! (*Continue even if child is still unable to replicate correct answer)*

Label

Say: Now, can you show where x/y goes on the squares?

***Answers correctly:** That's right! It looks like this (*show child appropriate PLACE remed.*).

***If circles without labeling:** Can you put the fraction where it goes?

***If labels without drawing circle:** Can you put the circle where it goes?

***Answers incorrectly:** Actually, you show where x/y is by drawing a circle around the part you shaded and writing the fraction above the circle. See, like this (*show appropriate PLACE remed.*) (*Present child with new worksheet for that fraction PLACE task*) Now, can you make these squares look like this? (*Show PLACE remed*). (*Keep Remediation page out while child completes task)* Great! (*Continue even if child is still unable to replicate correct answer*).

-Repeat -

Section C. Script of the cross-word puzzle control condition.

Please try to be as consistent as possible when testing each child.

Say: Now we are going to play a game where we are going to fill in crosswords. Have you ever done a crossword before? (wait for child to answer)

> **If they have:** Great! We are going to use these clues (*point to clue binder)*to solve the puzzles on this page. Let's start with the first one here. *(Read the prompt and have them answer)*
>
> **If they have not:** That's okay! You can use the clues from this page to decide what word will go in the empty spaces. Let's try the first one together. *(Read the prompt and have them answer)*During the puzzle please continue to interact with the child. This includes reading clues to them, suggesting parts of the puzzle to work on, and offering to provide "hints". Move from one crossword to another as the child finishes, and allow them to "pinch in" and "pinch out" of portions of the puzzle. Discontinue crosswords at 18 minutes, regardless of whether the child finishes the puzzle.
>
> If child wants to do puzzle on their own: Allow them to work on puzzle, but continue to engage by helping them check the puzzle (with key) and giving feedback ("I think that looks right, now how about this one?").
>
> If child is struggling/frustrated with puzzle: Offer child the word bank to use, and help them to focus on one item at a time ("Hmmm, why don't we try this one. (*Read item to child)* what do you think that might be?")

Section D. Coding scheme of the number line estimation and the area model estimation tasks.

For each trial on the number line estimation task, trained research assistants used Adobe Photoshop 2017 (Faulkner & Chavez, 2017) to measure in millimeters where participants placed a hashmark on the number line. Figure 1 illustrates common examples of coding instances, with dashed blue lines indicating where the research assistants would code the hashmark placement. In cases where more than one hashmark was drawn, research assistants measured the hashmark that the participant indicated as the answer by circling it (e.g., Figure 1A). In cases where more than one hashmark was drawn but no marks were circled, the trial was marked as "un-codable" (e.g., Figure 1B). In cases where hashmarks were drawn to the left or right of the line, the trail was marked as "un-codable" (e.g., Figure 1C). In cases where the only hashmark was especially thick (e.g., Figure 1D) or a hashmark crossed the number line multiple times (e.g., Figure 1E), the midpoint between the farthest left and farthest right portions of the hashmark was coded.

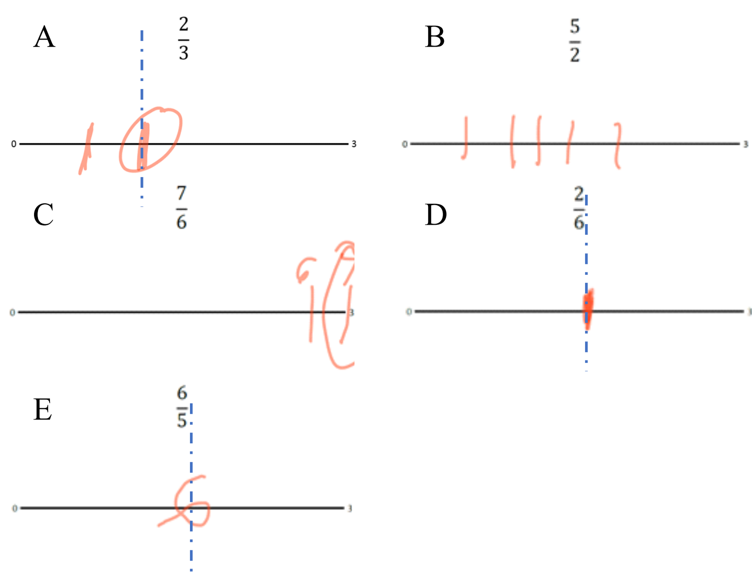

Figure 1. Examples of coding instances of the number line estimation task.

For each trial on the area model estimation task, trained research assistants measured the total area of all squares that the participant shaded in pixels using Adobe Photoshop 2017 (Faulkner & Chavez, 2017). Figure 2 illustrates common examples of coding instances. Blue lines indicate how the research assistants would code the instance. When participants segmented the squares and shaded in one or more segments, the area of the entire segments in which the participants shaded were coded, and the segments in which the participants did not shade were not as coded (e.g., Figure 2A). When participants did not segment the squares, the research assistants traced around the shaded area in each square and coded the total shaded area (e.g., Figure 2B). When participants shaded beyond the square border, the shading outside of the square border was coded (e.g., Figures 2C and 2D).

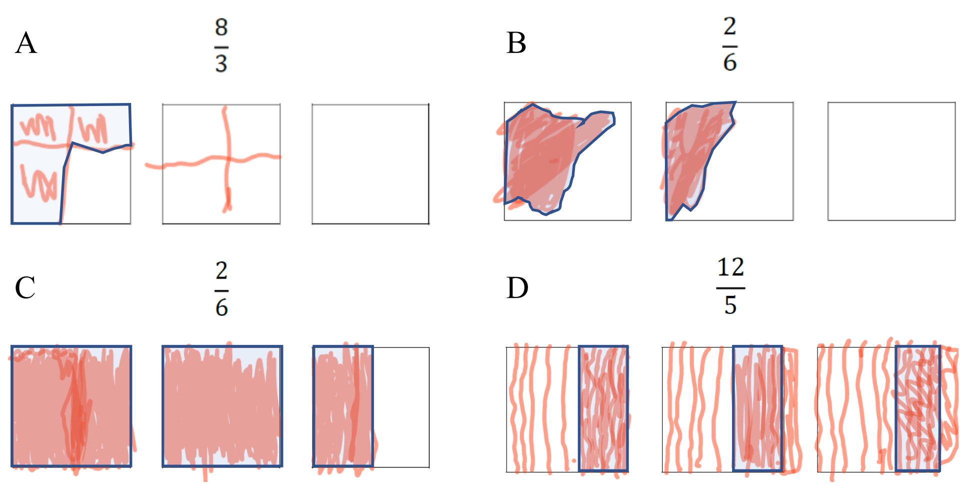

Figure 2. Examples of coding instances of the area model estimation task

Section E. Table E1. Fraction pairs used in the magnitude comparison task.

Table E1

Fraction Pairs in Magnitude Comparison

Item	Whole number strategy category	Fraction types	In larger fraction, the numerator is ...	In larger fraction, the denominator is ...	Pretest accuracy	Immediate posttest accuracy	Delayed posttest accuracy
6/2 vs. 7/6	Inconsistent	Improper vs. Improper	Smaller	Smaller	0.28	0.29	0.31
9/4 vs. 6/2	Inconsistent	Improper vs. Improper	Smaller	Smaller	0.31	0.24	0.28
3/2 vs. 5/6	Inconsistent	Proper vs. Improper	Smaller	Smaller	0.26	0.30	0.30
5/6 vs. 4/3	Inconsistent	Proper vs. Improper	Smaller	Smaller	0.37	0.29	0.27
1/2 vs. 2/5	Inconsistent	Proper vs. Proper	Smaller	Smaller	0.37	0.32	0.33
2/6 vs. 1/2	Inconsistent	Proper vs. Proper	Smaller	Smaller	0.35	0.34	0.39
3/2 vs. 8/3	Consistent	Improper vs. Improper	Larger	Larger	0.74	0.76	0.76
9/5 vs. 4/3	Consistent	Improper vs. Improper	Larger	Larger	0.74	0.74	0.77
4/3 vs. 1/2	Consistent	Proper vs. Improper	Larger	Larger	0.78	0.78	0.77
4/5 vs. 9/6	Consistent	Proper vs. Improper	Larger	Larger	0.73	0.71	0.76
3/5 vs. 5/6	Consistent	Proper vs. Proper	Larger	Larger	0.76	0.75	0.76
5/6 vs. 2/4	Consistent	Proper vs. Proper	Larger	Larger	0.79	0.77	0.76
10/5 vs. 9/6	Ambiguous	Improper vs. Improper	Larger	Smaller	0.63	0.71	0.69
14/6 vs. 15/5	Ambiguous	Improper vs. Improper	Larger	Smaller	0.71	0.67	0.69
7/6 vs. 8/5	Ambiguous	Improper vs. Improper	Larger	Smaller	0.64	0.63	0.64
9/5 vs. 7/6	Ambiguous	Improper vs. Improper	Larger	Smaller	0.60	0.62	0.70
4/5 vs. 6/3	Ambiguous	Proper vs. Improper	Larger	Smaller	0.61	0.70	0.67
5/4 vs. 3/6	Ambiguous	Proper vs. Improper	Larger	Smaller	0.57	0.58	0.60
5/6 vs. 7/4	Ambiguous	Proper vs. Improper	Larger	Smaller	0.71	0.62	0.68
8/4 vs. 5/6	Ambiguous	Proper vs. Improper	Larger	Smaller	0.69	0.70	0.68
1/5 vs. 2/4	Ambiguous	Proper vs. Proper	Larger	Smaller	0.61	0.61	0.62
2/3 vs. 1/4	Ambiguous	Proper vs. Proper	Larger	Smaller	0.70	0.69	0.69
2/6 vs. 3/5	Ambiguous	Proper vs. Proper	Larger	Smaller	0.61	0.59	0.59
4/5 vs. 1/6	Ambiguous	Proper vs. Proper	Larger	Smaller	0.67	0.72	0.72

Section F. Descriptive statistics of children's performance on different types of problems on each measure at pretest and posttests.

Table F1

Means (and Standard Deviations) of Children's Percent Absolute Error of Trained vs. Untrained Fractions on the Number Line Estimation Task.

	NL Training (n = 40)	AM Training (n = 41)	CW Control (n = 38)
Pretest			
Trained	0.28 (0.11)	0.27 (0.10)	0.26 (0.12)
Untrained	0.27 (0.09)	0.27 (0.12)	0.28 (0.12)
Immediate posttest			
Trained	0.26 (0.10)	0.24 (0.11)	0.25 (0.09)
Untrained	0.23 (0.11)	0.25 (0.11)	0.25 (0.12)
Delayed posttest			
Trained	0.26 (0.09)	0.24 (0.10)	0.26 (0.10)
Untrained	0.27 (0.09)	0.22 (0.10)	0.26 (0.11)

Table F2

Means (and Standard Deviations) of Children's Percent Absolute Error of Proper vs. Improper Fractions on the Number Line Estimation Task.

	NL Training (n = 40)	AM Training (n = 41)	CW Control (n = 38)
Pretest			
Proper	0.30 (0.14)	0.28 (0.16)	0.30 (0.20)
Improper	0.25 (0.09)	0.26 (0.11)	0.24 (0.10)
Immediate posttest			
Proper	0.24 (0.15)	0.22 (0.15)	0.26 (0.16)
Improper	0.26 (0.10)	0.27 (0.11)	0.24 (0.08)
Delayed posttest			

Proper	0.28 (0.14)	0.22 (0.16)	0.27 (0.16)
Improper	0.25 (0.07)	0.24 (0.08)	0.25 (0.09)

Table F3

Means (and Standard Deviations) of Children's Percent Absolute Error of Trained vs. Untrained Fractions on the Area Model Estimation Task.

	NL Training (n = 40)	AM Training (n = 41)	CW Control (n = 38)
Pretest			
Trained	0.28 (0.09)	0.26 (0.11)	0.28 (0.13)
Untrained	0.23 (0.11)	0.24 (0.13)	0.24 (0.15)
Immediate posttest			
Trained	0.23 (0.11)	0.08 (0.09)	0.23 (0.13)
Untrained	0.21 (0.14)	0.07 (0.09)	0.22 (0.14)
Delayed posttest			
Trained	0.25 (0.12)	0.18 (0.14)	0.22 (0.13)
Untrained	0.21 (0.12)	0.14 (0.12)	0.20 (0.13)

Table F4

Means (and Standard Deviations) of Children's Percent Absolute Error of Proper vs. Improper Fractions on the Area Model Estimation Task.

	NL Training (n = 40)	AM Training (n = 41)	CW Control (n = 38)
Pretest			
Proper	0.15 (0.17)	0.18 (0.19)	0.22 (0.22)
Improper	0.37 (0.12)	0.33 (0.15)	0.30 (0.14)
Immediate posttest			
Proper	0.16 (0.17)	0.05 (0.06)	0.17 (0.18)
Improper	0.28 (0.16)	0.10 (0.13)	0.28 (0.16)
Delayed posttest			
Proper	0.16 (0.17)	0.10 (0.12)	0.15 (0.17)
Improper	0.31 (0.14)	0.22 (0.17)	0.28 (0.15)

Table F5

Means (and Standard Deviations) of Children's Accuracy of Different Types of Magnitude Comparison Problems.

	Proper vs. Proper	Improper vs. Improper	Proper vs. Improper
Pretest			
Immediate posttest	0.60 (0.24)	0.58 (0.21)	0.59 (0.21)
Delayed posttest	0.61 (0.23)	0.60 (0.22)	0.60 (0.22)

Note. At all testing time points, accuracy of all types of problems were significantly above chance (0.50), *p*s < .001. Accuracy on the three types of problems did not differ at any testing time point, *p*s > .533

Table F6

Means (and Standard Deviations) of Children's Accuracy of Different Types of Magnitude Comparison Problems.

	Whole Number Strategy Consistent	Whole Number Strategy Inconsistent	Whole Number Strategy Ambiguous
Pretest			
Immediate posttest	0.75 (0.33)	0.29 (0.35)	0.66 (0.30)
Delayed posttest	0.77 (0.32)	0.31 (0.35)	0.67 (0.30)

Note. At all testing time points, accuracy of all types of problems were significantly different from chance (0.50), *p*s < .001.

Table F7

Means (and Standard Deviations) of Children's Accuracy of Proper Fractions, Improper Fractions, and Fractions Equal to One on the Comparison to One Task.

	Proper Fractions	Improper Fractions	Fractions Equal to One
Pretest			
Immediate posttest	0.55 (0.39)	0.78 (0.28)	0.78 (0.33)
Delayed posttest	0.58 (0.38)	0.80 (0.26)	0.78 (0.34)

Section G. Detailed results of ANCOVA analyses.

Dependent variable (at immediate posttest if not specified)	Effect of Age	Effect of Corresponding Pretest Performance	Effect of Condition
PAE on area model estimation (all items)	F (1, 114) = 0.46, p = .499, η_p^2< .01	F (1, 114) = 28.78, p < .001, η_p^2= .20	F (2, 114) = 27.25, p < .001, η_p^2= .32
PAE on trained fractions on area model estimation	F (1, 114) = 1.93, p = .168, η_p^2=.02	F (1, 114) = 31.86, p < .001, η_p^2= .22	F (2, 114) = 25.76, p < .001, η_p^2= .31
PAE on untrained fractions on area model estimation	F (1, 114) < 0.01, p = .976, η_p^2< .01	F (1, 114) = 23.03, p < .001, η_p^2= .17	F (2, 114) = 21.77, p < .001, η_p^2 = .28
PAE on proper fractions on area model estimation	F (1, 114) = 0.04, p = .846, η_p^2 < .001	F (1, 114) = 30.37, p < .001, η_p^2 = .21	F (2, 114) = 10.08, p < .001, η_p^2 = .15
PAE on improper fractions on area model estimation	F (1, 113) = 1.31, p = .255, η_p^2 = .01	F (1, 113) = 47.60, p < .001, η_p^2 = .30	F (2, 113) = 29.06, p < .001, η_p^2 = .34
PAE on area model estimation at delayed posttest	F (1, 112) < 0.01, p = .942, η_p^2 < .01	F (1, 112) = 27.36, p < .001, η_p^2= .20	F (2, 112) = 4.91, p = .009, η_p^2= .08
PAE on number line estimation (all items)	F (1, 108) = 0.22, p = .641, η_p^2< .01	F (1, 108) = 27.68, p < .001, η_p^2= .20	F (2, 108) = 0.44, p = .644, η_p^2= .01
PAE on trained fractions on number line estimation	F (1, 108) = 0.15, p = .702, η_p^2 < .01.	F (1, 108) = 17.75, p < .001, η_p^2 = .14.	F (2, 108) = 0.53, p = .593, η_p^2 = .01.
PAE on untrained fractions on number line estimation	F (1, 108) = 0.28, p = .600, η_p^2 < .01.	F (1, 108) = 22.76, p < .001, η_p^2 = .17.	F (2, 108) = 1.02, p = .363, η_p^2 = .02.
PAE on proper fractions on number line estimation	F (1, 108) = 0.11, p = .740, η_p^2 < .01.	F (1, 108) = 27.19, p < .001, η_p^2 = .20.	F (2, 108) = 1.45, p = .240, $\boldsymbol{\eta_p^2}$ = .03.
PAE on improper fractions on number line estimation	F (1, 108) = 0.14, p = .709, η_p^2 < .01.	F (1, 108) = 17.62, p < .001, η_p^2 = .14.	F (2, 108) = 0.42, p = .655, $\boldsymbol{\eta_p^2}$ = .01.
Accuracy on magnitude comparison (all items)	F (1, 113) = 0.23, p = .632, η_p^2 < .01	F (1, 113) = 50.04, p < .001, η_p^2= .31	F (2, 113) = 1.36, p = .261, η_p^2= .02

Accuracy on magnitude comparison of ambiguous items	*F* (1, 113) = 0.04, *p* = .838, η_p^2 < .01	*F* (1, 113) = 39.76, *p* < .001, η_p^2= .26	*F* (2, 113) = 0.17, *p* = .847, η_p^2< .01.
Accuracy on comparison to one (all items)	*F* (1, 108) = 0.52, *p* = .964, η_p^2 < .01	*F* (2, 108) = 48.77, *p* = .964, η_p^2 = .31	*F* (2, 108) = 0.04, *p* = .964, η_p^2 < .01